PI APPROXIMATION DAY MATHQUIZ
First Round *on* 04/07/2024

Total marks: 35 (5 in Part A and 30 in Part B)

Instructions:

<u>Part A</u>

(i) This part contains knowledge type problems related to mathematics.

(ii) There are 3 problems. Problem A1 carries 1 mark. Problems A2 and A3 carry 2 marks each. Total marks in Part A is 5.

(iii) The answers of Part A problems are of one word or a short answer of 5-6 words, if reason is asked.

(iv) Phonetically correct spellings are accepted if the answer is a person's name. However, in case of a tie, correct spellings will be preferred.

(v) There is only one starred problem in Part A which is A2. Starred problems will be used to break ties.

<u>Part B</u>

(i) This part contains problems from Algebra, Geometry, Number Theory and Combinatorics.

(ii) There are 10 problems. Problems B1 to B4 carry 2 marks each. Problems B5 to B8 carry 3 marks each. Problems B9 and B10 carry 5 marks each. Total marks in Part B is 30.

(iii) The answers of Part B problems are integers from 00-99.

(iv) If the answer is a one-digit integer, you are advised to put a 0 before it. For example, if the answer is 7, you are advised to write 07 and if the answer is 0, you are advised to write 00.

(v) A problem Bn is starred if n is an even number, i.e., problems B2, B4, B6, B8 and B10 are starred. Starred problems will be used to break ties.

The test starts at 8:00 pm and ends at 9:00 pm sharp. Submit the answers on or before 8:58 to avoid any internet issues at the final moment, because there is no option of automatic submission. You are advised to use 7-8 minutes for Part A problems and the remaining time for Part B problems. Do not panic and enjoy the problems!

All the best!

PROBLEMS

Part A

A1: Renowned Indian statistician X' was the architect of a five-year plan introduced by the Indian government. 'X' played a crucial role in formulating the plan and directing resources toward enhancing productivity and investment sectors for long-term optimization. The concept behind the plan was originally proposed by Soviet economist G.A. Feldman in 1928, whose Neo-Marxist model served as an inspiration for 'X'. The plan was structured around the Harrod-Domar Model, aiming to boost the country's agricultural sector and address the challenges posed by the partition and World War II. The plan sought to rejuvenate the nation's economy post-independence, with Rs. 48 billion allocated for its implementation. During this period, India experienced economic stability, supplemented by foreign loans to support the plan's execution. Although the projected growth rate for this plan was 4.5%, the foreign exchange crisis, growing population, and rising prices led to the plan not achieving its goal and landing at a GDP of 4.3%. Identify 'X'.

A2:⋆ In 1865, English author, poet, mathematician and photographer, 'X' published a beloved children's novel, 'Y', which follows the fantastical adventures of a young girl who falls down a rabbit hole into a strange and wonderful world. It is seen as an example of the literary nonsense genre. The book has never been out of print and has been translated into over 170 languages and has become a cultural icon, inspiring countless adaptations, parodies, and references in popular culture. Below is an AI-generated image, generated by Meta AI, depicting the main character in 'Y'.

Within the academic discipline of mathematics, 'X' worked primarily in the fields of geometry, linear and matrix algebra, mathematical logic, and recreational mathematics, producing nearly a dozen books under his real name. His talent as a mathematician won him the Christ Church Mathematical Lectureship. Despite early unhappiness, 'X' remained at Christ Church, University of Oxford, in various capacities, until his death, including that of Sub-Librarian of the Christ Church library. Identify 'X' (writing his pen name is also fine) and 'Y'.

A3: 'X' is an ancient mathematical work comprising of 13 books, authored by the Greek mathematician 'Y' around 300 BC. This work contains a systematic collection of definitions, postulates, theorems, constructions, and proofs. The books cover plane and solid geometry, elementary number theory, and incommensurable lines. 'X' is the oldest extant large-scale deductive treatment of mathematics. It has proven instrumental in the development of logic and modern science, and its logical rigor was not surpassed until the 19th century. 'X' was also one of the very earliest mathematical works to be printed after the invention of the printing press and has been estimated to be second only to the Bible in the number of editions published since the first printing in 1482, the number reaching well over one thousand. Identify 'X' and 'Y'.

<u>Part B</u>

B1: Let

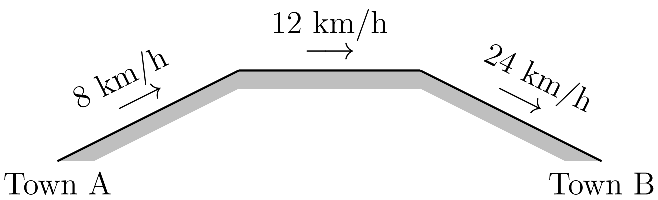
$$N = \sqrt{2024 + 2023\sqrt{2024 + 2023\sqrt{2024 + 2023\sqrt{\dots}}}},$$

then it turns out that N is a positive integer. What is the sum of the digits of N?

B2:$\star$ Find the number of pairs of positive integers (x, y) which satisfy the equation

$$20x + 24y = 2024.$$

B3: The road between Town A and Town B consists of three segments: an uphill section, a horizontal section, and a downhill section, each of equal length. A cyclist travels from Town A to Town B at uniform speeds of 8 km/h uphill, 12 km/h on the horizontal section, and 24 km/h downhill. What is the cyclist's average speed for the entire journey?



B4:$\star$ Let $a_1 = 2025$, and for $n \geq 2$,

$$a_1 + a_2 + \cdots + a_n = n^2 a_n.$$

What is the value of $2024a_{2024}$?

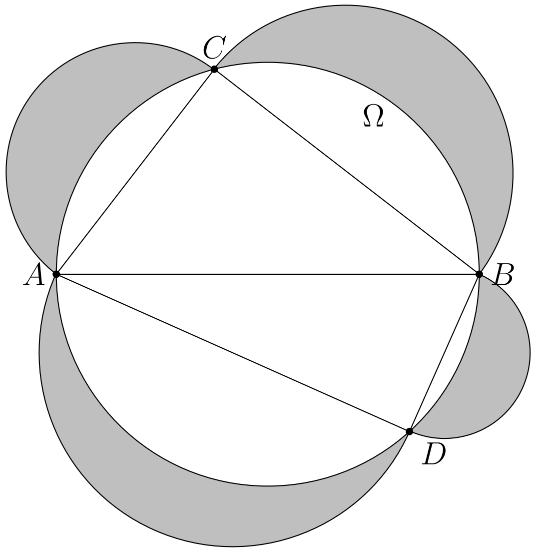
B5: A collection of subsets of a non-empty set S is said to have the *intersection property* if for any two subsets A and B in the collection, $A \cap B$ is non-empty. Let $S = \{1, 2, \dots, 2024\}$ be the set of the first 2024 positive integers. Let $\mathcal{C}$ be the collection of subsets of S such that $\mathcal{C}$ has the intersection property. It turns out that the maximum number of elements that $\mathcal{C}$ can have is 2^n, where n is a positive integer. What is the sum of the digits of n?

B6:$\star$ Lata writes down the following 2024-digit integer on the board:

$$20242024\dots2024.$$

Asha removes the first m digits and the last n digits of the above number so that the sum of the remaining digits is 2026. Find the sum of the digits of $m + n$.

B7: Given a right angled triangle ABC, right angled at C. Let Ω be the circumcircle of ΔABC. Let D be a point on Ω, **not** on the same semicircular arc (made by AB) as C. Semicircles are drawn on sides AC, BC, BD and AD, as shown in the figure. If the area of ΔABC is 24 square units and the area of the shaded region is 42 square units, then what is the area of ΔABD (in square units)?



B8:$\star$ Let S be the set of positive integers n for which $n^2 + 11n + 24$ is a perfect square. How many elements does S have?

B9: Let $P(x)$ be a polynomial of degree 2022 such that $P(n) = \dfrac{n}{1+n}$ for $n = 0, 1, 2, \ldots, 2022$. Find the value of $P(2024)$.

Hint: Try to construct a polynomial that is 0 on all of $0, 1, 2, \ldots, 2022$.

B10:$\star$ A collection of circles placed on the plane is called *beautifully placed* if any pair of circles intersects at two points and no point lies on three circles. Suppose a collection of 2024 circles is beautifully placed, and it divides the plane into N regions. Find the number formed by the last two digits of N.

SOLUTIONS

Part A

A1: Prasanta Chandra Mahalanobis.
A2: Charles Dodgson (Lewis Carroll), Alice's Adventures in Wonderland.
A3: The Elements, Euclid.

Part B

B1: 08
Observe that

$$N^2 = 2024 + 2023N.$$

This reduces the given equation to the equation

$$N^2 - 2023N - 2024 = 0,$$

or,

$$(N - 2024)(N + 1) = 0.$$

Therefore, $N = 2024$. (Note that $N = -1$ is not possible because from the given equation, it is clear that N should be positive.) Thus, the sum of the digits of N is $2 + 0 + 2 + 4 = \boxed{8}$. ■

B2: 17
Note that

$$20x + 24y = 2024 \iff 5x + 6y = 506.$$

The positive integer solutions of the given equation are

$$(x, y) = (4, 81), (10, 76), (16, 71), \ldots, (100, 1).$$

Now we need to find how many solutions are there. Consider the possible values of x, which are $4, 10, 16, \ldots, 100$. This is an arithmetic progression with first term 4 and common difference 6. If there are n terms, then we have $100 = 4 + (n - 1)6$, which gives $n = 17$. Therefore, there are $\boxed{17}$ pairs of positive integers (x, y) which satisfy the given equation. ■

B3: 12
Since each segment has equal length, we take the length of each segment to be d km. Then the total distance is $3d$ km. The formula for time taken is distance travelled divided by the speed. Thus, for the uphill section, the time taken is $\frac{d}{8}$, for the horizontal section it is $\frac{d}{12}$ and for the downhill section it is $\frac{d}{24}$. So the total time taken is $\frac{d}{8} + \frac{d}{12} + \frac{d}{24} = \frac{6d}{24} = \frac{d}{4}$ hours. Thus, the required average speed for the entire journey is total distance travelled divided by the total time taken, which is $\frac{3d}{\frac{d}{4}} = \boxed{12}$ km/h. ■

B4: 02
Here,

$$a_n = (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = n^2 a_n - (n-1)^2 a_{n-1}.$$

Combining all the a_n terms together, we have

$$(n^2 - 1)a_n = (n-1)^2 a_{n-1}.$$

Using the fact that $n^2 - 1 = (n-1)(n+1)$, we arrive at the equation

$$a_n = \frac{n-1}{n+1} a_{n-1}.$$

Therefore,

$$\begin{aligned} a_{2024} &= \frac{2023}{2025} \cdot a_{2023} \\ &= \frac{2023}{2025} \cdot \frac{2022}{2024} \cdot a_{2022} \\ &\ \vdots \\ &= \frac{\cancel{2023} \cdot \cancel{2022} \cdot \cancel{2021} \cdots \cancel{3} \cdot 2 \cdot 1}{2025 \cdot 2024 \cdot \cancel{2023} \cdots \cancel{5} \cdot \cancel{4} \cdot \cancel{3}} \cdot a_1 \\ &= \frac{2a_1}{2025 \cdot 2024} \end{aligned}$$

Since $a_1 = 2025$, so we have $2024 a_{2024} = \boxed{2}$. ■

B5: 07
Note that if $A \in \mathcal{C}$, then the complement $S \setminus A \notin \mathcal{C}$, because $\mathcal{C}$ has the intersection property. So at most half of all the subsets of S can belong to $\mathcal{C}$, i.e.,

$$|\mathcal{C}| \leq \frac{2^{2024}}{2} = 2^{2023}.$$

Therefore, $n = 2023$; so the sum of the digits of n is $2 + 0 + 2 + 3 = \boxed{7}$. ■

B6: 02
There are $\frac{2024}{4} = 506$ blocks of '2024' in the integer Lata wrote on the board. Since each 2024 contributes to a sum of $2 + 0 + 2 + 4 = 8$, so the sum of the digits of the integer is $506 \times 8 = 4048$. We want to remove some digits from the front and back of the number to get the sum 2026. So, we basically need to remove some digits having sum $4048 - 2026 = 2022$. Now, observe that

$$2022 = 252 \times 8 + 6.$$

This means that we need to remove 252 blocks of '2024' from the front or from the back. Also, we need to remove some more digits that have sum 6. This is possible if we remove a '24' from the back, or a '2' from the front and a '4' from the back. In any case, we shall have $m + n = 252 \times 4 + 2 = 1010$. So the sum of the digits of $m + n$ is $1 + 0 + 1 + 0 = \boxed{2}$.

B7: 18

The key fact to observe here is that the area of the cyclic quadrilateral $ADBC$ is equal to the area of the shaded region. In fact, the area of ΔABC is equal to the area of the shaded region above the diameter AB (and similarly the area of ΔABD is equal to the area of the shaded region below the diameter AB). You can prove this as follows:

Let $AB = c, BC = a, CA = b$. Then the area of the semicircle of AB is $\frac{\pi c^2}{2} = \frac{\pi(a^2+b^2)}{2}$, using the Pythagoras theorem $c^2 = a^2 + b^2$ in right triangle ABC. Also, the areas of semicircles of BC and AC are $\frac{\pi a^2}{2}$ and $\frac{\pi b^2}{2}$ respectively. Now, note that the area of the shaded region above AB is equal to the area of the semicircle of BC plus the area of the semicircle of AC plus the area of ΔABC minus the area of the semicircle of AB. Here, the sum of the areas of the semicircles of BC and AC is $\frac{\pi a^2}{2} + \frac{\pi b^2}{2} = \frac{\pi(a^2+b^2)}{2}$, which is the area of the semicircle of AB. This proves that the area of ΔABC is equal to the area of the shaded region above the diameter AB. A similar conclusion also holds for ΔABD because $\angle ADB$ is also a right angle, being on a semicircle.

Therefore, the area of the shaded region (which is 42 square units) is equal to the sum of the areas of ΔABC and ΔADC. Since the area of ΔABC is 24 square units, therefore, the area of ΔADC is equal to $42 - 24 = \boxed{18}$ square units. ■

B8: 01

We shall try to find all n for which $n^2 + 11n + 24$ is a perfect square. First, note that $n^2 + 11n + 24 = (n + 3)(n + 8)$. If $5 \mid n + 3$, then $n + 3 = 5k$ for some integer k, so that $n + 8 = (n + 3) + 5 = 5(k + 1)$.

However, $(n + 3)(n + 8) = 25k(k + 1)$, which is clearly not a perfect square. Therefore, $5 \nmid n + 3$, which means that $\gcd(n + 3, n + 8) = 1$. This implies that $n + 3$ and $n + 8$ should be both perfect squares.

So, let $n + 3 = k^2$ for some integer k, then $n + 8 \geq (k + 1)^2 = k^2 + 2k + 1 = (n + 3) + 2k + 1$. This implies that $k \leq 2$. $k = 1$ is obviously not possible because this gives $n = -2$. For $k = 2$, we get $n = 1$, which is such that $n^2 + 11n + 24 = 36$, a perfect square. Therefore, S is the singleton set $\{1\}$, i.e., S has only $\boxed{1}$ element. ■

B9: 00
Following the hint, we construct the polynomial $Q(x) = (1+x)P(x) - x$. Then the degree of $Q(x)$ is 2023, and $Q(0) = Q(1) = Q(2) = \cdots = Q(2022) = 0$. Therefore,

$$Q(x) = kx(x-1)(x-2)\cdots(x-2022)$$

for some constant k.

Therefore,

$$1 = Q(-1) = k(-1)(-2)(-3)\cdots(-2023) = -k \cdot 2023!,$$

and hence,

$$Q(2024) = k \cdot 2024! = \left(-\frac{1}{2023!}\right) \cdot 2024! = -2024.$$

Therefore,

$$P(2024) = \frac{Q(2024) + 2024}{2025} = \frac{-2024 + 2024}{2025} = \boxed{0}.$$

■

B10: 54
We denote by $P(n)$ the numbers of regions divided by n beautifully placed circles. We have $P(1) = 2, P(2) = 4, P(3) = 8, P(4) = 14, \ldots$ and from this we notice that

$$\begin{aligned}
P(1) &= 2, \\
P(2) &= P(1) + 2, \\
P(3) &= P(2) + 4, \\
P(4) &= P(3) + 6, \\
&\vdots \\
P(n) &= P(n-1) + 2(n-1).
\end{aligned}$$

Summing these equations, we obtain

$$P(n) = 2 + 2 + 4 + \cdots + 2(n-1) = 2 + n(n-1).$$

This formula can be shown by induction on n to hold true.
Base case: $n = 1$ is obvious.
Inductive step: Assume that the formula holds for $n = k \geq 1$, i.e., $P(k) = 2 + k(k-1)$. Consider $k+1$ circles, the $(k+1)$-th circle intersects k other

circles at $2k$ points (for each one, it cuts twice), which means that this circle is divided into $2k$ arcs, each of which divides the region it passes into two sub-regions. Therefore, we have in addition $2k$ regions, and so

$$P(k+1) = P(k) + 2k = 2 + k(k-1) + 2k = 2 + k(k+1).$$

The proof by induction is thus complete.
Using this result, put $n = 2024$ to get the number of regions:

$$N = 2 + 2024 \cdot (2024 - 1).$$

Clearly, N modulo 100 is 54 and therefore, the last number formed by the last two digits of N is $\boxed{54}$. ■