PI APPROXIMATION DAY MATHQUIZ 22/07/2022

Total marks: 60 (15 in Part A and 45 in Part B)

Instructions:

<u>Part A</u>

(i) This part contains knowledge type problems related to mathematics.

(ii) There are 10 problems. Problems A1 to A5 carry 1 mark each. Problems A6 to A10 carry 2 marks each. Total marks in Part A is 15.

(iii) The answers of Part A problems are of one word or a short answer of 5-6 words, if reason is asked.

(iv) Phonetically correct spellings are accepted if the answer is a person's name. However, in case of a tie, correct spellings will be preferred.

(v) A problem Am is starred if m is a positive multiple of 3 i.e., problems A3, A6 and A9 are starred. Starred problems will be used to break ties.

<u>Part B</u>

(i) This part contains problems from Algebra, Geometry, Number Theory and Combinatorics.

(ii) There are 15 problems. Problems B1 to B4 carry 2 marks each. Problems B5 to B13 carry 3 marks each. Problems B14 and B15 carry 5 marks each. Total marks in Part B is 45.

(iii) The answers of Part B problems are integers from 00-99.

(iv) If the answer is a one-digit integer, you are advised to put a 0 before it. For example, if the answer is 7, you are advised to write 07 and if the answer is 0, you are advised to write 00.

(v) A problem Bn is starred if n is a term of the arithmetic progression with first term 2 and common difference 3 i.e., problems B2, B5, B8, B11 and B14 are starred. Starred problems will be used to break ties.

The test starts at 8:30 pm and ends at 10:00 pm sharp. Submit the answers at 9:58 or 9:59 pm to avoid any problem as there is no system for automatic submission. You are advised to use 20 minutes for Part A problems and remaining time for Part B problems. Do not panic and enjoy the problems.

All the best!

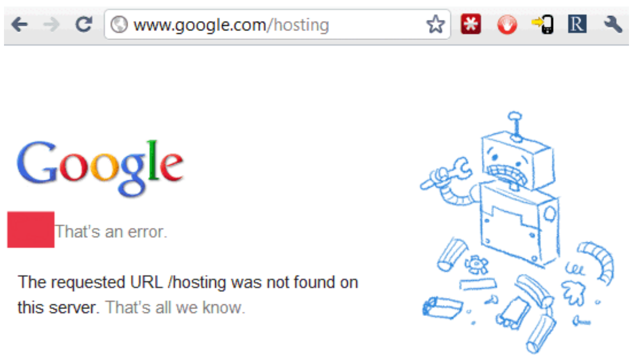
PROBLEMS

Part A

A1: In the stamp shown below, identify the name of the mathematician hidden in blue.

A2: Identify this American cyclist who also has a PhD in mathematics.

A3:⋆ The image below shows an error while loading a webpage.



Which number written before "That's an error." is hidden in red?

A4: 'X' is a high-quality typesetting software, which includes features for the production of technical and scientific documentation. The literary meaning of 'X' is "a thick white liquid that is produced by some plants and trees especially rubber trees". Identify 'X'.

A5: In the Bible, the verse 23 of the Old Testament, 1 Kings, Chapter 7 describes a vessel built at the order of King Solomon.

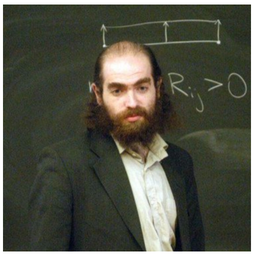
<u>Old Testament, 1 Kings, 7:23</u>
Also he made a molten sea of ten cubits from brim to brim,
round in compass, and five cubits the height thereof;
and a line of thirty cubits did compass it round about.

What is the mathematical fallacy of this?
A6:⋆ When 'X' disclosed her true identity to 'Y', he replied:
How can I describe my astonishment and admiration on seeing my esteemed correspondent M. Le Blanc metamorphosed into this celebrated person ... when a woman, because of her sex, our customs and prejudices, encounters infinitely more obstacles than men in familiarising herself with [number theory's] knotty problems, yet overcomes these fetters and penetrates that which is most hidden, she doubtless has the noblest courage, extraordinary talent, and superior genius.
Identify 'X' and 'Y'.
A7: 'X' was a very well known mathematician, known for his work in game theory, differential geometry and partial differential equations. 'X' suffered from schizophrenia during 1959-70 and returned to academic work by the mid-1980s after treatment. 'Y' is a film on the life of 'X', based on a book written by Sylvia Nasar of the same name. Identify 'X' and 'Y'.
A8: In *The Simpson's* episode "The Wizard of Evergreen Terrace", 'X' writes the equation $3987^{12} + 4365^{12} = 4472^{12}$ on a blackboard, which appears to be a counterexample to 'Y'. The equation is wrong, but it appears to be correct if entered in a calculator with 10 significant figures. Identify 'X' and 'Y'.
A9:⋆ Have a read of the following.

Sir, I bear a rhyme
excelling
In mystic force and magic spelling
Celestial sprites
elucidate
All my own striving
can't relate
Or locate they who
can cogitate
And so finally
terminate.

What punny name is given to these kind of "literary undertakings"? What do they help us do?

A10: Identify this mathematician.



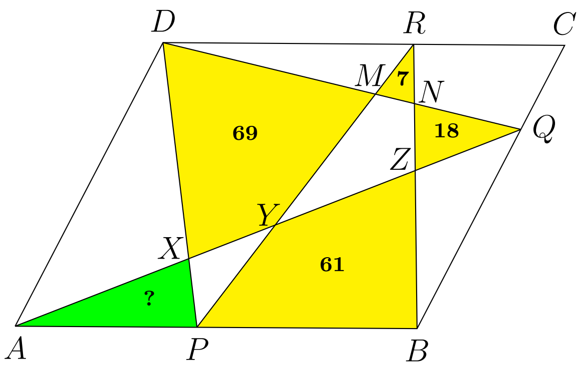
Why was he in the news in 2006?

<u>**Part B**</u>

B1: What is the remainder when $2^{2016} \cdot 3^{2018} \cdot 7^{2022}$ is divided by 43?

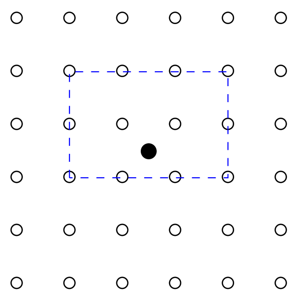
B2:$\star$ Consider $S_1 = x + y$ as the first step. At each subsequent step, x gets replaced by $2y$ and y gets replaced by x. Thus, $S_2 = 2y + x$, $S_3 = 2x + 2y$, $S_4 = 4y + 2x$ and so on. Define T_n as the sum of the coefficients of x and y in S_n for all $n \in \mathbb{N}$. For example, $T_4 = 4 + 2 = 6$. What is the unit digit of T_{2022}?

B3: $ABCD$ is a parallelogram. P, Q and R are points on sides AB, BC and CD respectively. Let AQ intersect DP, PR and BR at X, Y and Z respectively. Let DQ intersect PR and BR at M and N respectively. Suppose the areas of triangles RMN and QNZ be 7 and 18 sq. units respectively, and the areas of quadrilaterals $BPYZ$ and $DXYM$ be 61 and 69 sq. units respectively. Find the area of the triangle AXP (in sq. units).
(Areas are shown in figure below. Figure is not to scale.)



B4: Find the number of pairs of positive integers (x, y) satisfying the equation $x^3 - (y!)^2 = 2022$.

B5:$\star$ Consider a 6 by 6 square array of points with a black dot in the centre as shown in the figure. In how many ways can a rectangle be constructed such that its vertices are among the points of the array and the black dot lies inside it? (Note that a square is also a rectangle. A possible rectangle is shown in blue.)



B6: Let $x_1, x_2, \cdots, x_{50}$ be positive integers such that $x_i + x_{i+1} = 2022$ for $1 \leq i \leq 49$. Let $N = x_{25} + x_{50}$. Find the sum of the digits of N.
B7: A positive integer n is said to be *symmetric* if the binary representation of n reads the same backwards as forwards. For example, $21 = (10101)_2$ and $51 = (110011)_2$ are symmetric, whereas $13 = (1101)_2$ is not symmetric. How many positive integers less than 2^9 are symmetric?
B8:$\star$ Consider the polynomial equation $x^6 - 6x^5 + px^4 + qx^3 + rx^2 + sx + 1 = 0$, where p, q, r, s are real numbers. It is given that all the roots of this equation are positive real numbers (not necessarily distinct). What is the value of $p + q + r$?
B9: What is the integer part of the fraction

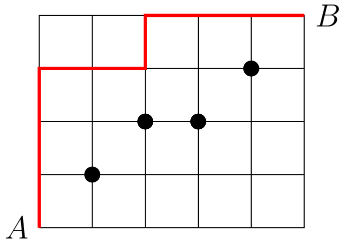
$$\frac{1}{\frac{1}{2016} + \frac{1}{2017} + \cdots + \frac{1}{2039}}?$$

B10: Let $a_0, a_1, a_2, \cdots, a_6$ be real numbers such that

$$(1 + z)^6 = a_0 + a_1 z + a_2 z^2 + \cdots + a_6 z^6$$

for all $z \in \mathbb{C}$. What is the value of $(a_0 - a_2 + a_4 - a_6)^2 + (a_1 - a_3 + a_5)^2$?
B11:$\star$ Let $f(x)$ be a polynomial whose coefficients are from the set $\{0, 1, 2\}$. If $f(3) = 76$, find the value of $f(2)$.
B12: In how many ways can 2^{2022} be expressed as the sum of four squares of non-negative integers (the integers need not be distinct)?

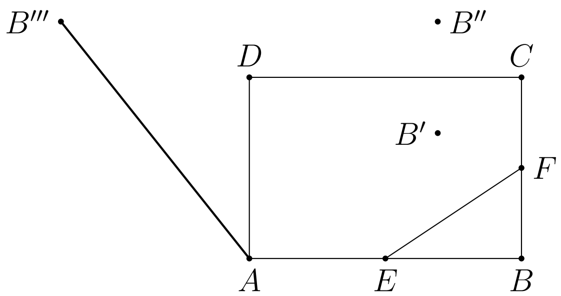
B13: The map of a town is shown below. It is allowed to travel only "east" and "north", and not through the black spots. How many different ways are there to travel from A to B? (A possible path is shown in red)



B14:$\star$ The number of ways to place 6 non-attacking rooks in a 6 by 6 chessboard such that no rooks are placed in the forbidden positions (which are denoted by $\times$) is N. Find the sum of the digits of N.
(A rook is a chess piece which can move horizontally or vertically on a chessboard without jumping, but not diagonally.)

$\times$	$\times$				
		$\times$	$\times$		
				$\times$	$\times$

B15: $ABCD$ is a rectangle with $AB = 6$ units and $BC = 4$ units. E and F are the mid-points of sides AB and BC respectively. B' is the reflection of B about EF, B'' is the reflection of B' about CD and B''' is the reflection of B'' about AD (see figure below). If $AB''' = \sqrt{\frac{m}{n}}$ units, where m and n are positive integers and $\gcd(m, n) = 1$. What is the sum of the digits of $m + n$?



SOLUTIONS

Part A
A1: Leonhard Euler.
A2: Anna Kiesenhofer.
A3: 404.
A4: LaTeX.
A5: It gives the value of π as 3 ($\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{30}{10} = 3$).
A6: Sophie Germain, Carl Friedrich Gauss.
A7: John Nash, A Beautiful Mind.
A8: Homer Simpson, Fermat's Last Theorem.
A9: Piems (π+Poem). They help us memorize π.
A10: Grigori Perelmen. He declined the Fields Medal.

Part B
B1: 09
$2^{2016} \cdot 3^{2018} \cdot 7^{2022} = (2 \cdot 3 \cdot 7)^{2016} \cdot 3^2 \cdot 7^6 = 42^{2016} \cdot (3 \cdot 7^3)^2 \equiv (-1)^{2016} \cdot 1029^2 \equiv (-3)^2 \equiv 9 \pmod{43}$.
B2: 02
Here, $T_1 = 2^0 + 2^0$, $T_3 = 2^1 + 2^1$, $T_5 = 2^2 + 2^2$ and so on. Therefore, $T_{2n-1} = 2^{n-1} + 2^{n-1}$ and $T_{2n} = 2^n + 2^{n-1} = 3 \cdot 2^{n-1}$.
Thus, $T_{2022} = 3 \cdot 2^{1010} = 3 \cdot (2^4)^{252} \cdot 2^2 = 3 \cdot 16^{252} \cdot 4 \equiv 3 \cdot 6 \cdot 4 \equiv 2 \pmod{10}$.
B3: 19
Let $[*]$ denote the area of $*$. Let $[DAX] = x$ and $[MYZN] = y$ sq. units.
Note that $[ADP] + [PRB] = [DAQ] = \frac{1}{2}[ABCD]$.
So, $(x + [AXP]) + (7 + y + 61) = x + 69 + y + 18 \Rightarrow [AXP] = 19$ sq. units.
B4: 00
Considering $y \geq 3$, $(y!)^2 \equiv 0 \pmod 9$, which gives $x^3 \equiv 2022 \equiv 6 \pmod 9$. But cubes are congruent to 0,1,8 modulo 9. Thus, $y < 3$.
For $y = 1$ and 2, $x^3 = 2023$ and 2026 respectively, which are both impossible. Thus, no such pairs of positive integers (x, y) exist.
B5: 81
Notice that a rectangle is uniquely determined by a pair of opposite vertices. The top right vertex can be chosen in 9 ways and the bottom left vertex can be chosen in 9 ways so that it contains the black dot. So the rectangle can overall be chosen in 81 ways.
B6: 06
Note that $x_i + x_{i+1} = 2022$ and $x_{i+1} + x_{i+2} = 2022$. This gives $x_i = x_{i+2}$. So all the integers with odd subscripts are equal to one another and all the integers with even subscripts are also equal to one another. So $x_{25} = x_1$ and $x_{50} = x_2$. Thus $N = x_1 + x_2 = 2022$. So the sum of the digits of N is 6.

B7: 30

In binary representation, 2^9 is 10^9. Thus, in binary, we look for symmetric positive integers less than 10^9. For such 1-digit and 2-digit numbers, there are 1_2 and 11_2 only. Such 3-digit and 4-digit numbers are of the form $1a1_2$ and $1aa1_2$ respectively, where $a \in \{0,1\}$. Such 5-digit and 6-digit numbers are of the form $1aba1_2$ and $1abba1_2$ respectively, where $a, b \in \{0,1\}$. Such 7-digit and 8-digit numbers are of the form $1abcba1_2$ and $1abccba1_2$ respectively, where $a, b, c \in \{0,1\}$. Thus, the number of symmetric positive integers less than 2^9 is $2 \times 1 + 2 \times 2^1 + 2 \times 2^2 + 2 \times 2^3 = 30$.

B8: 10

Let the roots of the equation be $r_1, r_2, \cdots, r_6$.

Thus we have, $r_1 + r_2 + \cdots + r_6 = 6$ and $r_1 r_2 \cdots r_6 = 1$. Since all the roots are positive real numbers and $\frac{r_1+r_2+\cdots+r_6}{6} = 1 = r_1 r_2 \cdots r_6$, we have AM=GM. Therefore, $r_1 = r_2 = \cdots = r_6 = 1$. Thus, $p = \binom{6}{2} = 15$, $q = -\binom{6}{3} = -20$, $r = \binom{6}{4} = 15$ and $s = -\binom{6}{5} = -6$. So, $p + q + r = 15 - 20 + 15 = 10$.

B9: 84

We have,

$$\frac{1}{\frac{1}{2016}+\frac{1}{2017}+\cdots+\frac{1}{2039}} > \frac{1}{\frac{1}{2016}+\frac{1}{2016}+\cdots+\frac{1}{2016}\,(24\,\text{terms})} = \frac{1}{24\times\frac{1}{2016}} = 84$$

and

$$\frac{1}{\frac{1}{2016}+\frac{1}{2017}+\cdots+\frac{1}{2039}} < \frac{1}{\frac{1}{2040}+\frac{1}{2040}+\cdots+\frac{1}{2040}\,(24\,\text{terms})} = \frac{1}{24\times\frac{1}{2040}} = 85$$

Therefore,

$$84 < \frac{1}{\frac{1}{2016}+\frac{1}{2017}+\cdots+\frac{1}{2039}} < 85$$

So, the integer part of the given fraction is 84.

B10: 64

Putting $z = i$ we get $(1+i)^6 = a_0 - a_2 + a_4 - a_6 + i(a_1 - a_3 + a_5)$.

Putting $z = -i$ we get $(1-i)^6 = a_0 - a_2 + a_4 - a_6 - i(a_1 - a_3 + a_5)$.

So, $(a_0 - a_2 + a_4 - a_6)^2 + (a_1 - a_3 + a_5)^2 = \{(1+i)^6(1-i)^6\} = 2^6 = 64$.

B11: 27

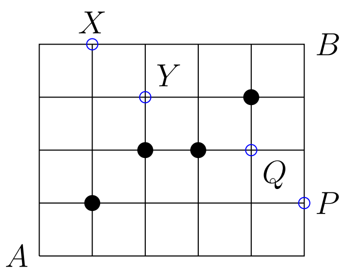
If $f(x) = ax^3 + bx^2 + cx + d$, then $f(3) = 3^3 a + 3^2 b + 3c + d = (abcd)_3$ because $a, b, c, d \in \{0,1,2\}$. Now since $76 = (2211)_3$, so $a = 2, b = 2, c = 1, d = 1$ and thus, $f(x) = 2x^3 + 2x^2 + x + 1$ giving $f(2) = 27$.

B12: 02

Let $2^{2022} = a^2+b^2+c^2+d^2$. The number of odd integers in the set $\{a, b, c, d\}$ must be even. If there are two odd integers then $a^2+b^2+c^2+d^2 \equiv 2 \pmod 4$. If all the four numbers are odd then $a^2 + b^2 + c^2 + d^2 \equiv 4 \pmod 8$. So none of the numbers a, b, c, d is odd. Then let $a = 2a_1, b = 2b_1, c = 2c_1, d = 2d_1$. This gives $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 2^{2020}$. Continuing this argument we arrive at four numbers s, t, u, v such that $s^2 + t^2 + u^2 + v^2 = 2^2$. This gives possible expression as $2^2 = 2^2 + 0 + 0 + 0 = 1 + 1 + 1 + 1$. So there are only two ways in which 2^{2022} can be expressed as the sum of four squares and these are $2^{2022} = 2^{2022} + 0 + 0 + 0 = 2^{2020} + 2^{2020} + 2^{2020} + 2^{2020}$.

B13: 14

Points P, Q, X, Y are named as shown in the map below.



There are 4 ways to travel from A to P and 1 way to travel from P to B.
There are 3 ways to travel from A to Q and 1 way to travel from Q to B.
There are 3 ways to travel from A to X and 1 way to travel from X to B.
There are 2 ways to travel from A to Y and 2 ways to travel from Y to B.
So, the required number of ways is $4 \times 1 + 3 \times 1 + 3 \times 1 + 2 \times 2 = 14$.

B14: 06

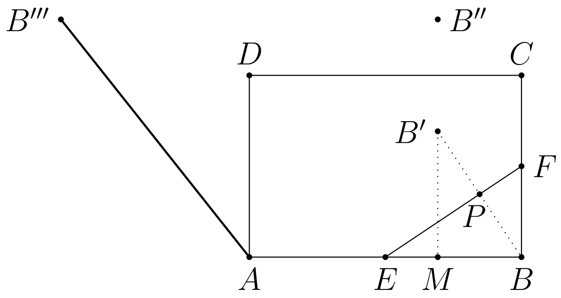
We shall use inclusion and exclusion principle to solve the the problem. The number of ways to place 6 non-attacking rooks in a 6 by 6 chess board without any condition is 6!. This is because the first rook can be place in any of the 6 places of the first row, the second rook can be placed in any of the 5 places of the second row (except the place below the first rook) and so on.

Let r_k denote the number of ways of putting 6 non-attacking rooks in a 6 by 6 chessboard such that k of them are placed in the forbidden positions. Note that in our problem for $k \geq 4$, $r_k = 0$. For r_1, the rook in the forbidden place can be placed in 6 ways (as there are 6 forbidden places) and the rest 5 rooks can be placed anywhere on the chessboard in 5! ways. So $r_1 = 6 \times 5! = 6!$. For r_2, The two rooks must be placed in either row 1,2 or 3 and this can be done in $\binom{3}{2} = 3$ ways and in each of these rows we have two choices of forbidden positions. The other four rooks can be placed in 4! ways. So $r_2 = 3 \times 2^2 \times 4! = 288$. Again for r_3, each of the rooks must be placed in the first three rows and there are two forbidden position for each rook. The other three rooks can be placed in the rest of the board in 3! ways. So $r_3 = 2^3 \times 3! = 48$.

So, the total number of ways is $6! - 6! + 288 - 48 = 240$. Therefore, the sum of digits of N is $2 + 4 + 0 = 6$.

B15: 17

Join BB'. Let $P = EF \cap BB'$ and $B'M \perp AB$.



Let $A = (0,0), B = (6,0), C = (6,4), D = (0,4)$.
By Pythagoras theorem, $EF = \sqrt{3^2 + 2^2} = \sqrt{13}$.
Clearly, $\Delta BPF \sim \Delta EBF \sim \Delta B'MB$.
So we have, $\dfrac{BP}{BF} = \dfrac{EB}{EF} = \dfrac{B'M}{B'B} \Rightarrow \dfrac{BP}{2} = \dfrac{3}{\sqrt{13}} = \dfrac{B'M}{B'B}$.
Thus, $BP = \dfrac{6}{\sqrt{13}}$ and hence $BB' = 2BP = \dfrac{12}{\sqrt{13}}$ and $B'M = \dfrac{36}{13}$.
Also we have, $\dfrac{EB}{BF} = \dfrac{B'M}{MB} \Rightarrow \dfrac{3}{2} = \dfrac{\frac{36}{13}}{MB} \Rightarrow MB = \dfrac{24}{13}$.
Therefore, $AM = 6 - \dfrac{24}{13} = \dfrac{54}{13}$. So, $B' = \left(\dfrac{54}{13}, \dfrac{36}{13}\right)$.
Thus, $B'' = \left(\dfrac{54}{13}, 4 + 4 - \dfrac{36}{13}\right) = \left(\dfrac{54}{13}, \dfrac{68}{13}\right)$ and $B''' = \left(-\dfrac{54}{13}, \dfrac{68}{13}\right)$.
Therefore, $AB''' = \sqrt{\dfrac{54^2 + 68^2}{13^2}} = \sqrt{\dfrac{7450}{13^2}} = \sqrt{\dfrac{580}{13}}$.
So, $m = 580, n = 13$. Thus, $m + n = 593$ and the sum of the digits is 17.