Quiz 1 answers

1. (a) scalar multiple

(b) $2022 e_1 + e_2 = \left(2022 - \frac{1}{r}\right) e_1 + \frac{1}{r}(e_1 + r e_2)$ Answer: $\begin{bmatrix} 2022 - \frac{1}{r} \\ \frac{1}{r} \end{bmatrix}$

2. blank: $A(x-y) = 0$

I $\Leftrightarrow$ Rows indep (None)

(None) Rows span $\mathbb{R}^c$ $\Leftrightarrow$ II

(None) Cols indep $\Leftrightarrow$ II

I $\Leftrightarrow$ Cols span $\mathbb{R}^r$ (None)

I $\Leftrightarrow$ Pivot in each row (None)

(None) Pivot in each col $\Leftrightarrow$ II

I $\Rightarrow$ $r \leq c$ (None)

(None) $c \leq r$ $\Leftarrow$ I

3. (a) The three nonzero rows of the matrix on the board, which is in row echelon form (not reduced REF but that doesn't matter)

(b) $\{v_1, v_3, v_4\}$ pivot cols of given matrix

(c) dim Nul(M) = 2 A basis: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -17 \\ 0 \\ 11 \\ -7 \\ 1 \end{bmatrix}$

4. (a) $E_{ij} E_{kl} = \delta_{jk} E_{il}$

(b) $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

(c) M = any solution of a homogeneous system of n^2 equations (equalities for n^2 entries of AM and MA) in n^2 variables (entries of M). We can solve this in the usual way by row reduction. Solutions will form a vector space (e.g. in (b) the dim of solution space is 2).

5. (a) $\begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r+1} \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -r \\ 0 & 1 \end{bmatrix}$

(c) $\left[\begin{array}{ccc|ccc} 1 & 2021 & 2022+r \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - 2021R_2} \begin{bmatrix} 1 & 0 & 1+r \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 - (1+r)R_3]{R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \xrightarrow{\text{same}} \begin{bmatrix} 1 & -2021 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{same}} \begin{bmatrix} 1 & -2021 & -(1+r) \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ Answer

HW 2 answers

2.2 + 3.1) (a) yes, $\{E_{ii}, E_{ij}+E_{ji} (i<j)\}$ is one basis

(b) No (O is not invertible) (c) Yes

3.3) Let $ax^2 + b\cos x + ce^x = 0$ <u>function</u> for $a,b,c \in \mathbb{R}$, i.e. equation is true for each real value of x.

Substitute suitable values for x to deduce a, b, c must all be 0 by solving a system of eqn.

You can play other games e.g. take $\lim_{x\to-\infty}$ (why valid) to first get $a=0$, then $b=0$ and hence $c=0$ (how?)

3.7) Let $e_1, \dots, e_m$ be the standard basis of $\mathbb{R}^m$ and

let $f_1, \dots, f_m$ be the standard basis of $\mathbb{R}^n$.

See that $e_i f_j^t = E_{ij}$ (matrix units) and these form a basis of $\mathbb{R}^{m\times n}$.

Fix i and j. Then $(1\le i\le m,\ 1\le j\le n)$

$$e_i = \sum_{k=1}^{m} a_k X_k \quad (\text{as } X_k \text{ span } \mathbb{R}^m) \text{ and}$$

$$f_j = \sum_{l=1}^{n} b_l Y_k \quad (\text{as } Y_l \text{ span } \mathbb{R}^n)$$

Now $e_i f_j^t = \left(\sum_k a_k X_k\right)\left(\sum_l b_l Y_l\right)^t = \sum_{k,l} a_k b_l X_k Y_l^t$ is in the span of the given set $X_k Y_l^t$

So $X_i Y_j^t$ is a set of mn vectors that span a v.space of dim mn. Thus the given set is a basis.

4.1) (a) Show $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[\text{reduction}]{\text{row}} Id$ (b) Solve $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

4.3) Sketch: write $B' = (w_1, w_2, \dots, w_n)$ = arbitrary basis. Write

$w_i = a_{1i} v_1 + a_{2i} v_2 + \dots + a_{ni} v_n$

Matrix $A = (a_{ij})$ is an invertible matrix. Symbolically in terms of hypervectors we have

$[w_1 \cdots w_n] = [v_1 \cdots v_n]\, A$

Given operations (i), (ii), (iii) amount to using A = an elementary matrix

As any invertible A can be written as a product of elementary matrices, the result follows.

6) $$A \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & 2 & 0 & 13 & 7/2 \\ 0 & 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Row operations preserve row space and do not change (in)dependence of columns.

$\therefore$ Basis of row space of A = the two nonzero rows of R

Basis of col space of A = 1^{st} and 3^{rd} columns of A (cols of R with pivots)

$Ax=0 \Leftrightarrow Rx=0$. General solution $= x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-13\\0\\3\\1\\0\end{bmatrix} + x_5\begin{bmatrix}-7/2\\0\\-1/2\\0\\1\end{bmatrix}$ These three vectors are indep (see slots 2,4,5) $\Rightarrow$ a basis of null space

7) REF(A) has at most one pivot in each row and at most one pivot in each col.

From table $\begin{cases} \text{(ii)} \Leftrightarrow \text{RREF(A) has a pivot in each column (i.e. no free variables)} \\ \text{(iii)} \Leftrightarrow \text{RREF(A) has a pivot in each row} \end{cases}$

Thus any two of i, ii, iii imply $RREF(A)=Id$ which implies all three statements.

8)
$$\begin{tikzcd}
 & c \arrow[dr, Rightarrow, "1"] & & & & \\
 & & g \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & & & \\
a \arrow[uur, Rightarrow, "\text{obvious}"] \arrow[r, Rightarrow, "0"] \arrow[ddr, Rightarrow, "\text{obvious}"'] & b \arrow[rr, Leftrightarrow, "\text{problem}", "7"'] & & e \arrow[r, Rightarrow, "3"] & f \arrow[r, Rightarrow, "4"] & a \\
 & & h \arrow[ul, Leftrightarrow] \arrow[ur, Leftrightarrow] & & & \\
 & d \arrow[ur, Rightarrow, "2"] & & & &
\end{tikzcd}$$

0 If A is invertible, $x \mapsto A^{-1}x$ is the inverse function of $x \mapsto Ax$.

1 Assume (c). If $Ax=0$ then $LAx=0$ i.e. $Ix=x=0$. So $x \mapsto Ax$ is injective. Use problem 7.

2 Assume (d). Then $ARb=Ib=b$. So $x=Rb$ solves $Ax=b$. Thus $x \mapsto Ax$ is surjective. Use problem 7.

3 Assume (e). Then $E_k \cdots E_1 A = I$, where $E = E_k \cdots E_1$ is a product of elementary matrices E_i.
Therefore $A = E_1^{-1} \cdots E_k^{-1}$ because E_i are invertible. Each E_i^{-1} is also an elementary matrix (encoding the reverse of the row operation corresponding to E_i), giving (f).

4 Product of invertible matrices is invertible: $(XY)^{-1} = Y^{-1}X^{-1}$ by inspection

HW 3 answers

4.1c + 4.2ab) straightforward $B' = BP$ e.g. in 4.2a $[e_1+e_2, e_1-e_2] = [e_1, e_2]\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

1.4) There is exactly one linearly indep row, (say i^{th} row $= [a_1 \cdots a_n]$) (equivalently, column) and all others are scalar multiples of it. Then

$$A = \begin{bmatrix} \lambda_1 a_1 & \cdots & \lambda_1 a_n \\ \lambda_2 a_1 & \cdots & \lambda_2 a_n \\ \vdots & & \\ a_1 & \cdots & a_n \\ \vdots & & \\ \lambda_m a_1 & \cdots & \lambda_m a_n \end{bmatrix} = \underbrace{\begin{bmatrix}\lambda_1 \\ \vdots \\ 1 \\ \vdots \\ \lambda_m\end{bmatrix}}_{m\times 1} \underbrace{[a_1, \cdots, a_n]}_{1\times n}$$

$\rightarrow i^{th}$ row

Both vectors can be scaled by nonzero scalars reciprocal to each other.

2.1) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} = \sum a_{ij} e_{ij}$ $\qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix} = \sum b_{ij} e_{ij}$

M	AMB	
e_{11}	$\begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12}\end{bmatrix}$	$= \underline{a_{11}b_{11}}\, e_{11} + \underline{a_{11}b_{12}}\, e_{12} + \underline{a_{21}b_{11}}\, e_{21} + \underline{a_{21}b_{12}}\, e_{22}$
e_{pq}	$A e_{pq} B$	$= \underbrace{a_{1p} b_{q1} e_{11}}_{(a_{1p}e_{1p})(e_{pq})(b_{q1}e_{q1})} + a_{1p} b_{q2} e_{12} + a_{2p} b_{q1} e_{21} + a_{2p} b_{q2} e_{22}$

First column of desired matrix

$$\text{Desired matrix} = \begin{array}{c|c} & \begin{array}{cccc} e_{11} & e_{12} & e_{21} & e_{22} \end{array} \\ \hline \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array} & \begin{bmatrix} a_{11}b_{11} & a_{11}b_{21} & a_{12}b_{11} & a_{12}b_{21} \\ a_{11}b_{12} & a_{11}b_{22} & a_{12}b_{12} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{21} & a_{22}b_{11} & a_{22}b_{21} \\ a_{21}b_{12} & a_{21}b_{22} & a_{22}b_{12} & a_{22}b_{22} \end{bmatrix} \end{array}$$

2.4) Sketch: Doing an elementary row (respectively, column) operation on A is equivalent to multiplying A on the left (respectively, right) by an elementary matrix. Thus it is enough to show that $A \xrightarrow[\text{col ops}]{\text{row and}} \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right]$

Now $A \xrightarrow{\text{row op}} R = RREF(A)$. Now kill each nonpivot entry r_{ij} by using the pivot in i^{th} row: if this $1 = r_{ik}$ (must have $k<j$), then do $Col_j - r_{ij} Col_k$. This operation does not change any other entry in R as $r_{ik}=1$ is the only nonzero entry in Col_k of R.

Now switch columns to move all zero columns to the right, past all pivot cols.

3.1) There are two cases depending on parity of n.

- If $n=2k$ is even, $\ker T = \{(x_1, x_2, \cdots, x_k, -x_k, \cdots, -x_1)^t \mid x_i \in \mathbb{R}\}$ which has dimension $k = \frac{n}{2}$ (e.g. $e_1-e_n, e_2-e_{n-1}, \cdots, e_k - e_{k+1}$ is a basis.) Then by dimension formula $\dim(\operatorname{im} T) = n-k = k = \frac{n}{2}$

- If $n = 2k+1$ is odd, $\ker T = \{(x_1, x_2, \cdots, x_k, 0, -x_k, \cdots, -x_1)^t \mid x_i \in \mathbb{R}\}$ $\operatorname{Dim}(\ker T) = k = \frac{n-1}{2}$ ($e_1 - e_n, \cdots\cdots, e_k - e_{k-2}$ being a basis.) Then by dimension formula $\operatorname{Dim}(\operatorname{im} T) = n-k = \frac{n+1}{2}$

3.3) Claim: There is a vector $v \in V$ such that $T(v) \neq cv$ for any scalar c.

Proof: If not, pick a basis v_1, v_2 and let $T(v_1) = c_1 v_1$, $T(v_2) = c_2 v_2$. If $c_1 = c_2 = c$ then for any $v \in V$, let $v = \alpha v_1 + \beta v_2$: $T(v) = T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2) = c(\alpha v_1 + \beta v_2) = cv$. This contradicts the hypothesis that $T \neq c\,Id$. Thus $c_1 \neq c_2$. Now $T(v_1 + v_2) = c_1 v_1 + c_2 v_2$, which is not a scalar multiple of $v_1 + v_2$. This proves the claim.

Pick v as in the claim. As v & Tv are lin indep and V has dimension 2, $\{v, Tv\}$ is a basis of V. To write the matrix of T wrt this basis, we write

$$\begin{cases} T(v) = 0 \cdot v + 1 \cdot T(v) \\ T(T(v)) = a \cdot v + b\, T(v) \end{cases}$$

Desired matrix =

$$\begin{array}{cc} \begin{array}{cc} v & T(v) \end{array} & \\ \begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix} & \begin{array}{c} v \\ T(v) \end{array} \end{array}$$

7) (a), (b) are straightforward e.g. (1) by definition $T_1 + T_2(v) := T_1(v) + T_2(v)$. $T_1 + T_2$ is linear because $T_1 + T_2(cv + v') = T_1(cv+v') + T_2(cv+v')$ and now use linearity of T_1 & T_2 with commutativity of $+$ in W

(2) Addition & scalar multiplication in $\operatorname{Hom}(V, W)$ satisfy desired properties due to the same properties in W.

To show equality of functions $f=g$ means showing $f(x)=g(x)$ for each $x \in$ domain

(c) $(S\circ(T_1+T_2))(v) = S((T_1+T_2)(v))$ by definition of $\circ$

$= S(T_1(v)+T_2(v))$ by definition of T_1+T_2

$= S(T_1(v)) + S(T_2(v))$ by linearity of S

$= (S\circ T_1)(v) + (S\circ T_2)(v)$

$= ((S\circ T_1)+(S\circ T_2))(v)$ by definition of $+$ in $\mathrm{Hom}(V,U)$

$\Rightarrow \quad S\circ(T_1+T_2) = (S\circ T_1)+(S\circ T_2)$

Similarly $\quad (T_1+T_2)\circ R = (T_1\circ R)+(T_2\circ R)$

These translate into distributivity of matrix multiplication over matrix $+$.

8) (a) $V \xrightarrow{T} W \xrightarrow{S} U$

$\mathrm{Im}(T) = T(V) \subset W$

$\therefore \mathrm{Im}(ST) = S(T(V)) \subset S(W)$

subspace containments

$$\begin{array}{ccc} S(T(V)) & \subset & S(W) \\ \downarrow{\scriptstyle \dim} & & \downarrow{\scriptstyle \dim} \\ \mathrm{rank}(ST) & \le & \mathrm{rank}(S) \end{array}$$

$\Rightarrow \quad \mathrm{rank}(ST) \le \mathrm{rank}(S)$

Also $\mathrm{rank}(ST) = \left(\text{rank of the map } T(V) \xrightarrow{S} W\right) \le \dim(\text{domain})$

$\| \;\; \mathrm{rank}(T)$

by dimension formula applied here

(b) each col of AB = lin combination of cols of A

$\Rightarrow$ col space of $AB \subset$ col space of A

$\Rightarrow \quad \mathrm{rank}(AB) \le \mathrm{rank}(A)$

each row of AB = lin combination of rows of B

$\Rightarrow$ row space of $AB \subset$ row space of A

$\Rightarrow \quad \mathrm{rank}(AB) \le \mathrm{rank}(B) \quad$ (row rank = col rank)

(c) Under bijection matrices $\longleftrightarrow$ lin maps, multiplication $\longleftrightarrow$ composition, column rank $\longleftrightarrow$ rank = dim(image)

HW4 answers

5.1) intersection 0 and they span $\mathbb{R}^{n\times n}$ (how?)

5.2) "scalar matrices" $\{cI \mid c\in\mathbb{R}\}$

6.1) Eventually constant sequences. One nonexample $(a_1, a_2, \cdots)$ where $a_i = i$

Chapter 1

4.3) Answer: $n+1$ (by induction)

4.6) (Sketch) Use our formula (complete expansion) to see that entries from B cannot contribute (due to the 0 block below A) and remaining terms can be grouped to get product $(\det A)(\det D)$. [Lengths of permutation in A part and D part add up.]

6.1) straightforward.

6.2) If A^{-1} has integer entries, use $\det(AA^{-1}) = \det(A)\det(A^{-1})$

If $\det(A) = \pm 1$, then $\pm$ cofactor/adj of A is A^{-1} by Thm 1.6.9

M.7)

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} \xrightarrow{R_4 - aR_3,\ R_3 - aR_2,\ R_2 - aR_1} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b^2-ab & c^2-ac & d^2-ad \\ 0 & b^3-ab^2 & c^3-ac^2 & d^3-ad^2 \end{vmatrix} \rightarrow (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & b & c & d \\ 0 & b^2 & c^2 & d^2 \end{vmatrix}$$

$(b-a)(c-a)(d-a)$ times this det

It works better to use a^{n-1} to clear a^n, a^{n-2} to clear a^{n-1}, $\cdots$, a to clear a^2, 1 to clear a than to use top 1 to clear all powers of a.

(c) Want $P(t) = a_0 + a_1 t + \cdots + a_n t^n$ with $P(t_i) = b_i$ $i = 0, \cdots, n$

This is equivalent to solving the following for $a_0, \cdots, a_n$.

$$\begin{bmatrix} 1 & t_0 & \cdots & t_0^n \\ \vdots & \vdots & & \vdots \\ 1 & t_n & \cdots & t_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

The coeff matrix is invertible by Vandermonde determinant. So there is a unique solution.

[Also see Artin example 1.2.22 and Lagrange interpolation formula]

See HW 2 #6

8) Basis of domain $\mathbb{R}^5$ = $\underbrace{e_1, e_3}_{\text{std basis vectors}}$, basis vectors of Nul(A)

Basis of codomain $\mathbb{R}^3$ = cols 1 and 3 of A, $\begin{bmatrix}1\\0\\0\end{bmatrix}$ (or any $v \notin$ col(A))