PI APPROXIMATION DAY MATHQUIZ 22/07/2023

Total marks: 60 (15 in Part A and 45 in Part B)

Instructions:

Part A

(i) This part contains knowledge type problems related to mathematics.

(ii) There are 10 problems. Problems A1 to A5 carry 1 mark each. Problems A6 to A10 carry 2 marks each. Total marks in Part A is 15.

(iii) The answers of Part A problems are of one word or a short answer of 5-6 words, if reason is asked.

(iv) Phonetically correct spellings are accepted if the answer is a person's name. However, in case of a tie, correct spellings will be preferred.

(v) A problem Am is starred if m is a positive multiple of 3 i.e., problems A3, A6 and A9 are starred. Starred problems will be used to break ties.

Part B

(i) This part contains problems from Algebra, Geometry, Number Theory and Combinatorics.

(ii) There are 15 problems. Problems B1 to B4 carry 2 marks each. Problems B5 to B13 carry 3 marks each. Problems B14 and B15 carry 5 marks each. Total marks in Part B is 45.

(iii) The answers of Part B problems are integers from 00-99.

(iv) If the answer is a one-digit integer, you are advised to put a 0 before it. For example, if the answer is 7, you are advised to write 07 and if the answer is 0, you are advised to write 00.

(v) A problem Bn is starred if n is a term of the arithmetic progression with first term 2 and common difference 3 i.e., problems B2, B5, B8, B11 and B14 are starred. Starred problems will be used to break ties.

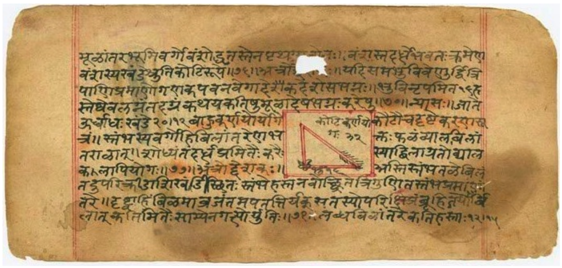
The test starts at 8:30 pm and ends at 10:00 pm sharp. Submit the answers at 9:58 or 9:59 pm to avoid any problem as there is no system for automatic submission. You are advised to use 20 minutes for Part A problems and remaining time for Part B problems. Do not panic and enjoy the problems.

All the best!

PROBLEMS

Part A

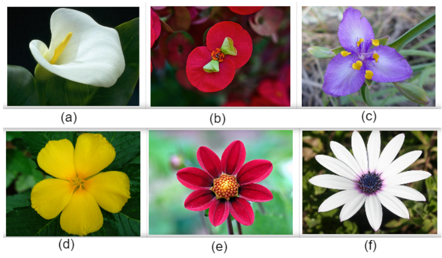
A1: Name the widely used theorem that is described in the following sloka.



A2: The following formula for calculating the value of π is attributed to which mathematician?

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

A3:$\star$ Name the famous sequence which corresponds to the following image.



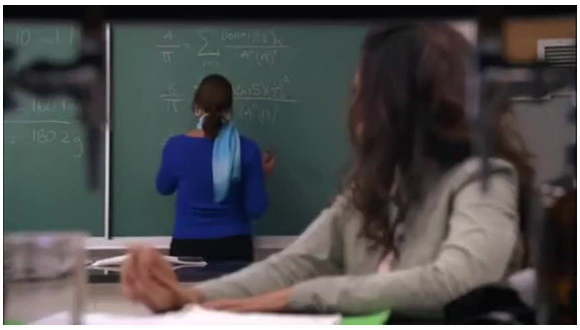
A4: This is an alternative method of solving quadratic equations which was popularised by ‘X’ in 2019.

1. If you find r and s with sum $-B$ and product C, then $x^2 + Bx + C = (x - r)(x - s)$, and they are all the roots
2. Two numbers sum to $-B$ when they are $-\frac{B}{2} \pm u$
3. Their product is C when $\frac{B^2}{4} - u^2 = C$
4. Square root always gives valid u
5. Thus $-\frac{B}{2} \pm u$ work as r and s, and are all the roots

‘X’ is an American professor of mathematics at the Carnegie Mellon University. He earned his B.S. in Mathematics from California Institute of Technology and continued to study in Cambridge University. ‘X’ is the national coach of the United States’ IMO team. Identify ‘X’.

A5: Identify this Assamese mathematician who expired recently. (He held the esteemed position of Professor Emiritus of Mathematics at Guwahati University and was an expert in Plasma Physics, Relativity and Graph Theory.)

A6:⋆ The following scene appears in the insanely popular Disney movie 'X':



The second equation written on the board is

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(42n+5)\left(\frac{1}{2}\right)_n^3}{64^n (n!)^3}$$

However, the student points out that the equation should read sixteen over π. The teacher initially replied that it was impossible but then using a calculator somehow figures out that the student is right.

This movie moment also now figures in a paper published in the American Mathematical Monthly. An Assamese Mathematician 'Y', along with Bruce C. Berndt, and Heng Huat Chan provide a survey of Ramanujan's series for $1/\pi$ and start off with the formulas that play a part in the movie 'X'. 'Y' is known for his work related to the mathematics of Ramanujan. 'Y' obtained his bachelor's degree in mathematics from Cotton College, Guwahati in 1992, graduated with a masters in mathematics from the Indian Institute of Technology, Kanpur in 1995 and a Ph.D. in mathematics from Tezpur University in 2001. Following a short stint at Assam University, Silchar, ‘Y’ has been a member of the faculty at Tezpur University since 1997, becoming full professor in 2009. For a period of one year in 2006-07, he was a visitor at the University of Illinois, Urbana-Champaign working with Bruce C. Berndt.

Identify ‘X’ and ‘Y’.

A7: There was an argument between two mathematicians over who has first invented the branch ‘X’ in mathematics, which has wide applications in several areas. The question was a major intellectual controversy, which began simmering in 1699 and broke out in full force in 1711. The modern consensus is that the two men developed their ideas independently. ‘Y’ was one of the two mathematicians, who served as the Lucasian professor of mathematics at Cambridge University and is widely known for his work in physics. Identify ‘X’ and ‘Y’.

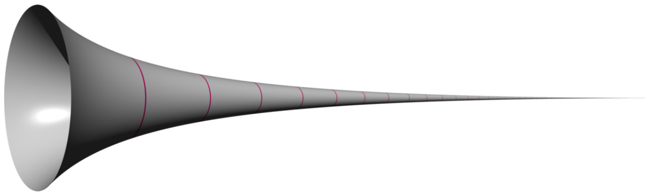
A8: Read the following probability puzzle:

Suppose you are on a game show, being asked to choose between three doors. Behind one door is a car; behind the others, goats. You pick a door (say door 1) and the host, who knows what's behind all the doors, opens one of the other two doors (say door 2), which has a goat. He then says to you,“Do you want to pick door 3?” Is it to your advantage to switch your choice?

Interestingly, it turns out that switching the choice has a $\frac{2}{3}$ probability of winning the car, whereas sticking with the initial choice has only a $\frac{1}{3}$ probability. So it is better to switch the choice and choose door 3. This puzzle is widely known as the ‘X’ problem and is loosely based on the American television game show ‘Y’. (‘X’ was the original host of the show.) Identify ‘X’ and ‘Y’.

A9:⋆ Look at the image below:

It is a painting attributed to the Italian Renaissance artist Leonardo da Vinci. The subject matter of the work is drawn from Luke 1.26-39. It depicts the angel ‘X’ announcing to Mary that she would conceive miraculously and give birth to a son to be named Jesus and called “the Son of God”, whose reign would never end. The image below shows the 3D illustration of a geometric figure named after ‘X’:



The properties of this figure were first studied by the Italian physicist and mathematician ‘Y’. Identify ‘X’ and ‘Y.

A10: ‘X’ is a deterministic primality-proving algorithm created and published by three computer scientists at the Indian Institute of Technology, Kanpur. The algorithm was the first one which is able to determine in polynomial time, whether a given number is prime or composite and this without relying on mathematical conjectures such as the generalized Riemann hypothesis. The algorithm is shown below:

Input: integer $n > 1$.

1. Check if n is a perfect power: if $n = a^b$ for integers $a > 1$ and $b > 1$, output *composite*.
2. Find the smallest r such that $\mathrm{ord}_r(n) > (\log_2 n)^2$. (if r and n are not coprime, then skip this r)
3. For all $2 \leq a \leq \min(r, n-1)$, check that a does not divide n: If $a|n$ for some $2 \leq a \leq \min(r, n-1)$, output *composite*.
4. If $n \leq r$, output *prime*.
5. For $a = 1$ to $\left\lfloor \sqrt{\varphi(r)} \log_2(n) \right\rfloor$ do

   if $(X+a)^n \neq X^n + a \pmod{X^r - 1, n}$, output *composite*;
6. Output *prime*.

An Assamese mathematician and computer scientist ‘Y’ was one among those three computer scientists to have developed the test. Since 2008, ‘Y’ has been working with the Microsoft Research Lab India as a researcher. A picture of ‘Y’ is shown below:

Identify ‘X’ and ‘Y’.

Part B

B1: Find the unit digit of $2023^1 + 2023^2 + 2023^3 + \cdots + 2023^{2023}$.

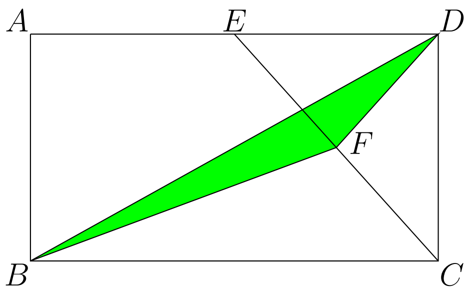
B2:$\star$ Let α and β be two roots of the equation $x^2 - \sqrt{2}x + 2 = 0$. Then $\alpha^{14} + \beta^{14} = -N$, where N is a 3-digit number. Determine the value of the sum of the digits of N.

B3: Let a and b be two positive integers that satisfy

$$\frac{a}{b} = \frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \cdots + \frac{1}{49 \times 50 \times 51},$$

where $\gcd(a, b) = 1$. Find the sum of the digits of $a + b$.

B4: In the diagram below, $ABCD$ is a rectangle. E is the midpoint of AD. F is the midpoint of EC. Let $[*]$ denote the area of $*$. Suppose $[ABCD] = 120$ square units, find $[BDF]$.



B5:$\star$ Let $f(x)$ be a polynomial of degree 2021 such that $f(k) = \frac{13}{k}$ for $k = 1, 2, 3, \ldots, 2022$. Find the value of $2023 \times f(2023)$.

B6: For a positive integer n, $s(n)$ denotes the sum of the digits of n. A positive integer n is said to be *cute* if $s(n) \mid n$. For example, 111 is cute but 123 is not. How many 2-digit cute positive integers are there?

B7: Let $l_1, l_2, \ldots, l_{100}$ be consecutive terms of an arithmetic progression with common difference d, and $w_1, w_2, \ldots, w_{100}$ be consecutive terms of another arithmetic progression with common difference D, where $dD = 10$. Let R_i be a rectangle with length l_i, width w_i and area A_i, where $i = 1, 2, \ldots, 100$. If $A_{51} - A_{50} = 1000$, then $A_{20} - A_{10} = p$. What is the sum of the digits of p?

B8:$\star$ Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Suppose n is the number of ordered pairs (A, B), where A and B are subsets of S and A is a proper subset of B. (Note that A can be the empty set.) Find the sum of the digits of n.

B9: For a positive integer n, $s(n)$ denotes the sum of the digits of n. A positive integer n is said to be 9-*like* if $s(n) = 9$. Let N be the number of 9-like positive integers up to 2023. Find the sum of the digits of N.

B10: Let T_n denote the n^{th} triangular number, i.e., $T_n = \frac{n(n+1)}{2}$. Let S be the set of positive integers k such that

$$T_{2k} \,\Big|\, \sum_{i=1}^{2k}(-1)^i T_i^2.$$

Find the number of elements in S.

B11:$\star$ Odd integers starting from 1 are grouped as follows:

$$(1), (3, 5), (7, 9, 11), (13, 15, 17, 19), \ldots,$$

where the n^{th} group consists of n odd integers. How many odd integers are there in the group 2023 belongs to?

B12: Define

$$a_{n+1} = \frac{1}{1 + a_n}$$

for every integer $n \geq 1$. Suppose $a_1 = a_{2023}$. Then find the sum of the squares of all the possible values of a_1.

B13: If

$$\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) = \frac{\sqrt{a}}{b},$$

where $\gcd(a, b) = 1$. Then find ab.

B14:$\star$ Given a seven degree polynomial P with real coefficients such that $P(x) = x \cdot 2^{x-1}$ for $x = 0, 1, 2, 3, 4, 5, 6, 7$. Find the value of $|P(-1)|$.

B15: Find the product of the digits of the smallest integer n such that

$$\sqrt{\frac{(n+1)(2n+1)}{6}}$$

is an integer for $n \geq 2$.

SOLUTIONS

Part A

A1: Baudhayana Theorem (or Pythagoras Theorem).
A2: Srinivasa Ramanujan.
A3: Fibonacci Sequence $(1, 2, 3, 5, 8, 13, \dots)$.
A4: Po-Shen Loh.
A5: Dr. Bhaben Chandra Kalita.
A6: High School Musical, Nayandeep Deka Baruah.
A7: Calculus, Sir Isaac Newton.
A8: Monty Hall, Let's Make a Deal.
A9: Gabriel, Evangelista Torricelli.
A10: AKS primality test, Neeraj Kayal.

Part B

B1: 09
We have, $2023^1 + 2023^2 + 2023^3 + \cdots + 2023^{2023} \equiv 3^1 + 3^2 + 3^3 + \cdots + 3^{2023} \pmod{10}$. Now observe that modulo 10, $3^{4k} \equiv 1$, $3^{4k+1} \equiv 3$, $3^{4k+2} \equiv 9$ and $3^{4k+3} \equiv 7$, for $k = 0, 1, 2, \dots$. Therefore, the sum $3+9+7+1 = 20 \equiv 0 \pmod{10}$ and hence it does not contribute anything to the sum. We have extra three terms remaining, which will contribute to the sum– they are 3^{2021}, 3^{2022} and 3^{2023}, which are 3, 9 and 7 respectively. Therefore, the required unit digit is $3 + 9 + 7$ modulo 10, i.e., 9.
B2: 11
Here, $\alpha + \beta = \sqrt{2}$ and $\alpha\beta = 2$.. Putting $x = \alpha$ in the equation, we have

$$\alpha^2 + 2 = \sqrt{2}\alpha.$$

Squaring, we get

$$\alpha^4 + 4\alpha^2 + 4 = 2\alpha^2 \implies \alpha^4 + 4 = -2\alpha^2.$$

Squaring again, we get

$$\alpha^8 + 8\alpha^4 + 16 = 4\alpha^4 \implies \alpha^8 + 16 = -4\alpha^4.$$

Multiplying α^6 both sides, we get

$$\alpha^{14} + 16\alpha^6 = -4\alpha^{10} \implies \alpha^{14} = -4\alpha^{10} - 16\alpha^6 = -4\alpha^6(\alpha^4 + 4).$$

Using $\alpha^4 + 4 = -2\alpha^2$, we get $\alpha^{14} = 8\alpha^8$. Similarly, $\beta^{14} = 8\beta^8$.
Therefore,

$$\alpha^{14} + \beta^{14} = 8(\alpha^8 + \beta^8) = 8\left((\alpha^4 + \beta^4)^2 - 2(\alpha\beta)^4\right).$$

Now,

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2 = \left((\alpha + \beta)^2 - 2ab\right)^2 - 2(\alpha\beta)^2 = -4,$$

and hence,

$$\alpha^{14} + \beta^{14} = 8\left((-4)^2 - 2(2)^4\right) = -128 = -N.$$

Therefore, the sum of the digits of N is $1 + 2 + 8 = 11$.

B3: 13

Observe that

$$\frac{a}{b} = \sum_{n=1}^{49} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{49} \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right).$$

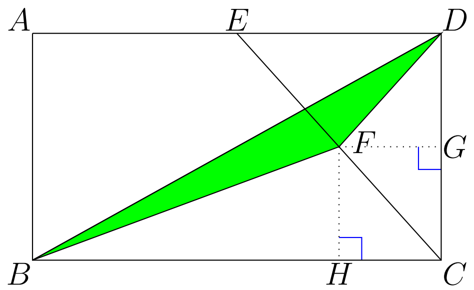
This is a telescoping sum and

$$\frac{a}{b} = \frac{1}{2} - \frac{1}{50 \times 51} = \frac{1275 - 1}{2550} = \frac{1274}{2550} = \frac{637}{1275}.$$

Therefore, $a + b = 1912$ and hence sum of the digits of $a + b$ is 13.

B4: 15

We draw perpendiculars FG to CD and FH to BC, as shown in the figure below.



Let the sides of $ABCD$ be $2a$ and $2b$. Therefore,

$$[ABCD] = 4ab = 120 \implies ab = 30.$$

Since $FG \parallel ED$ in ΔCED and F is the midpoint of EC, so by Thales' Theorem, G is the midpoint of CD. i.e., $CG = GD = b$. Therefore,

$$\begin{aligned}[BDF] &= [BDC] - [DFC] - [BFC] \\ &= 60 - \frac{1}{2}(2b)\left(\frac{a}{2}\right) - \frac{1}{2}(2a)b \\ &= 60 - \frac{1}{2}(30) - 30 = 15 \text{ sq. units}\end{aligned}$$

B5: 26

Let $g(x) = xf(x) - 13$, which is a polynomial of degree 2022. Since $f(k) = \frac{13}{k}$ for $k = 1, 2, 3, \ldots, 2022$, hence

$$g(1) = g(2) = \cdots = g(2022) = 0.$$

Therefore,

$$g(x) = \lambda(x-1)(x-2)\cdots(x-2022).$$

But

$$g(0) = \lambda(-2022!) = -13 \implies \lambda = \frac{13}{2022!}.$$

Therefore,

$$\begin{aligned}
&g(2023) = 2023 \times f(2023) - 13 \\
\implies &\lambda \times 2022! = 2023 \times f(2023) - 13 \\
\implies &13 = 2023 \times f(2023) - 13 \\
\implies &2023 \times f(2023) = 26.
\end{aligned}$$

B6: 23

Consider a 2-digit number $\overline{ab} = 10a + b$. If $\overline{ab}$ is cute, then

$$a + b \mid 10a + b \implies a + b \mid 10a + b - (a + b) \implies a + b \mid 9a.$$

<u>Case 1</u>: If $3 \nmid (a + b)$, then $(a + b) \mid a$, and thus $b = 0$. We have, $\{10, 20, 40, 50, 70, 80\}$.
<u>Case 2:</u> If $3 \mid (a + b)$ but $9 \nmid (a + b)$, then $(a + b) \mid 3a$ and $1 \leq \frac{3a}{a+b} \leq 3$. If $\frac{3a}{a+b} = 1$, then $2a = b$, we have $\{12, 24, 48\}$. If $\frac{3a}{a+b} = 2$, then $a = 2b$, we have $\{21, 42, 84\}$. If $\frac{3a}{a+b} = 3$, then $b = 0$, we have $\{30, 60\}$.
<u>Case 3:</u> If $9 \mid (a + b)$, then $a + b = 9$ or 18. If $a + b = 18$, then $a = b = 9$, which is impossible. If $a + b = 9$, then we have 9 cute numbers $\{18, 27, 36, 45, 54, 63, 72, 81\}$.
Therefore, in total there are 23 2-digit cute positive integers.

B7: 11

Given,

$$\begin{aligned}
&A_{51} - A_{50} = l_{51}w_{51} - l_{50}w_{50} = 1000 \\
\implies &(l_1 + 50d)(w_1 + 50D) - (l_1 + 49d)(w_1 + 49D) = 1000 \\
\implies &l_1w_1 + 50l_1D + 50w_1d + 50^2dD - l_1w_1 - 49l_1D - 49w_1d - 49^2dD = 1000 \\
\implies &l_1d + w_1D + 99dD = 1000 \\
\implies &l_1d + w_1D = 100 - 99dD = 100 - 990 = 10.
\end{aligned}$$

Also,

$$\begin{aligned} p = A_{20} - A_{10} = l_{20}w_{20} - l_{19}w_{19} \\ &= (l_1 + 19d)(w_1 + 19D) - (l_1 + 9d)(w_1 + 9D) \\ &= 10l_1 d + 10w_1 D + 280dD \\ &= 10(l_1 d + w_1 D) + 2800 = 100 + 2800 = 2900. \end{aligned}$$

Therefore, the sum of the digits of p is 11.

B8: 16

Since B cannot be empty, the number of elements in B is between 1 to 7. After picking B with k elements, there are $2k-1$ possible subsets of B which qualifies for A. Thus the total number of possibilities is:

$$\begin{aligned} n = \sum_{k=1}^{7} \binom{7}{k}(2^k - 1) &= \sum_{k=0}^{7} \binom{7}{k}(2^k - 1) \\ &= \sum_{k=0}^{7} \binom{7}{k} 2^k - \sum_{k=0}^{7} \binom{7}{k} \\ &= (2+1)^7 - 2^7 \\ &= 2187 - 128 = 2059, \end{aligned}$$

and the sum of the digits of n is 16.

B9: 03

We have the following cases for 9-like positive integers n:

<u>Case 1:</u> If $n < 1000$, write $n = \overline{abc}$. Then,

$$a + b + c = 9;\ a, b, c \in \{0, 1, \ldots, 9\}$$

<u>Case 2:</u> If $1000 \le n < 2000$, write $n = \overline{1abc}$. Then,

$$a + b + c = 8;\ a, b, c \in \{0, 1, \ldots, 8\}$$

<u>Case 3:</u> If $2000 \le n \le 2023$, then only $n = 2007$ and $n = 2016$ are 9-like.

Then the total number of 9-like positive integers is:

$$N = \binom{9+3-1}{9} + \binom{8+3-1}{8} + 2 = 55 + 45 + 2 = 102$$

and the sum of the digits of N is 3.

B10: 00

We have,

$$T_{n+1} - T_n = \frac{n+1}{2}(n + 2 - n) = n + 1$$

and

$$T_{n+1} + T_n = \frac{n+1}{2}(n+2+n) = (n+1)^2.$$

So,

$$T_{n+1}^2 - T_n^2 = (n+1)^3.$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{2k}(-1)^i T_i^2 &= (2k)^3 + (2k-2)^3 + \cdots + 2^3 \\
&= 8(k^3 + (k-1)^3 + \cdots + 1^3) \\
&= 2k^2(k+1)^2
\end{aligned}$$

Now $T_{2k} = k(2k+1)$. So we must have

$$\begin{aligned}
& 2k+1 \mid 2k(k+1)^2 \\
\Longrightarrow\ & 2k+1 \mid (k^2+k)(2k+2) \\
\Longrightarrow\ & 2k+1 \mid (k^2+k)(2k+1) + k^2 + k \\
\Longrightarrow\ & 2k+1 \mid k^2+k \\
\Longrightarrow\ & 2k+1 \mid 4(k^2+k) - (2k+1)^2 \\
\Longrightarrow\ & 2k+1 \mid -1
\end{aligned}$$

This gives $2k+1 = \pm 1 \Rightarrow k = 0, -1$, which is not possible as k is positive integer. So the number of elements in S is 0.

B11: 45

2023 is the 1012^{th} odd number. If 2023 is in the group $k+1$, then

$$\begin{aligned}
& 1 + 2 + \cdots + k < 1012 \leq 1 + 2 + \cdots + (k+1) \\
\Longrightarrow\ & \frac{k(k+1)}{2} < 1012 \leq \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Thus, we get $k = 44$. So 2023 is in the group $k+1 = 45$ (and that group has 45 odd integers).

B12: 03

Here,

$$a_2 = \frac{1}{1+a_1}, a_3 = \frac{1}{1+a_2} = \frac{1}{1+\frac{1}{1+a_1}} = \frac{1+a_1}{2+a_1}.$$

Also,

$$a_4 = \frac{2+a_1}{3+2a_1}, a_5 = \frac{3+2a_1}{5+3a_1} \text{ and so on.}$$

In general,

$$a_n = \frac{F_n + F_{n-1}a_1}{F_{n+1} + F_n a_1}$$

where F_n denotes the n^{th} Fibonacci number starting from $F_1 = 0$, i.e., $F_1 = 0, F_2 = 1, F_{n+1} = F_n + F_{n-1}$.
Therefore,

$$\begin{aligned} a_{2023} = & \frac{F_{2023} + F_{2022}a_1}{F_{2024} + F_{2023}a_1} = a_1 \\ & \implies F_{2023} + F_{2022}a_1 = F_{2024}a_1 + F_{2023}a_1^2 \\ & \implies F_{2023}a_1^2 - F_{2023} + (F_{2024-F_{2022}})a_1 = 0 \\ & \implies (a_1^2 + a_1 - 1)F_{2023} = 0 \\ & \implies a_1^2 + a_1 - 1 = 0 \end{aligned}$$

If the roots of this equation are α and β, then they are the only possible values of a_1. We want to find $\alpha^2 + \beta^2$. But

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-1)^2 - 2(-1) = 3.$$

B13: 14
Let $\theta = \frac{2n\pi}{7}$. We have,

$$\begin{aligned} & 4\theta = 2n\pi - 3\theta \\ \implies & \sin 4\theta = -\sin 3\theta \\ \implies & 2\sin 2\theta \cos 2\theta = 4\sin^3\theta - 3\sin\theta \\ \implies & 4\sin\theta\cos\theta(1 - 2\cos^2\theta) = 4\sin^3\theta - 3\sin\theta \\ \implies & 4\cos\theta(1 - 2\cos^2\theta) = 4\sin^2\theta - 3 \end{aligned}$$

Squaring both sides, we have

$$\begin{aligned} & 16(1 - \sin^2\theta)(1 - 2\sin^2\theta)^2 = (4\sin^2\theta - 3)^2 \\ \implies & 64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7 = 0 \end{aligned}$$

This is a cubic equation in $\sin^2\theta$ where $\theta = \frac{2n\pi}{7}$, $n = 1, 2, 3, 4, 5, 6$. But note that $\sin^2\left(\frac{2\pi}{7}\right) = \sin^2\left(\frac{12\pi}{7}\right)$, $\sin^2\left(\frac{4\pi}{7}\right) = \sin^2\left(\frac{10\pi}{7}\right)$,$\sin^2\left(\frac{6\pi}{7}\right) = \sin^2\left(\frac{8\pi}{7}\right)$. So the roots of the cubic can be written as $\sin^2\left(\frac{2\pi}{7}\right), \sin^2\left(\frac{4\pi}{7}\right), \sin^2\left(\frac{8\pi}{7}\right)$. Also using $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$, we can show that $\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)\sin\left(\frac{2\pi}{7}\right) = 0$. Therefore,

$$\left(\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)\right)^2 = \frac{112}{64} + 0,$$

and hence

$$\sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right)=\frac{\sqrt{7}}{2}$$

So, $ab = 14$.

B14: 04

The main idea behind the problem is a binomial identity. Consider the following sum.

$$\binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{n-1}=2^{n-1}$$
$$\implies \frac{1}{n}\binom{n}{1}+\frac{2}{n}\binom{n}{2}+\frac{3}{n}\binom{n}{3}+\cdots+\frac{n}{n}\binom{n}{n}=2^{n-1}$$
$$\implies 1\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}=n2^{n-1}$$

Let

$$Q(x)=x+2\times\frac{x(x-1)}{2!}+3\times\frac{x(x-1)(x-2)}{3!}+4\times\frac{x(x-1)(x-2)(x-3)}{4!}$$
$$+5\times\frac{x(x-1)(x-2)(x-3)(x-4)}{5!}+6\times\frac{x(x-1)(x-2)(x-3)(x-4)(x-5)}{6!}$$
$$+7\times\frac{x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)}{7!}$$

Clearly $P(x)-Q(x)$ is zero for $x = 0, 1, 2, 3, 4, 5, 6, 7$. Since both P and Q are of degree seven, we must have $P(x) = Q(x)$ for all x as otherwise $P - Q$ which is a seven degree polynomial will have eight roots, giving

$$|P(-1)| = |Q(-1)| = |-4| = 4.$$

B15: 63

Let

$$\sqrt{\frac{(n+1)(2n+1)}{6}}=m$$

and so

$$(n+1)(2n+1)=6m^2.$$

Therefore, $6 \mid (n+1)(2n+1)$ and hence $n \equiv 1$ or $5 \pmod 6$.

Case 1: $n = 6k + 5$.

Then $m^2 = (k + 1)(12k + 11)$. Since $\gcd(k + 1, 12k + 11) = 1$, each one of them must be squares, say $k+1 = a^2$ and $12k+11 = b^2$ for positive integers a and b. Then $12a^2 = b^2 + 1$. But since $12a^2 \equiv 0 \pmod 4$ and $b^2 + 1 \equiv 1$ or

2 (mod 4), so this case cannot happen.
Case 2: $n = 6k + 1$.
Then $m^2 = (3k+1)(4k+1)$. Since $\gcd(3k+1, 4k+1) = 1$, each one of them must be squares, say $3k + 1 = a^2$ and $4k + 1 = b^2$ for positive integers a and b. Then $3b^2 = (2a - 1)(2a + 1)$. Observe that in the left-hand side, every prime factor except 3 has an even power. So neither $2a - 1$ nor $2a + 1$ can be a prime other than 3. If $a = 1$, then $b = 1$ and $n = 1$. So we consider $a \geq 2$. The next smallest suitable value for a is 13. When $a = 13$, we have $3b^2 = 25 \times 27$ and so $b = 15$ implying that $k = 56$ and so $n = 6k + 1 = 337$. And the product of the digits of 337 is 63.