CHENNAI MATHEMATICAL INSTITUTE

B.Sc. Analysis-2

End-term Examination, 2023, Aug-Nov

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Part A

The problems in this section are either already discussed in the class or a slight variation of that. Give the required proof completely with all details. You can score up to a maximum of 50 marks from this section.

1. If $U \subseteq \mathbb{R}^n$ is open and connected, then show that U is path connected. 10

2. Answer (with proofs) whether the set of all real invertible matrices in $M_n(\mathbb{R})$ and the set of all complex invertible matrices in $M_n(\mathbb{C})$ are path connected ? 10

3. Let X be a compact metric space. Let A be a subalgebra of $C(X)_{\mathbb{R}}$. Assume that A separates points of X. Show that the closure of A in $C(X)$ with respect to $\|\cdot\|_\infty$ is either $C(X)$ or there exists an $x_0 \in X$ such that $A = \{f \in C(X) : f(x_0) = 0\}$. ((You can assume Stone-Weistrass theorem.) 10

4. Let $k \in C([0,1] \times [0,1])$. Define $T_k : C([0,1]) \mapsto C([0,1])$ by

$$(T_k f)(x) = \int_0^1 k(x,y) f(y) dy.$$

Show that the set $\{T_k f : \|f\|_{\infty} \leq 1\}$ is totally bounded in $C([0,1])$. 10

5. Let H be a separable Hilbert space. Show that an orthonormal set $\{e_i\}_{i \in I}$ (where I is countable) is maximal if and only if $x = \sum_{i \in I} \langle x, e_i \rangle e_i$ for all $x \in H$. 12

6. Assuming uniform boundedness principle prove the existence of a function whose Fourier series does not converge at any given point. 20

7. Let $f : [0, 2\pi] \mapsto \mathbb{C}$ is continuously differentiable, prove that the Fourier sum $S_N(f)$ converges to f uniformly as $|N| \to \infty$. 10

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Part B

You may use any result proved in the class or in assignments. You can score up to a maximum of 75 marks from this section.

1. Let X and Y be compact metric spaces and $f : X \mapsto Y$ a continuous surjection such that for all $y \in Y$, $f^{-1}(y)$ is connected. Show that for every connected subset $C \subseteq Y$, $f^{-1}(C)$ is connected. 25

2. Let $F : [0,1] \times [0,1] \mapsto \mathbb{R}$ be continuous and satisfy

$$\int_0^1 \int_0^1 F(x,y) f(x) g(y) dx dy = 0,$$

for all $f, g \in C([0,1])$. Show that F identically equals to 0. 15

3. Let $K \subset L^2([-\pi,\pi])$ be a compact subset. Prove that for any given $\epsilon \geq 0$, there exists an $N \in \mathbb{N}$, so that for any $f \in K$, $|n| \geq N$, $|\hat{f}(n)| < \epsilon$. 20

4. Prove either (i) or (ii)

(i) Let

$$\Delta_f(r) = \sup_{s,t \in [-\pi,\pi], |s-t|<r} |f(s) - f(t)|.$$

Let $f \in C^0([-\pi,\pi])$ satisfies

$$\int_{-\pi}^{\pi} \frac{\Delta_f(r)}{|r|} dr < \infty.$$

Then show that the Fourier series $\{S_n(f) : n \in \mathbb{Z}\}$ onverges uniformly to f. 22

(ii) Let $f \in C^0([-\pi,\pi])$ which satisfies for some $C > 0$, $\alpha \in (0,1]$,

$$|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all $x, y \in [-\pi,\pi]$. Show that the Fourier series $\{S_n(f) : n \in \mathbb{Z}\}$converges uniformly to f. 16

5. Assuming (Part A, Problem 6), prove that there exists uncountably many continuous functions on $[0,2\pi]$, whose Fourier series diverge on a dense G_δ subset of $[0,2\pi]$. 20

6. Let $f \in C^0([0,2\pi])$. For $a > 0$, define

$$(A_a f)(t) = \sum_{n=-\infty}^{\infty} e^{-a|n|} \hat{f}(n) e^{int}, \quad \forall t \in [0,2\pi]).$$

Prove that $A_a f \mapsto f$ uniformly as $a \mapsto 0$. 35

7. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (Hint: Consider $f(x) = x^2$ on $[-\pi,\pi]$.) 20