Analysis II Assignment 3

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Solution 1:

Given, f, g are continuous functions satisfying

$$\int_0^1 f(t)t^n dt = \int_0^1 g(t)t^n dt$$

for all integers $n \geq 0$. Therefore,

$$\int_0^1 (f(t) - g(t))t^n dt = 0,$$

and hence

$$\int_0^1 (f(t) - g(t))p(t) dt = 0$$

for all polynomials $p(t) \in \mathbb{R}[t]$. Let $h(t) = f(t) - g(t)$ and $\epsilon > 0$. Then, since h is continuous on $[0, 1]$, by Weierstrass approximation theorem, there exists a polynomial p such that $|h(t) - p(t)| < \epsilon$ for all t in $[0, 1]$.

We shall prove that $h(t) = 0 \ \forall \ t \in [0, 1]$. We have,

$$\begin{aligned}
\int_0^1 \left(h(t)^2\right) dt &= \left|\int_0^1 \left(h(t)^2\right) dt\right| \\
&= \left|\int_0^1 h(t)(h(t) - p(t)) dt\right| \\
&< \left|\int_0^1 h(t) \cdot \epsilon dt\right| \\
&\leq \epsilon \cdot \int_0^1 |h(t)| dt.
\end{aligned}$$

Since $h(t) = f(t) - g(t)$ is continuous on $[0, 1]$, therefore it is bounded on $[0, 1]$. Hence,

$$\int_0^1 |h(t)| dt \leq N$$

for some $N > 0$. Thus,

$$\int_0^1 \left(h(t)^2\right) dt < \epsilon$$

for all $\epsilon > 0$. Therefore,

$$\int_0^1 \left(h(t)^2\right) dt = 0,$$

and hence $h(t) = 0$, i.e., $f = g$ as required.

Solution 3:

Given set is $\mathcal{S} = \{z : z = \exp(i \sin t), t \in \mathbb{R}\} \subseteq \mathbb{C}$. We claim that $\mathcal{S}$ is path connected and hence connected. Consider two elements $z_1 = \exp(i \sin t_1), z_2 = \exp(i \sin t_2) \in \mathcal{S}$. Define $f : [0, 1] \to \mathbb{C}$ by $f(x) = \exp\left((1 - x) \sin t_1 + x \sin t_2\right)$. Then clearly f is continuous and $f(0) = \exp(i \sin t_1) = z_1$, $f(1) = \exp(i \sin t_2) = z_2$. Also, we have,

$$\min\{\sin t_1, \sin t_2\} \leq (1 - x) \sin t_1 + x \sin t_2 \leq \max\{\sin t_1, \sin t_2\}.$$

This gives,

$$-1 \leq (1 - x) \sin t_1 + x \sin t_2 \leq 1.$$

Therefore, there exists $x_t \in \mathbb{R}$ such that

$$\sin x_t = (1-t)\sin t_1 + t\sin t_2.$$

Thus, we have shown that $f(t) = e^{i\sin x_t} \in \mathcal{S}$ and hence S is path connected. Hence, S is connected.