Analysis I, Final Exam
23 November, 2022
Total Points: 40, Duration: 3 hours

Start the solution of each problem on a new sheet and put them in the correct order at the end

- $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of rational numbers and real numbers, respectively.

1. (1+2+2+3 points) Determine true or false. You have to justify your answers for nonzero marks.
   (a) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function whose derivative is positive everywhere. Then f is injective (one-one).
   (b) Let $f : [a, b] \to \mathbb{R}$ be a bounded function such that the function f^2 is Riemann integrable. Then f is also Riemann integrable.
   (c) Let $f : [a, b] \to \mathbb{R}$ be a bounded function such that the function $|f|$ is Riemann integrable. Then f is also Riemann integrable.
   (d) Composition of Riemann integrable functions is Riemann integrable. **Hint:** Use the function in Problem 5

2. (7 points) Let a_n, b_n be real numbers such that the series $\sum a_n^2$ and $\sum b_n^2$ both converge. Show that the series $\sum a_n b_n$ converges absolutely.

3. (7 points) For any positive integer n, show that a real polynomial[1] of degree n has at most n roots.
   Hint: Use the Mean Value Theorem.

4. (1+7 points)
   (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Let Y be a dense[2] subset of $\mathbb{R}$ such that $f(y) = g(y)$ for all $y \in Y$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions for all $x, y \in \mathbb{R}$:
   $$f(0) = 0, f(1) = 1, f(xy) = f(x)f(y), f(x + y) = f(x) + f(y).$$
   Show that f is the identity function.

5. (8 points) Consider the function $f : [-1, 1] \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} 0, \text{ if } x \notin \mathbb{Q}, \\ 1, \text{ if } x = 0, \\ \frac{1}{n}, \text{ if } 0 \neq x = \frac{m}{n} \ (m, n \text{ are coprime integers}) \end{cases}$$
   Show that f is Riemann integrable and compute $\int_{-1}^{1} f(x)\, dx$.

6. (2 points) Write the statement of your favourite theorem in this course (full statement with all the hypotheses, and no proof).

[1] A real polynomial of degree n is an expression $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_n, \ldots, a_0$ are real numbers with $a_n \neq 0$ and x is an indeterminate. A root of $f(x)$ is a real number λ such that $f(\lambda) = 0$.
[2] A subset Y of a metric space X is *dense* if the closure of Y is X.