PI APPROXIMATION DAY MATHQUIZ
First Round *on* 09/07/2025

Total marks: 35 (5 in Part A and 30 in Part B)

Instructions:

Part A

(i) This part contains knowledge type problems related to mathematics.

(ii) There are 3 problems. Problem A1 carries 1 mark. Problems A2 and A3 carry 2 marks each. Total marks in Part A is 5.

(iii) The answers of Part A problems are of one word or a short answer of 5-6 words, if reason is asked.

(iv) Phonetically correct spellings are accepted if the answer is a person's name. However, in case of a tie, correct spellings will be preferred.

(v) There is only one starred problem in Part A which is A2. Starred problems will be used to break ties.

Part B

(i) This part contains problems from Algebra, Geometry, Number Theory and Combinatorics.

(ii) There are 10 problems. Problems B1 to B4 carry 2 marks each. Problems B5 to B8 carry 3 marks each. Problems B9 and B10 carry 5 marks each. Total marks in Part B is 30.

(iii) The answers of Part B problems are integers from 00-99.

(iv) If the answer is a one-digit integer, you are advised to put a 0 before it. For example, if the answer is 7, you are advised to write 07 and if the answer is 0, you are advised to write 00.

(v) A problem Bn is starred if n is an even number, i.e., problems B2, B4, B6, B8 and B10 are starred. Starred problems will be used to break ties.

The test starts at 8:00 pm and ends at 9:00 pm sharp. Submit the answers on or before 8:58 to avoid any internet issues at the final moment, because there is no option of automatic submission. You are advised to use 4-5 minutes for Part A problems and the remaining time for Part B problems. Do not panic and enjoy the problems!

All the best!

PROBLEMS

Part A

A1: A mathematician named 'X' described a famous thought experiment about a hotel with an infinite number of rooms, numbered 1, 2, 3, and so on. In this hotel, every room is occupied, but 'X' showed that:

1. You can still accommodate one more guest by moving the guest in room 1 to room 2, room 2 to room 3, and so forth.
2. You can also accommodate any finite number of new guests by shifting the occupants accordingly.
3. Surprisingly, you can even accommodate infinitely many new guests, just as many as were already in the hotel, by a clever reassignment of rooms: move each current guest from room k to room $2k-1$ (the odd-numbered rooms), freeing up all the even-numbered rooms; then place the new guests $n_1, n_2, n_3, \ldots$ into rooms 2, 4, 6, and so on.

Who is the mathematician 'X' famous for this idea?

A2:$\star$ In a university class, the professor is teaching about conditional statements. She begins by explaining that a conditional statement has the form:

"If P, then Q."

Here, P is the hypothesis, and Q is the conclusion. She proceeds by introducing the following conditional statement.

"If a number is divisible by 4 (P), then it is even (Q)."

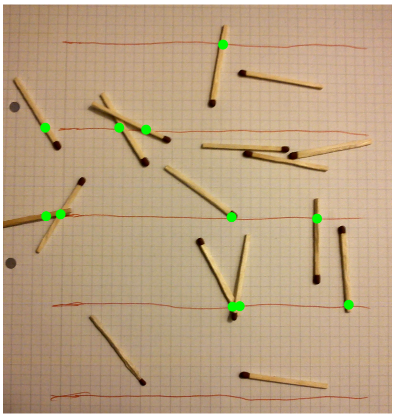
The professor then introduces two related statements derived from the original conditional:

1. Statement X: "If a number is even (Q), then it is divisible by 4 (P)."
2. Statement Y: "If a number is not even ($\neg Q$), then it is not divisible by 4 ($\neg P$)."

She explains that statement X is formed by swapping the hypothesis and conclusion of the original statement and that statement Y is formed by negating both the hypothesis and conclusion of the original statement and reversing their order.

The professor points out that Y always has the same truth value as the original conditional, i.e., Y is always equally true or false as the original statement. However, statement X does not necessarily share this property and may be false even when the original statement is true. What are the mathematical names of statements X and Y?

A3: In a university lab, students are performing a famous probability experiment. The setup involves dropping a collection of thin sticks (or matches) onto a sheet of paper ruled with equally spaced parallel lines. In their experiment, each match is the length of 9 squares on the grid, and the parallel lines are also spaced 9 squares apart. After throwing 17 matches at random, they observe that 11 of the matches have landed such that they cross one of the parallel lines (marked by the green points), as shown in the image below:



The professor asks them to compute the value of the fraction

$$\frac{2 \cdot l \cdot n}{t \cdot h},$$

where l is the length of each stick, n is the total number of throws, t is the spacing between the lines, and h is the number of hits (sticks crossing a line). Substituting $l = 9$, $n = 17$, $t = 9$, and $h = 11$, they compute

$$\frac{2 \cdot l \cdot n}{t \cdot h} = \frac{2 \cdot 9 \cdot 17}{9 \cdot 11} = \frac{306}{99} \approx 3.1.$$

This experiment is based on a famous problem in geometric probability posed in the 18th century. What is being estimated in this experiment, and who was the mathematician that first proposed this problem?

Part B

B1: Find the number of ordered triples of non-negative integers (a, b, c) such that

$$abc + ab + bc + ca + a + b + c = 2025.$$

B2:$\star$ How many integer pairs (x, y) satisfy the equation

$$20x + 25y + 17 = xy?$$

B3: A 6-digit number $abcdef$ is called a *half-palindrome* if either the first three digits abc or the last three digits def form a palindrome. What percentage of all 6-digit numbers are half-palindromes?

B4:$\star$ Let $x_1, x_2, \ldots, x_{2020}$ be real numbers such that

$$5^{x_1} = 6,\ 6^{x_2} = 7,\ 7^{x_3} = 8, \ldots, 2024^{x_{2020}} = 2025.$$

What is the value of

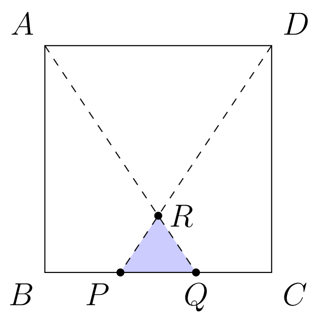
$$\sqrt{5^{x_1 x_2 \cdots x_{2020}}}?$$

B5: Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example: $\lfloor 2.5 \rfloor = 2$ and $\lfloor 3 \rfloor = 3$. Consider the sequence

$$\left\lfloor \frac{1}{1} \right\rfloor, \left\lfloor \frac{1}{2} \right\rfloor, \left\lfloor \frac{2}{2} \right\rfloor, \left\lfloor \frac{1}{3} \right\rfloor, \left\lfloor \frac{2}{3} \right\rfloor, \left\lfloor \frac{3}{3} \right\rfloor, \ldots$$

where for each positive integer k, the terms $\left\lfloor \frac{n}{k} \right\rfloor$ are listed for $n = 1, 2, \ldots, k$. Compute the sum of the first 2025 terms of this sequence.

B6:$\star$ In the figure below, $ABCD$ is a unit square (i.e., with side length 1 unit) with points P, Q on side BC such that $BP = PQ = QC$. Let R be the point of intersection of AQ and DP. The area of $\triangle PQR$ is $\frac{1}{N}$ square units, where N is a positive integer. Find N.



B7: Let n be a natural number. At every step, you are allowed to perform either of the following two operations on n:

1. Replace n with $2n$.
2. Replace n with $n - 5$ (only if $n > 5$).

For example, starting with $n = 8$, it is possible to reach 11 in two steps:

$$8 \to 8 \times 2 = 16 \to 16 - 5 = 11.$$

Starting from $n = 20$, determine the *minimum* number of steps required to reach 2025.

B8:$\star$ A computer generated the following sequence of fractions:

$$\frac{2}{9}, \frac{4}{11}, \frac{6}{13}, \frac{8}{15}, \ldots, \frac{2018}{2025}.$$

Let N be the number of fractions from the above sequence, which are in simplest form. What is the sum of the digits of N?

B9: A class is divided into three groups: Red, Green, and Blue. For a yes/no general knowledge question:

1. The teacher answers correctly with probability α; $\alpha \neq \frac{1}{2}$.
2. A student in the Red, Green, and Blue groups answers correctly with probabilities β, γ, and δ, respectively.

A student is selected uniformly at random from the class. The probability that this student's answer agrees with the teacher's answer is $\frac{1}{2}$.

It is known that the ratio of the number of students in the Green group to the number in the Blue group is $3:2$. The ratio of the number of students in the Blue group to the number in the Red group can be expressed as

$$\frac{p\beta - 2}{5 - q\gamma - r\delta},$$

where p, q, r are positive integers. Find $p + q + r$.

B10:$\star$ Let N be a three-digit integer that does not end in zero. Define $R(N)$ as the three-digit integer obtained by reversing the digits of N. For example, $R(137) = 731$. How many such integers N exist for which $R(N) = 2N + 5$?

SOLUTIONS

Part A

A1: David Hilbert.

A2: ‘converse’ and ‘contrapositive’.

A3: π, Buffon. (Check it out!)

Part B

B1: 09

Note that

$$(a+1)(b+1)(c+1) = abc + ab + bc + ca + a + b + c + 1.$$

Thus, we can write the given equation as

$$(a+1)(b+1)(c+1) = 2026.$$

Since $1\times2\times1013$. So the possible solutions (a, b, c) are $(1, 2, 1013)$, $(1, 1013, 2)$, $(2, 1, 1013)$, $(2, 1013, 1)$, $(1013, 1, 2)$, $(1013, 2, 1)$, $(1, 1, 2026)$, $(1, 2026, 1)$ and $(2026, 1, 1)$, i.e., $\boxed{9}$ solutions. ■

B2: 08

We can write the given equation as $xy - 20x - 25y + 500 = 517$. The LHS factors neatly to give

$$(x - 25)(y - 20) = 517.$$

This equation implies that $(x - 25)$ and $(y - 20)$ are integer divisors of 517. Since $517 = 11 \times 47$, the divisors of 517 are $\pm1, \pm11, \pm47, \pm517$. Each choice of a divisor d for $(x - 25)$ determines x and y uniquely. Indeed, from $(x - 25)(y - 20) = 517$, we have

$$x = d + 25, \quad y = \frac{517}{d} + 20.$$

As there are 8 divisors of 517, we obtain $\boxed{8}$ integer pairs (x, y) satisfying the original equation. These solutions are

$$(26, 537),\ (36, 67),\ (72, 31),\ (542, 21),$$
$$(24, -477),\ (14, -23),\ (-22, -7),\ (-492, -1).$$

■

B3: 19

There are a total of $9 \times 10^5 = 900000$ six-digit numbers.

We first count the number of 6-digit numbers where the first three digits abc form a palindrome. A 3-digit palindrome has the form aba, where $a \in \{1, \ldots, 9\}$ and $b \in \{0, \ldots, 9\}$. This gives $9 \times 10 = 90$ possible palindromic

abc. Since the last three digits def can be arbitrary, there are $10^3 = 1000$ choices for def. Thus, the total number of 6-digit numbers where abc is a palindrome is $90 \times 1000 = 90000$. Similarly, the total number of 6-digit numbers where def is a palindrome is $90 \times 1000 = 90000$.

Some numbers have both abc and def palindromic. There are 90 possible palindromic abc and 100 possible palindromic def, giving $90 \times 100 = 9000$ such numbers.

By the inclusion-exclusion principle, the total number of half-palindromic 6-digit numbers is

$$90000 + 90000 - 9000 = 171000.$$

The required percentage is

$$\frac{171000}{900000} \times 100 = \boxed{19}\%.$$

■

B4: 45

Observe that

$$\begin{aligned} 5^{x_1 x_2 \cdots x_{2020}} &= \left(5^{x_1}\right)^{x_2 x_3 \cdots x_{2020}} \\ &= 6^{x_2 x_3 \cdots x_{2020}} \\ &= \left(6^{x_2}\right)^{x_3 x_4 \cdots x_{2020}} \\ &= 7^{x_3 x_4 \cdots x_{2020}} \\ &\vdots \\ &= 2023^{x_{2019} x_{2020}} \\ &= 2024^{x_{2020}} = 2025. \end{aligned}$$

Therefore,

$$\sqrt{5^{x_1 x_2 \cdots x_{2020}}} = \sqrt{2025} = \boxed{45}.$$

■

B5: 63

Each group of terms corresponding to k has exactly k terms, i.e., group k has terms

$$\left\lfloor \frac{1}{k} \right\rfloor, \left\lfloor \frac{2}{k} \right\rfloor, \ldots, \left\lfloor \frac{k}{k} \right\rfloor$$

Thus, the total number of terms in the first k groups is:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

Observe that for $k = 63$, $\frac{k(k+1)}{2} = 2016$, so the sum of the first 2025 terms of the given sequence is:

$$\text{sum of the first 2016 terms } + \left\lfloor \frac{1}{2025} \right\rfloor + \left\lfloor \frac{2}{2025} \right\rfloor + \cdots + \left\lfloor \frac{9}{2025} \right\rfloor.$$

Note that for $1 \le r < k$, $\left\lfloor \frac{r}{k} \right\rfloor = 0$ and $\left\lfloor \frac{k}{k} \right\rfloor = 1$. So, the sum of the first 2016 terms is just 63, and obviously

$$\left\lfloor \frac{1}{2025} \right\rfloor = \left\lfloor \frac{2}{2025} \right\rfloor = \cdots = \left\lfloor \frac{9}{2025} \right\rfloor = 0.$$

Thus, the final sum is $\boxed{63}$. ■

B6: 24

Let S be a point on BC such that $RS \perp BC$. Then $RS \parallel DC$, which implies that triangles PRS and PDC are similar. This gives

$$\frac{RS}{DC} = \frac{PS}{PC}.$$

We have, $DC = 1$ and $PC = \frac{2}{3}$. Also, $PS = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$, because $PR = QR$ by symmetry and S is the mid-point of PQ. Thus,

$$\frac{RS}{1} = \frac{1/6}{2/3}.$$

Therefore, $RS = \frac{1}{4}$ and so the area of $\triangle PQR$ is $\frac{1}{2} \cdot PQ \cdot RS = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{24}$. Thus, $N = \boxed{24}$. ■

B7: 12

To solve this, it is helpful to work backward from the target number 2025. Since the only operations available are doubling and subtracting 5, their inverses are halving (if the number is even) and adding 5, respectively.

Starting at 2025, note that 2025 is odd, so halving is not possible. Thus, the only possible inverse operation is to add 5. This gives: $2025 + 5 = 2030$. Now, 2030 is even, so we may consider halving: $2030 \div 2 = 1015$. Since 1015 is odd, we again add 5 to obtain 1020. Then, we halve 1020 to get 510, halve again to get 255, add 5 to get 260, halve to 130, halve to 65, add 5 to 70, halve to 35, add 5 to 40, and finally halve to 20. This gives the following reverse sequence: $20 \to 40 \to 35 \to 70 \to 65 \to 130 \to 260 \to 255 \to 510 \to 1020 \to 1015 \to 2030 \to 2025$. Reading this in forward order, the steps are

$$20 \times 2 = 40,\ 40 - 5 = 35,\ 35 \times 2 = 70,\ 70 - 5 = 65,\ 65 \times 2 = 130,$$

$$130 \times 2 = 260,\ 260 - 5 = 255,\ 255 \times 2 = 510,\ 510 \times 2 = 1020,\ 1020 - 5 = 1015,$$

$$1015 \times 2 = 2030, \; 2030 - 5 = 2025.$$

Therefore, the minimum number of steps required is $\boxed{12}$. ■

B8: 19

The sequence of fractions has numerators $2, 4, 6, \ldots, 2018$ and denominators $9, 11, 13, \ldots, 2025$. The numerators form an arithmetic sequence $2n$ for $n = 1, 2, 3, \ldots$. Similarly, the denominators increase by 2 starting from 9, giving the general formula $2n + 7$. Hence, the nth term of the sequence is $\frac{2n}{2n+7}$ and there are $\frac{2018}{2} = 1009$ terms in total.

For the nth term $\frac{2n}{2n+7}$, consider $\gcd(2n, 2n+7)$. Using the property $\gcd(x, y) = \gcd(x, y - x)$, we find

$$\gcd(2n, 2n+7) = \gcd(2n, 7).$$

Since 7 is prime, $\gcd(2n, 7) = 1$ precisely when $2n$ is not divisible by 7. As 2 and 7 are relatively prime, $2n$ is divisible by 7 if and only if n itself is divisible by 7. Therefore, the fraction $\frac{2n}{2n+7}$ is not in simplest form when n is a multiple of 7.

Among the first 1009 positive integers, the number of multiples of 7 is $\lfloor 1009/7 \rfloor = 144$. These correspond to the fractions not in simplest form. The remaining $1009 - 144 = 865$ fractions are in simplest form. So $N = 865$ and the sum of the digits of N is $8 + 6 + 5 = \boxed{19}$. ■

B9: 14

Let r, g, b be the number of students in groups Red, Green and Blue respectively. A student agrees with the teacher's answer in two scenarios: when both are correct, or when both are incorrect. Therefore, in the case of Red group, the probability of agreement is $\alpha\beta + (1-\alpha)(1-\beta)$, and the probability of selecting a student from the red group is $\frac{r}{r+g+b}$.

Considering all three groups, the total probability that a randomly chosen student agrees with the teacher is

$$\frac{r}{r+g+b}\left[\alpha\beta + (1-\alpha)(1-\beta)\right] + \frac{g}{r+g+b}\left[\beta\gamma + (1-\beta)(1-\gamma)\right] + \frac{b}{r+g+b}\left[\gamma\delta + (1-\gamma)(1-\delta)\right] = \frac{1}{2},$$

which gets simplified as

$$\frac{(r+b+g)(1-\alpha) + (2\alpha - 1)(r\beta + b\gamma + g\delta)}{r+b+g} = \frac{1}{2}.$$

Simplifying further, we get

$$(2\alpha - 1)\left(\frac{r\beta + b\gamma + g\delta}{r+b+g}\right) = \frac{2\alpha - 1}{2}.$$

Now since $\alpha \neq \frac{1}{2}$, we get

$$\frac{r\beta + b\gamma + g\delta}{r + b + g} = \frac{1}{2}.$$

Let $\frac{r}{b} = x$. We can use the fact that $g : b = 3 : 2$ to find the value of $b : r$, as follows:

$$\begin{aligned}
&\frac{\frac{r}{b}\beta + \gamma + \frac{g}{b}\delta}{\frac{r}{b} + 1 + \frac{g}{b}} = \frac{1}{2}, \\
\implies &\frac{x\beta + \gamma + \frac{3}{2}\delta}{x + 1 + \frac{3}{2}} = \frac{1}{2} \\
\implies &2x\beta + 2\gamma + 3\delta = x + \frac{5}{2} \\
\implies &(2\beta - 1)x = \frac{5}{2} - 2\gamma - 3\delta \\
\implies &x = \frac{5 - 4\gamma - 6\delta}{4\beta - 2}.
\end{aligned}$$

Thus,

$$\frac{b}{r} = \frac{4\beta - 2}{5 - 4\gamma - 6\delta},$$

and hence $p = 4, q = 4, r = 6$. Therefore, $p + q + r = \boxed{14}$. ■

B10: 01

Let $N = 100a + 10b + c$ and $R(N) = 100c + 10b + c$ such that $a, c \in \{1, 2, \ldots, 9\}$ and $b \in \{0, 1, 2, \ldots, 9\}$. From $R(N) = 2N + 5$, we get

$$100c + 10b + a = 2(100a + 10b + c) + 5.$$

Simplifying, we get

$$98c = 199a + 10b + 5.$$

Since the LHS is even, a must be odd, so that the RHS becomes even as well. Thus, $a \in \{1, 3, 5, 7, 9\}$. Observe that for $a \geq 5$, $199a + 10b + 5 \geq 1000 > 98c$ for any $c \in \{1, 2, \ldots, 9\}$. Therefore, we just need to check for $a = 1, 3$.

For $a = 1$, we have $98c = 204 + 10b$. Since the RHS ends in 4, c can be 3 or 8, but for $c = 8$, the LHS becomes larger than the RHS. For $c = 3$, we have $b = 9$. Therefore, one such N is 193. For $a = 3$, we have $98c = 602 + 10b$. Since the RHS ends in 2, c can be 4 or 9. For $c = 4$, the LHS is smaller than the RHS and for $c = 9$, the LHS is larger than the RHS. Therefore, only $\boxed{1}$ such N exists, which is 193. ■