Mangled Children

Quantum exploration of a classical epistemic puzzle

Krasimir Georgiev

January 19, 2016

Introduction

The Muddy children puzzle is a well-known epistemic puzzle that illustrates some non-trivial and at first sight even paradoxical features of the dynamics of knowledge. It can however be easily modeled as a relational epistemic model and analyzed completely using basic dynamic epistemic logic. Quantum physics is normally perceived as a notoriously unintuitive discipline. It lacks connections with people's real world intuitions and predicts bizarre, paradoxical behaviours. In this report we explore quantum analogues of the Muddy children puzzle. These variants sometimes exhibit bizarre behaviours, and we use epistemic techniques and concepts to shed some light on the reasons behind them.

Muddy children

The classical Muddy children puzzle goes as follows: several clever (logically omniscient) children are playing with mud; here comes the father and tells them (publicly announces): some of you have mud on their foreheads. Then he proceeds to ask the same question: do you know if you have mud on your forehead; each time the children all answer (publicly announce) at the same time truthfully. Each child can see everyone else's foreheads, except for its own. During the first several rounds, everyone answers negatively. But then, at some round, exactly the muddy children answer positively. This is seemingly paradoxical: by listening to the same answers of the same question over and over again, the children come to learn more.

For n children, we may model the puzzle by an epistemic model where the worlds are all the strings over $\{0, 1\}$ of length n, where 1 at position i denotes that the i-th child has mud on its forehead (and 0 that it's clean). The indistinguishability relation R_i for the i-th child contains all the pairs of strings that possibly differ only at the i-th position. Suppose that u of the children are actually muddy. The public announcement of the father effectively deletes the world 0^n. At round $r = 1, 2, \ldots, u-1$, the public announcements of the children delete all the worlds having r ones. At round $r = u$, since all worlds with a lower number of ones are deleted, all the muddy children can distinguish the real world, so they give a positive answer.

Quantum reading

A postulate of quantum physics is that quantum states can be modelled as unit vectors $s \in \mathcal{H}, |s| = 1$ in some Hilbert space. This is equivalent (up to a sign, which is insignificant with respect to measurements) with the one dimensional subspaces interpretation of quantum states, and it more easy to work with numerically.

In the quantum world, observations are both limited and limiting. Only the "testable" properties (corresponding to closed subspaces of the Hilbert space) can possibly be observed, and observing them leads to a real, ontic "collapse" of the quantum system as a projection into the testable property subspace. On top of that the result of the observation is probabilistic, making most of the implicit information in the quantum state inaccessible for observers.

These basic quantum principles force us to reconsider the basic ingredients of the Muddy children puzzle if we hope to achieve a quantum version of it. Since the detection of mud amounts to an observation of the quantum system, which collapses it, there may be no definite presence or absence of mud on the children's foreheads at all. In general, the forehead might exist in a superposition of the presence and absence of mud, that is additionally entangled with the foreheads of the other children. Since the observation updates the initial quantum state irreversibly, it no longer makes much sense to talk about the absence or presence of mud on any child's forehead in general. A possible adaptation is: to say "if you observe your forehead, it is possible that you will see some mud". Another variation might be: "if you observe your forehead, you will surely see some mud", or even "you will observe your forehead being muddy with probability $P \in [0, 1]$". Let us see what these mean in terms of the states of the system. Let $s \in \mathcal{H}$ be unit vector corresponding to the real state of the system and let $m \in \mathcal{H}$ be the unit vector corresponding to subspace of the forehead observation. The first interpretation corresponds to the non-orthogonality of s and m: $s \not\perp m$. The second corresponds to the perpendicularity of s and m, and the third corresponds to the following: in an orthonormal basis for $\mathcal{H}$ containing m, the state s will have a coefficient (amplitude) $\alpha \in \mathbb{C}$ along m for which the probability is the square of the amplitude: $P = |\alpha|^2$. In this version, we also need to assume that there are infinitely many copies of the initial quantum system, and after performing infinitely many observations, the probability in the limit for a muddy forehead will reach P.

Another issue that pops up is the simultaneous observations. Since an observation has an ontic effect, it is in general impossible for all observers to observe the system at the same time. Only in specific cases (as in the next section) will it be possible to combine two observable properties and to get an observable property. So in general, the interpretation of disjunction might be problematic, because it might lead to a non-observable property. We may need to introduce a total order between the observations, so that the system first collapses to the subspace of the first observation first, then to the subspace of the next, and so on. Note that this holds only for the quantum observation part of the action of a child; it is still possible for them, after having made all observations, to answer to the question posed by the father classically simultaneously (at least up to the epistemic characteristics of their answers). For convenience, we will number the children by $1, 2, \ldots, n$ and we will assume that they make their observations in this order.

TODO Also the question "do you know if you are dirty" needs an adaptation. It is "after doing your quantum observation, do you know if you did an observation of your forehead, you will be dirty".

Qubit mud

Let us consider a situation in which there are just $n = 2$ children, Alice and Bob and there is a qubit for the mud on each one's forehead. Alice's forehead has two possible observable states: $|a\rangle$, in which case her forehead is clean, and $|A\rangle$, in which case she is dirty. Similarly for Bob the states are $|b\rangle$ and $|B\rangle$. A general state of this system is a unit vector in a four-dimensional Hilbert space $\mathcal{H}$ having the form $s = \delta\,|ab\rangle + \beta\,|aB\rangle + \alpha\,|Ab\rangle + \gamma\,|AB\rangle$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. Assume that this is common knowledge, so our initial epistemic model contains all these states as possible worlds. Also, since observation has an ontic effect, the kids don't really look at each other's foreheads, so the indistinguishability relation R_i is the full relation for both of them.

Then comes the quantum omniscient father (he is cool because he can see the complex coefficients along the basis vectors *without* disturbing the system), and tells them: some of you has mud on its forehead. In this particular case, there is a yet another interesting quantum interpretation of this statement: that the coefficient δ is zero. Observe that this is a different interpretation than any of the interpretations discussed in the previous section. Here, since the property of absence of mud on both foreheads actually corresponds to the basis vector $|ab\rangle$, it is testable, and we give the rough interpretation as: "well kids, in this particular situation, as you can see, you may in principle observe both foreheads at the same time, and if you did that, you'll surely not observe two clean ones". This public announcement effectively deletes the states having $\delta \neq 0$, leaving us with an epistemic model containing states that are the unit vectors of the form $s = \beta\,|aB\rangle + \alpha\,|Ab\rangle + \gamma\,|AB\rangle$.

Now the father asks the children if they know if they are dirty (after a quantum observation) and Alice observes Bob's forehead. By doing so, she collapses the system to a state that is consistent to the answer she gets. In detail:

- She observes that Bob is muddy, which might happen with probability $|\beta|^2 + |\gamma|^2$. In this case, the system collapses to the state $s' = \frac{\beta|aB\rangle+\gamma|AB\rangle}{|\beta|^2+|\gamma|^2}$. She confuses all the states of this form, and she answers "no" to the question of the father. TODO: BOB?
- She observes that Bob isn't muddy, which might happen with probability $|\alpha|^2$. In this case, the system collapses to the state $s'' = \frac{\alpha|aB\rangle}{|\alpha|^2}$, which is a unit complex multiple of the basis vector $|aB\rangle$. She of course confuses all the states of this form, but she knows that the only possible outcome of measuring her dirtiness at any state of the form s'' must be $|a\rangle$, so she now knows that she is dirty and answers "yes" to the question of the father. TODO: BOB?

Shared mud

Spin mud