

# Satisfiability with Equivalences in Agreement, Part 2

Krasimir Georgiev

September 1, 2016

# Two-variable Logic

- ▶ The two-variable logic  $\mathcal{L}^2$  is the fragment of first-order logic featuring only the variables  $x$  and  $y$  (with formal equality and restricted to just unary and binary predicate symbols).
- ▶ “resource restriction”
- ▶ connections with modal logic

# Satisfiability Results

- ▶  $\mathcal{L}^2$  has the finite model property [Mortimer, 1975] and its (finite) satisfiability problem is NEXPTIME-complete [Grädel, Kolaitis, Vardi, 1997].

# Satisfiability Results

- ▶  $\mathcal{L}^2$  has the finite model property [Mortimer, 1975] and its (finite) satisfiability problem is NEXPTIME-complete [Grädel, Kolaitis, Vardi, 1997].
- ▶  $\mathcal{L}^2\text{1E}$  has the finite model property and its satisfiability problem is NEXPTIME-complete [Kieroński, 2005].

# Satisfiability Results

- ▶  $\mathcal{L}^2$  has the finite model property [Mortimer, 1975] and its (finite) satisfiability problem is NEXPTIME-complete [Grädel, Kolaitis, Vardi, 1997].
- ▶  $\mathcal{L}^2\text{1E}$  has the finite model property and its satisfiability problem is NEXPTIME-complete [Kieroński, 2005].
- ▶  $\mathcal{L}^2\text{2E}$  lacks the finite model property and both its satisfiability and finite satisfiability are N2EXPTIME-complete [Kieroński, Michaliszyn, Pratt-Hartmann, Tendera, 2014],

# Satisfiability Results

- ▶  $\mathcal{L}^2$  has the finite model property [Mortimer, 1975] and its (finite) satisfiability problem is NEXPTIME-complete [Grädel, Kolaitis, Vardi, 1997].
- ▶  $\mathcal{L}^2 1E$  has the finite model property and its satisfiability problem is NEXPTIME-complete [Kieroński, 2005].
- ▶  $\mathcal{L}^2 2E$  lacks the finite model property and both its satisfiability and finite satisfiability are N2EXPTIME-complete [Kieroński, Michaliszyn, Pratt-Hartmann, Tendera, 2014],
- ▶ Both the satisfiability and finite satisfiability problems for  $\mathcal{L}^2 eE$  are undecidable for  $e \geq 3$  [Kieroński, Otto, 2005].

# Satisfiability Results

- ▶  $\mathcal{L}^2$  has the finite model property [Mortimer, 1975] and its (finite) satisfiability problem is NEXPTIME-complete [Grädel, Kolaitis, Vardi, 1997].
- ▶  $\mathcal{L}^2 1E$  has the finite model property and its satisfiability problem is NEXPTIME-complete [Kieroński, 2005].
- ▶  $\mathcal{L}^2 2E$  lacks the finite model property and both its satisfiability and finite satisfiability are N2EXPTIME-complete [Kieroński, Michaliszyn, Pratt-Hartmann, Tendera, 2014],
- ▶ Both the satisfiability and finite satisfiability problems for  $\mathcal{L}^2 eE$  are undecidable for  $e \geq 3$  [Kieroński, Otto, 2005].
- ▶ **We show that  $\mathcal{L}^2 eE_{\text{refine}}$  has the finite model property and its satisfiability problem is in NEXPTIME for  $e \geq 1$ .**

# Overview

Two-variable Logic

Types

Type Realizability

Type Realizability with Equivalences in Refinement



# Scott Normal Form

## Theorem (Scott, 1962)

*There is a polynomial-time reduction  $\text{sctr} : \mathcal{L}^2 \rightarrow \mathcal{L}^2$  which reduces every sentence  $\varphi$  to a sentence  $\text{sctr } \varphi$  in Scott normal form:*

$$\forall \mathbf{x} \forall \mathbf{y} (\alpha_0(\mathbf{x}, \mathbf{y}) \vee \mathbf{x} = \mathbf{y}) \wedge \bigwedge_{1 \leq i \leq m} \forall \mathbf{x} \exists \mathbf{y} (\alpha_i(\mathbf{x}, \mathbf{y}) \wedge \mathbf{x} \neq \mathbf{y}),$$

*where  $m \geq 1$ , the formulas  $\alpha_i$  are quantifier-free and may use fresh unary predicate symbols, such that  $\varphi$  and  $\text{sctr } \varphi$  are satisfiable over the same domains of cardinality at least 2.*

# Scott Normal Form

$$\forall \mathbf{x} \forall \mathbf{y} (\alpha_0(\mathbf{x}, \mathbf{y}) \vee \mathbf{x} = \mathbf{y}) \wedge \bigwedge_{1 \leq i \leq m} \forall \mathbf{x} \exists \mathbf{y} (\alpha_i(\mathbf{x}, \mathbf{y}) \wedge \mathbf{x} \neq \mathbf{y})$$

Idea: if  $\psi$  is a subformula of  $\varphi$  of the form  $Qx\alpha(x, y)$ , where  $\alpha$  is quantifier-free, then

$$\varphi \text{ and } \varphi' \wedge \forall y (\mathbf{p}(y) \leftrightarrow Qx\alpha(x, y))$$

are satisfiable over the same domains, where  $\mathbf{p}$  is a new unary symbol and  $\varphi'$  is obtained from  $\varphi$  by replacing the subformula  $\psi$  by  $\mathbf{p}(y)$ .

# Classified Signatures

$$\forall \mathbf{x} \forall \mathbf{y} (\alpha_0(\mathbf{x}, \mathbf{y}) \vee \mathbf{x} = \mathbf{y}) \wedge \bigwedge_{1 \leq i \leq m} \forall \mathbf{x} \exists \mathbf{y} (\alpha_i(\mathbf{x}, \mathbf{y}) \wedge \mathbf{x} \neq \mathbf{y})$$

- ▶ replace  $\alpha_i, i \in [1, m]$  with fresh binary predicate *message* symbols  $\mathbf{m}_i$  with interpretation  $\forall \mathbf{x} \forall \mathbf{y} (\mathbf{m}_i(\mathbf{x}, \mathbf{y}) \leftrightarrow \alpha_i(\mathbf{x}, \mathbf{y}))$
- ▶ A *classified signature*  $\langle \Sigma, \bar{\mathbf{m}} \rangle$  consists of a signature  $\Sigma$  together with a sequence of distinct binary predicate symbols  $\bar{\mathbf{m}} = \mathbf{m}_1 \mathbf{m}_2 \dots \mathbf{m}_m$  from  $\Sigma$ .
- ▶ A *structure*  $\mathfrak{A}$  for  $\langle \Sigma, \bar{\mathbf{m}} \rangle$  is a structure for  $\Sigma$  satisfying:

$$\bigwedge_{1 \leq i \leq m} \forall \mathbf{x} \exists \mathbf{y} (\mathbf{m}_i(\mathbf{x}, \mathbf{y}) \wedge \mathbf{x} \neq \mathbf{y})$$

# Classified Signatures

- ▶ The (finite) *classified satisfiability problem* is: given a classified signature  $\langle \Sigma, \bar{\mathbf{m}} \rangle$  and a quantifier-free formula  $\alpha(\mathbf{x}, \mathbf{y})$ , is there a  $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structure  $\mathfrak{A}$  such that  $\mathfrak{A} \models \forall \mathbf{x} \forall \mathbf{y} \alpha(\mathbf{x}, \mathbf{y})$ .
- ▶ Scott tells us how (finite) satisfiability reduces to (finite) classified satisfiability.

# Types

Let  $\Sigma = \langle \mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^n \rangle$  be a predicate signature.

- ▶ A 1-type  $\pi$  over  $\Sigma$  is a maximal consistent set of literals featuring only the variable  $\mathbf{x}$  (in model theory known as atomic type).
- ▶ A 2-type  $\tau$  over  $\Sigma$  is maximal consistent set of literals featuring the variable symbols  $\mathbf{x}$  and  $\mathbf{y}$  and including  $(\mathbf{x} \neq \mathbf{y})$ .

$$\text{tp}_{\mathbf{x}}\tau \bullet \xrightarrow[\tau^{-1}]{\tau} \bullet \text{tp}_{\mathbf{y}}\tau$$

- ▶ If  $\tau$  is a 2-type, the  $\mathbf{x}$ -type  $\text{tp}_{\mathbf{x}}\tau$  is the 1-type consisting of those literals featuring only the variable  $\mathbf{x}$ ,
- ▶ the *inverse*  $\tau^{-1}$  is the 2-type obtained from swapping  $\mathbf{x}$  and  $\mathbf{y}$  in the literals of  $\tau$
- ▶ and  $\text{tp}_{\mathbf{y}}\tau = \text{tp}_{\mathbf{x}}(\tau^{-1})$ .

# Type Instances

- ▶ A *type instance*  $T$  over  $\langle \Sigma, \bar{m} \rangle$  is a nonempty set of 2-types that is closed under inversion.
- ▶ The type instance  $T[\mathfrak{A}]$  of a  $\langle \Sigma, \bar{m} \rangle$ -structure  $\mathfrak{A}$  is:

$$T[\mathfrak{A}] = \left\{ \text{tp}^{\mathfrak{A}}[a, b] \mid a \in A, b \in A \setminus \{a\} \right\},$$

where  $\text{tp}^{\mathfrak{A}}[a, b]$  is the 2-type realized by  $(a, b)$  in  $\mathfrak{A}$ .

- ▶ Type instances are typically exponentially bigger than the classified signature.

# Type Realizability

- ▶ The *(finite) type realizability problem* is: given a classified signature  $\langle \Sigma, \bar{\mathbf{m}} \rangle$  and a type instance  $T$  over  $\langle \Sigma, \bar{\mathbf{m}} \rangle$ , is there a (finite)  $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structure  $\mathfrak{A}$  such that  $T[\mathfrak{A}] = T$ .
- ▶ We aim to show that the type realizability problem for  $\mathcal{L}^2$  is in  $\text{NPTIME}$ .
- ▶ (Finite) classified satisfiability reduces in nondeterministic exponential time to (finite) type realizability: just guess a type instance consisting of 2-types consistent with  $\alpha(x, y)$ .

# Kings and Workers

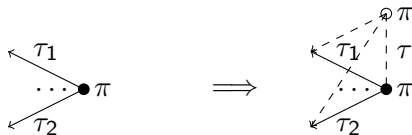
- ▶ Let  $T$  be a type instance over  $\langle \Sigma, \bar{m} \rangle$ . The 1-types of  $T$  are  $\Pi_T = (\text{tp}_x \upharpoonright T)$ .
- ▶ A 1-type  $\kappa \in \Pi_T$  is a *king type* if no  $\tau \in T$  connects  $\kappa$  with itself. The set of king types is  $K_T$ .
- ▶ The remaining 1-types are the *worker types*.
- ▶ In models of  $T$ , king types are realized uniquely, while worker types are realized by at least two elements.



# Worker Copies

If  $\mathfrak{A}$  is a model for  $T$ , any worker element can be copied:

$$\pi \in W_T, \tau \in T, \text{tp}_x \tau = \text{tp}_y \tau = \pi$$



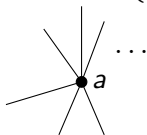
This doesn't work for kings!

$$T[\pi] = \begin{cases} \Pi_T & \text{if } \pi \text{ is a worker type} \\ \Pi_T \setminus \{\pi\} & \text{otherwise, if it is a king type} \end{cases}$$

# Star-types

The *star-type* of  $a \in A$  is:

$$\text{stp}^{\mathfrak{A}}[a] = \left\{ \text{tp}^{\mathfrak{A}}[a, b] \mid b \in A \setminus \{a\} \right\}$$



Gives reason why the existential condition is satisfied locally at  $a$ .



image © 2014 Asia Citro, reproduced with permission from

<http://www.funathomewithkids.com/2014/07/magic-foaming-treasure-stars.html>

# Star-types

A *star-type*  $\sigma$  over  $T$  is a nonempty subset  $\sigma \subseteq T$  satisfying:

$(\sigma x)$  If  $\tau, \tau' \in \sigma$ , then  $\text{tp}_x \tau = \text{tp}_x \tau'$ . Denote  $\text{tp}_x \tau$  for any  $\tau \in \sigma$  by  $\pi = \text{tp}_x \sigma$ .

$(\sigma \pi y)$  If  $\pi' \in T[\pi]$ , then some  $\tau \in \sigma$  has  $\text{tp}_y \tau = \pi'$ .

$(\sigma \kappa y)$  If  $\kappa' \in T[\pi] \cap K_T$  and if  $\tau, \tau' \in \sigma$  have  $\text{tp}_y \tau = \text{tp}_y \tau' = \kappa'$ , then  $\tau = \tau'$ .

$(\sigma m)$  If  $m \in \bar{m}$ , then some  $\tau \in \sigma$  has  $m(x, y) \in \tau$ .

Then:

$(\sigma \kappa y')$  If  $\kappa' \in T[\pi] \cap K_T$ , then a unique  $\tau \in \sigma$  has  $\text{tp}_y \tau = \kappa$ .

# Certificates

A *certificate*  $\mathcal{S}$  for  $\mathbb{T}$  is a nonempty set of star-types satisfying:

- $(\mathcal{S}\tau)$  If  $\tau \in \mathbb{T}$ , then some  $\sigma \in \mathcal{S}$  has  $\tau \in \sigma$ , that is there is a star-type containing each 2-type.
- $(\mathcal{S}\kappa)$  If  $\kappa \in K_{\mathbb{T}}$  and if  $\sigma, \sigma' \in \mathcal{S}$  have  $\text{tp}_{\mathbf{x}}\sigma = \text{tp}_{\mathbf{x}}\sigma' = \kappa$ , then  $\sigma = \sigma'$ .

Then:

- $(\mathcal{S}\pi)$  If  $\pi \in \Pi_{\mathbb{T}}$ , then some  $\sigma \in \mathcal{S}$  has  $\text{tp}_{\mathbf{x}}\sigma = \pi$ .
- $(\mathcal{S}\kappa')$  If  $\kappa \in K_{\mathbb{T}}$ , then a unique  $\sigma \in \mathcal{S}$  has  $\text{tp}_{\mathbf{x}}\sigma = \pi$ .

# Certificate Extraction

A star-type for every  $\tau \in \mathbb{T}$  is sufficient!

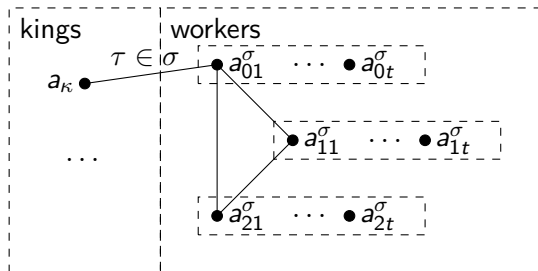
- ▶ Let  $\mathfrak{A}$  be a model for  $\mathbb{T}$
- ▶ For every  $\tau \in \mathbb{T}$  choose  $a_\tau, b_\tau \in A$  such that  $\text{tp}^{\mathfrak{A}}[a_\tau, b_\tau] = \tau$
- ▶  $\mathcal{S} = \left\{ \text{stp}^{\mathfrak{A}}[a_\tau] \mid \tau \in \mathbb{T} \right\}$  is a *polynomial* certificate

# Certificate Expansion

## Theorem

*Let  $S$  be a certificate for the type instance  $T$  and let  $t \geq |T|$ .  
Then  $T$  has a finite model in which every worker type is realized at least  $t$  times.*

# Construction



- ▶ single element for each king
- ▶ 3 blocks of  $t$  elements for each worker star-type
- ▶ king-to-element determined by the star-type of the element
- ▶ worker-to-worker between consecutive blocks
- ▶ completion to a full structure

# Summary

- ▶ Type realizability for  $\mathcal{L}^2$  is in  $\text{NPTIME}$
- ▶  $\mathcal{L}^2$  has the finite model property and its satisfiability problem is in  $\text{NEXPTIME}$



# Strategy

- ▶ Consider the two-variable logic with a single builtin equivalence symbol  $\mathcal{L}^2\text{E}$
- ▶ Equivalence classes are structures for the *simpler*  $\mathcal{L}^2$
- ▶ Exploit the previous result to ensure *classes are “consistent”* and figure out how to *glue them together*
- ▶ Make sure the argument is suitable for induction to get to  $\mathcal{L}^2\text{eE}_{\text{refine}}$

# Analogue

$$\mathcal{L}^2 \iff \mathcal{L}^2 eE_{\text{refine}}$$

models  $\iff$  nobly distinguished models

elements  $\iff$  galaxies

kings  $\iff$  nobles

workers  $\iff$  peasants

star-types  $\iff$  locally consistent cosmic spectrums

certificates  $\iff$  certificates

# Galaxies and Cosmos

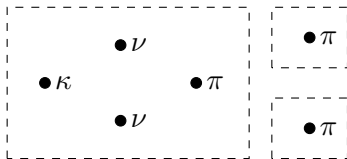
Let  $\mathfrak{A}$  be a  $\mathcal{L}^21E$ -structure for the type instance  $T$ .

- ▶ classes of  $\mathfrak{A}$  are the *galaxies* of  $\mathfrak{A}$
- ▶  $\mathfrak{A}$  is *the cosmos*
- ▶  $\tau \in T$  is *galactic* if  $\mathbf{e}(\mathbf{x}, \mathbf{y}) \in \tau$ ,  $T^g \subseteq T$
- ▶ otherwise  $\tau$  is *cosmic*,  $T^c \subseteq T$

# Noble and Peasant Types

- ▶  $\nu \in \Pi_T$  is *noble* if no cosmic  $\tau$  has  $\text{tp}_x\tau = \text{tp}_y\tau = \nu$ ; the set of noble types is  $N_T$
- ▶  $\pi \in \Pi_T$  is *peasant* if it is not noble; the set of peasant types is  $\Pi_T$
- ▶ kings are noble
- ▶ peasants are workers
- ▶ a galaxy is *noble* if it realizes a noble type
- ▶ a galaxy is *peasant* if it realizes only peasant types

# Example



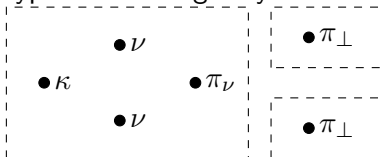
Noble galaxies may contain peasants!

# Noble Distinguishability

- ▶ A structure is *nobly distinguished* if every noble galaxy realizes only noble types

# Noble Distinguishability

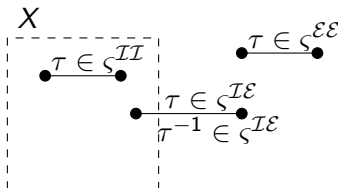
- ▶ A structure is *nobly distinguished* if every noble galaxy realizes only noble types
- ▶ Reduction: tag peasants in a noble galaxy with some noble type from that galaxy



# Cosmic Spectrums

- ▶ Let  $\mathfrak{A}$  be a nobly distinguished model for  $\mathbb{T}$  having at least 2 galaxies
- ▶ The cosmic spectrum of a galaxy  $X \subset A$  is

$$\text{csp}^{\mathfrak{A}}[X] = (\varsigma^{\mathcal{II}}, \varsigma^{\mathcal{IE}}, \varsigma^{\mathcal{EI}}, \varsigma^{\mathcal{EE}})$$





# Cosmic Spectrums

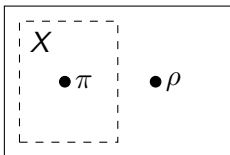
A cosmic spectrum  $\varsigma = (\varsigma^{\mathcal{II}}, \varsigma^{\mathcal{IE}}, \varsigma^{\mathcal{EI}}, \varsigma^{\mathcal{EE}})$  over  $T$  is a tuple satisfying:

- ( $\varsigma^{\mathcal{II}}$ ) The set of *internal types*  $\varsigma^{\mathcal{II}} \subseteq T^g$  is a set of galactic types that is closed under inversion.
- ( $\varsigma^{\mathcal{IE}}$ ) The set of *boundary types*  $\varsigma^{\mathcal{IE}} \subseteq T^c$  is a nonempty set of cosmic types.
- ( $\varsigma^{\mathcal{EI}}$ ) The set of *inverted boundary types* is:  $\varsigma^{\mathcal{EI}} = \{\tau^{-1} \mid \tau \in \varsigma^{\mathcal{IE}}\}$ .
- ( $\varsigma^{\mathcal{EE}}$ ) The set of *external types*  $\varsigma^{\mathcal{EE}} \subseteq T$  is a set of 2-types that is closed under inversion.
- ( $\varsigma^T$ ) We require that  $T = \varsigma^{\mathcal{II}} \cup \varsigma^{\mathcal{IE}} \cup \varsigma^{\mathcal{EI}} \cup \varsigma^{\mathcal{EE}}$ .
- ( $\varsigma^{\text{NP}}$ ) The (nonempty) set  $\text{Tp}_x \varsigma = (\text{tp}_x \upharpoonright \varsigma^{\mathcal{IE}})$  is the set of *internal 1-types* of  $\varsigma$ . We require that  $\text{Tp}_x \varsigma \subseteq N_T$  or  $\text{Tp}_x \varsigma \subseteq P_T$ .

# Locally consistent cosmic spectrums

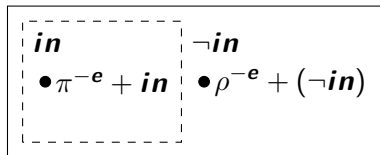
- ▶ The *spectral type instance*  $T^\varsigma$  of the  $\mathcal{L}^2 eE_{\text{refine}}$ -cosmic spectrum  $\varsigma$  is  $T^\varsigma = T_{II}^\varsigma \cup T_{IE}^\varsigma \cup T_{EI}^\varsigma \cup T_{EE}^\varsigma$ , where  $T_{\mathcal{X}\mathcal{Y}}^\varsigma = \{\tau_{\mathcal{X}\mathcal{Y}} \mid \tau \in \varsigma^{\mathcal{X}\mathcal{Y}}\}$ , where  $\tau_{\mathcal{X}\mathcal{Y}}$  works by “removing  $e$  and tagging the ends with a new predicate symbol *in*”.
- ▶  $\varsigma$  is *locally consistent* if  $T^\varsigma$  is realizable over *the simpler*  $\mathcal{L}^2(e-1)E_{\text{refine}}$

$\mathfrak{A} : \mathcal{L}^2 eE_{\text{refine}}$



$\Rightarrow$

$\mathfrak{A}' : \mathcal{L}^2(e-1)E_{\text{refine}}$



# Certificates

A *certificate*  $\mathcal{S}$  for the type instance  $T$  is a nonempty set of locally consistent cosmic spectrums satisfying:

$(\mathcal{S}T^c)$  If  $\tau \in T^c$  then some  $\varsigma \in \mathcal{S}$  has  $\tau \in \varsigma^{\mathcal{IE}}$ .

$(\mathcal{S}T^g)$  If  $\tau \in T^g$  then some  $\varsigma \in \mathcal{S}$  has  $\tau \in \varsigma^{\mathcal{II}}$ .

$(\mathcal{S}\nu)$  If  $\nu \in N_T$  and  $\varsigma, \varsigma' \in \mathcal{S}$  have  $\nu \in \text{Tp}_x \varsigma$  and  $\nu \in \text{Tp}_x \varsigma'$ , then  $\varsigma' = \varsigma$ .

# Certificate Expansion

## Theorem

*Let  $S$  be a certificate for the type instance  $T$  over the  $\mathcal{L}^2\text{eE}_{\text{refine}}$ -classified signature  $\langle \Sigma, \bar{m} \rangle$ . Then  $T$  has a finite model in which each worker type is realized at least  $t$  times.*

- ▶ Type realizability for  $\mathcal{L}^2\text{eE}_{\text{refine}}$  is in  $\text{NPTIME}$
- ▶ Satisfiability for  $\mathcal{L}^2\text{eE}_{\text{refine}}$  is in  $\text{NEXPTIME}$