Satisfiability with Equivalences in Agreement

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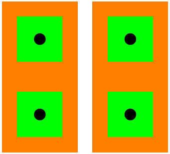
- *refinement* if $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_e$
- *global agreement* if it forms a chain, that is $E_{\nu(1)} \subseteq E_{\nu(2)} \subseteq \cdots \subseteq E_{\nu(e)}$ for some permutation ν of $[1, e]$
- *local agreement* if the equivalence classes of any point form a chain, that is for any $a \in A$ there is some permutation ν of $[1, e]$ such that $E_{\nu(1)}[a] \subseteq E_{\nu(2)}[a] \subseteq \cdots \subseteq E_{\nu(e)}$



equivalence class of E_1

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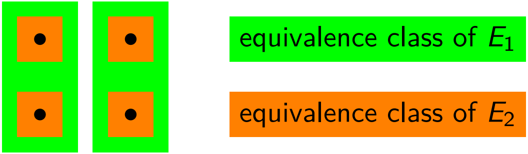

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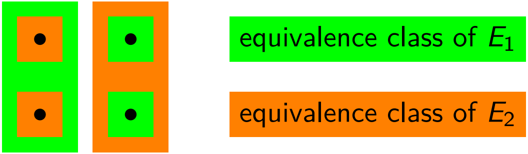

Notation

$\mathcal{L}_p^v e\mathrm{E}_{\mathsf{a}}$

- $\mathcal{L}$ is the first-order predicate logic with equality featuring only unary and binary predicate symbols
- v bounds the number of variables
- e specifies the number of built-in equivalence symbols
- $\mathsf{a} \in \{\mathrm{refine}, \mathrm{local}, \mathrm{global}\}$ specifies an agreement condition
- if $p = 0$ only constantly many additional unary predicate symbols are allowed
- if $p = 1$ only additional unary predicate symbols are allowed

Examples

- $\mathcal{L}_1$ is the monadic fragment
- $\mathcal{L}_0 1\mathrm{E}$ is the fragment of a single equivalence
- $\mathcal{L}^2 2\mathrm{E}_{\mathsf{local}}$ is the two-variable fragment of two equivalences in local agreement

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Goal

In this work, we investigate the ***computational complexity*** of the ***satisfiability*** for the ***monadic*** and the ***two-variable fragment*** in the presence of ***equivalences in agreement***.

Reductions

Refinement is the easiest condition to work with, so we define polynomial time reductions to its satisfiability problem:

$$\mathrm{SAT}\text{-}\mathcal{L}_p^v e\mathrm{E}_{\mathsf{global}} \leq_{\mathrm{m}}^{\mathrm{PT{\scriptsize IME}}} \mathrm{SAT}\text{-}\mathcal{L}_p^v e\mathrm{E}_{\mathsf{refine}}$$
$$\mathrm{SAT}\text{-}\mathcal{L}_p^v e\mathrm{E}_{\mathsf{local}} \leq_{\mathrm{m}}^{\mathrm{PT{\scriptsize IME}}} \mathrm{SAT}\text{-}\mathcal{L}_p^v e\mathrm{E}_{\mathsf{refine}}.$$

Central to these reductions is the notion of ***levels***.

Levels

The *level sequence* $L_1, L_2, \ldots, L_e$ of a sequence $E_1, E_2, \ldots, E_e$ of equivalence relations on A in local agreement is defined by:

$$L_m = \bigcap \{E_{i_1} \cup E_{i_2} \cup \cdots \cup E_{i_m} \mid 1 \leq i_1 < i_2 < \cdots < i_m \leq e\}.$$

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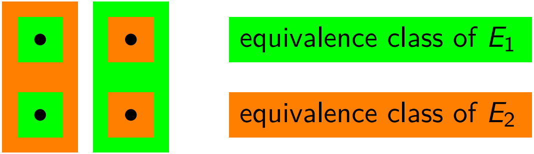
Remark

The level sequence is a sequence of equivalence relations on A in refinement.



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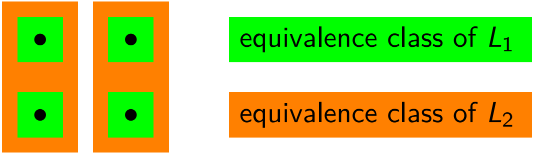

Monadic Fragments

It is known that

- $\mathrm{SAT}\text{-}\mathcal{L}_1$ is in $\mathrm{NE{\small XP}T{\small IME}}$ [Löwenheim 1915] and is $\mathrm{NE{\small XP}T{\small IME}}$-hard
- $\mathrm{SAT}\text{-}\mathcal{L}_0\mathrm{1E}$ is $\mathrm{PS{\small PACE}}$-complete
- $\mathrm{SAT}\text{-}\mathcal{L}_0\mathrm{2E}$ is undecidable [Janiczak 1953]

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- $\mathrm{SAT}\text{-}\mathcal{L}_0 1\mathrm{E}$ is $\mathrm{PS{\scriptstyle PACE}}$-complete
- $\mathrm{SAT}\text{-}\mathcal{L}_0 2\mathrm{E}$ is undecidable [Janiczak 1953]

We show that

- $\mathrm{SAT}\text{-}\mathcal{L}_1 1\mathrm{E}$ is $\mathrm{N2E{\scriptstyle XP}T{\scriptstyle IME}}$-complete
- In general, $\mathrm{SAT}\text{-}\mathcal{L}_1 e\mathrm{E}_{\mathsf{refine}}$ is $\mathrm{N}(e+1)\mathrm{E{\scriptstyle XP}T{\scriptstyle IME}}$-complete for $e \geq 1$

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Approach: "work at the quantifier rank level of abstraction"

- for the upper bound, we use Ehrenfeucht-Fraïssé games to bound the size of a minimal model of a satisfiable formula
- for hardness, we reduce a version of the domino tiling problem to satisfiability