Satisfiability with Agreement and Counting

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Glossary

$|A|$ the cardinality of A. 1
$\wp A$ the powerset of A. 1
$\wp^+ A$ the set of nonempty subsets of A. 1
$\wp^\kappa A$ the set of subsets of A of cardinality κ. 1
$A \times B$ the cartesian product of A and B. 1
$\operatorname{dom} R$ the domain of R. 1
$\operatorname{ran} R$ the range of R. 1
R^{-1} the inverse of R. 1
$R \restriction S$ the restriction of R to S. 1
$R[a]$ the R-successors of a. 1
$S \circ R$ the composition of S and R. 1
id_A the identity on A. 1
$f : A \to B$ a total function from A to B. 1
$f : A \hookrightarrow B$ an injective function from A into B. 1
$f : A \twoheadrightarrow B$ a surjective function from A onto B. 1
$f : A \leftrightarrow B$ a bijective function between A and B. 1
$f : A \rightsquigarrow B$ a partial function from A to B. 1
$f(a) \simeq b$ f is defined at a with value b. 1
$f(a) \simeq \perp$ f is not defined at a. 1
ch_S^A characteristic function. 1
$\|\mathrm{A}\|$ the length of A. 1
$\langle a, b, c\rangle$ a sequence. 1
ε the empty sequence. 1
$\mathrm{A} + \mathrm{B}$ the concatenation of A and B. 1
$\mathrm{A} - \mathrm{B}$ A without the elements of B. 1
$\mathbb{N}$ the natural numbers. 1
$\mathbb{N}^+$ the positive natural numbers. 1
$[n, m]$ the discrete interval between n and m. 1
log the base-2 logarithm. 2
$\mathrm{v} \prec \mathrm{w}$ lexicographically smaller. 2
$\mathbb{S}_n$ the set of permutations of $[1, n]$. 2
$\exp_a^e(x)$ tetration. 2
Ω an alphabet. 2
$w = w_1 w_2 \ldots w_n$ a word. 2
Ω^* the set of words over Ω. 2
Ω^+ the set of nonempty words over Ω. 2
Ω^n the set of words of length n over Ω. 2
$\mathbb{B}$ the bits. 2
$\mathbb{B}^+$ the bitstrings. 2
$\|n\|$ the bitsize of n. 2
$\overline{n}$ the binary encoding of n. 2
$\underline{\mathrm{b}}$ the number encoded by b. 2
N_t the largest t-bit number. 2
$\mathbb{B}_t$ the t-bit numbers. 2
$\Omega_{\mathcal{C}}$ the symbol alphabet. 2
$\mathcal{V}$ the variable symbols. 3
$\boldsymbol{x}$ the first variable symbol. 3
$\boldsymbol{y}$ the second variable symbol. 3
$\boldsymbol{z}$ the third variable symbol. 3
Σ a predicate signature. 3
$\boldsymbol{p}_i$ a predicate symbol. 3
$\operatorname{ar} \boldsymbol{p}_i$ the arity of $\boldsymbol{p}_i$. 3
$\mathcal{A}t[\Sigma]$ the atomic formulas over Σ. 3
$\mathcal{L}it[\Sigma]$ the literals over Σ. 3
$\mathcal{C}[\Sigma]$ the first-order formulas with counting quantifiers over Σ. 3
$\mathcal{L}[\Sigma]$ the first-order formulas over Σ. 3
$\operatorname{vars} \varphi$ the variables occurring φ. 3
$\operatorname{fvars} \varphi$ the variables freely occurring φ. 3
$\mathcal{L}^v[\Sigma]$ the v-variable first-order formulas over Σ. 3
$\mathcal{C}^v[\Sigma]$ the v-variable first-order formulas with counting quantifiers over Σ. 3

$\operatorname{qr}\varphi$ the quantifier rank of φ. 3
$\mathcal{L}_r[\Sigma]$ the r-rank first-order formulas over Σ. 4
$\mathcal{C}_r[\Sigma]$ the r-rank first-order formulas with counting quantifiers over Σ. 4
$\mathcal{L}^v_r[\Sigma]$ the r-rank v-variable first-order formulas over Σ. 4
$\mathcal{C}^v_r[\Sigma]$ the r-rank v-variable first-order formulas with counting quantifiers over Σ. 4
$\mathfrak{A}$ a structure. 4
$\varphi^{\mathfrak{A}}$ interpretation of φ in $\mathfrak{A}$. 5
SAT-$\mathcal{K}$ the satisfiable sentences of $\mathcal{K}$. 5
FIN-SAT-$\mathcal{K}$ the finitely satisfiable sentences of $\mathcal{K}$. 6
$\varphi \equiv \psi$ logically equivalent formulas. 6
$\mathfrak{A} \equiv \mathfrak{B}$ elementary equivalent structures. 6
$\mathfrak{A} \equiv_r \mathfrak{B}$ r-rank equivalent structures. 6
$\mathfrak{A} \equiv^v \mathfrak{B}$ v-variable equivalent structures. 6
$\mathfrak{A} \equiv^v_r \mathfrak{B}$ r-rank v-variable equivalent structures. 6
$\mathfrak{p}$ parital isomorphism. 6
$G_r(\mathfrak{A}, \mathfrak{B})$ the r-round Ehrenfeucht-Fraïssé game. 6
$\Pi[\Sigma]$ the set of 1-types over Σ. 7
$\mathrm{T}[\Sigma]$ the set of 1-types over Σ. 7
τ^{-1} the inverse of the type τ. 7
$\operatorname{tp}_{\boldsymbol{x}}\tau$ the $\boldsymbol{x}$-type of τ. 7
$\operatorname{tp}_{\boldsymbol{y}}\tau$ the $\boldsymbol{y}$-type of τ. 7
$\tau \parallel \tau'$ parallel 2-types. 7
$\operatorname{tp}^{\mathfrak{A}}[a]$ the 1-type of a in $\mathfrak{A}$. 7
$\pi^{\mathfrak{A}}$ the interpretation of the 1-type π in $\mathfrak{A}$. 7
$\operatorname{tp}^{\mathfrak{A}}[a, b]$ the 1-type of a in $\mathfrak{A}$. 7
$\tau^{\mathfrak{A}}$ the interpretation of the 2-type τ in $\mathfrak{A}$. 7
PTime complexity class. 8
$A \leq^{\text{PTime}}_{\text{m}} B$ A is polynomial-time reducible to B. 9
$A =^{\text{PTime}}_{\text{m}} B$ A and B are polynomial-time equivalent. 9
B a bit setup. 11
$[\boldsymbol{u}\text{:data}]^{\mathfrak{A}}$ $\boldsymbol{u}$-data at $\mathfrak{A}$. 11
$[\boldsymbol{u}\text{:eq-}d](\boldsymbol{x})$ $\boldsymbol{u}$-data at $\boldsymbol{x}$ is d. 11
$[\boldsymbol{u}\text{:eq}](\boldsymbol{x}, \boldsymbol{y})$ $\boldsymbol{u}$-data equal at $\boldsymbol{x}$ and $\boldsymbol{y}$. 11
$[\boldsymbol{u}\text{:eq-}01](\boldsymbol{x}, \boldsymbol{y})$ $\boldsymbol{u}$-data at $\boldsymbol{x}$ and $\boldsymbol{y}$ is 0 and 1. 11
$[\boldsymbol{u}\text{:eq-}10](\boldsymbol{x}, \boldsymbol{y})$ $\boldsymbol{u}$-data at $\boldsymbol{x}$ and $\boldsymbol{y}$ is 1 and 0. 11
C a counter setup. 11
$[\text{C:data}]^{\mathfrak{A}}$ C-data at $\mathfrak{A}$. 12
$[\text{C:eq-}d](\boldsymbol{x})$ C-data at $\boldsymbol{x}$ is d. 12
$[\text{C:eq}](\boldsymbol{x}, \boldsymbol{y})$ C-data equal at $\boldsymbol{x}$ and $\boldsymbol{y}$. 12
$[\text{C:less}](\boldsymbol{x}, \boldsymbol{y})$ C-data at $\boldsymbol{x}$ less than C-data at $\boldsymbol{y}$. 12
$[\text{C:succ}](\boldsymbol{x}, \boldsymbol{y})$ C-data at $\boldsymbol{y}$ succeeds C-data at $\boldsymbol{x}$. 12
$[\text{C:less}]d(\boldsymbol{x})$ C-data at $\boldsymbol{x}$ less than d. 13
$[\text{C:betw-}d\text{-}e](\boldsymbol{x})$ C-data at $\boldsymbol{x}$ between d and e. 13
$[\text{C:allbetw-}d\text{-}e]$ C-data between d and e. 13
$[\text{V}(p)\text{:data}]^{\mathfrak{A}}a$ the value of the p-th counter at a. 13
$[\text{V:data}]^{\mathfrak{A}}$ the V-data at a. 13
$[\text{V:eq-v}](\boldsymbol{x})$ the V-data at $\boldsymbol{x}$. 13
$[\text{V}(pq)\text{:at-}i\text{-eq}](\boldsymbol{x})$ equal i-th bits at p and q at $\boldsymbol{x}$. 14
$[\text{V}(pq)\text{:at-}i\text{-eq-}01](\boldsymbol{x})$ equal i-th bits at p and q are 0 and 1. 14
$[\text{V}(pq)\text{:at-}i\text{-eq-}10](\boldsymbol{x})$ equal i-th bits at p and q are 1 and 0. 14
$[\text{V}(pq)\text{:eq}](\boldsymbol{x})$ equal p and q V-data at $\boldsymbol{x}$. 14
$[\text{V}(pq)\text{:less}](\boldsymbol{x})$ V-data at p less than at q. 14
$[\text{V}(pq)\text{:succ}](\boldsymbol{x})$ V-data at q succeeds the data at p. 14
$[\text{P:alldiff}]$ P-data at different positions is different. 15
$[\text{P:perm}]$ P-data is a permutation. 15
$\mathscr{E}E$ the set of equivalence classes of E. 17
$[\boldsymbol{e}\text{:refl}]$ $\boldsymbol{e}$ is reflexive. 17
$[\boldsymbol{e}\text{:symm}]$ $\boldsymbol{e}$ is symmetric. 17
$[\boldsymbol{e}\text{:trans}]$ $\boldsymbol{e}$ is transitive. 17
$[\boldsymbol{e}\text{:equiv}]$ $\boldsymbol{e}$ is transitive. 17
$[\boldsymbol{d}, \boldsymbol{e}\text{:refine}]$ refinement. 18

[$\boldsymbol{d}, \boldsymbol{e}$:global] global agreement. 18
[$\boldsymbol{d}, \boldsymbol{e}$:local] local agreement. 18
[$\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$:refine] symbols in refinement. 19
[$\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$:global] symbols in global agreement. 19
[$\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$:local] symbols in local agreement. 19
$\Lambda_p^v e\mathrm{E}_{\mathsf{a}}$ logic notation. 21
[P:alleq] P-data equal everywhere. 22
[P:globperm] P-data is a global permutation. 22
[L:eg-i]$(\boldsymbol{x}, \boldsymbol{y})$ global refinement induced by levels. 23
gtr translation of global agreement to refinement. 23
[E:chperm]$^{\mathfrak{A}}$ characteristic E-permutation in $\mathfrak{A}$. 25
[L:fixperm] fixed permutation condition. 26
[L:locperm] local agreement condition. 27
[L:el-i] local refinement induced by levels. 27
ltr translation of local agreement to refinement. 28
g granularity. 29
G granularity color setup. 29
[Γ:d] finer equivalence granularity formula. 30
grtr granularity translation. 30
[Σ:cell]$(\boldsymbol{x}, \boldsymbol{y})$ Σ-cell formula. 33
$\mathcal{O}$ organ-equivalence relation. 34
O sub-organ-equivalence relation. 35
[D:Data]$^{\mathfrak{A}}$ D-Data. 39
[D:Zero]$(\boldsymbol{x})$ zero D-Data at $\boldsymbol{x}$. 39
[D:Largest]$(\boldsymbol{x})$ maximum D-Data at $\boldsymbol{x}$. 40
$\boldsymbol{m}_i$ message symbols. 47
$\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ classified signature. 48
$\pi \sim^{\mathrm{T}} \pi'$ connectable 1-types. 49

1 Introduction

1.1 Notation

The cardinal number $|A|$ is the *cardinality* of the set A. The set $\wp A$ is the *powerset* of A. The set $\wp^+ A = \wp A \setminus \{\varnothing\}$ is the *set of nonempty subsets* of A. If κ is a cardinal number, the set $\wp^\kappa A = \{S \in \wp A \mid |S| = \kappa\}$ is the *κ-powerset* of A. The *cartesian product* of A and B is $A \times B$. The sets A and B *properly intersect* if $A \cap B \neq \varnothing$, $A \setminus B \neq \varnothing$ and $B \setminus A \neq \varnothing$.

If R is a binary relation, its *domain* is $\operatorname{dom} R$ and its *range* is $\operatorname{ran} R$. The *inverse* of $R \subseteq A \times B$ is

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\} .$$

If S is a set and $R \subseteq A \times B$, the *restriction* of R to S is

$$R \upharpoonright S = \{(a, b) \in R \mid a \in S\} .$$

If $R \subseteq A \times B$ is a binary relation and $a \in A$, the *R-successors* of a are

$$R[a] = \{b \in B \mid (a, b) \in R\} .$$

If $S \subseteq B \times C$ and $R \subseteq A \times B$ are two binary relations, their *composition* is

$$S \circ R = \{(a, c) \in A \times C \mid (\exists b \in B)(a, b) \in R \wedge (b, c) \in S\} .$$

A *function* is formally just a functional relation. The *identity function* on A is id_A. A *total* function from A to B is denoted $f : A \to B$. A *injective* function from A into B is denoted $f : A \hookrightarrow B$. A *surjective* function from A onto B is denoted $f : A \twoheadrightarrow B$. A *bijective* function between A and B is denoted $f : A \leftrightarrow B$. A *partial* function from A to B is denoted $f : A \rightsquigarrow B$. If $f : A \rightsquigarrow B$ is a partial function and $a \in A$, the notation $f(a) \simeq b$ means that f is defined at a and its value is b; the notation $f(a) \simeq \bot$ means that f is not defined at a. If $S \subseteq A$, the *characteristic function* of S in A is $\mathrm{ch}_S^A : A \to \{0, 1\}$.

A *sequence* is formally just a function with domain an ordinal number. If A is a sequence, its *length* $\|\mathrm{A}\|$ is just the domain of A. The sequence consisting of the elements a, b and c in that order is $\langle a, b, c\rangle$. The *empty sequence* is ε. A *finite sequence* is a sequence of finite length. If A and B are two sequences, their *concatenation* is $\mathrm{A} + \mathrm{B}$, and the sequence obtained from A by dropping all elements of B is $\mathrm{A} - \mathrm{B}$.

The set of *natural numbers* is $\mathbb{N} = \{0, 1, \ldots\}$. The set of *positive natural numbers* is $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. If $n, m \in \mathbb{N}$ are natural numbers, the *discrete interval* $[n, m]$ between n

and m is
$$[n, m] = \begin{cases} \{n, n+1, \dots, m\} & \text{if } n \leq m \\ \varnothing & \text{otherwise.} \end{cases}$$
The function $\log$ is the *base-2 logarithm*.

An n-vector $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n) \in \mathbb{N}^n$ is just a tuple of natural numbers. The n-vector v is *lexicographically smaller* [1] than the n-vector w (written $\mathrm{v} \prec \mathrm{w}$) if there is a position $p \in [1, n]$ such that $\mathrm{v}_p < \mathrm{w}_p$ and $\mathrm{v}_q = \mathrm{w}_q$ for all $q \in [p+1, n]$.

The set of *n-permutations* of $[1, n]$ is $\mathbb{S}_n$. We think of an n-permutation ν as an n-vector $\nu = (\nu(1), \nu(2), \dots, \nu(n))$.

A function $f : \mathbb{N} \to \mathbb{N}$ is *polynomially bounded* if there is a polynomial p and a number $n_0 \in \mathbb{N}$ such that $f(n) \leq p(n)$ for all $n \geq n_0$. The function f is *exponentially bounded* if there is a polynomial p and a number $n_0 \in \mathbb{N}$ such that $f(n) \leq 2^{p(n)}$ for all $n \geq n_0$. We are going to use these terms implicitly with respect to quantities that depend on one another. For example, the cardinality of $\mathbb{S}_n$ is exponentially bounded by n.

Define the *tetration* operation $\exp_a^e(x)$ by $\exp_a^0(x) = x$ and $\exp_a^{e+1}(x) = a^{\exp_a^e(x)}$, so $\exp_a^e(x) = a^{a^{\cdot^{\cdot^{a^x}}}}$ is a tower of e exponentiations.

An *alphabet* Ω is just a nonempty set. The elements of Ω are *characters*. A *word* $w = w_1 w_2 \dots w_n$ is a finite sequence of characters. The set of words over Ω is Ω^*. The set of nonempty words over Ω is $\Omega^+ = \Omega^* \setminus \{\varepsilon\}$. If $n \in \mathbb{N}$, the set of words of length n over Ω is Ω^n.

The set of *bits* is $\mathbb{B} = \{0, 1\}$. The set of *bitstrings* is $\mathbb{B}^+$. The bitstrings are read right-to-left, that is the bitstring $\mathrm{b} = 10$ has first character 0. If $t < u \in \mathbb{N}^+$, the t-bit bitstrings $\mathbb{B}^t$ are embedded into the u-bit bitstrings $\mathbb{B}^u$ by appending leading zeroes. If $n \in \mathbb{N}$, the *bitsize* $\|n\|$ of n is:
$$\|n\| = \begin{cases} 1 & \text{if } n = 0 \\ \lfloor \log n \rfloor + 1 & \text{otherwise.} \end{cases}$$
If $n \in \mathbb{N}$, the *binary encoding* of n is $\overline{n} \in \mathbb{B}^{\|n\|}$. If $\mathrm{b} \in \mathbb{B}^t$, the *number encoded by* b is $\underline{\mathrm{b}}$. The *largest t-bit number* is $N_t = 2^t - 1$. The set of *t-bit numbers* is $\mathbb{B}_t = [0, N_t]$.

1.2 Syntax

The *symbol alphabet* for the first-order logic with counting quantifiers is
$$\Omega_{\mathcal{C}} = \left\{ \neg, \wedge, \vee, \rightarrow, \leftrightarrow; \exists, \forall; =; (, , ,); {}^{\leq}, {}^{=}, {}^{\geq}, {}^{0}, {}^{1} \right\}.$$
The propositional connectives are listed in decreasing order of precedence. The *negation* $\neg$ is unary; the *disjunction* $\vee$, *conjunction* $\wedge$ and *equivalence* $\leftrightarrow$ are left-associative; the

[1] the higher positions to the right are more significant; it may *look like* this ordering is the anti-lexicographic one, for example $(1, 1, 0) \prec (0, 0, 1)$.

implication $\rightarrow$ is right-associative. The quantifiers bind as strong as the negation. Note that we consider logics with *formal equality* $=$.

A *counting quantifier* is a word over $\Omega_{\mathcal{C}}$ of the form $\exists^{\leq \overline{m}}$ or $\exists^{=\overline{m}}$ or $\exists^{\geq \overline{m}}$, where $m \in \mathbb{N}$ and $\overline{m} \in \mathbb{B}^{+}$ is the binary encoding of m. Note that this encoding of the counting quantifiers is *succinct*. As we note in Remark 1, this succinct representation allows for exponentially small counting formulas compared to their pure first-order equivalents. We denote the counting quantifiers by $\exists^{\leq m}$, $\exists^{=m}$ and $\exists^{\geq m}$, that is, we omit the encoding notation for m.

The sequence $\mathcal{V} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \rangle$ is a countable sequence of distinct *variable symbols*. We pay special attention to $\boldsymbol{x} = \boldsymbol{v}_1$, $\boldsymbol{y} = \boldsymbol{v}_2$ and $\boldsymbol{z} = \boldsymbol{v}_3$, the *first*, *second* and *third* variable symbol, respectively.

A *predicate signature* $\Sigma = \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_s \rangle$ is a finite sequence of distinct *predicate symbols* $\boldsymbol{p}_i$ together with their *arities* $\operatorname{ar} \boldsymbol{p}_i \in \mathbb{N}^{+}$. A predicate signature is *unary* or *monadic* if all of its predicate symbols have arity 1. A predicate signature is *binary* if all of its predicate symbols have arity 1 or 2. For the purposes of this work we will not be considering constant and function symbols—constant symbols can be simulated by a fresh unary predicate symbol having the intended interpretation of being true at a unique element; presence of function symbols on the other hand leads quite easily to undecidable satisfiability problems. By convention $\Omega_{\mathcal{C}}$, $\mathcal{V}$ and Σ are disjoint.

Let Σ be a predicate signature. The set of *atomic formulas* $\mathcal{A}t[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\alpha ::= (x = y) \mid p(x_1, x_2, \ldots, x_n)$$

for $x, y \in \mathcal{V}$, $p \in \Sigma$, $n = \operatorname{ar} p$ and $x_1, x_2, \ldots, x_n \in \mathcal{V}$.

The set of *literals* $\mathcal{L}it[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\lambda ::= \alpha \mid (\neg \alpha).$$

The set of *first-order formulas with counting quantifiers* $\mathcal{C}[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\begin{aligned}\varphi ::= \alpha \mid (\neg\varphi) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi) \mid (\exists x \varphi) \mid (\forall x \varphi) \mid \\ \mid (\exists^{\leq m} x \varphi) \mid (\exists^{=m} x \varphi) \mid (\exists^{\geq m} x \varphi)\end{aligned}$$

for $x \in \mathcal{V}$ and $m \in \mathbb{N}$.

The set of *first-order formulas* $\mathcal{L}[\Sigma] \subset \mathcal{C}[\Sigma]$ over Σ consists of the formulas that do not feature a counting quantifier.

The set of variables occurring in φ is $\operatorname{vars} \varphi \subset \mathcal{V}$. The set of variables freely occurring in φ is $\operatorname{fvars} \varphi \subset \mathcal{V}$. A formula φ is a *sentence* if $\operatorname{fvars} \varphi = \varnothing$. For $v \in \mathbb{N}$, a formula φ is a *v-variable formula* if $\operatorname{vars} \varphi \subseteq \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_v\}$. The set of v-variable first-order formulas over Σ is $\mathcal{L}^v[\Sigma]$. The set of v-variable first-order formulas with counting quantifiers over Σ is $\mathcal{C}^v[\Sigma]$.

If $\varphi \in \mathcal{C}[\Sigma]$, the *quantifier rank* $\operatorname{qr} \varphi \in \mathbb{N}$ of φ is defined as follows. If φ matches:

- $(x = y)$, then $\operatorname{qr}\varphi = 0$
- $p(x_1, x_2, \ldots, x_n)$, then $\operatorname{qr}\varphi = 0$
- $(\neg\psi)$, then $\operatorname{qr}\varphi = \operatorname{qr}\psi$
- $\psi_1 \oplus \psi_2$ for $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\operatorname{qr}\varphi = \max(\operatorname{qr}\psi_1, \operatorname{qr}\psi_2)$
- $(\exists x\psi)$ or $(\forall x\psi)$, then $\operatorname{qr}\varphi = 1 + \operatorname{qr}\psi$
- $(\exists^{\leq m} x\psi)$ or $(\exists^{=m} x\psi)$, then $\operatorname{qr}\varphi = m + 1 + \operatorname{qr}\psi$
- $(\exists^{\geq m} x\psi)$, then $\operatorname{qr}\varphi = m + \operatorname{qr}\psi$.

An *r-rank formula* is a formula having quantifier rank r. The set of r-rank first-order formulas over Σ is $\mathcal{L}_r[\Sigma]$. The set of r-rank first-order formulas with counting quantifiers over Σ is $\mathcal{C}_r[\Sigma]$. The set of r-rank v-variable first-order formulas over Σ is $\mathcal{L}_r^v[\Sigma]$. The set of r-rank v-variable first-order formulas with counting quantifiers over Σ is $\mathcal{C}_r^v[\Sigma]$.

If φ is a formula and $x_1, x_2, \ldots, x_n \in \mathcal{V}$ are distinct variables, we use the notation $\varphi(x_1, x_2, \ldots, x_n)$, a *focused formula*, to show that we are interested in the free occurrences of the variables x_i in φ. If $\varphi(x_1, x_2, \ldots, x_n)$ is a focused formula and $y_1, y_2, \ldots, y_n \in \mathcal{V}$, then $\varphi(y_1, y_2, \ldots, y_n)$ denotes the formula φ where all free occurrences of x_i are replaced by y_i. The notation $\varphi = \varphi(x_1, x_2, \ldots, x_n)$ means that $\operatorname{fvars}\varphi \subseteq \{x_1, x_2, \ldots, x_n\}$.

We will omit unnecessary brackets in formulas.

1.3 Semantics

If Σ is a predicate signature, a *Σ-structure* $\mathfrak{A}$ consists of a nonempty set A (the *domain* of $\mathfrak{A}$), together with a relation $p^{\mathfrak{A}} \subseteq A^{\operatorname{ar} p}$ (the *interpretation* of p at $\mathfrak{A}$) for every predicate symbol $p \in \Sigma$. A structure is *finite* if its domain is finite. We omit the standard definition of semantic notions. Seldom it will be useful to consider *structures with possibly empty domain*. We will be explicit when this is the case. If $\mathfrak{A}$ is a structure and $B \subseteq A$ there is a substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with possibly empty domain B. We call it the substructure induced by B and denote it $(\mathfrak{A} \restriction B)$.

Note that the interpretation of the counting quantifiers is clear: $\exists^{\leq m} x\varphi$ means that "at most m elements satisfy φ"; $\exists^{=m} x\varphi$ means that "exactly m elements satisfy φ"; $\exists^{\geq m} x\varphi$ means that "at least m elements satisfy φ".

The *standard translation* $\operatorname{st} : \mathcal{C}[\Sigma] \to \mathcal{L}[\Sigma]$ of first-order formulas with counting quantifiers to logically equivalent first-order formulas is defined as follows. If φ matches:

- $(x = y)$ or $p(x_1, x_2, \ldots, x_n)$, then $\operatorname{st}\varphi = \varphi$
- $(\neg\psi)$, then $\operatorname{st}\varphi = (\neg \operatorname{st}\psi)$
- $(\psi_1 \oplus \psi_2)$ for $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\operatorname{st}\varphi = (\operatorname{st}\psi_1 \oplus \operatorname{st}\psi_2)$
- $(\mathsf{Q}x\psi)$ for $\mathsf{Q} \in \{\exists, \forall\}$, then $\operatorname{st}\varphi = (\mathsf{Q}x \operatorname{st}\psi)$

- $(\exists^{\leq m} x \psi(x))$ or $(\exists^{=m} x \psi(x))$ or $(\exists^{\geq m} x \psi(x))$, then let

$$\theta_{\leq} = \forall y_1 \forall y_2 \ldots \forall y_m \forall y_{m+1} \left(\bigwedge_{1 \leq i \leq m+1} \operatorname{st} \psi(y_i) \to \bigvee_{1 \leq i < j \leq m+1} y_i = y_j \right)$$

$$\theta_{\geq} = \exists y_1 \exists y_2 \ldots \exists y_m \left(\bigwedge_{1 \leq i \leq m} \operatorname{st} \psi(y_i) \wedge \bigwedge_{1 \leq i < j \leq m} y_i \neq y_j \right)$$

where $y_1, y_2, \ldots, y_{m+1}$ are distinct variable symbols not occurring in φ. The formula $\theta_{\leq}$ asserts that there are at most m distinct values satisfying ψ. The formula $\theta_{\geq}$ asserts that there are at least m distinct values satisfying ψ. If $\varphi = (\exists^{\leq m} x \psi(x))$, then $\operatorname{st} \varphi = \theta_{\leq}$. If $\varphi = (\exists^{=m} x \psi(x))$, then $\operatorname{st} \varphi = (\theta_{\leq} \wedge \theta_{\geq})$. If $\varphi = (\exists^{\geq m} x \psi(x))$, then $\operatorname{st} \varphi = \theta_{\geq}$.

Remark 1. *The translation of a first-order formula with counting quantifiers φ to a logically equivalent first-order formula $\psi = \operatorname{st} \varphi$ preserves quantifier rank. However, the resulting formula ψ may have exponentially larger length.*

A *predicate signature with intended interpretations* Σ is formally a predicate signature together with an *intended interpretation condition* $\mathcal{A}$, which is formally a class of Σ-structures. A Σ-structure $\mathfrak{A}$ is then just an element of $\mathcal{A}$. That is, when we speak about a predicate signature with intended interpretations, we are considering the logics strictly over the class of structures respecting the intended interpretation condition. The semantic concepts are relativised appropriately in this context. For example, if $\Sigma = \langle \boldsymbol{e} \rangle$ is a predicate signature consisting of the single binary predicate symbol $\boldsymbol{e}$, having intended interpretation as an equivalence, then the Σ-formula $\forall \boldsymbol{x} \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{x})$ is logically valid. From now on, we will use the term *predicate signature* as *predicate signature with possible intended interpretations*.

The predicate signature Σ' is an *enrichment* of the predicate signature Σ if Σ' contains all predicate symbols of Σ and respects their intended interpretation in Σ. A Σ'-structure $\mathfrak{A}'$ is an enrichment of the Σ-structure $\mathfrak{A}$ if they have the same domain and the same interpretation of the predicate symbols of Σ. The basic semantic significance of enrichment is that if $\varphi(x_1, x_2, \ldots, x_n)$ is a Σ-formula and $a_1, a_2, \ldots, a_n \in A$, then $\mathfrak{A} \vDash \varphi(a_1, a_2, \ldots, a_n)$ iff $\mathfrak{A}' \vDash \varphi(a_1, a_2, \ldots, a_n)$. If $\mathfrak{A}'$ is an enrichment of $\mathfrak{A}$ then $\mathfrak{A}$ is a *reduct*[2] of $\mathfrak{A}'$.

If $\varphi(x_1, x_2, \ldots, x_n)$ is a focused formula, the interpretation of φ in $\mathfrak{A}$ is

$$\varphi^{\mathfrak{A}} = \{(a_1, a_2, \ldots, a_n) \in A^n \mid \mathfrak{A} \vDash \varphi(a_1, a_2, \ldots, a_n)\}.$$

If Σ is a predicate signature and φ is a Σ-sentence, then φ is *satisfiable* if there is a Σ-structure that is a model for φ; φ is *finitely satisfiable* if there is a *finite* Σ-structure that is a model for φ. If $\mathcal{K} \subseteq \mathcal{C}[\Sigma]$ is a family of formulas over the predicate signature Σ, the set of *satisfiable sentences* is SAT-$\mathcal{K} \subseteq \mathcal{K}$ and the set of *finitely satisfiable sentences*

[2] or why not *empoverishment*?

is FIN-SAT-$\mathcal{K} \subseteq \mathcal{K}$. The family $\mathcal{K}$ has the *finite model property* if SAT-$\mathcal{K}$ = FIN-SAT-$\mathcal{K}$. By the Löwenheim-Skolem theorem, every satisfiable sentence φ has a finite or countable model (assuming the intended interpretation condition of the predicate signature is first-order-definable). In this work the intended interpretation conditions of the predicate signatures will always be first-order-definable formula and we will silently assume that all structures are either finite or countable.

Two Σ-sentences φ and ψ are *logically equivalent* (written $\varphi \equiv \psi$) if they have the same models.

Two Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementary equivalent* (written $\mathfrak{A} \equiv \mathfrak{B}$) if they satisfy the same first-order sentences (hence also the same first-order sentences with counting quantifiers). The structures $\mathfrak{A}$ and $\mathfrak{B}$ are *r-rank equivalent* (written $\mathfrak{A} \equiv_r \mathfrak{B}$) if they satisfy the same r-rank first-order sentences. The structures $\mathfrak{A}$ and $\mathfrak{B}$ are *v-variable equivalent* (written $\mathfrak{A} \equiv^v \mathfrak{B}$) if they satisfy the same v-variable first-order sentences. The structures $\mathfrak{A}$ and $\mathfrak{B}$ are *r-rank v-variable equivalent* (written $\mathfrak{A} \equiv_r^v \mathfrak{B}$) if they satisfy the same r-rank v-variable first-order sentences.

1.4 Games

Logic games capture structure equivalence. Let Σ be a predicate signature and let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures. A *partial isomorphism* $\mathfrak{p} : A \rightsquigarrow B$ from $\mathfrak{A}$ to $\mathfrak{B}$ is a partial mapping that is an isomorphism between the induced substructures $(\mathfrak{A} \restriction \operatorname{dom} \mathfrak{p})$ and $(\mathfrak{B} \restriction \operatorname{ran} \mathfrak{p})$.

Let $r \in \mathbb{N}^+$. The *r-round Ehrenfeucht-Fraïssé game* $G_r(\mathfrak{A}, \mathfrak{B})$ is a two-player game, played with a pair of pebbles, one for each structure. The two players are Spoiler and Duplicator. Initially the pebbles are off the structures. During each round, Spoiler picks a pebble and a structure and places it on some element in that structure. Duplicator responds by picking the other pebble and placing it on some element in the other structure. Thus during round i the players play a pair of elements $a_i \mapsto b_i \in A \times B$. Collect the sequences of played elements $\bar{a} = \langle a_1, a_2, \dots, a_r \rangle$ and $\bar{b} = \langle b_1, b_2, \dots, b_r \rangle$. Duplicator wins the match if the relation $\bar{a} \mapsto \bar{b} = \{a_1 \mapsto b_1, a_2 \mapsto b_2, \dots, a_r \mapsto b_r\} \subseteq A \times B$, built from the pairs of elements in each round, is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Ehrenfeucht's theorem says that Duplicator has a winning strategy for $G_r(\mathfrak{A}, \mathfrak{B})$ iff $\mathfrak{A} \equiv_r \mathfrak{B}$. Fraïssé's theorem gives a back-and-forth characterization of the winning strategy for Duplicator [1, ch. 2]:

Theorem 1. *Suppose that $(\mathfrak{I}_0, \mathfrak{I}_1, \dots, \mathfrak{I}_r)$ is a sequence of nonempty sets of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ with the following properties:*

1. *For every $i < r$, $\mathfrak{p} \in \mathfrak{I}_i$ and $a \in A$, there is $\mathfrak{q} \in \mathfrak{I}_{i+1}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $a \in \operatorname{dom} \mathfrak{q}$.*

2. *For every $i < r$, $\mathfrak{p} \in \mathfrak{I}_i$ and $b \in B$, there is $\mathfrak{q} \in \mathfrak{I}_{i+1}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $b \in \operatorname{ran} \mathfrak{q}$.*

Then $\mathfrak{A} \equiv_r \mathfrak{B}$.

1.5 Types

Let $\Sigma = \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_s \rangle$ be a predicate signature. A 1-*type* π over Σ is a maximal consistent set of literals featuring only the variable symbol $\boldsymbol{x}$[3]. The set of 1-types over Σ is $\Pi[\Sigma]$. Note that consistency here is relativised by the intended interpretations of the predicate signature. For example if Σ contains the binary predicate symbol $\boldsymbol{e}$ with intended interpretation as an equivalence, then every 1-type over Σ includes the literal $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{x})$. Also note that the cardinality of a 1-type over Σ is polynomially bounded by the length s of Σ and the cardinality of $\Pi[\Sigma]$ is exponentially bounded by s.

A 2-*type* τ over Σ is a maximal consistent set of literals featuring only the variable symbols $\boldsymbol{x}$ and $\boldsymbol{y}$ and including the literal $(\boldsymbol{x} \neq \boldsymbol{y})$. The set of 2-types over Σ is $\mathrm{T}[\Sigma]$. Again, consistency is relativised by the intended interpretation of the predicate signature. For example, if Σ contains the binary predicate symbol $\boldsymbol{e}$ with intended interpretation as an equivalence, then if $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \in \tau$, then $\boldsymbol{e}(\boldsymbol{y}, \boldsymbol{x}) \in \tau$. Again, the cardinality of a 2-type over Σ is polynomially bounded by s and the cardinality of $\mathrm{T}[\Sigma]$ is exponentially bounded by s.

If $\tau \in \mathrm{T}[\Sigma]$, the *inverse* τ^{-1} of τ is the 2-type obtained from τ by swapping the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ in every literal. The $\boldsymbol{x}$-*type* of τ is the 1-type $\mathrm{tp}_{\boldsymbol{x}} \tau$ consisting of all the literals of τ featuring only the variable symbol $\boldsymbol{x}$. Similarly, the $\boldsymbol{y}$-*type* of τ is the 1-type $\mathrm{tp}_{\boldsymbol{y}} \tau$ consisting of all the literals of τ featuring only the variable symbol $\boldsymbol{y}$, that is replaced by $\boldsymbol{x}$. For instance we have the identity $\mathrm{tp}_{\boldsymbol{x}} \tau^{-1} = \mathrm{tp}_{\boldsymbol{y}} \tau$. We say that τ *connects* the 1-types $\mathrm{tp}_{\boldsymbol{x}} \tau$ and $\mathrm{tp}_{\boldsymbol{y}} \tau$ and we refer to $\mathrm{tp}_{\boldsymbol{x}} \tau$ and $\mathrm{tp}_{\boldsymbol{y}} \tau$ as the *endpoints* of τ. Two 2-types τ, τ' are *parallel* if they have the same endpoints, that is if $\mathrm{tp}_{\boldsymbol{x}} \tau = \mathrm{tp}_{\boldsymbol{x}} \tau'$ and $\mathrm{tp}_{\boldsymbol{y}} \tau = \mathrm{tp}_{\boldsymbol{y}} \tau'$. We use the notation $\tau \parallel \tau'$ to denote parallel 2-types.

If $\mathfrak{A}$ is a Σ-structure and $a \in A$, the 1-*type of* a *in* $\mathfrak{A}$ is

$$\mathrm{tp}^{\mathfrak{A}}[a] = \{\lambda(\boldsymbol{x}) \in \mathcal{L}it[\Sigma] \mid \mathfrak{A} \models \lambda(a)\}\,.$$

If $\mathrm{tp}^{\mathfrak{A}}[a] = \pi$, we say that the 1-type π is *realized* by a in $\mathfrak{A}$. The interpretation of the 1-type π in $\mathfrak{A}$ is the set of elements realizing π:

$$\pi^{\mathfrak{A}} = \left\{a \in A \;\middle|\; \mathrm{tp}^{\mathfrak{A}}[a] = \pi\right\}.$$

If $a \neq b \in A$, the 2-*type of* (a, b) *in* $\mathfrak{A}$ is

$$\mathrm{tp}^{\mathfrak{A}}[a, b] = \{\lambda(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{L}it[\Sigma] \mid \mathfrak{A} \models \lambda(a, b)\}\,.$$

We do not define a 2-type in case $a = b$. If $\mathrm{tp}^{\mathfrak{A}}[a, b] = \tau$, we say that the 2-type τ is *realized* by (a, b) in $\mathfrak{A}$. The interpretation of the 2-type τ in $\mathfrak{A}$ is the set of pairs realizing τ:

$$\tau^{\mathfrak{A}} = \left\{(a, b) \in A \times A \;\middle|\; a \neq b \wedge \mathrm{tp}^{\mathfrak{A}}[a, b] = \tau\right\}.$$

[3] this is different than the commonly used notion of type in model theory, where types are sets of general formulas, not just literals

1.6 Normal forms

In two-variable logics, a common technique of reducing formula quantifier rank while preserving satisfiability is Skolemization [2]: Let φ be a $\mathcal{L}^2$-sentence. By replacing universally quantified subformulas $\forall x \psi$ by twofold existential negations $\neg\exists x \neg\psi$, without loss of generality assume that only existential quantifiers occur in φ. Consider a subformula ψ of φ that has the lowest possible nontrivial quantifier rank 1. Then $\psi = \psi(y) = \exists x \alpha(x, y)$, where the formula α is quantifier-free, $\{x, y\} = \{\boldsymbol{x}, \boldsymbol{y}\}$ and y may or may not necessarily occur freely in α. Introduce a new unary predicate symbol $\boldsymbol{u}_\psi$ with the intended interpretation $\forall y(\boldsymbol{u}_\psi(y) \leftrightarrow \exists x \alpha(x, y))$ and let φ' be the formula obtained from φ by replacing the subformula ψ by $\boldsymbol{u}_\psi(y)$. The original formula φ is equisatisfiable with $\varphi_1 = \forall y(\boldsymbol{u}_\psi(y) \leftrightarrow \exists x \alpha(x, y)) \wedge \varphi'$ in a strinct sense, that is any model for φ can be $\boldsymbol{u}_\psi$-enriched into a model for φ_1 and any model for φ_1 is a model for φ. By repeating this process linearly many times, we can bring the formula to a form where the quantifier rank is at most 2 [3, 2]:

Theorem 2 (Scott)**.** *There is a polynomial-time reduction* $\operatorname{sctr} : \mathcal{L}^2 \to \mathcal{L}^2$ *which reduces every sentence* φ *to a sentence* $\operatorname{sctr}\varphi$ *in* Scott normal form*:*

$$\forall \boldsymbol{x} \forall \boldsymbol{y}(\alpha_0(\boldsymbol{x}, \boldsymbol{y}) \vee \boldsymbol{x} = \boldsymbol{y}) \wedge \bigwedge_{1 \le i \le m} \forall \boldsymbol{x} \exists \boldsymbol{y}(\alpha_i(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{x} \ne \boldsymbol{y}),$$

where the formulas α_i *are quantifier-free and use at most linearly many new unary predicate symbols. The sentences* φ *and* $\operatorname{sctr}\varphi$ *are satisfiable over the same domains. Moreover the length* $\operatorname{sctr}\varphi$ *is linear in the length of* φ*.*

A completely analogous normal form can be described for the two-variable fragment with counting quantifiers [4]:

Theorem 3 (Pratt-Hartmann)**.** *There is a polynomial-time reduction* $\operatorname{prtr} : \mathcal{C}^2 \to \mathcal{C}^2$ *with reduces every sentence* φ *to a sentence* $\operatorname{prtr}\varphi$ *in the form:*

$$\forall \boldsymbol{x} \forall \boldsymbol{y}(\alpha_0(\boldsymbol{x}, \boldsymbol{y}) \vee \boldsymbol{x} = \boldsymbol{y}) \wedge \bigwedge_{1 \le i \le m} \forall \boldsymbol{x} \exists^{=M_i} \boldsymbol{y}(\alpha_i(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{x} \ne \boldsymbol{y}),$$

where the formulas α_i *are quantifier-free and may use linearly many new unary and binary predicate symbols. Let* $M = \max\{M_1, M_2, \dots, M_m\}$*. Then* φ *and* $\operatorname{prtr}\varphi$ *are satisfiable over the same domains of cardinality greater than* M*. Moreover the length* $\operatorname{prtr}\varphi$ *is linear in the length of* φ*.*

1.7 Complexity

We denote the complexity classes PTime $= \textsc{Time}[\operatorname{poly}(n)] = \bigcup_{c \in \mathbb{N}^+} \textsc{Time}[n^c]$, NPTime, PSpace, ExpTime and NExpTime. For $e \in \mathbb{N}^+$, the e-exponential deterministic and nondeterministic time classes are eExpTime $= \textsc{Time}[\exp_2^e(\operatorname{poly}(n))]$ and NeExpTime. The complexity class Elementary is the union of the complexity classes eExpTime for $e \in \mathbb{N}^+$.

The Grzegorczyk hierarchy $\mathcal{E}^i$ for $i \in \mathbb{N}$ orders the primitive recursive functions by means of the power of recursion needed. The *basic functions* are the zero function $\mathrm{zero}(n) = 0$, the successor function $\mathrm{succ}(n) = n + 1$ and the projection functions $\mathrm{proj}_i^u(n_1, n_2, \dots, n_u) = n_i$. If $u, v \in \mathbb{N}$, $f : \mathbb{N}^u \to \mathbb{N}$ and $g_1, g_2, \dots, g_u : \mathbb{N}^v \to \mathbb{N}$ are functions, their *superposition* is the function $h : \mathbb{N}^v \to \mathbb{N}$ defined by $h(\bar{n}) = f(g_1(\bar{n}), g_2(\bar{n}), \dots, g_u(\bar{n}))$ for $\bar{n} \in \mathbb{N}^v$. If $u \in \mathbb{N}$, $f : \mathbb{N}^u \to \mathbb{N}$ and $g : \mathbb{N}^{u+2} \to \mathbb{N}$, their *primitive recursion* is the function $h : \mathbb{N}^{u+1} \to \mathbb{N}$ defined by:

$$h(\bar{n}, 0) = f(\bar{n})$$
$$h(\bar{n}, i + 1) = g(\bar{n}, i, h(\bar{n}, i))$$

for $\bar{n} \in \mathbb{N}^u$. For $i \in \mathbb{N}$, define the function E_i by $E_0(n) = n + 1$ and

$$E_{i+1}(n) = E_i^n(2) = \underbrace{E_i(E_i(\dots E_i}_{n}(2))).$$

For $i \in \mathbb{N}$, the i-th level of the Grzegorczyk hierarchy $\mathcal{E}^i$ as the least set of functions containing the basic functions, the functions E_k for $k \in [0, i]$ and closed under superposition and *limited primitive recursion*, that is a primitive recursion $h : \mathbb{N}^{u+1}$ of the functions $f : \mathbb{N}^u \to \mathbb{N}$, $g : \mathbb{N}^{u+2} \to \mathbb{N}$, $f, g \in \mathcal{E}^i$, such that there is a function $b : \mathbb{N}^{u+1} \to \mathbb{N}$, $b \in \mathcal{E}^i$ bounding h: $h(\bar{n}) \le b(\bar{n})$ for all $n \in \mathbb{N}^{u+1}$. A decision problem $A \subseteq \Omega^*$ is in some level of the Grzegorczyk hierarchy just in case its characteristic function occurs at that level. The primitive recursive functions are partitioned by the Grzegorczyk hierarchy. The complexity class Elementary coincides with the third level of the Grzegorczyk hierarchy $\mathcal{E}^3$.

If $A \subseteq \Omega_1^*$ and $B \subseteq \Omega_2^*$ are decision problems, the problem A is *many-one polynomial-time reducible* to B (written $A \le_{\mathrm{m}}^{\mathrm{PTime}} B$) if there is a polynomial-time algorithm $f : \Omega_1^* \to \Omega_2^*$ such that $a \in A$ iff $f(a) \in B$. Similar reductions are defined analogously. The decision problems A and B are *many-one polynomial-time equivalent* (written $A =_{\mathrm{m}}^{\mathrm{PTime}} B$) if $A \le_{\mathrm{m}}^{\mathrm{PTime}} B$ and $B \le_{\mathrm{m}}^{\mathrm{PTime}} A$.

A decision problem is *hard* for a complexity class if any decision problem of that complexity class is polynomial-time reducible to it. A decision problem is *complete* for a complexity class if it is hard for that class and contained in that class.

We will need the following standard domino tiling problem [5, p. 403]: A *domino system* is a triple $D = (T, H, V)$, where $T = [1, k]$ is a finite set of *tiles* and $H, V \subseteq T \times T$ are *horizontal* and *vertical matching relations*. A *tiling* of $m \times m$ for a domino system D with *initial condition* $c^0 = \langle t_1^0, t_2^0, \dots, t_n^0 \rangle$, where $n \le m$, is a mapping $t : [1, m] \times [1, m] \to T$ such that:

- $(t(i, j), t(i + 1, j)) \in H$ for all $i \in [1, m - 1]$ and $j \in [1, m]$
- $(t(i, j), t(i, j + 1)) \in V$ for all $i \in [1, m]$ and $j \in [1, m - 1]$
- $t(i, 1) = t_i^0$ for all $i \in [1, n]$.

It is well-known [6, 7] that there exists a domino system D_0 for which:

- the problem asking whether there exists a tiling of $m \times m$ with initial condition c^0 of length n, where $m = n$, is NPTime-complete.
- the problem asking whether there exists a tiling of $m \times m$ with initial condition c^0 of length n, where $m = 2^n$, is NExpTime-complete.
- the problem asking whether there exists a tiling of $m \times m$ with initial condition c^0 of length n, where $m = 2^{2^n}$, is N2ExpTime-complete.
- the argument extends to arbitrary exponential towers: the problem asking whether there exists a tiling of $m \times m$ with initial condition c^0 of length n, where $m = \exp_2^e(n)$ is NeExpTime-complete.

2 Counter setups

2.1 Bits

A *bit setup* $\mathrm{B} = \langle \boldsymbol{u} \rangle$ is a predicate signature consisting of a single unary predicate symbol $\boldsymbol{u}$.

Definition 1. *Let $\mathfrak{A}$ be a* B*-structure. Define the function* $[\boldsymbol{u}\text{:data}]^{\mathfrak{A}} : A \to \mathbb{B}$ *by:*

$$[\boldsymbol{u}\text{:data}]^{\mathfrak{A}} a = \begin{cases} 1 & \text{if } \mathfrak{A} \vDash \boldsymbol{u}(a) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2. *Let $d \in \mathbb{B}$. Define the quantifier-free $\mathcal{L}^1[\mathrm{B}]$-formula* $[\boldsymbol{u}\text{:}\mathsf{eq}\text{-}d](\boldsymbol{x})$ *by:*

$$[\boldsymbol{u}\text{:}\mathsf{eq}\text{-}d](\boldsymbol{x}) = \begin{cases} \boldsymbol{u}(\boldsymbol{x}) & \text{if } d = 1 \\ \neg \boldsymbol{u}(\boldsymbol{x}) & \text{otherwise.} \end{cases}$$

If $\mathfrak{A}$ is a B-structure, $a \in A$ and $d \in \mathbb{B}$, then $\mathfrak{A} \vDash [\boldsymbol{u}\text{:}\mathsf{eq}\text{-}d](a)$ iff $[\boldsymbol{u}\text{:data}]^{\mathfrak{A}} a = d$.

Definition 3. *Define the quantifier-free $\mathcal{L}^2[\mathrm{B}]$-formulas* $[\boldsymbol{u}\text{:}\mathsf{eq}](\boldsymbol{x}, \boldsymbol{y})$, $[\boldsymbol{u}\text{:}\mathsf{eq}\text{-}01](\boldsymbol{x}, \boldsymbol{y})$ *and* $[\boldsymbol{u}\text{:}\mathsf{eq}\text{-}10](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$\begin{aligned} [\boldsymbol{u}\text{:}\mathsf{eq}](\boldsymbol{x}, \boldsymbol{y}) &= \boldsymbol{u}(\boldsymbol{x}) \leftrightarrow \boldsymbol{u}(\boldsymbol{y}) \\ [\boldsymbol{u}\text{:}\mathsf{eq}\text{-}01](\boldsymbol{x}, \boldsymbol{y}) &= \neg \boldsymbol{u}(\boldsymbol{x}) \wedge \boldsymbol{u}(\boldsymbol{y}) \\ [\boldsymbol{u}\text{:}\mathsf{eq}\text{-}10](\boldsymbol{x}, \boldsymbol{y}) &= \boldsymbol{u}(\boldsymbol{x}) \wedge \neg \boldsymbol{u}(\boldsymbol{y}). \end{aligned}$$

If $\mathfrak{A}$ is a B-structure and $a, b \in A$, then:

- $\mathfrak{A} \vDash [\boldsymbol{u}\text{:}\mathsf{eq}](a, b)$ iff $[\boldsymbol{u}\text{:data}]^{\mathfrak{A}} a = [\boldsymbol{u}\text{:data}]^{\mathfrak{A}} b$
- $\mathfrak{A} \vDash [\boldsymbol{u}\text{:}\mathsf{eq}\text{-}01](a, b)$ iff $[\boldsymbol{u}\text{:data}]^{\mathfrak{A}} a = 0$ and $[\boldsymbol{u}\text{:data}]^{\mathfrak{A}} b = 1$
- $\mathfrak{A} \vDash [\boldsymbol{u}\text{:}\mathsf{eq}\text{-}10](a, b)$ iff $[\boldsymbol{u}\text{:data}]^{\mathfrak{A}} a = 1$ and $[\boldsymbol{u}\text{:data}]^{\mathfrak{A}} b = 0$.

2.2 Counters

A *t-bit counter setup* for $t \in \mathbb{N}^+$ is a predicate signature $\mathrm{C} = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_t \rangle$ consisting of t distinct unary predicate symbols $\boldsymbol{u}_i$.

Definition 4. *Let $\mathfrak{A}$ be a* C*-structure. Define the function* $[\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}} : A \to \mathbb{B}_t$ *by:*

$$[\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}} a = \sum_{1 \le i \le t} 2^{i-1} [\boldsymbol{u}_i{:}\mathrm{data}]^{\mathfrak{A}} a.$$

Definition 5. *Let $d \in \mathbb{B}_t$ be a t-bit number. Define the quantifier-free $\mathcal{L}^1[\mathrm{C}]$-formula* $[\mathrm{C}{:}\mathsf{eq}\text{-}d](\boldsymbol{x})$ *by:*

$$[\mathrm{C}{:}\mathsf{eq}\text{-}d](\boldsymbol{x}) = \bigwedge_{1 \le i \le t} [\boldsymbol{u}_i{:}\mathsf{eq}\text{-}\bar{d}_i](\boldsymbol{x}).$$

If $\mathfrak{A}$ is a C-structure, $a \in A$ and $d \in \mathbb{B}_t$, then $\mathfrak{A} \vDash [\mathrm{C}{:}\mathsf{eq}\text{-}d](a)$ iff $[\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}} a = d$.

If A is a nonempty set and $\mathrm{data} : A \to \mathbb{B}_t$ is any function, there is a C-structure $\mathfrak{A}$ over A such that $[\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}} = \mathrm{data}$.

Definition 6. *Define the quantifier-free $\mathcal{L}^2[\mathrm{C}]$-formula* $[\mathrm{C}{:}\mathsf{eq}](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$[\mathrm{C}{:}\mathsf{eq}](\boldsymbol{x}, \boldsymbol{y}) = \bigwedge_{1 \le i \le t} [\boldsymbol{u}_i{:}\mathsf{eq}](\boldsymbol{x}, \boldsymbol{y}).$$

If $\mathfrak{A}$ is a C-structure and $a, b \in A$, then $\mathfrak{A} \vDash [\mathrm{C}{:}\mathsf{eq}](a, b)$ iff $[\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}} a = [\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}} b$.

The bitstring $\mathrm{a} \in \mathbb{B}^t$ encodes a number less than the number encoded by the bitstring $\mathrm{b} \in \mathbb{B}^t$, if they differ and at least position where they are different $j \in [1, t]$ the bitstring a has value 0 and the bitstring b has value 1, that is, iff there is a position $j \in [1, t]$ such that the following two conditions hold:

$$\mathrm{a}_j = 0 \text{ and } \mathrm{b}_j = 1 \tag{Less1}$$

$$\mathrm{a}_k = \mathrm{b}_k \text{ for all } k \in [j+1, t]. \tag{Less2}$$

Definition 7. *Define the quantifier-free $\mathcal{L}^2[\mathrm{C}]$-formula* $[\mathrm{C}{:}\mathsf{less}](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$[\mathrm{C}{:}\mathsf{less}](\boldsymbol{x}, \boldsymbol{y}) = \bigvee_{1 \le j \le t} [\boldsymbol{u}_j{:}\mathsf{eq}\text{-}01](\boldsymbol{x}, \boldsymbol{y}) \wedge \bigwedge_{j < k \le t} [\boldsymbol{u}_k{:}\mathsf{eq}](\boldsymbol{x}, \boldsymbol{y}).$$

If $\mathfrak{A}$ is a C-structure and $a, b \in A$, then $\mathfrak{A} \vDash [\mathrm{C}{:}\mathsf{less}](a, b)$ iff $[\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}} a < [\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}} b$.

The bitstring $\mathrm{b} \in \mathbb{B}^t$ encodes the successor of the number encoded by the bitstring a if there is a position $j \in [1, t]$ such that the following four conditions hold:

$$\mathrm{a}_j = 0 \text{ and } \mathrm{b}_j = 1 \tag{Succ1}$$

$$\mathrm{a}_i = 1 \text{ for all } i \in [1, j-1] \tag{Succ2}$$

$$\mathrm{b}_i = 0 \text{ for all } i \in [1, j-1] \tag{Succ3}$$

$$\mathrm{a}_k = \mathrm{b}_k \text{ for all } k \in [j+1, t]. \tag{Succ4}$$

Definition 8. *Define the quantifier-free $\mathcal{L}^2[\mathrm{C}]$-formula* $[\mathrm{C}{:}\mathsf{succ}](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$[\mathrm{C}{:}\mathsf{succ}](\boldsymbol{x}, \boldsymbol{y}) = \bigvee_{1 \le j \le t} [\boldsymbol{u}_j{:}\mathsf{eq}\text{-}01](\boldsymbol{x}, \boldsymbol{y}) \wedge \bigwedge_{1 \le i < j} [\boldsymbol{u}_i{:}\mathsf{eq}\text{-}10](\boldsymbol{x}, \boldsymbol{y}) \wedge \bigwedge_{j < k \le t} [\boldsymbol{u}_k{:}\mathsf{eq}](\boldsymbol{x}, \boldsymbol{y}).$$

If $\mathfrak{A}$ is a C-structure and $a, b \in A$, then:

$$\mathfrak{A} \vDash [\mathrm{C}\textbf{:}\mathsf{succ}](a, b) \text{ iff } [\mathrm{C}\textbf{:}\mathrm{data}]^{\mathfrak{A}} b = 1 + [\mathrm{C}\textbf{:}\mathrm{data}]^{\mathfrak{A}} a.$$

Definition 9. *Let $d \in \mathbb{B}_t$. Define the quantifier-free $\mathcal{L}^1[\mathrm{C}]$-formula $[\mathrm{C}\textbf{:}\mathsf{less}]d(\boldsymbol{x})$ by:*

$$[\mathrm{C}\textbf{:}\mathsf{less}\text{-}d](\boldsymbol{x}) = \bigvee_{1 \le j \le t} \neg \boldsymbol{u}_j(\boldsymbol{x}) \wedge \neg[\boldsymbol{u}_j\textbf{:}\mathsf{eq}\text{-}\bar{d}_j](\boldsymbol{x}) \wedge \bigwedge_{j < k \le t} [\boldsymbol{u}_k\textbf{:}\mathsf{eq}\text{-}\bar{d}_k](\boldsymbol{x}).$$

If $\mathfrak{A}$ is a C-structure, $a \in A$ and $d \in \mathbb{B}_t$, then $\mathfrak{A} \vDash [\mathrm{C}\textbf{:}\mathsf{less}\text{-}d](a)$ iff $[\mathrm{C}\textbf{:}\mathrm{data}]^{\mathfrak{A}} a < d$.

Definition 10. *Let $d \le e \in \mathbb{B}_t$. Define the quantifier-free $\mathcal{L}^1[\mathrm{C}]$-formula $[\mathrm{C}\textbf{:}\mathsf{betw}\text{-}d\text{-}e](\boldsymbol{x})$ by:*

$$[\mathrm{C}\textbf{:}\mathsf{betw}\text{-}d\text{-}e](\boldsymbol{x}) = \neg[\mathrm{C}\textbf{:}\mathsf{less}\text{-}d](\boldsymbol{x}) \wedge ([\mathrm{C}\textbf{:}\mathsf{less}\text{-}e](\boldsymbol{x}) \vee [\mathrm{C}\textbf{:}\mathsf{eq}\text{-}e](\boldsymbol{x})).$$

If $\mathfrak{A}$ is a C-structure, $a \in A$ and $d \le e \in \mathbb{B}_t$, then

$$\mathfrak{A} \vDash [\mathrm{C}\textbf{:}\mathsf{betw}\text{-}d\text{-}e](a) \text{ iff } d \le [\mathrm{C}\textbf{:}\mathrm{data}]^{\mathfrak{A}} a \le e.$$

Definition 11. *Let $d \le e \in \mathbb{B}_t$. Define the $\mathcal{L}^1[\mathrm{C}]$-sentence $[\mathrm{C}\textbf{:}\mathsf{allbetw}\text{-}d\text{-}e]$ by:*

$$[\mathrm{C}\textbf{:}\mathsf{allbetw}\text{-}d\text{-}e] = \forall \boldsymbol{x} [\mathrm{C}\textbf{:}\mathsf{betw}\text{-}d\text{-}e](\boldsymbol{x}).$$

If $\mathfrak{A}$ is a C-structure and $d \le e \in \mathbb{B}_t$, then $\mathfrak{A} \vDash [\mathrm{C}\textbf{:}\mathsf{betw}\text{-}d\text{-}e]$ iff $d \le [\mathrm{C}\textbf{:}\mathrm{data}]^{\mathfrak{A}} a \le e$ for all $a \in A$.

2.3 Vectors

Let $n, t \in \mathbb{N}^+$. Recall the set of *n-dimensional t-bit vectors* is $\mathbb{B}_t^n$. An *n-dimensional t-bit vector setup* is a predicate signature $\mathrm{V} = \langle \boldsymbol{u}_{11}, \boldsymbol{u}_{12}, \ldots, \boldsymbol{u}_{nt} \rangle$ of (nt) distinct unary predicate symbols. The *counter setup* $\mathrm{V}(p)$ of V at position $p \in [1, n]$ is $\mathrm{V}(p) = \langle \boldsymbol{u}_{p1}, \boldsymbol{u}_{p2}, \ldots, \boldsymbol{u}_{pt} \rangle$.

Definition 12. *Let $\mathfrak{A}$ be a V-structure and $a \in A$. We refer to $[\mathrm{V}(p)\textbf{:}\mathrm{data}]^{\mathfrak{A}} a$ as the* value of the p-th counter at a. *Define the function $[\mathrm{V}\textbf{:}\mathrm{data}]^{\mathfrak{A}} : A \to \mathbb{B}_t^n$ by:*

$$[\mathrm{V}\textbf{:}\mathrm{data}]^{\mathfrak{A}} a = \Big([\mathrm{V}(1)\textbf{:}\mathrm{data}]^{\mathfrak{A}} a, [\mathrm{V}(2)\textbf{:}\mathrm{data}]^{\mathfrak{A}} a, \ldots, [\mathrm{V}(n)\textbf{:}\mathrm{data}]^{\mathfrak{A}} a \Big).$$

Definition 13. *Let $\mathrm{v} = (d_1, d_2, \ldots, d_n) \in \mathbb{B}_t^n$ be an n-dimensional t-bit vector. Define the quantifier-free $\mathcal{L}^1[\mathrm{V}]$-formula $[\mathrm{V}\textbf{:}\mathsf{eq}\text{-}\mathrm{v}](\boldsymbol{x})$ by:*

$$[\mathrm{V}\textbf{:}\mathsf{eq}\text{-}\mathrm{v}](\boldsymbol{x}) = \bigwedge_{1 \le p \le n} [\mathrm{V}(p)\textbf{:}\mathsf{eq}\text{-}d_p](\boldsymbol{x}).$$

If $\mathfrak{A}$ is a V-structure, $a \in A$ and $\mathrm{v} \in \mathbb{B}_t^n$, then $\mathfrak{A} \vDash [\mathrm{V}\textbf{:}\mathsf{eq}\text{-}\mathrm{v}](a)$ iff $[\mathrm{V}\textbf{:}\mathrm{data}]^{\mathfrak{A}} a = \mathrm{v}$.

If $\mathfrak{A}$ is a nonempty set and $\mathrm{data} : A \to \mathbb{B}_t^n$ is any function, then there is a V-structure $\mathfrak{A}$ over A such that $[\mathrm{V}\textbf{:}\mathrm{data}]^{\mathfrak{A}} = \mathrm{data}$.

Definition 14. *Let $p, q \in [1, n]$ and let $i \in [1, t]$. Define the quantifier-free $\mathcal{L}^1[\mathrm{V}]$-formulas $[\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}](\boldsymbol{x})$, $[\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}\text{-}01](\boldsymbol{x})$ and $[\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}\text{-}10](\boldsymbol{x})$ by:*

$$\begin{aligned}
[\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}](\boldsymbol{x}) &= \boldsymbol{u}_{pi}(\boldsymbol{x}) \leftrightarrow \boldsymbol{u}_{qi}(\boldsymbol{x}) \\
[\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}\text{-}01](\boldsymbol{x}) &= \neg\boldsymbol{u}_{pi}(\boldsymbol{x}) \wedge \boldsymbol{u}_{qi}(\boldsymbol{x}) \\
[\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}\text{-}10](\boldsymbol{x}) &= \boldsymbol{u}_{pi}(\boldsymbol{x}) \wedge \neg\boldsymbol{u}_{qi}(\boldsymbol{x}).
\end{aligned}$$

If $\mathfrak{A}$ is a V-structure and $a \in A$, then:

- $\mathfrak{A} \models [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}](a)$ iff $[\boldsymbol{u}_{pi}\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a = [\boldsymbol{u}_{qi}\mathbf{:}\mathrm{data}]^{\mathfrak{A}}$, that is the values of the i-th bit at positions p and q at a are equal
- $\mathfrak{A} \models [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}\text{-}01](a)$ iff $[\boldsymbol{u}_{pi}\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a = 0$ and $[\boldsymbol{u}_{qi}\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a = 1$, that is the i-th bit at position p at a is 0 and the i-th bit at position q at a is 1
- $\mathfrak{A} \models [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}\text{-}10](a)$ iff $[\boldsymbol{u}_{pi}\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a = 1$ and $[\boldsymbol{u}_{qi}\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a = 0$, that is the i-th bit at position p at a is 1 and the i-th bit at position q at a is 0.

Definition 15. *Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[\mathrm{V}]$-formula $[\mathrm{V}(pq)\mathbf{:}\mathsf{eq}](\boldsymbol{x})$ by:*

$$[\mathrm{V}(pq)\mathbf{:}\mathsf{eq}](\boldsymbol{x}) = \bigwedge_{1 \le i \le t} [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}](\boldsymbol{x}).$$

If $\mathfrak{A}$ is a V-structure and $a \in A$, then:

$$\mathfrak{A} \models [\mathrm{V}(pq)\mathbf{:}\mathsf{eq}](a) \text{ iff } [\mathrm{V}(p)\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a = [\mathrm{V}(q)\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a.$$

Definition 16. *Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[\mathrm{V}]$-formula $[\mathrm{V}(pq)\mathbf{:}\mathsf{less}](\boldsymbol{x})$ by:*

$$[\mathrm{V}(pq)\mathbf{:}\mathsf{less}](\boldsymbol{x}) = \bigvee_{1 \le j \le t} [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}j\text{-}\mathsf{eq}\text{-}01](\boldsymbol{x}) \wedge \bigwedge_{j < k \le t} [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}k\text{-}\mathsf{eq}](\boldsymbol{x}).$$

If $\mathfrak{A}$ is a V-structure and $a \in A$, then:

$$\mathfrak{A} \models [\mathrm{V}(pq)\mathbf{:}\mathsf{less}](a) \text{ iff } [\mathrm{V}(p)\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a < [\mathrm{V}(q)\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a.$$

Definition 17. *Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[\mathrm{V}]$-formula*

$$\begin{aligned}
[\mathrm{V}(pq)\mathbf{:}\mathsf{succ}](\boldsymbol{x}) = \bigvee_{1 \le j \le t} \bigwedge_{1 \le i < j} [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}i\text{-}\mathsf{eq}\text{-}10](\boldsymbol{x}) \wedge [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}j\text{-}\mathsf{eq}\text{-}01](\boldsymbol{x}) \wedge \\
\bigwedge_{j < k \le t} [\mathrm{V}(pq)\mathbf{:}\mathsf{at}\text{-}k\text{-}\mathsf{eq}](\boldsymbol{x}).
\end{aligned}$$

If $\mathfrak{A}$ is a V-structure and $a \in A$, then:

$$\mathfrak{A} \models [\mathrm{V}(pq)\mathbf{:}\mathsf{succ}](a) \text{ iff } [\mathrm{V}(q)\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a = 1 + [\mathrm{V}(p)\mathbf{:}\mathrm{data}]^{\mathfrak{A}}a.$$

2.4 Permutations

Let $n \in \mathbb{N}^+$. An *n-permutation setup* $\mathrm{P} = \langle \boldsymbol{u}_{11}, \boldsymbol{u}_{12}, \ldots, \boldsymbol{u}_{nt} \rangle$ is just an n-dimensional t-bit vector setup, where $t = \|n\|$ is the bitsize of n. Recall that the set $\mathbb{S}_n$ of all permutations of $[1, n]$ is a subset of $\mathbb{B}_t^n$.

Definition 18. *Define the quantifier-free* $\mathcal{L}^1[\mathrm{P}]$*-sentence* $[\mathrm{P}\mathbf{:}\mathsf{alldiff}]$ *by:*

$$[\mathrm{P}\mathbf{:}\mathsf{alldiff}] = \forall \boldsymbol{x} \bigwedge_{1 \le p < q \le n} \neg [\mathrm{P}(pq)\mathbf{:}\mathsf{eq}](\boldsymbol{x}).$$

If $\mathfrak{A}$ is a P-structure then $\mathfrak{A} \vDash [\mathrm{P}\mathbf{:}\mathsf{alldiff}]$ iff $[\mathrm{P}(p)\mathbf{:}\mathrm{data}]^{\mathfrak{A}} a \neq [\mathrm{P}(q)\mathbf{:}\mathrm{data}]^{\mathfrak{A}} a$ for all $a \in A$ and $p \neq q \in [1, n]$.

Definition 19. *Define the quantifier-free* $\mathcal{L}^1[\mathrm{P}]$*-sentence* $[\mathrm{P}\mathbf{:}\mathsf{perm}]$ *by:*

$$[\mathrm{P}\mathbf{:}\mathsf{perm}] = [\mathrm{P}\mathbf{:}\mathsf{betw}\text{-}1\text{-}n] \wedge [\mathrm{P}\mathbf{:}\mathsf{alldiff}].$$

If $\mathfrak{A}$ is a P-structure then $\mathfrak{A} \vDash [\mathrm{P}\mathbf{:}\mathsf{perm}]$ iff $[\mathrm{P}\mathbf{:}\mathrm{data}]^{\mathfrak{A}} a \in \mathbb{S}_n$ for all $a \in A$.

If A is a nonempty set and $\mathrm{data} : A \to \mathbb{S}_n$ is any function, then there is a P-structure $\mathfrak{A} \vDash [\mathrm{P}\mathbf{:}\mathsf{perm}]$ over A such that $[\mathrm{P}\mathbf{:}\mathrm{data}]^{\mathfrak{A}} = \mathrm{data}$.

3 Equivalence relations

An *equivalence relation* $E \subseteq A \times A$ on A is a relation that is reflexive, symmetric and transitive. The set of *equivalence classes* of E is $\mathscr{E}E = \{E[a] \mid a \in A\}$.

Let $\mathrm{E} = \langle \boldsymbol{e} \rangle$ be a predicate signature consisting of a single binary predicate symbol $\boldsymbol{e}$. Define the $\mathcal{L}^2[\mathrm{E}]$-sentence $[\boldsymbol{e}\text{:refl}]$ by:

$$[\boldsymbol{e}\text{:refl}] = \forall \boldsymbol{x} \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{x}).$$

Define the $\mathcal{L}^2[\mathrm{E}]$-sentence $[\boldsymbol{e}\text{:symm}]$ by:

$$[\boldsymbol{e}\text{:symm}] = \forall \boldsymbol{x} \forall \boldsymbol{y} \left(\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{e}(\boldsymbol{y}, \boldsymbol{x}) \right).$$

Define the $\mathcal{L}^3[\mathrm{E}]$-sentence $[\boldsymbol{e}\text{:trans}]$ by:

$$[\boldsymbol{e}\text{:trans}] = \forall \boldsymbol{x} \forall \boldsymbol{y} \forall \boldsymbol{z} \left(\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{e}(\boldsymbol{y}, \boldsymbol{z}) \to \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{z}) \right).$$

Define the $\mathcal{L}^3[\mathrm{E}]$-sentence $[\boldsymbol{e}\text{:equiv}]$ by:

$$[\boldsymbol{e}\text{:equiv}] = [\boldsymbol{e}\text{:refl}] \wedge [\boldsymbol{e}\text{:symm}] \wedge [\boldsymbol{e}\text{:trans}].$$

Let $\mathfrak{A}$ be an E-structure and let $E = \boldsymbol{e}^{\mathfrak{A}}$. Then E is reflexive iff $\mathfrak{A} \models [\boldsymbol{e}\text{:refl}]$; E is symmetric iff $\mathfrak{A} \models [\boldsymbol{e}\text{:symm}]$; E is transitive iff $\mathfrak{A} \models [\boldsymbol{e}\text{:trans}]$; E is an equivalence on A iff $\mathfrak{A} \models [\boldsymbol{e}\text{:equiv}]$. It can be shown that transitivity and equivalence cannot be defined in the two-variable fragment with counting $\mathcal{C}^2[\mathrm{E}]$.

3.1 Two equivalence relations in agreement

Definition 20. *Let $\langle D, E \rangle$ be a sequence of two equivalence relations on A. The relation D is* finer *than the relation E if every equivalence class of D is a subset of some equivalence class of E. Equivalently, $D \subseteq E$. Equivalently,*

$$(\forall a \in A)(\forall b \in A) \left(D(a, b) \to E(a, b) \right).$$

If D is finer than E, then E is coarser *than D. The sequence $\langle D, E \rangle$ is a sequence of equivalence relations on A in* refinement *if D is finer E.*

The sequence $\langle D, E \rangle$ is a sequence of equivalence relations in global agreement *if either D is finer than E or E is finer than D.*

The sequence $\langle D, E \rangle$ is a sequence of equivalence relations in local agreement *if for every $a \in A$, either $D[a] \subseteq E[a]$ or $E[a] \subseteq D[a]$. Equivalently, no two equivalence classes $E[a]$ and $D[b]$ properly intersect. Equivalently,*

$$(\forall a \in A) \left((\forall b \in A) \left(D(a, b) \to E(a, b) \right) \vee (\forall b \in A) \left(E(a, b) \to D(a, b) \right) \right).$$

Let $\mathrm{E} = \langle \boldsymbol{d}, \boldsymbol{e} \rangle$ be a predicate signature consisting of the two binary predicate symbols $\boldsymbol{d}$ and $\boldsymbol{e}$. Let $\mathfrak{A}$ is an E-structure and suppose that $\boldsymbol{d}$ and $\boldsymbol{e}$ are interpreted in $\mathfrak{A}$ as equivalence relations on A. Let $D = \boldsymbol{d}^{\mathfrak{A}}$ and $E = \boldsymbol{e}^{\mathfrak{A}}$ be the interpretations of the two symbols.

Definition 21. *Define the $\mathcal{L}^2[\mathrm{E}]$-sentence $[\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{refine}]$ by:*

$$[\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{refine}] = \forall \boldsymbol{x} \forall \boldsymbol{y} \left(\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \right).$$

Then $\langle D, E \rangle$ is in refinement iff $\mathfrak{A} \vDash [\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{refine}]$.

Definition 22. *Define the $\mathcal{L}^2[\mathrm{E}]$-sentence $[\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{global}]$ by:*

$$[\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{global}] = [\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{refine}] \vee [\boldsymbol{e}, \boldsymbol{d}{:}\mathsf{refine}].$$

Then $\langle D, E \rangle$ is in global agreement iff $\mathfrak{A} \vDash [\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{global}]$.

Definition 23. *Define the $\mathcal{L}^2[\mathrm{E}]$-sentence $[\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{local}]$ by:*

$$[\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{local}] = \forall \boldsymbol{x} \left(\forall \boldsymbol{y} \left(\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \right) \vee \forall \boldsymbol{y} \left(\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \right) \right).$$

Then $\langle D, E \rangle$ is in global agreement iff $\mathfrak{A} \vDash [\boldsymbol{d}, \boldsymbol{e}{:}\mathsf{local}]$.

Lemma 1. *If $\langle D, E \rangle$ is a sequence two equivalence relations on A, then it is in local agreement iff $L = D \cup E$ is an equivalence relation on A.*

Proof. The union of two equivalence relations on A is a reflexive and symmetric relation.

First suppose that D and E are in local agreement. We claim that L is transitive. Let $a, b, c \in A$ be such that $(a, b) \in L$ and $(b, c) \in L$. Since D and E are in local agreement, without loss of generality $D[b] \subseteq E[b]$. Since $(a, b) \in L$, either $a \in D[b] \subseteq E[b]$ or $a \in E[b]$. Similarly $c \in E[b]$. Therefore $(a, c) \in E \subseteq L$.

Next suppose that L is an equivalence relation, let $b \in A$ and assume towards a contradiction that $D[b] \nsubseteq E[b]$ and $E[b] \nsubseteq D[b]$. There is some $a \in D[b] \setminus E[b]$ and $c \in E[b] \setminus D[b]$. Then $(a, b) \in D \subseteq L$ and $(b, c) \in E \subseteq L$, hence $(a, c) \in L$. Without loss of generality $(a, c) \in E$. Since $c \in E[b]$, we have $a \in E[b]$ — a contradiction. $\square$

3.2 Many equivalence relations in agreement

Let e be a positive natural number.

Definition 24. *Let $\langle E_1, E_2, \ldots, E_e \rangle$ be a sequence of equivalence relations on A.*

The sequence is in refinement *if $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_e$.*

The sequence is in global agreement *if the equivalence relations form a chain under inclusion, that is for all $i, j \in [1, e]$, either $E_i \subseteq E_j$ or $E_j \subseteq E_i$. Equivalently, there is a (not necessarily unique) permutation $\nu \in \mathbb{S}_e$ such that $E_{\nu(1)} \subseteq E_{\nu(2)} \subseteq \cdots \subseteq E_{\nu(e)}$.*

The sequence is in local agreement *if for every element $a \in A$ the equivalence classes $E_1[a], E_2[a], \ldots, E_e[a]$ form a chain under inclusion. Equivalently, no two equivalence classes $E_i[a]$ and $E_j[b]$ properly intersect.*

Let $\mathrm{E} = \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e \rangle$ be a predicate signature consisting of e binary predicate symbols. Let $\mathfrak{A}$ be an E-structure and suppose that the symbols $\boldsymbol{e}_i$ are interpreted as equivalence relations on A. Let $E_i = \boldsymbol{e}_i^{\mathfrak{A}}$ for $i \in [1, e]$.

Definition 25. *Define the* $\mathcal{L}^2[\mathrm{E}]$*-sentence* $[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:refine}]$ *by:*

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:refine}] = \forall \boldsymbol{x} \forall \boldsymbol{y} \bigwedge_{1 \le i < e} (\boldsymbol{e}_i(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{e}_{i+1}(\boldsymbol{x}, \boldsymbol{y}))\,.$$

Then $\langle E_1, E_2, \ldots, E_e \rangle$ is in refinement iff $\mathfrak{A} \models [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:refine}]$.

Definition 26. *Define the* $\mathcal{L}^2[\mathrm{E}]$*-sentence* $[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:global}]$ *by:*

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:global}] = \bigvee_{\nu \in \mathbb{S}_e} [\boldsymbol{e}_{\nu(1)}, \boldsymbol{e}_{\nu(2)}, \ldots, \boldsymbol{e}_{\nu(e)}\textsf{:refine}].$$

Then $\langle E_1, E_2, \ldots, E_e \rangle$ is in global agreement iff $\mathfrak{A} \models [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:global}]$. Note that the length of the formula $[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:global}]$ grows exponentially as e grows.

Definition 27. *Define the* $\mathcal{L}^2[\mathrm{E}]$*-sentence* $[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:local}]$ *by:*

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:local}] = \forall \boldsymbol{x} \bigvee_{\nu \in \mathbb{S}_e} \forall \boldsymbol{y} \bigwedge_{1 \le i < e} (\boldsymbol{e}_{\nu(i)}(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{e}_{\nu(i+1)}(\boldsymbol{x}, \boldsymbol{y})).$$

Then $\langle E_1, E_2, \ldots, E_e \rangle$ is in local agreement iff $\mathfrak{A} \models [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:local}]$. Note that the length of the formula $[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:local}]$ grows exponentially as e grows.

Let $E = \langle E_1, E_2, \ldots, E_e \rangle$ be a sequence of equivalence relations on A.

Theorem 4. *The sequence E is in local agreement iff the union $\cup S$ of any nonempty subsequence $S \subseteq E$ is an equivalence relation on A.*

Proof. First suppose that the equivalence relations E_i are in local agreement. We show that the union $\cup S$ of arbitrary nonempty subsequence $S = \{E_{i(1)}, E_{i(2)}, \ldots, E_{i(s)}\}$, where $1 \le i(1) < i(2) < \cdots < i(s) \le e$, is an equivalence relation by induction on s, the length of S. If $s = 1$ this claim is trivial. Suppose $s > 1$. By the induction hypothesis, $D = \cup\{E_{i(1)}, E_{i(2)}, \ldots, E_{i(s-1)}\}$ is an equivalence relation on A. We claim that D and $E_{i(s)}$ are in local agreement. Indeed, let $a \in A$ be arbitrary and consider $D[a] = E_{i(1)}[a] \cup E_{i(2)}[a] \cup \cdots \cup E_{i(s-1)}[a]$ and $E_{i(s)}[a]$. Since all equivalences E_k are in local agreement, either $E_{i(s)}[a] \subseteq E_{i(j)}[a]$ for some $j \in [1, s-1]$, or $E_{i(j)}[a] \subseteq E_{i(s)}[a]$ for all $j \in [1, s-1]$. In the first case $E_{i(s)}[a] \subseteq D[a]$; in the second case $D[a] \subseteq E_{i(s)}[a]$. Thus D and $E_{i(s)}$ are in local agreement. By Lemma 1, $\cup S = D \cup E_{i(s)}$ is an equivalence relation on A.

Next suppose that the equivalences are not in local agreement. There is an element $a \in A$ such that $\{E_i[a] \mid i \in [1, e]\}$ is not a chain. There are $i, j \in [1, e]$ such that $E_i[a] \not\subseteq E_j[a]$ and $E_j[a] \not\subseteq E_i[a]$. Thus E_i and E_j are not in local agreement. By Lemma 1, the union $E_i \cup E_j$ is not an equivalence relation on A. ☐

Suppose that the sequence $E = \langle E_1, E_2, \ldots, E_e \rangle$ is in local agreement.

Definition 28. *An* index set *is an element* $\mathrm{I} \in \wp^{+}[1,e]$*. Define* $(E \restriction \cdot) : \wp^{+}[1,e] \to \wp^{+}E$ *by:*

$$(E \restriction \mathrm{I}) = \{E_i \mid i \in \mathrm{I}\}\,.$$

That is, $(E \restriction \mathrm{I})$ *just collects the equivalences having indices from* I.

The level sequence $L = \langle L_1, L_2, \ldots, L_e\rangle$ *of the sequence* E *is defined as follows. For* $k \in [1,e]$*:*

$$L_k = \cap \left\{ \cup(E \restriction \mathrm{I}) \;\middle|\; \mathrm{I} \in \wp^{k}[1,e] \right\}.$$

Remark 2. *All* L_k *are equivalence relations on* A*.*

Proof. Let $k \in [1,e]$ and let $\mathrm{K} \in \wp^{k}[1,e]$ be any k-index set. By Theorem 4, $\cup(E \restriction \mathrm{K})$ is an equivalence relation on A. Since intersection of equivalence relations on A is again an equivalence relation on A, the level $L_k = \cap \left\{ \cup(E \restriction \mathrm{K}) \;\middle|\; \mathrm{K} \in \wp^{k}[1,e] \right\}$ is an equivalence relation on A. □

Remark 3. *The level sequence* $L = \langle L_1, L_2, \ldots, L_e\rangle$ *is a sequence of equivalence relations on* A *in refinement.*

Proof. Let $i < j \in [1,e]$. Let $\mathrm{J} \in \wp^{j}[1,e]$ be any j-index set. We claim that $L_i \subseteq \cup(E \restriction \mathrm{J})$. Indeed, choose some i-index set $\mathrm{I} \subset \mathrm{J}$. By the definition of L_i we have $L_i \subseteq \cup(E \restriction \mathrm{I}) \subseteq \cup(E \restriction \mathrm{J})$. Hence $L_i \subseteq \cap\{\cup(E \restriction \mathrm{J}) \mid \mathrm{J} \in \wp^{j}[1,e]\} = L_j$. □

Let $a \in A$. Since the sequence $E = \langle E_1, E_2, \ldots, E_e\rangle$ is in local agreement, there is a permutation $\nu \in \mathbb{S}_e$ such that:

$$E_{\nu(1)}[a] \subseteq E_{\nu(2)}[a] \subseteq \cdots \subseteq E_{\nu(e)}[a]. \tag{3.1}$$

Lemma 2. *If* $\nu \in \mathbb{S}_e$ *is a permutation satisfying eq. (3.1), then* $L_{\nu^{-1}(i)}[a] = E_i[a]$ *for all* $i \in [1,e]$*.*

Proof. Let $k = \nu^{-1}(i)$, so $\nu(k) = i$. We claim that $L_k[a] = E_i[a]$. First, consider the k-index set $\mathrm{K} = \{\nu(1), \nu(2), \ldots, \nu(k)\}$. By the definition of L_k, followed by eq. (3.1), we have $L_k[a] \subseteq \cup(E \restriction \mathrm{K})[a] = E_{\nu(k)}[a] = E_i[a]$. Next, let $\mathrm{K} \subseteq \wp^{k}[1,e]$ be any k-index set. By the pigeonhole principle, there is some $k' \geq k$ such that $k' \in \mathrm{K}$. By eq. (3.1) we have:

$$E_i[a] = E_{\nu(k)}[a] \subseteq E_{\nu(k')}[a] \subseteq \cup(E \restriction \mathrm{K})[a].$$

Hence $E_i[a] \subseteq \cap \left\{ \cup(E \restriction \mathrm{K})[a] \;\middle|\; \mathrm{K} \in \wp^{k}[1,e] \right\} = L_k[a]$. □

4 Reductions

We restrict our attention to binary predicate signatures, consisting of unary and binary predicate symbols only. To denote various logics with builtin equivalence symbols, we use the notation

$$\Lambda_p^v e\mathrm{E}_{\mathsf{a}}$$

where:

- $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ is the *ground logic*
- v, if given, bounds the number of variables
- e, if given, bounds the number of builtin equivalence symbols
- $\mathsf{a} \in \{\mathsf{refine}, \mathsf{global}, \mathsf{local}\}$, if given, gives the agreement condition between the builtin equivalence symbols
- p, the *signature power*, specifies constraints on the signature:
  - if $p = 0$, the signature consists of only constantly many unary predicate symbols in addition to the builtin equivalence symbols
  - if $p = 1$, the signature consists of unboundedly many unary predicate symbols in addition to the builtin equivalence symbols
  - if p is not given, the signature consists of unboundedly many unary and binary predicate symbols in addition to the builtin equivalence symbols. This is the commonly investigated fragment with respect to satisfiability of the two-variable logics with or without counting quantifiers.

For example $\mathcal{L}_1$ is the monadic first-order logic, featuring only unary predicate symbols. $\mathcal{L}_0 1\mathrm{E}$ is the first-order logic of a single equivalence relation. $\mathcal{C}^2$ is the two-variable logic with counting quantifiers, featuring unary and binary predicate symbols. $\mathcal{L}^2 2\mathrm{E}$ is the two-variable logic, featuring unary, binary predicate symbols and two builtin equivalence symbols. $\mathcal{C}_1^2 2\mathrm{E}_{\mathsf{local}}$ is the two-variable logic with counting quantifiers, featuring unary predicate symbols and two builtin equivalence symbols in local agreement. $\mathcal{L}_1\mathrm{E}_{\mathsf{global}}$ is the monadic first-order logic featuring many equivalence symbols in global agreement.

When we working with a concrete logic, for example $\mathcal{C}^2 2\mathrm{E}_{\mathsf{local}}$, we implicitly assume an appropriate generic predicate signature Σ for it. In this case, there are two builtin equivalence symbols $\boldsymbol{d}$ and $\boldsymbol{e}$ in Σ and in addition Σ contains arbitrary many unary and binary predicate symbols. The *intended interpretation* of the builtin equivalence symbols is fixed by an appropriate condition θ. In this case:

$$\theta = [\boldsymbol{d}\text{:equiv}] \wedge [\boldsymbol{e}\text{:equiv}] \wedge [\boldsymbol{d}, \boldsymbol{e}\text{:local}].$$

Note that the interpretation condition might in general be a first-order formula outside the logic in interest, as in this case, since for instance $[\boldsymbol{d}\text{:equiv}]$ uses the variables $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ and the logic $\mathcal{C}^2 2\mathrm{E}_{\mathsf{local}}$ is a two-variable logic. Recall that when talking about semantics, we include the intended interpretation condition in the definition of Σ-structures.

4.1 Global agreement to refinement

In this section we demonstrate how (finite) satisfiability in logics featuring builtin equivalence symbols in global agreement reduces to (finite) satisfiability in logics featuring builtin equivalence symbols in refinement. Our strategy is to encode the permutation of the builtin equivalence symbols in global agreement that turns them in refinement into a permutation setup.

Fix an arbitrary ground logic $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ and think of Σ as a predicate signature for the logics $\Lambda e\mathrm{E}_{\mathsf{global}}$ and $\Lambda e\mathrm{E}_{\mathsf{refine}}$. The e builtin equivalence symbols of Σ are $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$.

Let φ be a $\Lambda[\Sigma]$-sentence. The class of $\Lambda e\mathrm{E}_{\mathsf{refine}}$-structures satisfying φ coincides with the class of $\Lambda e\mathrm{E}_{\mathsf{global}}$-structures satisfying:

$$\varphi \wedge [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\text{:refine}].$$

Hence:

$$(\text{FIN-})\text{SAT-}\Lambda e\mathrm{E}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\mathrm{PTIME}} (\text{FIN-})\text{SAT-}\Lambda e\mathrm{E}_{\mathsf{global}}.$$

Since the length of the formula $[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\text{:refine}]$ grows polynomially as e grows:

$$(\text{FIN-})\text{SAT-}\Lambda\mathrm{E}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\mathrm{PTIME}} (\text{FIN-})\text{SAT-}\Lambda\mathrm{E}_{\mathsf{global}}.$$

Consider the opposite direction. Let $\mathrm{P} = \langle \boldsymbol{u}_{11}, \boldsymbol{u}_{12}, \ldots, \boldsymbol{u}_{et} \rangle$ be an e-permutation setup (where $t = \|e\|$).

Definition 29. *Define the $\mathcal{L}^2[\mathrm{P}]$-sentence* $[\mathrm{P}\text{:alleq}]$ *by:*

$$[\mathrm{P}\text{:alleq}] = \forall \boldsymbol{x} \forall \boldsymbol{y} \bigwedge_{1 \leq i \leq e} [\mathrm{P}(i)\text{:eq}](\boldsymbol{x}, \boldsymbol{y}).$$

If $\mathfrak{A}$ is a P-structure, then $\mathfrak{A} \vDash [\mathrm{P}\text{:alleq}]$ iff $[\mathrm{P}\text{:data}]^{\mathfrak{A}} a = [\mathrm{P}\text{:data}]^{\mathfrak{A}} b$ for all $a, b \in A$. If A is a nonempty set and $\mathrm{v} \in \mathbb{B}_t^e$ is any e-dimensional t-vector, there is a P-structure $\mathfrak{A}$ over A such that $\mathfrak{A} \vDash [\mathrm{P}\text{:alleq}]$ and $[\mathrm{P}\text{:data}]^{\mathfrak{A}} a = \mathrm{v}$ for all $a \in A$.

Definition 30. *Define the $\mathcal{L}^2[\mathrm{P}]$-sentence* $[\mathrm{P}\text{:globperm}]$ *by:*

$$[\mathrm{P}\text{:globperm}] = [\mathrm{P}\text{:perm}] \wedge [\mathrm{P}\text{:alleq}].$$

If $\mathfrak{A}$ be a P-structure then $\mathfrak{A} \models [\mathrm{P}{:}\mathsf{globperm}]$ iff there is a permutation $\nu \in \mathbb{S}_e$ such that $[\mathrm{P}{:}\mathrm{data}]^{\mathfrak{A}} a = \nu$ for all $a \in A$.

If A be a nonempty set and $\nu \in \mathbb{S}_e$ is any permutation, there is a P-structure $\mathfrak{A}$ over A such that $\mathfrak{A} \models [\mathrm{P}{:}\mathsf{globperm}]$ and $[\mathrm{P}{:}\mathrm{data}]^{\mathfrak{A}} a = \nu$ for all $a \in A$.

Let $\mathrm{L} = \langle \boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_e \rangle + \mathrm{P}$ be a predicate signature consisting of the binary predicate symbols $\boldsymbol{l}_k$ in addition to the symbols from P.

Definition 31. *For $i \in [1, e]$, define the quantifier-free $\mathcal{L}^2[\mathrm{L}]$-formula $[\mathrm{L}{:}\mathsf{eg}\text{-}i](\boldsymbol{x}, \boldsymbol{y})$ by:*

$$[\mathrm{L}{:}\mathsf{eg}\text{-}i](\boldsymbol{x}, \boldsymbol{y}) = \bigwedge_{1 \le k \le e} \left([\mathrm{P}(k){:}\mathsf{eq}\text{-}i](\boldsymbol{x}) \to \boldsymbol{l}_k(\boldsymbol{x}, \boldsymbol{y}) \right).$$

Remark 4. *Let $\mathfrak{A}$ be an* L*-structure and suppose that $\mathfrak{A} \models [\mathrm{P}{:}\mathsf{globperm}]$ and that the binary symbols $\boldsymbol{l}_k$ are interpreted as equivalence relations on A in refinement. Recall that there is a permutation $\nu \in \mathbb{S}_e$ such that $[\mathrm{P}{:}\mathrm{data}]^{\mathfrak{A}} a = \nu$ for all $a \in A$. Then for all $i \in [1, e]$:*

$$[\mathrm{L}{:}\mathsf{eg}\text{-}i]^{\mathfrak{A}} = \boldsymbol{l}^{\mathfrak{A}}_{\nu^{-1}(i)}.$$

In particular, $\left\langle [\mathrm{L}{:}\mathsf{eg}\text{-}1]^{\mathfrak{A}}, [\mathrm{L}{:}\mathsf{eg}\text{-}2]^{\mathfrak{A}}, \ldots, [\mathrm{L}{:}\mathsf{eg}\text{-}e]^{\mathfrak{A}} \right\rangle$ is a sequence of equivalence relations on A in global agreement.

Proof. Let $k = \nu^{-1}(i)$, so $\nu(k) = i$ and $[\mathrm{P}(k){:}\mathrm{data}]^{\mathfrak{A}} a = i$. Since ν is a permutation, for every $k' \in [1, e]$:

$$\mathfrak{A} \models [\mathrm{P}(k'){:}\mathsf{eq}\text{-}i](a) \text{ iff } [\mathrm{P}(k'){:}\mathrm{data}]^{\mathfrak{A}} a = i \text{ iff } k' = k. \tag{4.1}$$

Let $a, b \in A$. First suppose that $\mathfrak{A} \models [\mathrm{L}{:}\mathsf{eg}\text{-}i](a, b)$. By eq. (4.1) we must have that $\mathfrak{A} \models [\mathrm{P}(k){:}\mathsf{eq}\text{-}i](a)$, hence $\mathfrak{A} \models \boldsymbol{l}_k(a, b)$.

Now suppose that $\mathfrak{A} \models \neg[\mathrm{L}{:}\mathsf{eg}\text{-}i](a, b)$. There is some $k' \in [1, e]$ such that:

$$\mathfrak{A} \models \neg \left([\mathrm{P}(k'){:}\mathsf{eq}\text{-}i](a) \to \boldsymbol{l}_{k'}(a, b) \right) \equiv [\mathrm{P}(k'){:}\mathsf{eq}\text{-}i](a) \wedge \neg \boldsymbol{l}_{k'}(a, b).$$

By eq. (4.1) we have $k' = k$, hence $\mathfrak{A} \models \neg \boldsymbol{l}_k(a, b)$. □

Let $\mathrm{E} = \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e \rangle$ be a predicate signature consisting of the binary predicate symbols $\boldsymbol{e}_i$. Let Σ be a predicate signature enriching E and not containing any symbols from L. Let $\Sigma' = \Sigma \cup \mathrm{L}$ and $\mathrm{L}' = \Sigma' - \mathrm{E}$.

Definition 32. *Define the syntactic operation $\mathrm{gtr} : \Lambda[\Sigma] \to \Lambda[\mathrm{L}']$ by:*

$$\mathrm{gtr}\, \varphi = \varphi' \wedge [\mathrm{P}{:}\mathsf{globperm}],$$

where φ' is obtained from φ by replacing all occurrences of a subformula of the form $\boldsymbol{e}_i(x, y)$ by the formula $[\mathrm{L}{:}\mathsf{eg}\text{-}i](x, y)$, where x and y are (not necessarily distinct) variables and $i \in [1, e]$.

Remark 5. *Let φ be a $\Lambda[\Sigma]$-formula and let $\mathfrak{A}$ be a Σ-structure. Suppose that $\mathfrak{A} \models \varphi$ and that the symbols $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$ are interpreted in $\mathfrak{A}$ as equivalence relations on A in global agreement. Then there is a Σ'-enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \models \operatorname{gtr}\varphi$ and that the symbols $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_e$ are interpreted in $\mathfrak{A}'$ as equivalence relations on A in refinement.*

Proof. There is a permutation $\nu \in \mathbb{S}_e$ such that $\boldsymbol{e}^{\mathfrak{A}}_{\nu(1)} \subseteq \boldsymbol{e}^{\mathfrak{A}}_{\nu(2)} \subseteq \cdots \subseteq \boldsymbol{e}^{\mathfrak{A}}_{\nu(e)}$. Consider an enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ to a Σ'-structure where $\boldsymbol{l}^{\mathfrak{A}'}_k = \boldsymbol{e}^{\mathfrak{A}}_{\nu(k)}$, so the interpretations of $\boldsymbol{l}_k$ in $\mathfrak{A}'$ are equivalence relations on A in refinement. We can interpret the unary predicate symbols from permutation setup P in $\mathfrak{A}'$ so that $\mathfrak{A}' \models [\mathrm{P{:}\mathsf{globperm}}]$ and $[\mathrm{P{:}data}]^{\mathfrak{A}} a = \nu$ for all $a \in A$. By Remark 4, for every $i \in [1, e]$:

$$[\mathrm{L{:}\mathsf{eg}}\text{-}i]^{\mathfrak{A}'} = \boldsymbol{l}^{\mathfrak{A}'}_{\nu^{-1}(i)} = \boldsymbol{e}^{\mathfrak{A}'}_{\nu(\nu^{-1}(i))} = \boldsymbol{e}^{\mathfrak{A}'}_i = \boldsymbol{e}^{\mathfrak{A}}_i.$$

Hence $\mathfrak{A}' \models \forall \boldsymbol{x} \forall \boldsymbol{y}\, (\boldsymbol{e}_i(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow [\mathrm{L{:}\mathsf{eg}}\text{-}i](\boldsymbol{x}, \boldsymbol{y}))$. Since $\mathfrak{A}' \models \varphi$ we have $\mathfrak{A}' \models \operatorname{gtr}\varphi$. ☐

Remark 6. *Let φ be a $\Lambda[\Sigma]$-formula and let $\mathfrak{A}$ be an L'-structure. Suppose that $\mathfrak{A} \models \operatorname{gtr}\varphi$ and that the symbols $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_e$ are interpreted in $\mathfrak{A}$ as equivalence relations on A in refinement. Then there is a Σ'-enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \models \varphi$ and that the symbols $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$ are interpreted as equivalence relations on A in global agreement in $\mathfrak{A}'$.*

Proof. Consider an enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ to a Σ'-structure where $\boldsymbol{e}^{\mathfrak{A}'}_i = [\mathrm{L{:}\mathsf{eg}}\text{-}i]^{\mathfrak{A}}$. By Remark 4, $\langle \boldsymbol{e}^{\mathfrak{A}'}_1, \boldsymbol{e}^{\mathfrak{A}'}_2, \ldots, \boldsymbol{e}^{\mathfrak{A}'}_e \rangle$ is a sequence of equivalence relations on A in global agreement. For every $i \in [1, e]$ we have $\mathfrak{A}' \models \forall \boldsymbol{x} \forall \boldsymbol{y} (\boldsymbol{e}_i(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow [\mathrm{L{:}\mathsf{eg}}\text{-}i](\boldsymbol{x}, \boldsymbol{y}))$ by definition. Since $\mathfrak{A}' \models \operatorname{gtr}\varphi$ we have $\mathfrak{A}' \models \varphi$. ☐

The last two remarks show that a $\Lambda e\mathrm{E}_{\mathsf{global}}$-formula φ has essentially the same models as the $\Lambda e\mathrm{E}_{\mathsf{refine}}$-formula $\operatorname{gtr}\varphi$, so we have shown:

Proposition 1. *The logic $\Lambda e\mathrm{E}_{\mathsf{global}}$ has the finite model property iff the logic $\Lambda e\mathrm{E}_{\mathsf{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent:* (FIN-)SAT-$\Lambda e\mathrm{E}_{\mathsf{global}} =^{\mathrm{PTIME}}_{\mathrm{m}}$ (FIN-)SAT-$\Lambda e\mathrm{E}_{\mathsf{refine}}$.

Since the relative size of $\operatorname{gtr}\varphi$ with respect to φ grows polynomially as e grows, we have shown:

Proposition 2. *The logic $\Lambda\mathrm{E}_{\mathsf{global}}$ has the finite model property iff the logic $\Lambda\mathrm{E}_{\mathsf{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent:* (FIN-)SAT-$\Lambda\mathrm{E}_{\mathsf{global}} =^{\mathrm{PTIME}}_{\mathrm{m}}$ (FIN-)SAT-$\Lambda\mathrm{E}_{\mathsf{refine}}$.

The reduction is two-variable first-order and uses additional (et) unary predicate symbols for the permutation setup P, so it is also valid for the two-variable fragments $\Lambda^2_0 e\mathrm{E}_{\mathsf{a}}$, $\Lambda^2_1 e\mathrm{E}_{\mathsf{a}}$ and $\Lambda^2_1\mathrm{E}_{\mathsf{a}}$ for $\mathsf{a} \in \{\mathsf{global}, \mathsf{refine}\}$ (but not for the fragment $\Lambda^2_0\mathrm{E}_{\mathsf{a}}$).

4.2 Local agreement to refinement

In this section demonstrate how (finite) satisfiability in logics featuring builtin equivalence symbols in local agreement reduces to (finite) satisfiability in logics featuring builtin equivalence symbols in refinement. Our strategy is to start with the level equivalences which form a refinement, and to encode a permutation specifying the local chain structure for every element in the structure.

Fix an arbitrary ground logic $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ and think of Σ as a predicate signature for the logics $\Lambda e\mathrm{E}_{\mathsf{local}}$ and $\Lambda e\mathrm{E}_{\mathsf{refine}}$. The e builtin equivalence symbols of Σ are $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$.

Let φ be a $\Lambda[\Sigma]$-sentence. The class of $\Lambda e\mathrm{E}_{\mathsf{refine}}$-structures satisfying φ coincides with the class of $\Lambda e\mathrm{E}_{\mathsf{local}}$-structures satisfying

$$\varphi \wedge [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:refine}].$$

Hence:

$$\text{(FIN-)SAT-}\Lambda e\mathrm{E}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\mathrm{PTIME}} \text{(FIN-)SAT-}\Lambda e\mathrm{E}_{\mathsf{local}}.$$

Since the size of the formula $[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e\textsf{:refine}]$ grows polynomially as e grows, we have:

$$\text{(FIN-)SAT-}\Lambda\mathrm{E}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\mathrm{PTIME}} \text{(FIN-)SAT-}\Lambda\mathrm{E}_{\mathsf{local}}.$$

Consider the opposite direction. Let $\mathrm{E} = \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e \rangle$ be a predicate signature consisting of the binary predicate symbols $\boldsymbol{e}_i$ (later, we will need these to be not necessarily interpreted as equivalences, but for now we will interpret them as such). Let $\mathfrak{A}$ be an E-structure and suppose that the symbols $\boldsymbol{e}_i$ are interpreted in $\mathfrak{A}$ as equivalence relations on A in local agreement. Let $E_i = \boldsymbol{e}_i^{\mathfrak{A}}$ for $i \in [1, e]$. Recall that for every $a \in A$ there is a permutation $\nu \in \mathbb{S}_e$ satisfying eq. (3.1):

$$E_{\nu(1)}[a] \subseteq E_{\nu(2)}[a] \subseteq \cdots \subseteq E_{\nu(e)}[a]. \tag{4.2}$$

Definition 33. *The* characteristic E-permutation *of a in $\mathfrak{A}$ is the lexicographically smallest permutation $\nu \in \mathbb{S}_e$ satisfying eq. (4.2). Define the function $[\mathrm{E}\textbf{:}\mathrm{chperm}]^{\mathfrak{A}} : A \to \mathbb{S}_e$ so that $[\mathrm{E}\textbf{:}\mathrm{chperm}]^{\mathfrak{A}}a$ is the characteristic E-permutation of a in $\mathfrak{A}$.*

Remark 7. *Let $a \in A$, $\nu = [\mathrm{E}\textbf{:}\mathrm{chperm}]^{\mathfrak{A}}a$ and $i < j \in [1, e]$. Suppose that $E_{\nu(i)}[a] = E_{\nu(j)}[a]$. Then $\nu(i) < \nu(j)$.*

Proof. Suppose not. For some $i < j \in [1, e]$ we have $\nu(i) \geq \nu(j)$. Since ν is a permutation and $i \neq j$, we have $\nu(i) > \nu(j)$. Since $E_{\nu(i)}[a] = E_{\nu(j)}[a]$, by eq. (4.2) we have $E_{\nu(k)} = E_{\nu(i)}$ for all $k \in [i, j]$. Consider the permutation $\mu \in \mathbb{S}_e$ defined by:

$$\mu(k) = \begin{cases} \nu(j) & \text{if } k = i \\ \nu(i) & \text{if } k = j \\ \nu(k) & \text{otherwise.} \end{cases}$$

Clearly, μ is a permutation satisfying eq. (4.2) that is lexicographically smaller than ν — a contradiction. ☐

Remark 8. *Let $a, b \in A$ and let $\alpha = [\text{E:chperm}]^{\mathfrak{A}} a$ and $\beta = [\text{E:chperm}]^{\mathfrak{A}} b$. Let $i \in [1, e]$ and suppose that $(a, b) \in E_i$. Then $\alpha^{-1}(i) = \beta^{-1}(i)$.*

Proof. Suppose not, so $\alpha^{-1}(i) \neq \beta^{-1}(i)$. Let $p = \alpha^{-1}(i)$ and $q = \beta^{-1}(i)$. Without loss of generality, suppose that $p < q$. Thus p is the position of i in the permutation α and $q > p$ is the position of i in the permutation β. By the pigeonhole principle, there is $k \in [1, e]$ that occurs after i in α and before j in β: $p < \alpha^{-1}(k)$ and $\beta^{-1}(k) < q$. Since β is the characteristic E-permutation of b in $\mathfrak{A}$, by eq. (4.2) we have $E_k[b] \subseteq E_i[b]$. Since $(a, b) \in E_i$, we have $E_k[b] \subseteq E_i[a]$. Since $E_k[b] \subseteq E_i[a]$ are equivalence classes, $E_k[a] \subseteq E_i[a]$. Since k occurs after i in α, which is the characteristic E-permutation of a in $\mathfrak{A}$, by eq. (4.2) we have $E_k[a] = E_i[a]$. By Remark 7, $i < k$. By the contrapositive of Remark 7, $E_k[b] = E_i[b]$ is impossible. Since k occurs before i in β, by eq. (4.2) we have $E_k[b] \subset E_i[b]$. Hence

$$E_k[b] \subset E_i[b] = E_i[a] = E_k[a]$$

— a contradiction — since the equivalence classes $E_k[b]$ and $E_k[a]$ are either equal or disjoint. □

Let $L = \langle L_1, L_2, \ldots, L_e \rangle$ be the levels of $E = \langle E_1, E_2, \ldots, E_e \rangle$. Recall that by Remark 3, the levels are equivalence relations on A in refinement.

Remark 9. *Let $a \in A$, $\alpha = [\text{E:chperm}]^{\mathfrak{A}} a$ and let $k \in [1, e]$. Then $L_k[a] = E_{\alpha(k)}[a]$.*

Proof. Since α satisfies eq. (4.2), by Lemma 2:

$$L_k[a] = L_{\alpha^{-1}(\alpha(k))}[a] = E_{\alpha(k)}[a].$$

□

Remark 10. *Let $a, b \in A$, $\alpha = [\text{E:chperm}]^{\mathfrak{A}} a$, $\beta = [\text{E:chperm}]^{\mathfrak{A}} b$ and $k \in [1, e]$. Suppose that $(a, b) \in L_k$. Then $\alpha(k) = \beta(k)$. That is, the elements connected at level k agree at position k in their characteristic permutations.*

Proof. By Remark 9, $L_k[a] = E_{\alpha(k)}[a]$, thus $(a, b) \in E_{\alpha(k)}$. By Remark 7,

$$k = \alpha^{-1}(\alpha(k)) = \beta^{-1}(\alpha(k)).$$

Hence $\beta(k) = \alpha(k)$. □

Let $\mathrm{P} = \langle \boldsymbol{u}_{11}, \boldsymbol{u}_{12}, \ldots, \boldsymbol{u}_{et} \rangle$ be an e-permutation setup. Let $\mathrm{L} = \langle \boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_e \rangle + \mathrm{P}$ be a predicate signature containing the binary predicate symbols $\boldsymbol{l}_k$ (not necessarily interpreted as equivalence relations) together with the symbols from P.

Definition 34. *Define the $\mathcal{L}^2[\mathrm{L}]$-sentence [L:fixperm] by:*

$$[\mathsf{L{:}fixperm}] = \forall \boldsymbol{x} \forall \boldsymbol{y} \bigwedge_{1 \leq k \leq e} (\boldsymbol{l}_k(\boldsymbol{x}, \boldsymbol{y}) \to [\mathrm{P}(k){:}\mathsf{eq}](\boldsymbol{x}, \boldsymbol{y}))\,.$$

Definition 35. *Define the* $\mathcal{L}^2[\mathrm{L}]$*-sentence* [L:locperm] *by:*

$$[\mathsf{L{:}locperm}] = [\mathsf{P{:}perm}] \wedge [\mathsf{L{:}fixperm}].$$

Remark 11. *Let* $\mathfrak{A}$ *be an* L*-structure and suppose that* $\mathfrak{A} \models [\mathsf{L{:}locperm}]$*. Let* $a, b \in A$*,* $k \in [1, e]$ *and suppose that* $\mathfrak{A} \models \boldsymbol{l}_k(a, b)$*. Let* $\alpha = [\mathrm{P{:}data}]^{\mathfrak{A}} a$ *and* $\beta = [\mathrm{P{:}data}]^{\mathfrak{A}} b$ *be the* e*-permutations at* a *and* b*,* encoded by the permutation setup P. *Then* $\alpha(k) = \beta(k)$*.*

Proof. Since $\mathfrak{A} \models [\mathsf{L{:}fixperm}]$ and $\mathfrak{A} \models \boldsymbol{l}_k(a, b)$, we have $\mathfrak{A} \models [\mathrm{P}(k){:}\mathsf{eq}](a, b)$, which means $\alpha(k) = \beta(k)$. □

Definition 36. *For* $i \in [1, e]$*, define the quantifier-free* $\mathcal{L}^2[\mathrm{L}]$*-formula* [L:el-$i$] *by:*

$$[\mathsf{L{:}el}\text{-}i](\boldsymbol{x}, \boldsymbol{y}) = \bigwedge_{1 \le k \le n} \left([\mathrm{P}(k){:}\mathsf{eq}\text{-}i](\boldsymbol{x}) \to \boldsymbol{l}_k(\boldsymbol{x}, \boldsymbol{y})\right).$$

Remark 12. *Let* $\mathfrak{A}$ *be an* L*-structure and suppose that* $\mathfrak{A} \models [\mathsf{L{:}locperm}]$ *and that the binary symbols* $\boldsymbol{l}_k$ *are interpreted in* $\mathfrak{A}$ *as equivalence relations on* A *in refinement. Define* $\nu : A \to \mathbb{S}_e$ *by* $\nu(a) = [\mathrm{P{:}data}]^{\mathfrak{A}} a$ *for* $a \in A$*. Let* $a \in A$ *be arbitrary. Then for all* $i \in [1, e]$*:*

$$[\mathsf{L{:}el}\text{-}i]^{\mathfrak{A}}[a] = \boldsymbol{l}^{\mathfrak{A}}_{\nu(a)^{-1}(i)}[a].$$

Proof. Let $E_i = [\mathsf{L{:}el}\text{-}i]^{\mathfrak{A}}$ and $L_i = \boldsymbol{l}_i^{\mathfrak{A}}$ for every $i \in [1, e]$. Let $i \in [1, e]$ be arbitrary. Let $\alpha = \nu(a)$ and $k = \alpha^{-1}(i)$, so $\alpha = [\mathrm{P{:}data}]^{\mathfrak{A}} a$ and $\alpha(k) = i$. We have to show that $E_i[a] = L_k[a]$. Since α is a permutation, for every $k' \in [1, e]$ we have:

$$\mathfrak{A} \models [\mathrm{P}(k'){:}\mathsf{eq}\text{-}i](a) \text{ iff } \alpha(k') = i \text{ iff } k' = k. \tag{4.3}$$

First, suppose $b \in E_i[a]$. Then $\mathfrak{A} \models [\mathsf{L{:}el}\text{-}i](a, b)$ and by eq. (4.3) we have $\mathfrak{A} \models \boldsymbol{l}_k(a, b)$, hence $b \in L_k[a]$.

Next, suppose $b \notin E_i[a]$. Then $\mathfrak{A} \models \neg[\mathsf{L{:}el}\text{-}i](a, b)$, so there is some $k' \in [1, e]$ such that $\mathfrak{A} \models \neg([\mathrm{P}(k'){:}\mathsf{eq}\text{-}i](a) \to \boldsymbol{l}_{k'}(a, b)) \equiv [\mathrm{P}(k'){:}\mathsf{eq}\text{-}i](a) \wedge \neg \boldsymbol{l}_{k'}(a, b)$. By eq. (4.3) we have $k' = k$. Hence $\mathfrak{A} \models \neg \boldsymbol{l}_k(a, b)$, so $b \notin L_k[a]$. □

Remark 13. *Let* $\mathfrak{A}$ *and* ν *are declared as in Remark 12. Then the sequence of interpretations* $\langle [\mathsf{L{:}el}\text{-}1]^{\mathfrak{A}}, [\mathsf{L{:}el}\text{-}2]^{\mathfrak{A}}, \ldots, [\mathsf{L{:}el}\text{-}e]^{\mathfrak{A}} \rangle$ *is a sequence of equivalence relations on* A *in local agreement.*

Proof. Let $E_i = [\mathsf{L{:}el}\text{-}i]^{\mathfrak{A}}$ and $L_i = \boldsymbol{l}_i^{\mathfrak{A}}$ for every $i \in [1, e]$. Let $i \in [1, e]$ be arbitrary. We check that E_i is reflexive, symmetric and transitive.

- For reflexivity, let $a \in A$. By Remark 12, $E_i[a] = L_k[a]$ for $k = \nu(a)^{-1}(i)$. But $L_k[a]$ is an equivalence class, hence $a \in L_k[a]$, so $(a, a) \in E_i$.
- For symmetry, let $a, b \in A$ and $(a, b) \in E_i$. Let $k = \nu(a)^{-1}(i)$ so that $i = \nu(k)$. By Remark 12, $E_i[a] = L_k[a]$. Thus $\mathfrak{A} \models \boldsymbol{l}_k(a, b)$ and by Remark 11, $i = \nu(a)(k) = \nu(b)(k)$. By Remark 12:

  $$E_i[b] = [\mathsf{L{:}el}\text{-}i]^{\mathfrak{A}}[b] = \boldsymbol{l}^{\mathfrak{A}}_{\nu(b)^{-1}(i)}[b] = L_k[b] = L_k[a].$$

  Since $a \in L_k[a] = E_i[b]$, we have $(b, a) \in E_i$.

- For transitivity, continue the argument for symmetry. Let $c \in E_i[b]$. Then $c \in E_i[b] = L_k[a] = E_i[a]$, thus $(a, c) \in E_i$.

By Remark 12, since the relations L_k are in refinement, we have that $E_1, E_2, \ldots, E_e$ are in local agreement. $\square$

Let $\mathrm{E} = \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e \rangle$ be a predicate signature consisting of binary predicate symbols. Let Σ be a predicate signature enriching E and not containing any symbols from L. Let $\Sigma' = \Sigma + \mathrm{L}$ and $\mathrm{L}' = \Sigma' - \mathrm{E}$.

Definition 37. *Define the syntactic operation* ltr $: \Lambda[\Sigma] \to \Lambda[\mathrm{L}']$ *by:*

$$\operatorname{ltr} \varphi = \varphi' \wedge [\mathrm{L}{:}\mathsf{locperm}],$$

where φ' is obtained from φ by replacing all occurrences of a subformula of the form $\boldsymbol{e}_i(x, y)$ by the formula $[\mathrm{L}{:}\mathsf{el}\text{-}i](x, y)$, where x and y are (not necessarily distinct) variable symbols and $i \in [1, e]$.

Remark 14. *Let φ be a $\Lambda[\Sigma]$-formula and let $\mathfrak{A}$ be a Σ-structure. Suppose that $\mathfrak{A} \models \varphi$ and that the symbols $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$ are interpreted in $\mathfrak{A}$ as equivalence relations on A in local agreement. Then there is a Σ'-enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \models \operatorname{ltr} \varphi$ and that the symbols $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_e$ are interpreted in $\mathfrak{A}'$ as equivalence relations on A in refinement.*

Proof. Since the binary symbols $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$ are interpreted as equivalence relations on A in local agreement in $\mathfrak{A}$, we may define the levels $L_1, L_2, \ldots, L_e \subseteq A \times A$ and the characteristic E-permutation mapping $\nu = [\mathrm{E}{:}\mathrm{chperm}]^{\mathfrak{A}} : A \to \mathbb{S}_e$. Consider an enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ where $\boldsymbol{l}_i^{\mathfrak{A}'} = L_i$. By Remark 3, L_i are equivalences on A in refinement. We interpret the unary symbols from the permutation setup P so that $[\mathrm{P}{:}\mathrm{data}]^{\mathfrak{A}'} a = \nu(a)$ for all $a \in A$. By Remark 10, $\mathfrak{A}' \models [\mathrm{L}{:}\mathsf{fixperm}]$. By Remark 12, followed by Lemma 2, for every $i \in [1, e]$ and $a \in A$ we have:

$$[\mathrm{L}{:}\mathsf{el}\text{-}i]^{\mathfrak{A}'}[a] = \boldsymbol{l}^{\mathfrak{A}'}_{\nu(a)^{-1}(i)}[a] = \boldsymbol{e}^{\mathfrak{A}'}_{\nu(a)(\nu(a)^{-1}(i))}[a] = \boldsymbol{e}_i^{\mathfrak{A}'}[a].$$

By Remark 13, the interpretations $[\mathrm{L}{:}\mathsf{el}\text{-}i]^{\mathfrak{A}'}$ are equivalence relations. Since the interpretation of the formula $[\mathrm{L}{:}\mathsf{el}\text{-}i]$ has the same classes as the interpretation of the symbol $\boldsymbol{e}_i$, we have $\mathfrak{A}' \models \forall \boldsymbol{x} \forall \boldsymbol{y}\, (\boldsymbol{e}_i(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow [\mathrm{L}{:}\mathsf{el}\text{-}i](\boldsymbol{x}, \boldsymbol{y}))$. Since $\mathfrak{A}' \models \varphi$ we have $\mathfrak{A}' \models \operatorname{ltr} \varphi$. $\square$

Remark 15. *Let φ be a $\Lambda[\Sigma]$-formula and let $\mathfrak{A}$ be an L'-structure. Suppose that $\mathfrak{A} \models \operatorname{ltr} \varphi$ and that the symbols $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_e$ are interpreted as equivalence relations on A in refinement in $\mathfrak{A}$. Then there is a Σ'-enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \models \varphi$ and that the binary symbols $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$ are interpreted as equivalence relations on A in global agreement in $\mathfrak{A}'$.*

Proof. Consider an enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ to a Σ'-structure where $\boldsymbol{e}_i^{\mathfrak{A}'} = [\mathrm{L}{:}\mathsf{el}\text{-}i]^{\mathfrak{A}}$. By Remark 13, $\boldsymbol{e}_i^{\mathfrak{A}'}$ are equivalence relations on A in local agreement. For every $i \in [1, e]$ we have $\mathfrak{A}' \models \forall \boldsymbol{x} \forall \boldsymbol{y}\, (\boldsymbol{e}_i(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow [\mathrm{L}{:}\mathsf{el}\text{-}i](\boldsymbol{x}, \boldsymbol{y}))$ by definition. Since $\mathfrak{A}' \models \operatorname{ltr} \varphi$ we have $\mathfrak{A}' \models \varphi$. $\square$

The last two remarks show that a $\Lambda e\mathrm{E}_{\mathsf{local}}$-formula φ has essentially the same models as the $\Lambda e\mathrm{E}_{\mathsf{refine}}$-formula $\operatorname{ltr}\varphi$, so we have shown:

Proposition 3. *The logic $\Lambda e\mathrm{E}_{\mathsf{local}}$ has the finite model property iff the logic $\Lambda e\mathrm{E}_{\mathsf{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent:* (FIN-)SAT-$\Lambda e\mathrm{E}_{\mathsf{local}} =_{\mathrm{m}}^{\mathrm{PTIME}}$ (FIN-)SAT-$\Lambda e\mathrm{E}_{\mathsf{refine}}$.

Since the relative size of $\operatorname{ltr}\varphi$ with respect to φ grows polynomially as e grows, we have shown:

Proposition 4. *The logic $\Lambda\mathrm{E}_{\mathsf{local}}$ has the finite model property iff the logic $\Lambda\mathrm{E}_{\mathsf{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent:* (FIN-)SAT-$\Lambda\mathrm{E}_{\mathsf{local}} =_{\mathrm{m}}^{\mathrm{PTIME}}$ (FIN-)SAT-$\Lambda\mathrm{E}_{\mathsf{refine}}$.

The reduction is two-variable first-order and uses additional (et) unary predicate symbols for the permutation setup P, so it is also valid for the two-variable fragments $\Lambda_0^2 e\mathrm{E}_{\mathsf{a}}$, $\Lambda_1^2 e\mathrm{E}_{\mathsf{a}}$ and $\Lambda_1^2\mathrm{E}_{\mathsf{a}}$ for $\mathsf{a} \in \{\mathsf{local}, \mathsf{refine}\}$ respectively.

4.3 Granularity

In this section we demonstrate how to replace the finest equivalence from a sequence of equivalences in refinement with a counter setup. This works if the structures are granular, that is, if the finest equivalence doesn't have many classes within a single bigger equivalence class.

Definition 38. *Let $\langle D, E\rangle$ be a sequence of two equivalence relations on A in refinement. Let $g \in \mathbb{N}^+$. The sequence is g-*granular *if every E-equivalence class includes at most g D-equivalence classes.*

Definition 39. *Let $g \in \mathbb{N}^+$ and let $\langle D, E\rangle$ be g-granular. The function $c : A \to [1, g]$ is a g-*granular coloring *for the sequence, if two E-equivalent elements have the same color iff they are D-equivalent. That is, for every $(a, b) \in E$ we have $c(a) = c(b)$ iff $(a, b) \in D$.*

Remark 16. *Let $g \in \mathbb{N}^+$ and let $\langle D, E\rangle$ be g-granular. Then there is a g-granular coloring for the sequence.*

Proof. Let X be an E-class. Since $D \subseteq E$ is g-granular, the set $S = \{D[a] \mid a \in X\}$ has cardinality at most g. Let $\imath : S \hookrightarrow [1, g]$ be any injective function. Define the color c on X as $c(a) = \imath(D[a])$. ☐

Remark 17. *Let $E \subseteq A \times A$ be an equivalence relation on A, $g \in \mathbb{N}^+$ and $c : A \to [1, g]$. Then there is an equivalence relation $D \subseteq E$ on A such that $\langle D, E\rangle$ is g-granular, having c as a g-granular coloring.*

Proof. Take $D = \{(a, b) \in E \mid c(a) = c(b)\}$. ☐

Definition 40. *Let $g \in \mathbb{N}^+$ and let $t = \|g\|$ be the bitsize of g. A g-*color setup *$\mathrm{G} = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_t\rangle$ is just a t-bit counter setup.*

Let $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ be a ground logic, $g \in \mathbb{N}^+$ and $\mathrm{G} = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_t \rangle$ be a g-color setup. Let Σ be a predicate signature containing the binary symbols $\boldsymbol{d}$ and $\boldsymbol{e}$ and not containing any symbols from G. Let $\Sigma' = \Sigma + \mathrm{G}$ and $\Gamma = \Sigma' - \{\boldsymbol{d}\}$.

Definition 41. *Define the quantifier-free $\mathcal{L}^2[\Gamma]$-formula $[\Gamma\mathsf{:d}](\boldsymbol{x}, \boldsymbol{y})$ by:*

$$[\Gamma\mathsf{:d}](\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \wedge [\mathrm{G}\mathsf{:eq}](\boldsymbol{x}, \boldsymbol{y}).$$

Definition 42. *Define the syntactic operation* $\mathrm{grtr} : \Lambda[\Sigma] \to \Lambda[\Gamma]$ *by:*

$$\mathrm{grtr}\,\varphi = \varphi' \wedge [\mathrm{G}\mathsf{:betw\text{-}1\text{-}}g],$$

where φ' is obtained from the formula φ by replacing all subformulas of the form $\boldsymbol{d}(x, y)$ by $[\Gamma\mathsf{:d}](x, y)$, where x and y are (not necessarily distinct) variable symbols.

Lemma 3. *Let $\mathfrak{A}$ be a Σ-structure and suppose that the sequence of symbols $\langle \boldsymbol{d}, \boldsymbol{e} \rangle$ is interpreted in $\mathfrak{A}$ as a g-granular sequence $\langle D, E \rangle$. Suppose that $\mathfrak{A} \vDash \varphi$. Then there is a Σ'-enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \vDash \mathrm{grtr}\,\varphi$.*

Proof. By Remark 16, there exists a g-granular coloring $c : A \to [1, g]$. We interpret the unary symbols in G so that $[\mathrm{G}\mathrm{:data}]^{\mathfrak{A}} = c$. Since $\mathfrak{A}'$ is an enrichment of $\mathfrak{A}$, we have $\mathfrak{A}' \vDash \varphi$. Let $a, b \in A$. Then $\mathfrak{A}' \vDash [\Gamma\mathsf{:d}](a, b)$ is equivalent to:

$$\mathfrak{A}' \vDash \boldsymbol{e}(a, b) \text{ and } \mathfrak{A}' \vDash [\mathrm{G}\mathsf{:eq}](a, b),$$

which is equivalent to:

$$(a, b) \in E \text{ and } [\mathrm{G}\mathrm{:data}]^{\mathfrak{A}'} a = [\mathrm{G}\mathrm{:data}]^{\mathfrak{A}'} b,$$

which, since $[\mathrm{G}\mathrm{:data}]^{\mathfrak{A}'} = c$ is a g-granular coloring, is equivalent to:

$$(a, b) \in D.$$

Hence $\mathfrak{A}' \vDash \forall \boldsymbol{x} \forall \boldsymbol{y} (\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow [\Gamma\mathsf{:d}](\boldsymbol{x}, \boldsymbol{y}))$ and since $\mathfrak{A}' \vDash \varphi$, we have $\mathfrak{A}' \vDash \mathrm{grtr}\,\varphi$. □

Lemma 4. *Let $\mathfrak{A}$ be a Γ-structure and suppose that the binary symbol $\boldsymbol{e}$ is interpreted in $\mathfrak{A}$ as an equivalence relation on A. Suppose that $\mathfrak{A} \vDash \mathrm{grtr}\,\varphi$. Then there is a Σ'-structure $\mathfrak{A}'$ enriching $\mathfrak{A}$ such that $\mathfrak{A}' \vDash \varphi$ and the sequence of binary symbols $\langle \boldsymbol{d}, \boldsymbol{e} \rangle$ is interpreted in $\mathfrak{A}'$ as a g-granular sequence $\langle D, E \rangle$.*

Proof. Since $\mathfrak{A} \vDash [\mathrm{G}\mathsf{:betw\text{-}1\text{-}}g]$, we have $[\mathrm{G}\mathrm{:data}]^{\mathfrak{A}} a \in [1, g]$ for all $a \in A$. Define $c : A \to [1, g]$ by $c(a) = [\mathrm{C}\mathrm{:data}]^{\mathfrak{A}} a$. By Remark 16, we can find $D \subseteq E$ such that the sequence $\langle D, E \rangle$ is g-granular, having c as a g-granular coloring. Consider the Σ'-structure $\mathfrak{A}'$, where $\boldsymbol{d}^{\mathfrak{A}'} = D$. Since $\mathfrak{A}'$ is an enrichment of $\mathfrak{A} \vDash \mathrm{grtr}\,\varphi$, we have $\mathfrak{A}' \vDash \mathrm{grtr}\,\varphi$. Let $a, b \in A$. Then $\mathfrak{A}' \vDash [\Gamma\mathsf{:d}](a, b)$ is equivalent to:

$$\mathfrak{A}' \vDash \boldsymbol{e}(a, b) \text{ and } \mathfrak{A}' \vDash [\mathrm{G}\mathsf{:eq}](a, b),$$

which is equivalent to:

$$(a, b) \in E \text{ and } c(a) = c(b),$$

which, since c is a g-granular coloring, is equivalent to:

$$(a, b) \in D.$$

Hence $\mathfrak{A}' \models \forall \boldsymbol{x} \forall \boldsymbol{y} \, (\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow [\Gamma\mathbf{:}\mathsf{d}](\boldsymbol{x}, \boldsymbol{y}))$ and since $\mathfrak{A}' \models \operatorname{grtr} \varphi$, we have $\mathfrak{A}' \models \varphi$. $\square$

5 Monadic logics

In this chapter we investigate questions about (finite) satisfiability of first-order sentences featuring unary predicate symbols and builtin equivalence symbols in refinement. Our strategy is to extract small substructures of structures and analyse them using Ehrenfeucht-Fraïssé games. It is known that:

- The monadic first-order logic $\mathcal{L}_1$ has the finite model property and its (finite) satisfiability problem is NExpTime-complete [8]
- The first-order logic of a single equivalence relation $\mathcal{L}_0 1\mathrm{E}$ has the finite model property and its (finite) satisfiability problem is PSpace-complete [6]
- The first-order logic of two equivalence relations $\mathcal{L}_0 2\mathrm{E}$ lacks the finite model property and both the satisfiability and finite satisfiability problems are undecidable [9].

Let $\mathrm{U}(u) = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_u \rangle$ be an unary predicate signature consisting of the unary predicate symbols $\boldsymbol{u}_i$. Let $\mathrm{E}(e) = \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_e \rangle$ be a binary predicate signature consisting of the builtin equivalence symbols $\boldsymbol{e}_j$ in refinement. Let $\Sigma(u, e) = \mathrm{U}(u) + \mathrm{E}(e)$, so $\Sigma(u, e)$ is a generic predicate signature for the monadic first-order logic $\mathcal{L}_1 e\mathrm{E}_{\mathsf{refine}}$.

5.1 Cells

Let $u, e \in \mathbb{N}$, $e \geq 1$ and $\Sigma = \Sigma(u, e) = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_u, \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_e \rangle$ be a predicate signature. Abbreviate the finest equivalence symbol $\boldsymbol{d} = \boldsymbol{e}_1$.

Definition 43. *Define the quantifier-free $\mathcal{L}^2[\Sigma]$-formula $[\Sigma{:}\mathsf{cell}](\boldsymbol{x}, \boldsymbol{y})$ by:*

$$[\Sigma{:}\mathsf{cell}](\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \wedge \bigwedge_{1 \leq i \leq u} (\boldsymbol{u}_i(\boldsymbol{x}) \leftrightarrow \boldsymbol{u}_i(\boldsymbol{y})).$$

If $\mathfrak{A}$ is a Σ-structure and $D = \boldsymbol{d}^{\mathfrak{A}}$, then the interpretation $C = [\Sigma{:}\mathsf{cell}]^{\mathfrak{A}} \subseteq A \times A$ is an equivalence relation on A that refines D. The cells *of $\mathfrak{A}$ are the equivalence classes of C. That is, a cell is a maximal set of D-equivalent elements satisfying the same $\boldsymbol{u}$-predicates.*

Remark 18. *Let $\mathfrak{A}$ be a Σ-structure, $r \in \mathbb{N}$, $\bar{a} = a_1 a_2 \dots a_r \in A^r$, $\bar{b} = b_1 b_2 \dots b_r \in A^r$ and a_i and b_i are in the same $\mathfrak{A}$-cell for all $i \in [1, r]$. Suppose that $a_i = a_j$ iff $b_i = b_j$ for all $i, j \in [1, r]$. Then $\bar{a} \mapsto \bar{b}$ is a partial isomorphism.*

Proof. Direct consequence of the fact that the cell equivalence relation refines the finest equivalence relation D and that the elements in the same cell satisfy the same $\boldsymbol{u}$-predicates. The equality condition ensures that the mapping is a bijection. □

Lemma 5. *Let $\mathfrak{A}$ be a Σ-structure and $r \in \mathbb{N}^+$. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \equiv_r \mathfrak{A}$ and every $\mathfrak{B}$-cell has cardinality at most r.*

Proof. Let $C \subseteq A \times A$ be the $\mathfrak{A}$-cell equivalence relation. Execute the following process: for every $\mathfrak{A}$-cell, if it has cardinality less than r, select all elements from that cell; otherwise select r distinct elements from that cell. Let $B \subseteq A$ be the set of selected elements and let $\mathfrak{B} = \mathfrak{A} \upharpoonright B$. By construction, every $\mathfrak{B}$-cell has cardinality at most r. We claim that $\mathfrak{A} \equiv_r \mathfrak{B}$. Let $h = C \cap (A \times B)$ relates elements from A with elements from B in the same cell. Note that for all $a \in A$:

$$|h[a]| = \min(|C[a]|, r). \tag{5.1}$$

For $i \in [0, r]$ let $\mathfrak{I}_i$ be the set of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ that have length i and that are included in h. The set $\mathfrak{I}_0$ is nonempty since it contains the empty partial isomorphism. We claim that the sequence $\mathfrak{I}_0, \mathfrak{I}_1, \ldots, \mathfrak{I}_r$ satisfies the back-and-forth conditions of Theorem 1. Let $i \in [0, r-1]$ and let

$$\mathfrak{p} = \bar{a} \mapsto \bar{b} = a_1 a_2 \ldots a_i \mapsto b_1 b_2 \ldots b_i \in \mathfrak{I}_i$$

be any partial isomorphism.

1. For the forth condition, let $a \in A$. We have to find some $b \in B$ such that $\bar{a}a \mapsto \bar{b}b \in \mathfrak{I}_{i+1}$. If $a = a_k$ for some $k \in [1, i]$, then $b = b_k$ is appropriate.

   Suppose that $a \neq a_k$ for all $k \in [1, i]$. Let $S \subseteq C[a]$ be the set of $\bar{a}$-elements in the same $\mathfrak{A}$-cell as a:
   $$S = \{a_k \in C[a] \mid k \in [1, i]\}.$$
   Note that $|S| \leq r - 1$ and $|C[a]| \geq |S| + 1$. By eq. (5.1), $|h[a]| \geq |S| + 1$. Hence there is an element $b \in h[a]$ that is distinct from b_k for all $k \in [1, i]$ and this b is appropriate.

2. For the back condition, let $b \in B$. We have to find some $a \in A$ such that $\bar{a}a \mapsto \bar{b}b \in \mathfrak{I}_{i+1}$. If $b = b_k$ for some $k \in [1, i]$, then $a = a_k$ is appropriate.

   Suppose that $b \neq b_k$ for all $k \in [1, i]$. Since $b \in h[b]$, $a = b$ is appropriate.

By Theorem 1, $\mathfrak{A} \equiv_r \mathfrak{B}$. $\square$

5.2 Organs

Let $u, e \in \mathbb{N}$, $e \geq 2$ and $\Sigma = \Sigma(u, e) = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_u, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e \rangle$ be a predicate signature. Abbreviate the finest two equivalence symbols $\boldsymbol{d} = \boldsymbol{e}_1$ and $\boldsymbol{e} = \boldsymbol{e}_2$.

Definition 44. *Let $\mathfrak{A}$ be a Σ-structure and let $D = \boldsymbol{d}^{\mathfrak{A}}$ and $E = \boldsymbol{e}^{\mathfrak{A}}$. Recall that the set of D-classes is $\mathscr{E}D$. Two D-classes $X, Y \in \mathscr{E}D$ are* organ-equivalent *if they are included in the same E-class (equivalently $X \times Y \subseteq E$), and the induced substructures $(\mathfrak{A} \upharpoonright X)$ and $(\mathfrak{A} \upharpoonright Y)$ are isomorphic. The organ-equivalence relation is $\mathcal{O} \subseteq \mathscr{E}D \times \mathscr{E}D$.*

Since D refines E, organ-equivalence is an equivalence relation on $\mathscr{E}D$. An organ *is an organ-equivalence-class. That is, an organ is a maximal set of isomorphic D-classes, included in the same E-class.*

For any two organ-equivalent D-classes $(X, Y) \in \mathcal{O}$, fix an isomorphism

$$\mathfrak{h}_{XY} : (\mathfrak{A} \upharpoonright X) \leftrightarrow (\mathfrak{A} \upharpoonright Y)$$

consistently, so that $\mathfrak{h}_{XX} = \mathrm{id}_X$, $\mathfrak{h}_{YX} = \mathfrak{h}_{XY}^{-1}$ and if $(Y, Z) \in \mathcal{O}$ then $\mathfrak{h}_{XZ} = \mathfrak{h}_{YZ} \circ \mathfrak{h}_{XY}$. Two elements $a, b \in A$ are sub-organ-equivalent *if $(D[a], D[b]) \in \mathcal{O}$ and $\mathfrak{h}_{D[a]D[b]}(a) = b$. Since the isomorphisms $\mathfrak{h}_{XY}$ are chosen consistently, sub-organ-equivalence $O \subseteq A \times A$ is an equivalence relation on A that refines E.*

Remark 19. *Let $\mathfrak{A}$ be a Σ-structure, $r \in \mathbb{N}$, $\bar{a} = a_1 a_2 \ldots a_r \in A^r$, $\bar{b} = b_1 b_2 \ldots b_r \in A^r$, a_i and b_i are sub-organ-equivalent for all $i \in [1, r]$. Suppose that $\mathfrak{A} \models \boldsymbol{d}(a_i, a_j)$ iff $\mathfrak{A} \models \boldsymbol{d}(b_i, b_j)$ for all $i, j \in [1, r]$. Then $\bar{a} \mapsto \bar{b}$ is a partial isomorphism.*

Proof. The condition about the finest equivalence symbol $\boldsymbol{d}$ ensures that the interpretation of $\boldsymbol{d}$ is preserved. Since sub-organ-equivalence relates isomorphic elements, the interpretation of the unary symbols and the formal equality is preserved. Since the sub-organ-equivalence $O \subseteq A \times A$ refines the second finest equivalence relation E, the interpretation of all remaining equivalence symbols $\boldsymbol{e}_j$ is preserved. □

Lemma 6. *Let $\mathfrak{A}$ be a Σ-structure and $r \in \mathbb{N}^+$. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \equiv_r \mathfrak{A}$ and every $\mathfrak{B}$-organ has cardinality at most r.*

Proof. Let $D = \boldsymbol{d}^{\mathfrak{A}}$, $E = \boldsymbol{e}^{\mathfrak{A}}$ and let $\mathcal{A} = \mathscr{E}D$ be the set of D-classes. Let $\mathcal{O} \subseteq \mathcal{A} \times \mathcal{A}$ be the $\mathfrak{A}$-organ-equivalence relation on $\mathcal{A}$. Execute the folloing process: for every $\mathfrak{A}$-organ, if it has cardinality at most r, select all D-classes from that organ; otherwise select r distinct D-classes from that organ (note that these will be isomorphic). Let $\mathcal{B} \subseteq \mathcal{A}$ be the set of selected D-classes. Let $B = \cup\mathcal{B} \subseteq A$ be the set of elements in the selected classes and let $\mathfrak{B} = (\mathfrak{A} \upharpoonright B)$. By construction, every $\mathfrak{B}$-organ has cardinality at most r. We claim that $\mathfrak{A} \equiv_r \mathfrak{B}$. Let $\mathcal{H} = \mathcal{O} \cap (\mathcal{A} \times \mathcal{B})$ relates the D-classes with the isomorphic D-classes from $\mathcal{B}$ in the same organ. Let h relates the elements of A with their isomorphic elements from B. Note that for all elements $a \in A$:

$$|h[a]| = \min(|\mathcal{O}[D[a]]|\, , r). \tag{5.2}$$

For $i \in [0, r]$ let $\mathfrak{I}_i$ be the set of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ that have length i and that are included in h. The set $\mathfrak{I}_0$ is nonempty since it contains the empty partial isomorphism. We claim that the sequence $\mathfrak{I}_0, \mathfrak{I}_1, \ldots, \mathfrak{I}_r$ satisfies the back-and-forth conditions of Theorem 1. Let $i \in [0, r-1]$ and let

$$\mathfrak{p} = \bar{a} \mapsto \bar{b} = a_1 a_2 \ldots a_i \mapsto b_1 b_2 \ldots b_i \in \mathfrak{I}$$

be any partial isomorphism.

1. For the forth condition, let $a \in A$. We have to find some $b \in B$ such that $\bar{a}a \mapsto \bar{b}b \in \mathfrak{I}_{i+1}$. If $a \in D[a_k]$ for some $k \in [1,i]$, then $b = \mathfrak{h}_{D[a_k]D[b_k]}(a)$ is appropriate.

   Suppose $a \notin D[a_k]$ for all $k \in [1,i]$. Let $\mathcal{S} \subseteq \mathcal{O}[D[a]]$ be the set of D-classes of $\bar{a}$-elements in the same $\mathfrak{A}$-organ as $D[a]$:

$$\mathcal{S} = \{D[a_k] \in \mathcal{O}[D[a]] \mid k \in [1,i]\}\,.$$

   Note that $|\mathcal{S}| \leq r-1$ and $|\mathcal{O}[D[a]]| \geq |\mathcal{S}|+1$. By eq. (5.2), $|h[a]| \geq |\mathcal{S}|+1$. Hence there is some $b \in h[a]$ such that $b \notin D[b_k]$ for all $k \in [1,i]$. This b is appropriate.

2. For the back condition, let $b \in B$. We have to find some $a \in A$ such that $\bar{a}a \mapsto \bar{b}b \in \mathfrak{I}$. If $b \in D[b_k]$ for some $k \in [1,i]$, then $a = \mathfrak{h}_{D[b_k]D[a_k]}(b)$ is appropriate.

   Suppose that $b \notin D[b_k]$ for all $k \in [1,i]$. Since $b \in h[b]$, $a = b$ is appropriate.

By Theorem 1, $\mathfrak{A} \equiv_r \mathfrak{B}$. □

5.3 Satisfiability

In this section we will employ the results on cells and organs to bound the size of a small substructure of a general structure.

Remark 20. *Let $u, e \in \mathbb{N}$, $e \geq 2$ and consider the predicate signature $\Sigma = \Sigma(u,e) = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_u, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e \rangle$. Abbreviate $\boldsymbol{d} = \boldsymbol{e}_1$ and $\boldsymbol{e} = \boldsymbol{e}_2$. Let $r \in \mathbb{N}^+$. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \equiv_r \mathfrak{A}$ and $\langle \boldsymbol{d}^{\mathfrak{B}}, \boldsymbol{e}^{\mathfrak{B}} \rangle$ is g-granular for $g = g(u,r) = r.((r+1)^{2^u} - 1)$. Furthermore, this $\mathfrak{B}$ has the property that every $\mathfrak{B}$-cell has cardinality at most r.*

Proof. By Lemma 5, there is $\mathfrak{B}' \subseteq \mathfrak{A}$ such that $\mathfrak{B}' \equiv_r \mathfrak{A}$ and every $\mathfrak{B}'$-cell has cardinality at most r. By Lemma 6, there is $\mathfrak{B} \subseteq \mathfrak{B}'$ such that $\mathfrak{B} \equiv_r \mathfrak{B}'$ and the $\mathfrak{B}$-organs have cardinality at most r. Let $D = \boldsymbol{d}^{\mathfrak{B}}$ and $E = \boldsymbol{e}^{\mathfrak{B}}$. Since every D-class includes at most 2^u cells and is nonempty and every cell has cardinality at most r, there are at most $\big((r+1)^{2^u} - 1\big)$ nonisomorphic D-classes in $\mathfrak{B}$. Since every E-class includes at most r isomorphic D-classes, we get that $\langle D, E \rangle$ is g-granular. □

Corollary 1. *Let $u, e \in \mathbb{N}$, $e \geq 2$ and consider $\Sigma = \Sigma(u,e)$. Let φ be a $\mathcal{L}[\Sigma]$-sentence having quantifier rank r. By Lemma 3 and Lemma 4 about granularity, the formula φ is essentially equisatisfiable with the formula $\operatorname{grtr}\varphi$, which is a $\Sigma(u + \|g(u,r)\|, e-1)$-sentence. Note that $\|g(u,r)\|$ is exponentially bounded by the length $\|\varphi\|$ of the formula. So we have a reduction:*

$$\text{(FIN-)SAT-}^{\mathcal{L}}1e\mathrm{E}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\textsc{ExpTime}} \text{(FIN-)SAT-}^{\mathcal{L}}1(e-1)\mathrm{E}_{\mathsf{refine}}.$$

If u is a constant independent of φ, then $\|g(u,r)\|$ is polynomially bounded by $\|\varphi\|$. So we have a reduction:

$$\text{(FIN-)SAT-}^{\mathcal{L}}0e\mathrm{E}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\textsc{PTime}} \text{(FIN-)SAT-}^{\mathcal{L}}1(e-1)\mathrm{E}_{\mathsf{refine}}.$$

Remark 21. *Let $u \in \mathbb{N}$ and consider $\Sigma = \Sigma(u,1) = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_u, \boldsymbol{d} \rangle$. Let $r \in \mathbb{N}^+$. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{A} \equiv_r \mathfrak{B}$ and $|B| \leq g.r.2^u$ for $g = g(u,r) = r.((r+1)^{2^u} - 1)$.*

Proof. Let $\Sigma' = \Sigma + \langle \boldsymbol{e} \rangle$ be an enrichment of Σ with the builtin equivalence symbols $\boldsymbol{e}$. Consider an enrichment $\mathfrak{A}'$ of $\mathfrak{A}$ to a Σ'-structure, where $\boldsymbol{e}^{\mathfrak{A}'} = A \times A$ is interpreted as the full relation on A. Then $\langle \boldsymbol{d}^{\mathfrak{A}'}, \boldsymbol{e}^{\mathfrak{A}'} \rangle$ is a sequence of equivalence relations on A in refinement. By Remark 20, there is $\mathfrak{B}' \subseteq \mathfrak{A}'$ such that $\mathfrak{B}' \equiv_r \mathfrak{A}'$ and $\langle \boldsymbol{d}^{\mathfrak{B}'}, \boldsymbol{e}^{\mathfrak{B}'} \rangle$ is g-granular. Consider the reduct $\mathfrak{B}$ of $\mathfrak{B}'$ to a Σ-structure. Let $D = \boldsymbol{d}^{\mathfrak{B}}$ and $E = \boldsymbol{e}^{\mathfrak{B}}$. Since every $\mathfrak{B}$-cell has cardinality at most r and every D-class includes at most 2^u cells, we have that every D-class has cardinality at most $r.2^u$. Since $\boldsymbol{e}$ was interpreted in $\mathfrak{A}$ as the full relation, it is also interpreted in $\mathfrak{B}$ as the full relation, so there is a single E-class—the whole domain B. Since the sequence $\langle D, E \rangle$ is g-granular, there are at most g D-classes, so $|B| \leq g.r.2^u$. ☐

Corollary 2. *The logic $\mathcal{L}_1$1E has the finite model property and its (finite) satisfiability problem is in* N2ExpTime.

Combinig Corollary 2 with Corollary 1, we get by induction on e:

Proposition 5. *For $e \in \mathbb{N}^+$, the logic $\mathcal{L}_1 e\mathrm{E}_{\mathsf{refine}}$ has the finite model property and its (finite) satisfiability problem is in* N$(e+1)$ExpTime.

By Proposition 1 and Proposition 3, the same holds for $\mathcal{L}_1 e\mathrm{E}_{\mathsf{global}}$ and $\mathcal{L}_1 e\mathrm{E}_{\mathsf{local}}$.

Proposition 6. *The logic $\mathcal{L}_1\mathrm{E}_{\mathsf{refine}}$ has the finite model property and its (finite) satisfiability problem is in the forth level of the Grzegorczyk hierarchy $\mathcal{E}^4$.*

By Proposition 2 and Proposition 4, the same holds for $\mathcal{L}_1\mathrm{E}_{\mathsf{global}}$ and $\mathcal{L}_1\mathrm{E}_{\mathsf{local}}$.

Proposition 7. *For $e \geq 2$, the logic $\mathcal{L}_0 e\mathrm{E}_{\mathsf{refine}}$ has the finite model property and its (finite) satisfiability problem is in* NeExpTime.

By Proposition 1 and Proposition 3, the same holds for $\mathcal{L}_0 e\mathrm{E}_{\mathsf{global}}$ and $\mathcal{L}_0 e\mathrm{E}_{\mathsf{local}}$.

5.4 Hardness with a single equivalence

In this section we show that the (finite) satisfiability of monadic first-order logic with a single equivalence symbol $\mathcal{L}_1$1E is N2ExpTime-hard by reducing the doubly exponential tiling problem to such satisfiability. Our strategy is to employ a counter setup of u unary predicate symbols to encode the exponentially many positions of a binary encoding of a doubly exponentially bounded quantity, encoding the coordinaes of a cell of the doubly exponential tiling square.

Consider the counter setup $\mathrm{C}(u) = \langle \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_u \rangle$ for $u \in \mathbb{N}^+$. Recall that the intention of a counter setup is to encode an arbitrary exponentially bounded value at every element of a structure. Let $\mathrm{D}(u) = \mathrm{C}(u) + \langle \boldsymbol{d} \rangle$ be a predicate signature enriching $\mathrm{C}(u)$ with the builtin equivalence symbol $\boldsymbol{d}$. We will define a system where every $\boldsymbol{d}$-equivalence class includes exponentially many cells. These cells will correspond to the exponentially many positions of the binary encoding of a doubly exponential value for the $\boldsymbol{d}$-class. The bit values at each cell position will be encoded by the cardinality of that cell: bit value 0 if the cardinality of the cell is 1 and bit value 1 if the cardinality is greater than 1. This will allow us to encode a doubly exponential value at each $\boldsymbol{d}$-class. Call the data $[\mathrm{C{:}data}]^{\mathfrak{A}}a$, encoded by the counter setup at a the *position* of a.

Let $\mathfrak{A}$ be a $\mathrm{D} = \mathrm{D}(u)$-structure.

Definition 45. *Define the quantifier-free* $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathrm{D{:}}\textsf{pos-eq}](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$[\mathrm{D{:}}\textsf{pos-eq}](\boldsymbol{x}, \boldsymbol{y}) = [\mathrm{C{:}}\textsf{eq}](\boldsymbol{x}, \boldsymbol{y}).$$

Then $\mathfrak{A} \models [\mathrm{D{:}}\textsf{pos-eq}](a, b)$ iff a and b are at the same positions (in possibly distinct $\boldsymbol{d}$-classes): $[\mathrm{C{:}data}]^{\mathfrak{A}}a = [\mathrm{C{:}data}]^{\mathfrak{A}}b$.

Definition 46. *Define the quantifier-rank-*1 $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathrm{D{:}}\textsf{bit-0}](\boldsymbol{x})$ *by:*

$$[\mathrm{D{:}}\textsf{bit-0}](\boldsymbol{x}) = \forall \boldsymbol{y}\, (\boldsymbol{d}(\boldsymbol{y}, \boldsymbol{x}) \wedge [\mathrm{D{:}}\textsf{pos-eq}](\boldsymbol{y}, \boldsymbol{x}) \rightarrow \boldsymbol{y} = \boldsymbol{x})\,.$$

Then $\mathfrak{A} \models [\mathrm{D{:}}\textsf{bit-0}](a)$ iff the cell of a has cardinality 1.

Definition 47. *Define the quantifier-rank-*1 $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathrm{D{:}}\textsf{bit-1}](\boldsymbol{x})$ *by:*

$$[\mathrm{D{:}}\textsf{bit-1}](\boldsymbol{x}) = \exists \boldsymbol{y}\, (\boldsymbol{d}(\boldsymbol{y}, \boldsymbol{x}) \wedge [\mathrm{D{:}}\textsf{pos-eq}](\boldsymbol{y}, \boldsymbol{x}) \wedge \boldsymbol{y} \neq \boldsymbol{x})\,.$$

Then $\mathfrak{A} \models [\mathrm{D{:}}\textsf{bit-1}](a)$ iff the cell of a has cardinality greater than 1.

Definition 48. *Define the quantifier-free* $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathrm{D{:}}\textsf{pos-zero}](\boldsymbol{x})$ *by:*

$$[\mathrm{D{:}}\textsf{pos-zero}](\boldsymbol{x}) = \bigwedge_{1 \le i \le u} \neg \boldsymbol{u}_i(\boldsymbol{x}).$$

Then $\mathfrak{A} \models [\mathrm{D{:}}\textsf{pos-zero}](a)$ iff the position of a is 0.

Definition 49. *Define the quantifier-free* $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathrm{D{:}}\textsf{pos-largest}](\boldsymbol{x})$ *by:*

$$[\mathrm{D{:}}\textsf{pos-largest}](\boldsymbol{x}) = \bigwedge_{1 \le i \le u} \boldsymbol{u}_i(\boldsymbol{x}).$$

Then $\mathfrak{A} \models [\mathrm{D{:}}\textsf{pos-largest}](a)$ iff the position of a is the largest u-bit number N_u.

Definition 50. *Define the quantifier-free* $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathrm{D{:}}\textsf{pos-less}](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$[\mathrm{D{:}}\textsf{pos-less}](\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \wedge [\mathrm{C{:}}\textsf{less}](\boldsymbol{x}, \boldsymbol{y}).$$

Then $\mathfrak{A} \models [\mathrm{D{:}}\textsf{pos-less}](a, b)$ iff a and b are in the same $\boldsymbol{d}$-class and the position of a is less than the position of b.

Definition 51. *Define the quantifier-free* $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathrm{D{:}}\textsf{pos-succ}](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$[\mathrm{D{:}}\textsf{pos-succ}](\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \wedge [\mathrm{C{:}}\textsf{succ}](\boldsymbol{x}, \boldsymbol{y}).$$

Then $\mathfrak{A} \models [\mathrm{D}{:}\textsf{pos-succ}](a,b)$ iff a and b are in the same $\boldsymbol{d}$-class and the position of b is the successor of the position of a.

Definition 52. *Define the closed $\mathcal{L}^2[\mathrm{D}]$-sentence $[\mathrm{D}{:}\textsf{pos-full}]$ by:*

$$[\mathrm{D}{:}\textsf{pos-full}] = \forall \boldsymbol{x} \exists \boldsymbol{y} \Big(\boldsymbol{d}(\boldsymbol{y},\boldsymbol{x}) \wedge [\mathrm{D}{:}\textsf{pos-zero}](\boldsymbol{y}) \Big) \wedge$$
$$\forall \boldsymbol{x} \Big(\neg [\mathrm{D}{:}\textsf{pos-largest}](\boldsymbol{x}) \to \exists \boldsymbol{y} [\mathrm{D}{:}\textsf{pos-succ}](\boldsymbol{x},\boldsymbol{y}) \Big).$$

The first part of this formula asserts that every $\boldsymbol{d}$-class has an element at position 0. The second part asserts that if a is an element at position p, that is not the largest possible, there exists an element b in the same $\boldsymbol{d}$-class at position $p+1$. Therefore in any model of $[\mathrm{D}{:}\textsf{pos-full}]$, every $\boldsymbol{d}$-class has 2^u cells. For example, in particular, every $\boldsymbol{d}$-class has cardinality at least 2^u. For the rest of the section, suppose that $\mathfrak{A} \models [\mathrm{D}{:}\textsf{pos-full}]$.

Definition 53. *For every u-bit number $p \in \mathbb{B}_u$, define the $\mathcal{L}^2[\mathrm{D}]$-formula $[\mathrm{D}{:}\textsf{pos-}p](\boldsymbol{x})$ recursively by:*

$$[\mathrm{D}{:}\textsf{pos-}0](\boldsymbol{x}) = [\mathrm{D}{:}\textsf{pos-zero}](\boldsymbol{x})$$

and for $p \in [0, N_u - 1]$:

$$[\mathrm{D}{:}\textsf{pos-}(p+1)](\boldsymbol{x}) = \exists \boldsymbol{y} \Big([\mathrm{D}{:}\textsf{pos-}p](\boldsymbol{y}) \wedge [\mathrm{D}{:}\textsf{pos-succ}](\boldsymbol{y},\boldsymbol{x}) \Big).$$

In this case, for the formula to be a two-variable formula, the formula $[\mathrm{D}{:}\textsf{pos-}p](\boldsymbol{y})$ is obtained from $[\mathrm{D}{:}\textsf{pos-}p](\boldsymbol{x})$ by swapping all occurences (not only the unbounded ones) of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$[1]. Note that the length of the formula $[\mathrm{D}{:}\textsf{pos-}p](\boldsymbol{x})$ grows linearly as p grows.

Then $\mathfrak{A} \models [\mathrm{D}{:}\textsf{pos-}p](a)$ iff p is the position of a.

Definition 54. *Let $\mathfrak{A}$ be a D-structure. Let $D = \boldsymbol{d}^{\mathfrak{A}}$. Define the function $[\mathrm{D}{:}\mathrm{Data}]^{\mathfrak{A}} : \mathscr{E}D \to \mathbb{B}^{2^u}$, assiging a 2^u-bit bitstring to any D-class X by:*

$$[\mathrm{D}{:}\mathrm{Data}]^{\mathfrak{A}}_p X = \begin{cases} 1 & \textit{if } [\mathrm{C}{:}\mathrm{data}]^{\mathfrak{A}}(a) = (p-1) \textit{ implies } \mathfrak{A} \models [\mathrm{D}{:}\textsf{bit-}1](a) \textit{ for all } a \in X \\ 0 & \textit{otherwise} \end{cases}$$

for $p \in [1, 2^u]$.

Definition 55. *Define the quantifier-rank-1 $\mathcal{L}^2[\mathrm{D}]$-formula $[\mathrm{D}{:}\textsf{Zero}](\boldsymbol{x})$ by:*

$$[\mathrm{D}{:}\textsf{Zero}](\boldsymbol{x}) = \forall \boldsymbol{y} \Big(\boldsymbol{d}(\boldsymbol{y},\boldsymbol{x}) \to [\mathrm{D}{:}\textsf{bit-}0](\boldsymbol{y}) \Big).$$

Then $\mathfrak{A} \models [\mathrm{D}{:}\textsf{Zero}](a)$ iff the data at the D-class of a encodes 0: $\underline{[\mathrm{D}{:}\mathrm{Data}]^{\mathfrak{A}} D[a]} = 0$.

[1] this is reminiscent to the process of defining a standard translation of modal logic to the two-variable first-order fragment

Definition 56. *Define the quantifier-rank-*1 $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathsf{D{:}Largest}](\boldsymbol{x})$ *by:*

$$[\mathsf{D{:}Largest}](\boldsymbol{x}) = \forall \boldsymbol{y}\Big(\boldsymbol{d}(\boldsymbol{y},\boldsymbol{x}) \to [\mathsf{D{:}bit}\text{-}1](\boldsymbol{y})\Big).$$

Then $\mathfrak{A} \models [\mathsf{D{:}Largest}](a)$ iff the data at the D-class of a encodes the largest 2^u-bit number: $\underline{[\mathrm{D{:}Data}]^{\mathfrak{A}} D[a]} = N_{2^u}$.

Definition 57. *Let $M \in \mathbb{B}_{2^u}$ be a t-bit number (where $t \le 2^u$). Define the* $\mathcal{L}^2[\mathrm{D}]$*-formula* $[\mathsf{D{:}Eq}\text{-}M](\boldsymbol{x})$ *by:*

$$\begin{aligned}[\mathsf{D{:}Eq}\text{-}M](\boldsymbol{x}) = \forall \boldsymbol{y}\Big(\boldsymbol{d}(\boldsymbol{y},\boldsymbol{x}) \to \bigwedge_{0\le p<t} \Big([\mathsf{D{:}pos}\text{-}p](\boldsymbol{y}) \to [\mathsf{D{:}bit}\text{-}(\overline{M}_{p+1})](\boldsymbol{y})\Big) \wedge \\ \forall \boldsymbol{x}\Big([\mathsf{D{:}pos}\text{-}(t-1)](\boldsymbol{y}) \wedge [\mathsf{D{:}pos\text{-}less}](\boldsymbol{y},\boldsymbol{x}) \to [\mathsf{D{:}bit}\text{-}0](\boldsymbol{x})\Big)\Big).\end{aligned}$$

The first part of this formula asserts that the bits at the first t positions of the $\boldsymbol{d}$-class of $\boldsymbol{x}$ encode the number M. The second part asserts that all the remaining bits at larger positions are zeroes. Note that the length of this formula is polynomially bounded by t, the bitsize of M. We have $\mathfrak{A} \models [\mathsf{D{:}Eq}\text{-}M](a)$ iff the data at the D-class of a encodes M: $\underline{[\mathrm{D{:}Data}]^{\mathfrak{A}} D[a]} = M$.

Definition 58. *Define the* $\mathcal{L}^6[\mathrm{D}]$*-formula* $[\mathsf{D{:}Less}](\boldsymbol{x},\boldsymbol{y})$ *by:*

$$\begin{aligned}[\mathsf{D{:}Less}](\boldsymbol{x},\boldsymbol{y}) = \exists \boldsymbol{x}'\exists \boldsymbol{y}'\Bigg(\boldsymbol{d}(\boldsymbol{x}',\boldsymbol{x}) \wedge \boldsymbol{d}(\boldsymbol{y}',\boldsymbol{y}) \wedge \\ \Big([\mathsf{D{:}pos\text{-}eq}](\boldsymbol{x}',\boldsymbol{y}') \wedge [\mathsf{D{:}bit}\text{-}0](\boldsymbol{x}') \wedge [\mathsf{D{:}bit}\text{-}1](\boldsymbol{y}')\Big) \wedge && \text{(Less1)} \\ \forall \boldsymbol{x}''\Big([\mathsf{D{:}pos\text{-}less}](\boldsymbol{x}',\boldsymbol{x}'') \to \exists \boldsymbol{y}''\Big(\boldsymbol{d}(\boldsymbol{y}'',\boldsymbol{y}') \wedge \\ [\mathsf{D{:}pos\text{-}eq}](\boldsymbol{y}'',\boldsymbol{x}'') \wedge ([\mathsf{D{:}bit}\text{-}0](\boldsymbol{y}'') \leftrightarrow [\mathsf{D{:}bit}\text{-}0](\boldsymbol{x}''))\Big)\Big)\Bigg). && \text{(Less2)}\end{aligned}$$

Then $\mathfrak{A} \models [\mathsf{D{:}Less}](a,b)$ iff $\underline{[\mathrm{D{:}Data}]^{\mathfrak{A}} D[a]} < \underline{[\mathrm{D{:}Data}]^{\mathfrak{A}} D[b]}$. By rearrangement and reusing variables, this can be also written using just three variables (but not using just two variables). Indeed, $[\mathsf{D{:}Less}](\boldsymbol{x},\boldsymbol{y})$ is logically equivalent to:

$$\begin{aligned}\exists \boldsymbol{z}\Bigg(\boldsymbol{d}(\boldsymbol{z},\boldsymbol{x}) \wedge \exists \boldsymbol{x}\Bigg(\boldsymbol{x} = \boldsymbol{z} \wedge \exists \boldsymbol{z}\Bigg(\boldsymbol{d}(\boldsymbol{z},\boldsymbol{y}) \wedge \exists \boldsymbol{y}\Bigg(\boldsymbol{y} = \boldsymbol{z} \wedge \\ \Big([\mathsf{D{:}pos\text{-}eq}](\boldsymbol{x},\boldsymbol{y}) \wedge [\mathsf{D{:}bit}\text{-}0](\boldsymbol{x}) \wedge [\mathsf{D{:}bit}\text{-}1](\boldsymbol{y})\Big) \wedge && \text{(Less1)} \\ \forall \boldsymbol{z}\Big([\mathsf{D{:}pos\text{-}less}](\boldsymbol{x},\boldsymbol{z}) \to \exists \boldsymbol{x}\Big(\boldsymbol{x} = \boldsymbol{z} \wedge \exists \boldsymbol{z}\Big(\boldsymbol{d}(\boldsymbol{z},\boldsymbol{y}) \wedge \exists \boldsymbol{y}\Big(\boldsymbol{y} = \boldsymbol{z} \wedge \\ [\mathsf{D{:}pos\text{-}eq}](\boldsymbol{y},\boldsymbol{x}) \wedge ([\mathsf{D{:}bit}\text{-}0](\boldsymbol{y}) \leftrightarrow [\mathsf{D{:}bit}\text{-}0](\boldsymbol{x}))\Big)\Big)\Big)\Big)\Bigg)\Bigg)\Bigg)\Bigg). && \text{(Less2)}\end{aligned}$$

Definition 59. *Define the* $\mathcal{L}^6[\mathrm{D}]$*-formula* $[\mathsf{D{:}Succ}](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$[\mathsf{D{:}Succ}](\boldsymbol{x}, \boldsymbol{y}) = \exists \boldsymbol{x}' \exists \boldsymbol{y}' \Bigg(\boldsymbol{d}(\boldsymbol{x}', \boldsymbol{x}) \wedge \boldsymbol{d}(\boldsymbol{y}', \boldsymbol{y}) \wedge$$

$$\Big([\mathsf{D{:}pos\text{-}eq}](\boldsymbol{x}', \boldsymbol{y}') \wedge [\mathrm{D{:}bit\text{-}0}](\boldsymbol{x}') \wedge [\mathrm{D{:}bit\text{-}1}](\boldsymbol{y}') \Big) \wedge \tag{Succ1}$$

$$\forall \boldsymbol{x}'' \Big([\mathsf{D{:}pos\text{-}less}](\boldsymbol{x}'', \boldsymbol{x}') \to [\mathrm{D{:}bit\text{-}1}](\boldsymbol{x}'') \Big) \wedge \tag{Succ2}$$

$$\forall \boldsymbol{y}'' \Big([\mathsf{D{:}pos\text{-}less}](\boldsymbol{y}'', \boldsymbol{y}') \to [\mathrm{D{:}bit\text{-}0}](\boldsymbol{y}'') \Big) \wedge \tag{Succ3}$$

$$\forall \boldsymbol{x}'' \Big([\mathsf{D{:}pos\text{-}less}](\boldsymbol{x}', \boldsymbol{x}'') \to \exists \boldsymbol{y}'' \Big(\boldsymbol{d}(\boldsymbol{y}'', \boldsymbol{y}') \wedge$$

$$[\mathsf{D{:}pos\text{-}eq}](\boldsymbol{y}'', \boldsymbol{x}'') \wedge ([\mathrm{D{:}bit\text{-}0}](\boldsymbol{y}'') \leftrightarrow [\mathrm{D{:}bit\text{-}0}](\boldsymbol{x}'')) \Big) \Big) \Bigg). \tag{Succ4}$$

By rearrangement and reusing variables, this can be also written using just three variables (but not using just two variables).

Then $\mathfrak{A} \models [\mathsf{D{:}Succ}](a, b)$ iff $\underline{[\mathrm{D{:}Data}]^{\mathfrak{A}} D[b]} = 1 + \underline{[\mathrm{D{:}Data}]^{\mathfrak{A}} D[a]}$.

Definition 60. *Define the* $\mathcal{L}^3[\mathrm{D}]$*-sentence* $[\mathsf{D{:}Full}]$ *by:*

$$[\mathsf{D{:}Full}] = \exists \boldsymbol{x} [\mathsf{D{:}Zero}](\boldsymbol{x}) \wedge \forall \boldsymbol{x} \Big(\neg [\mathsf{D{:}Largest}](\boldsymbol{x}) \to \exists \boldsymbol{y} [\mathsf{D{:}Succ}](\boldsymbol{x}, \boldsymbol{y}) \Big).$$

If $\mathfrak{A}$ satisfies $[\mathsf{D{:}Full}]$ then $\mathfrak{A}$ contains a $\boldsymbol{d}$-class of encoding any possible data: for every $M \in [0, N_{2^u}]$, there is a $\boldsymbol{d}$-class X such that $\underline{[\mathrm{D{:}Data}]^{\mathfrak{A}} X} = M$.

Definition 61. *Define the* $\mathcal{L}^4[\mathrm{D}]$*-formula* $[\mathsf{D{:}Eq}](\boldsymbol{x}, \boldsymbol{y})$ *by:*

$$[\mathsf{D{:}Eq}](\boldsymbol{x}, \boldsymbol{y}) = \forall \boldsymbol{x}' \forall \boldsymbol{y}' \Big(\boldsymbol{d}(\boldsymbol{x}', \boldsymbol{x}) \wedge \boldsymbol{d}(\boldsymbol{y}', \boldsymbol{y}) \wedge$$

$$[\mathsf{D{:}pos\text{-}eq}](\boldsymbol{x}', \boldsymbol{y}') \to ([\mathrm{D{:}bit\text{-}0}](\boldsymbol{x}') \leftrightarrow [\mathrm{D{:}bit\text{-}0}](\boldsymbol{y}')) \Big).$$

By rearrangement and reusing variables, this can be also written using just three variables (but not using just two variables).

Then $\mathfrak{A} \models [\mathsf{D{:}Eq}](\boldsymbol{x}, \boldsymbol{y})$ iff $[\mathrm{D{:}Data}]^{\mathfrak{A}} D[a] = [\mathrm{D{:}Data}]^{\mathfrak{A}} D[b]$.

Definition 62. *Define the* $\mathcal{L}^4[\mathrm{D}]$*-sentence* $[\mathsf{D{:}Alldiff}]$ *by:*

$$[\mathsf{D{:}Alldiff}] = \forall \boldsymbol{x} \forall \boldsymbol{y} \Big(\neg \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \to \exists \boldsymbol{x}' \exists \boldsymbol{y}' \Big(\boldsymbol{d}(\boldsymbol{x}', \boldsymbol{x}) \wedge \boldsymbol{d}(\boldsymbol{y}', \boldsymbol{y}) \wedge$$

$$[\mathsf{D{:}pos\text{-}eq}](\boldsymbol{x}', \boldsymbol{y}') \wedge \neg([\mathrm{D{:}bit\text{-}0}](\boldsymbol{x}') \leftrightarrow [\mathrm{D{:}bit\text{-}0}](\boldsymbol{y}')) \Big) \Big).$$

By rearrangement and reusing variables, this can be also written using just three variables (but not using just two variables).

If $\mathfrak{A}$ satisfies [D:Alldiff] then all D-classes in $\mathfrak{A}$ encode different data.

Recall from Section 1.7 that an instance of the *doubly exponential tiling problem* is an initial condition $c^0 = \langle t_1^0, t_2^0, \ldots, t_n^0 \rangle \subseteq T = [1,k]$ of tiles from the domino system $D_0 = (T, H, V)$, where $H, V \subseteq T \times T$ are the horizontal and vertical matching relations. We need to define a predicate signature capable enough to express a doubly exponential grid of tiles. Consider the predicate signature

$$\mathrm{D} = \left\langle \boldsymbol{u}_1^H, \boldsymbol{u}_2^H, \ldots, \boldsymbol{u}_n^H; \boldsymbol{u}_1^V, \boldsymbol{u}_2^V, \ldots, \boldsymbol{u}_n^V; \boldsymbol{u}_1^T, \boldsymbol{u}_2^T, \ldots, \boldsymbol{u}_k^T; \boldsymbol{d} \right\rangle .$$

It has the following relevant subsignatures:

- $\mathrm{D}^H = \left\langle \boldsymbol{u}_1^H, \boldsymbol{u}_2^H, \ldots, \boldsymbol{u}_n^H, \boldsymbol{d} \right\rangle$ encodes the horizontal index of a tile
- $\mathrm{D}^V = \left\langle \boldsymbol{u}_1^V, \boldsymbol{u}_2^V, \ldots, \boldsymbol{u}_n^V, \boldsymbol{d} \right\rangle$ encodes the vertical index of a tile
- $\mathrm{D}^{HV} = \left\langle \boldsymbol{u}_1^H, \boldsymbol{u}_2^H, \ldots, \boldsymbol{u}_n^H, \boldsymbol{u}_1^V, \boldsymbol{u}_2^V, \ldots, \boldsymbol{u}_n^V, \boldsymbol{d} \right\rangle$ encodes the combined horizontal and vertical index of a tile; we need this to define the full grid
- $\mathrm{D}^T = \left\langle \boldsymbol{u}_1^T, \boldsymbol{u}_2^T, \ldots, \boldsymbol{u}_k^T \right\rangle$ encodes the type of a tile.

Let $\mathfrak{A}$ be a D-structure satisfying [D^{HV}:pos-full] and let $D = \boldsymbol{d}^{\mathfrak{A}}$. The sentence

$$[\mathrm{D}^{HV}\textsf{:Full}] \wedge [\mathrm{D}^{HV}\textsf{:Alldiff}] \tag{5.3}$$

asserts that the D-classes form a doubly exponential grid. The sentence

$$\forall \boldsymbol{x} \Big(\bigwedge_{1 \le i \le k} \boldsymbol{u}_i^T(\boldsymbol{x}) \to \bigwedge_{i < j \le k} \neg \boldsymbol{u}_j^T(\boldsymbol{x}) \Big) \tag{5.4}$$

asserts that every element has a unique type. The sentence

$$\forall \boldsymbol{x} \forall \boldsymbol{y} \Big(\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \to \bigwedge_{1 \le i \le k} (\boldsymbol{u}_i^T(\boldsymbol{x}) \leftrightarrow \boldsymbol{u}_i^T(\boldsymbol{x})) \Big) \tag{5.5}$$

asserts that all elements in a D-class have the same type—the type of the tile corresponding to that D-class. For $j \in [1,n]$, the sentence

$$\forall \boldsymbol{x} \Big([\mathrm{D}^H\textsf{:Eq-}(j-1)](\boldsymbol{x}) \wedge [\mathrm{D}^V\textsf{:Zero}](\boldsymbol{x}) \to \boldsymbol{u}_{t_j^0}^T(\boldsymbol{x}) \Big) \tag{5.6}$$

encodes the initial segment in the first row of the square. The sentence

$$\forall \boldsymbol{x} \forall \boldsymbol{y} \Big([\mathrm{D}^H\textsf{:Succ}](\boldsymbol{x}, \boldsymbol{y}) \wedge [\mathrm{D}^V\textsf{:Eq}](\boldsymbol{x}, \boldsymbol{y}) \to \bigvee_{(i,j) \in H} \boldsymbol{u}_i^T(\boldsymbol{x}) \wedge \boldsymbol{u}_j^T(\boldsymbol{y}) \Big) \tag{5.7}$$

encodes the horizontal matching condition. The sentence

$$\forall \boldsymbol{x} \forall \boldsymbol{y} \Big([\mathrm{D}^V\textsf{:Succ}](\boldsymbol{x}, \boldsymbol{y}) \wedge [\mathrm{D}^H\textsf{:Eq}](\boldsymbol{x}, \boldsymbol{y}) \to \bigvee_{(i,j) \in V} \boldsymbol{u}_i^T(\boldsymbol{x}) \wedge \boldsymbol{u}_j^T(\boldsymbol{y}) \Big) \tag{5.8}$$

encodes the vertical matching condition.

Combining [D^{HV}:pos-full] with the formulas 5.3–5.8, we may encode an instance of the doubly exponential tiling problem as a (finite) satisfiability of a formula, so we have:

Proposition 8. *The (finite) satisfiability problem for the monadic first-order logic with a single equivalence symbol* $\mathcal{L}_1\mathrm{1E}$ *is* N2ExpTime*-hard. More precisely, even the three-variable fragment* $\mathcal{L}_1^3\mathrm{1E}$ *has this property.*

5.5 Hardness with many equivalences in refinement

The argument from the previous section can be iterated to yield the hardness of the (finite) satisfiability of the monadic first-order logic with several builtin equivalence symbols in refinement $\mathcal{L}_1 e\mathrm{E}_{\mathsf{refine}}$. Our strategy is to encode $(e{+}1)$-exponential numbers at every equivalence class of the coarsest relation by thinking of the e-exponential numbers at the classes of the second-to-coarsest relation as bit positions.

For $e \in \mathbb{N}^+$, consider the predicate signature $\mathrm{E}(e) = \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_e \rangle$ consisting of the builtin equivalence symbols $\boldsymbol{e}_i$ in refinement. Abbreviate the *coarsest* equivalence symbol $\boldsymbol{d} = \boldsymbol{e}_e$.

Definition 63. *Let* $e \in \mathbb{N}^+$*. An* e-exponential setup *is a uniform effective polynomial-time process for creating the following data structure. For every* $u \in \mathbb{N}^+$*, there is a predicate signature* $\mathrm{D}(e,u)$ *having length polynomial in* u*, consisting of unary predicate symbols and containing* $\mathrm{E}(e)$*. The following data is effectively defined:*

E1 There is a $\mathcal{L}^3[\mathrm{D}(e,u)]$*-sentence* $[\mathrm{D}(e,u){:}\mathsf{pos\text{-}full}]$*, whose length grows polynomially as* u *grows.*

E2 If $\mathfrak{A}$ *is a* $\mathrm{D}(e,u)$*-structure,* $\mathfrak{A} \vDash [\mathrm{D}(e,u){:}\mathsf{pos\text{-}full}]$ *and* $D = \boldsymbol{d}^{\mathfrak{A}}$*, then there is a function* $[\mathrm{D}(e,u){:}\mathrm{Data}]^{\mathfrak{A}} : \mathscr{E}D \to \mathbb{B}^{\exp_2^e(u)}$ *that assigns an* e*-exponential bitstring to every* D*-class.*

E3 There is a $\mathcal{L}^3[\mathrm{D}(e,u)]$*-formula* $[\mathrm{D}(e,u){:}\mathsf{Eq}](\boldsymbol{x},\boldsymbol{y})$ *whose length grows polynomially as* u *grows, such that for all* $a,b \in A$*:*

$$\mathfrak{A} \vDash [\mathrm{D}(e,u){:}\mathsf{Eq}](a,b) \text{ iff } \underline{[\mathrm{D}(e,u){:}\mathrm{Data}]^{\mathfrak{A}} D[a]} = \underline{[\mathrm{D}(e,u){:}\mathrm{Data}]^{\mathfrak{A}} D[b]}.$$

E4 There is a $\mathcal{L}^3[\mathrm{D}(e,u)]$*-formula* $[\mathrm{D}(e,u){:}\mathsf{Zero}](\boldsymbol{x})$*, whose length grows polynomially as* u *grows, such that for all* $a \in A$*:*

$$\mathfrak{A} \vDash [\mathrm{D}(e,u){:}\mathsf{Zero}](a) \text{ iff } \underline{[\mathrm{D}(e,u){:}\mathrm{Data}]^{\mathfrak{A}} D[a]} = 0.$$

E5 There is a $\mathcal{L}^3[\mathrm{D}(e,u)]$*-formula* $[\mathrm{D}(e,u){:}\mathsf{Largest}](\boldsymbol{x})$*, whose length grows polynomially as* u *grows, such that for all* $a \in A$*:*

$$\mathfrak{A} \vDash [\mathrm{D}(e,u){:}\mathsf{Largest}](a) \text{ iff } \underline{[\mathrm{D}(e,u){:}\mathrm{Data}]^{\mathfrak{A}} D[a]} = N_{\exp_2^e(u)} = \exp_2^{e+1}(u) - 1.$$

E6 There is a $\mathcal{L}^3[\mathrm{D}(e,u)]$*-formula* $[\mathrm{D}(e,u){:}\mathsf{Less}](\boldsymbol{x},\boldsymbol{y})$*, whose length grows polynomially as* u *grows, such that for all* $a,b \in A$*:*

$$\mathfrak{A} \vDash [\mathrm{D}(e,u){:}\mathsf{Less}](a,b) \text{ iff } \underline{[\mathrm{D}(e,u){:}\mathrm{Data}]^{\mathfrak{A}} D[a]} < \underline{[\mathrm{D}(e,u){:}\mathrm{Data}]^{\mathfrak{A}} D[b]}.$$

E7 There is a $\mathcal{L}^3[\mathrm{D}(e,u)]$*-formula* $[\mathrm{D}(e,u)\mathbf{:}\mathsf{Succ}](\boldsymbol{x},\boldsymbol{y})$*, whose length grows polynomially as* u *grows, such that for all* $a,b \in A$*:*

$$\mathfrak{A} \models [\mathrm{D}(e,u)\mathbf{:}\mathsf{Succ}](a,b) \text{ iff } \underline{[\mathrm{D}(e,u)\mathbf{:}\mathrm{Data}]^{\mathfrak{A}} D[b]} = \underline{[\mathrm{D}(e,u)\mathbf{:}\mathrm{Data}]^{\mathfrak{A}} D[a]} + 1.$$

E8 For every $\exp_2^e(u)$*-bit number* M*, there is a* $\mathcal{L}^3[\mathrm{D}(e,u)]$*-formula* $[\mathrm{D}(e,u)\mathbf{:}\mathsf{Eq}\text{-}M](\boldsymbol{x})$*, whose length grows polynomially as* u *and* M *grow, such that for all* $a \in A$*:*

$$\mathfrak{A} \models [\mathrm{D}(e,u)\mathbf{:}\mathsf{Eq}\text{-}M](a) \text{ iff } \underline{[\mathrm{D}(e,u)\mathbf{:}\mathrm{Data}]^{\mathfrak{A}} D[a]} = M.$$

The previous section defines a 1-exponential setup. Suppose that we have an e-exponential setup having predicate signature $\mathrm{D} = \mathrm{D}(e,u)$. Analogously to the previous section, we will describe an $(e+1)$-exponential setup $\mathrm{D}' = \mathrm{D}(e+1,u) = \mathrm{D} + \langle \boldsymbol{e} \rangle$ which is based on D, where $\boldsymbol{e} = \boldsymbol{e}_{e+1}$ is the new coarsest builtin equivalence symbol in D'. Define the following formulas:

$$\begin{aligned}
&[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}eq}](\boldsymbol{x},\boldsymbol{y}) = [\mathrm{D}\mathbf{:}\mathsf{Eq}](\boldsymbol{x},\boldsymbol{y})\\
&[\mathrm{D}'\mathbf{:}\mathsf{bit\text{-}0}](\boldsymbol{x}) = \forall \boldsymbol{y}(\boldsymbol{e}(\boldsymbol{y},\boldsymbol{x}) \wedge [\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}eq}](\boldsymbol{y},\boldsymbol{x}) \to \boldsymbol{d}(\boldsymbol{y},\boldsymbol{x}))\\
&[\mathrm{D}'\mathbf{:}\mathsf{bit\text{-}1}](\boldsymbol{x}) = \exists \boldsymbol{y}(\boldsymbol{e}(\boldsymbol{y},\boldsymbol{x}) \wedge [\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}eq}](\boldsymbol{y},\boldsymbol{x}) \wedge \neg \boldsymbol{d}(\boldsymbol{y},\boldsymbol{x}))\\
&[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}zero}](\boldsymbol{x}) = [\mathrm{D}\mathbf{:}\mathsf{Zero}](\boldsymbol{x})\\
&[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}largest}](\boldsymbol{x}) = [\mathrm{D}\mathbf{:}\mathsf{Largest}](\boldsymbol{x})\\
&[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}less}](\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{e}(\boldsymbol{x},\boldsymbol{y}) \wedge [\mathrm{D}\mathbf{:}\mathsf{Less}](\boldsymbol{x},\boldsymbol{y})\\
&[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}succ}](\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{e}(\boldsymbol{x},\boldsymbol{y}) \wedge [\mathrm{D}\mathbf{:}\mathsf{Succ}](\boldsymbol{x},\boldsymbol{y})\\
&[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}full}] = \forall \boldsymbol{x} \exists \boldsymbol{y}\Big(\boldsymbol{e}(\boldsymbol{y},\boldsymbol{x}) \wedge [\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}zero}](\boldsymbol{y})\Big) \wedge \qquad \text{(E1)}\\
&\quad \forall \boldsymbol{x}\Big(\neg[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}largest}](\boldsymbol{x}) \to \exists \boldsymbol{y}[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}succ}](\boldsymbol{x},\boldsymbol{y})\Big)\\
&[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}0}](\boldsymbol{x}) = [\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}zero}](\boldsymbol{x})\\
&[\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}}(p+1)](\boldsymbol{x}) = \exists \boldsymbol{y}\Big([\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}}p](\boldsymbol{y}) \wedge [\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}succ}](\boldsymbol{y},\boldsymbol{x})\Big)\\
&\quad \text{for } p \in [0, N_{\exp_2^e(u)} - 1].
\end{aligned}$$

Let $\mathfrak{A}$ be a D'-structure, $\mathfrak{A} \models [\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}full}]$ and let $E = \boldsymbol{e}^{\mathfrak{A}}$. Define the function $[\mathrm{D}'\mathbf{:}\mathrm{Data}]^{\mathfrak{A}} : \mathscr{E}E \to \mathbb{B}^{\exp_2^{e+1}(u)}$ assiging a $\exp_2^{e+1}(u)$-bit bitstring to any E-class X by:

$$[\mathrm{D}'\mathbf{:}\mathrm{Data}]_p^{\mathfrak{A}} X = \begin{cases} 1 \text{ if } \mathfrak{A} \models [\mathrm{D}'\mathbf{:}\mathsf{pos\text{-}}(p-1)](a) \text{ implies } \mathfrak{A} \models [\mathrm{D}'\mathbf{:}\mathsf{bit\text{-}1}](a) \text{ for all } a \in X \\ 0 \text{ otherwise} \end{cases} \qquad \text{(E2)}$$

for $p \in [1, \exp_2^{e+1}(u)]$.

Define the following formulas:

$$[\mathsf{D}'\textsf{:Eq}](\boldsymbol{x},\boldsymbol{y}) = \forall \boldsymbol{x}'\forall \boldsymbol{y}'\Big(\boldsymbol{e}(\boldsymbol{x}',\boldsymbol{x}) \wedge \boldsymbol{e}(\boldsymbol{y}',\boldsymbol{y}) \wedge [\mathsf{D}'\textsf{:pos-eq}](\boldsymbol{x}',\boldsymbol{y}') \to ([\mathsf{D}'\textsf{:bit-0}](\boldsymbol{x}') \leftrightarrow [\mathsf{D}'\textsf{:bit-0}](\boldsymbol{y}'))\Big) \tag{E3}$$

$$[\mathsf{D}'\textsf{:Zero}](\boldsymbol{x}) = \forall \boldsymbol{y}\Big(\boldsymbol{e}(\boldsymbol{y},\boldsymbol{x}) \to [\mathsf{D}'\textsf{:bit-0}](\boldsymbol{y})\Big) \tag{E4}$$

$$[\mathsf{D}'\textsf{:Largest}](\boldsymbol{x}) = \forall \boldsymbol{y}\Big(\boldsymbol{e}(\boldsymbol{y},\boldsymbol{x}) \to [\textsf{1:bit-D}'](\boldsymbol{y})\Big) \tag{E5}$$

$$\begin{aligned}[\mathsf{D}'\textsf{:Less}](\boldsymbol{x},\boldsymbol{y}) = \exists \boldsymbol{x}'\exists \boldsymbol{y}'\Bigg(&\boldsymbol{e}(\boldsymbol{x}',\boldsymbol{x}) \wedge \boldsymbol{e}(\boldsymbol{y}',\boldsymbol{y}) \wedge \\ &\Big([\mathsf{D}'\textsf{:pos-eq}](\boldsymbol{x}',\boldsymbol{y}') \wedge [\mathsf{D}'\textsf{:bit-0}](\boldsymbol{x}') \wedge [\mathsf{D}'\textsf{:bit-1}](\boldsymbol{y}')\Big) \wedge \\ &\forall \boldsymbol{x}''\Big([\mathsf{D}'\textsf{:pos-less}](\boldsymbol{x}',\boldsymbol{x}'') \to \exists \boldsymbol{y}''\Big(\boldsymbol{e}(\boldsymbol{y}'',\boldsymbol{y}') \wedge \\ &[\mathsf{D}'\textsf{:pos-eq}](\boldsymbol{y}'',\boldsymbol{x}'') \wedge ([\mathsf{D}'\textsf{:bit-0}](\boldsymbol{y}'') \leftrightarrow [\mathsf{D}'\textsf{:bit-0}](\boldsymbol{x}''))\Big)\Big)\Bigg)\end{aligned} \tag{E6}$$

$$\begin{aligned}[\mathsf{D}'\textsf{:Succ}](\boldsymbol{x},\boldsymbol{y}) = \exists \boldsymbol{x}'\exists \boldsymbol{y}'\Bigg(&\boldsymbol{e}(\boldsymbol{x}',\boldsymbol{x}) \wedge \boldsymbol{e}(\boldsymbol{y}',\boldsymbol{y}) \wedge \\ &\Big([\mathsf{D}'\textsf{:pos-eq}](\boldsymbol{x}',\boldsymbol{y}') \wedge [\mathsf{D}'\textsf{:bit-0}](\boldsymbol{x}') \wedge [\mathsf{D}'\textsf{:bit-1}](\boldsymbol{y}')\Big) \wedge \\ &\forall \boldsymbol{x}''\Big([\mathsf{D}'\textsf{:pos-less}](\boldsymbol{x}'',\boldsymbol{x}') \to [\mathsf{D}'\textsf{:bit-1}](\boldsymbol{x}'')\Big) \wedge \\ &\forall \boldsymbol{y}''\Big([\mathsf{D}'\textsf{:pos-less}](\boldsymbol{y}'',\boldsymbol{y}') \to [\mathsf{D}'\textsf{:bit-0}](\boldsymbol{y}'')\Big) \wedge \\ &\forall \boldsymbol{x}''\Big([\mathsf{D}'\textsf{:pos-less}](\boldsymbol{x}',\boldsymbol{x}'') \to \exists \boldsymbol{y}''\Big(\boldsymbol{e}(\boldsymbol{y}'',\boldsymbol{y}') \wedge \\ &[\mathsf{D}'\textsf{:pos-eq}](\boldsymbol{y}'',\boldsymbol{x}'') \wedge ([\mathsf{D}'\textsf{:bit-0}](\boldsymbol{y}'') \leftrightarrow [\mathsf{D}'\textsf{:bit-0}](\boldsymbol{x}''))\Big)\Big)\Bigg).\end{aligned} \tag{E7}$$

If $M \in \mathbb{B}_{\exp_2^{e+1}(u)}$ is an $\exp_2^{e+1}(u)$-bit number, let $t = \|M\|$ and define the formula:

$$\begin{aligned}[\mathsf{D}'\textsf{:Eq-}M](\boldsymbol{x}) = \forall \boldsymbol{y}\Bigg(&\boldsymbol{e}(\boldsymbol{y},\boldsymbol{x}) \to \bigwedge_{0\le p<t}\Big([\mathsf{D}'\textsf{:pos-}p](\boldsymbol{y}) \to [\mathsf{D}'\textsf{:bit-}\overline{M}_{p+1}](\boldsymbol{y})\Big) \wedge \\ &\forall \boldsymbol{x}\Big([\mathsf{D}'\textsf{:pos-}(t-1)](\boldsymbol{y}) \wedge [\mathsf{D}'\textsf{:pos-less}](\boldsymbol{y},\boldsymbol{x}) \to [\mathsf{D}'\textsf{:bit-0}](\boldsymbol{x})\Big)\Bigg).\end{aligned} \tag{E8}$$

This completes the definition of the $(e+1)$-exponential setup.

We can encode an instance of the $(e+1)$-exponential tiling problem into a (finite) satisfiability D-formula completely analogously to the previous section. Thus we have:

Proposition 9. *The (finite) satisfiability problem for the monadic first-order logic with e equivalence symbols in refinement $\mathcal{L}_1 e\mathrm{E}_{\mathsf{refine}}$ is* N$(e+1)$ExpTime*-hard. Even the three-variable fragment $\mathcal{L}_1^3 e\mathrm{E}_{\mathsf{refine}}$ has this property.*

By Proposition 1 and Proposition 3, the same holds for $\mathcal{L}_1^{(3)}e\mathrm{E}_{\mathsf{global}}$ *and* $\mathcal{L}_1^{(3)}e\mathrm{E}_{\mathsf{local}}$.

Proposition 10. *The (finite) satisfiability problem for the monadic first-order logic with many equivalence symbols in refinement* $\mathcal{L}_1\mathrm{E}_{\mathsf{refine}}$ *is* ELEMENTARY*-hard. Even the three-variable fragment* $\mathcal{L}_1^3\mathrm{E}_{\mathsf{refine}}$ *has this property.*

By Proposition 2 and Proposition 4, the same holds for $\mathcal{L}_1^{(3)}\mathrm{E}_{\mathsf{global}}$ *and* $\mathcal{L}_1^{(3)}\mathrm{E}_{\mathsf{local}}$.

6 Two-variable first-order logics

In this chapter we investigate quastions about the complexity of satisfiability and finite satisfiability of the two-variable first-order logic $\mathcal{L}^2$ with builtin equivalence relations in refinement. Recall that for this logic we are only interested in predicate signatures restricted to only unary and binary predicate symbols and the formal equality.

The base case for $\mathcal{L}^2$ and the general case of several *unrelated* builtin equivalence symbols have been studied. The following is known:

- The two-variable first-order logic $\mathcal{L}^2$ has the finite model property [10] and its (finite) satisfiability problem is NExpTime-complete [11].
- The two-variable first-order logic with a single builtin equivalence symbol $\mathcal{L}^2 1\mathrm{E}$ has the finite model property and its (finite) satisfiability problem is NExpTime-complete [12].
- The two-variable first-order logic with two *unrelated* builtin equivalence symbols $\mathcal{L}^2 2\mathrm{E}$ lacks the finite model property and both its satisfiability and finite satisfiability problems are N2ExpTime-complete [13].
- The satisfiability and finite satisfiability problems for the two-variable first-order logic with e builtin equivalence symbols $\mathcal{L}^2 e\mathrm{E}$ are both undecidable for $e \geq 3$ [14].

TODO: Summarise results!

6.1 Type realizibility

Recall from Section 1.6 about normal forms that every $\mathcal{L}^2$-sentence φ can be reduced in deterministic polynomial time to a sentence $\operatorname{sctr}\varphi$ in Scott normal form:

$$\forall \boldsymbol{x}\forall \boldsymbol{y}(\alpha_0(\boldsymbol{x},\boldsymbol{y}) \vee \boldsymbol{x} = \boldsymbol{y}) \wedge \bigwedge_{1\leq i\leq m} \forall \boldsymbol{x}\exists \boldsymbol{y}(\alpha_i(\boldsymbol{x},\boldsymbol{y}) \wedge \boldsymbol{x} \neq \boldsymbol{y}),$$

where $m \geq 1$, all the formulas α_i are quantifier-free and use at most linearly many new unary predicate symbols. The semantical connection between φ and $\operatorname{sctr}\varphi$ is that they are essentially equisatisfiable. More precisely, every model of φ of cardinality at least 2 can be enriched to a model for $\operatorname{sctr}\varphi$ and also every model of $\operatorname{sctr}\varphi$ (which by $m \geq 1$ must have cardinality at least 2) is a model for φ. We refer to α_0 as the *universal part*, or the $\forall\forall$-part of the formula $\operatorname{sctr}\varphi$ and to α_i as the *existential parts*, or the $\forall\exists$-parts of the formula, for $i \in [1, m]$.

6.1.1 Classified signatures

For any formula $\operatorname{sctr}\varphi$ in Scott normal form, we may replace its existential parts by fresh binary predicate symbols: for $i \in [1, m]$, let $\boldsymbol{m}_i$ be a fresh binary predicate symbol with the intended interpretation $\forall\boldsymbol{x}\forall\boldsymbol{y}(\boldsymbol{m}_i(\boldsymbol{x},\boldsymbol{y}) \leftrightarrow \alpha_i(\boldsymbol{x},\boldsymbol{y}))$. Since this is a universal sentence, it can be incorporated into the $\forall\forall$-part α_0 of the formula. We refer to the symbols $\boldsymbol{m}_i$ as the *message symbols*. Hence $\operatorname{sctr}\varphi$ can be transformed in deterministic polynomial time to the form:

$$\forall\boldsymbol{x}\forall\boldsymbol{y}(\alpha(\boldsymbol{x},\boldsymbol{y}) \vee \boldsymbol{x} = \boldsymbol{y}) \wedge \bigwedge_{1\le i\le m} \forall\boldsymbol{x}\exists\boldsymbol{y}(\boldsymbol{m}_i(\boldsymbol{x},\boldsymbol{y}) \wedge \boldsymbol{x} \neq \boldsymbol{y}), \tag{6.1}$$

where the $\forall\forall$-part α is quantifier-free and over an extended signature. For convenience, we make the existential parts of the formula part of the signature, so we can focus only on the universal part. The following is a term similar to the one defined in [4]:

Definition 64. *A* classified signature $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$ *for the two-variable first-order logic $\mathcal{L}^2$ is a predicate signature Σ together with a nonempty sequence $\bar{\boldsymbol{m}} = \boldsymbol{m}_1\boldsymbol{m}_2\ldots\boldsymbol{m}_m$ of distinct binary predicate symbols from Σ having intended interpretation*

$$\bigwedge_{1\le i\le m} \forall\boldsymbol{x}\exists\boldsymbol{y}(\boldsymbol{m}_i(\boldsymbol{x},\boldsymbol{y}) \wedge \boldsymbol{x} \neq \boldsymbol{y}). \tag{6.2}$$

That is, a classified signature *automatically includes* the $\forall\exists$-parts of formulas and $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$-structures *automatically satisfy* the $\forall\exists$-parts:

Definition 65. *A structure $\mathfrak{A}$ for the classified signature $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$ is a structure for the predicate signature Σ that satisfies the intended interpretation eq. (6.2) of the message symbols. Note that $\mathfrak{A}$ must have cardinality at least 2.*

Definition 66. *The* (finite) classified satisfiability problem for two-variable first-order logic *is: given a classified signature $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$ and a quantifier-free $\mathcal{L}^2[\Sigma]$-formula $\alpha(\boldsymbol{x},\boldsymbol{y})$, is there a (finite) $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$-structure $\mathfrak{A}$ satisfying eq. (6.1). Note that since $\mathfrak{A}$ is a $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$-structure, it must also satisfy eq. (6.2) and must have cardinality at least 2. Denote the classified satisfiability problem by* CL-SAT-$\mathcal{L}^2$ *and its finite version by* FIN-CL-SAT-$\mathcal{L}^2$.

Remark 22. *The problem of (finite) satisfiability reduces in nondeterministic polynomial time to the problem of (finite) classified satisfiability:*

$$\text{(FIN-)SAT-}\mathcal{L}^2 \le_{\mathrm{m}}^{\textsc{NPTime}} \text{(FIN-)CL-SAT-}\mathcal{L}^2.$$

Proof. Note that (finite) satisfiability in the class of models of cardinality 1 is trivially decidable in nondeterministic polynomial time — just guess the atomic 1-type (whose size is polynomially bounded by the size of the predicate signature) of the unique element of the structure and check (in deterministic polynomial time) that it satisfies the original formula.

Scott normal form shows that (finite) satisfiability in the class of models of cardinality at least 2 reduces in deterministic polynomial time to (finite) classified satisfiability.

Hence the following nondeterministic polynomial time procedure reduces an instance (Σ, φ) of the (finite) satisfiability problem to an instance $(\langle \Sigma', \bar{\boldsymbol{m}} \rangle, \alpha)$ of the (finite) classified satisfiability problem: First check if φ is satisfiable in the class of models of cardinality 1. If that is the case, then extend Σ to Σ' by adding a single message symbol $\boldsymbol{m}_1$ and let $\alpha = (\boldsymbol{x} = \boldsymbol{x})$ be a fixed predicate tautology. Otherwise transform φ into Scott normal form and let α be the universal part of that normal form. $\square$

6.1.2 Type instances

Definition 67. *A* type instance (Π, T) *over the classified signature* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *is a pair of a nonempty set of* 1*-types* $\Pi \subseteq \Pi[\Sigma]$ *and a nonempty set of* 2*-types* $\mathrm{T} \subseteq \mathrm{T}[\Sigma]$ *satisfying the following conditions:*

($\mathcal{T}i$) *The set of* 2*-types* T *is closed under inversion, that is* $\tau^{-1} \in \mathrm{T}$ *for every* $\tau \in \mathrm{T}$.

($\mathcal{T}c$) *Every* 2*-type* $\tau \in \mathrm{T}$ *connects* 1*-types from* Π*, that is* $\mathrm{tp}_{\boldsymbol{x}}\, \tau \in \Pi$ *and* $\mathrm{tp}_{\boldsymbol{y}}\, \tau \in \Pi$ *for every* $\tau \in \mathrm{T}$. *Equivalently, since* T *is closed under inversion, we require that* $(\mathrm{tp}_{\boldsymbol{x}} \restriction \mathrm{T}) \subseteq \Pi$.

Two 1*-types* $\pi, \pi' \in \Pi$ *are* connectable *(written* $\pi \sim^{\mathrm{T}} \pi'$*), if* $\pi = \mathrm{tp}_{\boldsymbol{x}}\, \tau$ *and* $\pi' = \mathrm{tp}_{\boldsymbol{y}}\, \tau$ *for some* $\tau \in \mathrm{T}$. *Connectability is symmetric since* T *is closed under inversion. However, connectability is not necessarily neither transitive nor reflexive: A* 1*-type* $\kappa \in \Pi$ *is a* king type *if* $\kappa \not\sim^{\mathrm{T}} \kappa$. *The set of king types is* $\mathrm{K}(\Pi, \mathrm{T})$. *A* 1*-type* $\pi \in \Pi$ *that is not a king type is a* peasant type *and the set of peasant types is* $\mathrm{P}(\Pi, \mathrm{T}) = \Pi \setminus \mathrm{K}(\Pi, \mathrm{T})$.

Remark 23. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *and let* $\mathrm{T}' \subseteq \mathrm{T}$ *be a nonempty subset of* 2*-types that is closed under inversion. Then* (Π, T') *is also a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *and* $\mathrm{K}(\Pi, \mathrm{T}) \subseteq \mathrm{K}(\Pi, \mathrm{T}')$ *and* $\mathrm{P}(\Pi, \mathrm{T}') \subseteq \mathrm{P}(\Pi, \mathrm{T})$.

Definition 68. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *and let* $\mathfrak{A}$ *be a* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*-structure. Then* $\mathfrak{A}$ realizes *(or is a model of) the type instance* (Π, T) *if the following conditions are satisfied:*

($\mathcal{R}\Pi$) $\mathrm{tp}^{\mathfrak{A}}[a] \in \Pi$ *for every* $a \in A$*, that is the* 1*-types realized in* $\mathfrak{A}$ *are from* Π.

($\mathcal{R}\mathrm{T}$) $\mathrm{tp}^{\mathfrak{A}}[a, b] \in \mathrm{T}$ *for every* $a \in A$ *and* $b \in A \setminus \{a\}$*, that is the* 2*-types realized in* $\mathfrak{A}$ *are from* T.

($\mathcal{R}\pi$) *If* $\pi \in \Pi$ *then some (possibly many)* $a \in A$ *realizes it. We don't require an analogous condition for* 2*-types.*

($\mathcal{R}\kappa$) *If* $\pi \in \Pi$ *is realized by a unique element* $a \in A$*, then* $\pi \in \mathrm{K}(\Pi, \mathrm{T})$ *is a king type.*

Remark 24. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *and let* $\mathfrak{A}$ *be a model for* (Π, T). *Then* Π *is the set of* 1*-types that are realized by some (possibly many) element in* $\mathfrak{A}$*,* $\mathrm{K}(\Pi, \mathrm{T})$ *is the set of* 1*-types that are realized by a unique element in* $\mathfrak{A}$ *and* $\mathrm{P}(\Pi, \mathrm{T})$ *is the set of* 1*-types that are realized by at least* 2 *elements in* $\mathfrak{A}$.

Proof. By Condition $(\mathcal{R}\Pi)$ and Condition $(\mathcal{R}\pi)$, Π is the set of 1-types that are realized by some element in $\mathfrak{A}$. By Condition $(\mathcal{R}\kappa)$ any 1-type that is realized by a unique element is a king type. Let κ be a king type. By Condition $(\mathcal{R}\pi)$ there is some $a \in A$ that realizes κ. Suppose towards a contradiction that some other $b \in A \setminus \{a\}$ also realizes κ. Then the 2-type $\tau = \mathrm{tp}^{\mathfrak{A}}[a, b]$ connects κ with itself. By Condition $(\mathcal{R}\mathrm{T})$ we have that $\tau \in \mathrm{T}$, hence $\kappa \sim^{\mathrm{T}} \kappa$ — a contradiction. Therefore $\mathrm{K}(\Pi, \mathrm{T})$ is the set of 1-types that are realized by a unique element in $\mathfrak{A}$, and so $\mathrm{P}(\Pi, \mathrm{T}) = \Pi \setminus \mathrm{K}(\Pi, \mathrm{T})$ is the set of 1-types that are realized by at least 2 elements in $\mathfrak{A}$. □

Remark 25. *Let* (Π, T) *be a type instance over* Σ*, let* $\mathrm{T}' \subseteq \mathrm{T}$ *be a nonempty subset of* 2*-types that is closed under inversion. Note that* $\mathrm{K}(\Pi, \mathrm{T}) \subseteq \mathrm{K}(\Pi, \mathrm{T}')$ *according to Remark 23. Suppose that* $\mathrm{K}(\Pi, \mathrm{T}') \subseteq \mathrm{K}(\Pi, \mathrm{T})$*, or equivalently that* $\mathrm{K}(\Pi, \mathrm{T}') = \mathrm{K}(\Pi, \mathrm{T})$*. Then any model* $\mathfrak{A}'$ *for* (Π, T') *is also a model for* (Π, T)*.*

Proof. We verify the conditions for $\mathfrak{A}'$ to be a model for the type instance (Π, T):

$(\mathcal{R}\Pi)$ follows from Condition $(\mathcal{R}\Pi)$ for the (Π, T')-model $\mathfrak{A}'$.

$(\mathcal{R}\mathrm{T})$ follows from Condition $(\mathcal{R}\mathrm{T})$ for the (Π, T')-model $\mathfrak{A}'$.

$(\mathcal{R}\pi)$ follows from Condition $(\mathcal{R}\pi)$ for the (Π, T')-model $\mathfrak{A}'$.

$(\mathcal{R}\kappa)$ follows from Condition $(\mathcal{R}\kappa)$ for the (Π, T')-model $\mathfrak{A}'$ together with the assumption $\mathrm{K}(\Pi, \mathrm{T}') = \mathrm{K}(\Pi, \mathrm{T})$.

□

Definition 69. *Let* $\mathfrak{A}$ *be a* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*-structure.* The type instance $\Pi\mathrm{T}[\mathfrak{A}] = (\Pi, \mathrm{T})$ *of* $\mathfrak{A}$ *is defined by:*

$$\Pi = \left\{ \mathrm{tp}^{\mathfrak{A}}[a] \;\middle|\; a \in A \right\}$$
$$\mathrm{T} = \left\{ \mathrm{tp}^{\mathfrak{A}}[a, b] \;\middle|\; a \in A, b \in A \setminus \{a\} \right\}.$$

That is, Π *is the set of* 1*-types realized in* $\mathfrak{A}$ *and* T *is the set of* 2*-types realized in* $\mathfrak{A}$*.*

Remark 26. *Then* $\Pi\mathrm{T}[\mathfrak{A}] = (\Pi, \mathrm{T})$ *is indeed a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *and* $\mathfrak{A}$ *is a model for* (Π, T)*.*

Proof. That Π and T are nonempty follows from the observation that the cardinality of $\mathfrak{A}$ is at least 2. First we verify the conditions for (Π, T) to be a type instance over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$.

$(\mathcal{T}i)$ If $\tau \in \mathrm{T}$, then $\tau = \mathrm{tp}^{\mathfrak{A}}[a, b]$ for some $a \neq b \in A$, hence $\tau^{-1} = \mathrm{tp}^{\mathfrak{A}}[b, a] \in \mathrm{T}$, so T is closed under inversion.

$(\mathcal{T}c)$ If $\tau \in \mathrm{T}$, then $\tau = \mathrm{tp}^{\mathfrak{A}}[a, b]$, hence $\mathrm{tp}_{\boldsymbol{x}} \tau = \mathrm{tp}^{\mathfrak{A}}[a] \in \Pi$, so $(\mathrm{tp}_{\boldsymbol{x}} \restriction \mathrm{T}) \subseteq \Pi$.

Next we verify the conditions for $\mathfrak{A}$ to be a model for the type instance (Π, T).

($\mathcal{R}\Pi$) follows by definition of Π.

($\mathcal{R}\mathrm{T}$) follows by definition of T.

($\mathcal{R}\pi$) follows by definition of Π.

($\mathcal{R}\kappa$) Let $\pi \in \Pi$ be realized by a a unique element $a \in A$. Then there is no $b \in A \setminus \{a\}$ such that $\mathrm{tp}^{\mathfrak{A}}[b] = \pi$, so no $\tau \in \mathrm{T}$ connects π with itself, therefore π is a king type.

□

Remark 27. *Let (Π, T) be a type instance over Σ, let $\mathfrak{A}$ be a model for (Π, T) and let $(\Pi', \mathrm{T}') = \Pi\mathrm{T}[\mathfrak{A}]$ be the type instance of $\mathfrak{A}$. Then $\Pi' = \Pi$, $\mathrm{T}' \subseteq \mathrm{T}$ and $\mathrm{K}(\Pi, \mathrm{T}') = \mathrm{K}(\Pi, \mathrm{T})$.*

Proof. Immediate by Remark 24. □

Remark 28. *Let (Π, T) be a type instance over Σ, let $\mathfrak{A}$ be a model for (Π, T) and let $\mathfrak{B}$ be a Σ-structure. If $\Pi\mathrm{T}[\mathfrak{B}] = \Pi\mathrm{T}[\mathfrak{A}]$ then $\mathfrak{B}$ is also a model for (Π, T).*

Proof. By Remark 27 we have $\Pi\mathrm{T}[\mathfrak{A}] = \Pi\mathrm{T}[\mathfrak{B}] = (\Pi, \mathrm{T}')$ where $\mathrm{T}' \subseteq \mathrm{T}$ and $\mathrm{K}(\Pi, \mathrm{T}') = \mathrm{K}(\Pi, \mathrm{T})$. Then by Remark 25 any model for (Π, T') is also a model for (T, T). □

6.1.3 Type realizability

Definition 70. *The* (finite) type realizability problem *is the following: given a classified signature $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ and a type instance (Π, T) over Σ, is there a (finite) $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure that realizes (Π, T). Denote the type realizability problem for $\mathcal{L}^2$ by* TP-REALIZ-$\mathcal{L}^2$ *and its finite version by* FIN-TP-REALIZ-$\mathcal{L}^2$.

We begin the study of the type realizability problem by reducing the classified satisfiability problem to it.

Remark 29. *Let $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ be a classified signature and let $\alpha(\boldsymbol{x}, \boldsymbol{y})$ be a quantifier-free $\mathcal{L}^2[\Sigma]$-formula. Define the following sets of types:*

- *$\Pi^\alpha \subseteq \Pi[\Sigma]$ is the set of those 1-types consistent with $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and the intended interpretation eq. (6.2).*
- *$\mathrm{T}^\alpha \subseteq \mathrm{T}[\Sigma]$ is the set of those 2-types τ such that both τ and τ^{-1} are consistent with $\alpha(\boldsymbol{x}, \boldsymbol{y})$ and the intended interpretation eq. (6.2).*

Then a $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure $\mathfrak{A}$ is a classified model for $\alpha(\boldsymbol{x}, \boldsymbol{y})$ iff there are some subsets $\Pi \subseteq \Pi^\alpha$ and $\mathrm{T} \subseteq \mathrm{T}^\alpha$ such that (Π, T) is a type instance over Σ and $\mathfrak{A}$ realizes (Π, T).

Recall that the number of possible 1-types or 2-types over Σ is exponentially bounded by the length s of Σ and that the cardinality of a 1-type or a 2-type over Σ is polynomially bounded by s. Hence we can reduce the (finite) classified satisfiability problem to the (finite) type realizability problem in nondeterministic exponential time:

$$\text{(FIN-)CL-SAT-}\mathcal{L}^2 \leq_{\mathrm{m}}^{\textsc{NExpTime}} \text{(FIN-)TP-REALIZ-}\mathcal{L}^2.$$

Definition 71. *We say that* (T, Π) *is a type instance over* the classified signature $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *if* (T, Π) *is a type instance over* Σ *and with the implicit requirement that we are interested only in models of* (T, Π) *that are* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*-structures, that is the intended interpretation of a type instance over a classified signature is the class of classified structures. Note that this is exactly the content of the type realizability problem.*

The next definition characterizes the 2-types emitted by an element in a model of a type instance.

Definition 72. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*. A* star-type $\sigma \subseteq \mathrm{T}$ *over* (Π, T) *is a nonempty set of* 2*-types satisfying the following conditions:*

($\sigma \boldsymbol{x}$) *If* $\tau, \tau' \in \sigma$*, then* $\operatorname{tp}_{\boldsymbol{x}} \tau = \operatorname{tp}_{\boldsymbol{x}} \tau'$*, that is the* $\boldsymbol{x}$*-type of every element of* σ *is the same.*

Denote the $\boldsymbol{x}$*-type of any element of* σ *by* $\pi = \operatorname{tp}_{\boldsymbol{x}} \sigma$*.*

($\sigma \kappa \boldsymbol{x}$) *If* $\pi = \kappa \in \mathrm{K}(\Pi, \mathrm{T})$ *is a king type, then no* $\tau \in \sigma$ *has* $\operatorname{tp}_{\boldsymbol{y}} \tau = \kappa$*.*

($\sigma \kappa \boldsymbol{y}$) *If* $\kappa \in \mathrm{K}(\Pi, \mathrm{T}) \setminus \{\pi\}$ *is any king type distinct from* π *(note that* π *may possibly be a peasant type), then a unique* $\tau \in \sigma$ *has* $\operatorname{tp}_{\boldsymbol{y}} \tau = \kappa$*.*

($\sigma \boldsymbol{m}$) *If* $\boldsymbol{m} \in \bar{\boldsymbol{m}}$*, then* $\boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y}) \in \tau$ *for some (possibly many)* $\tau \in \sigma$*.*

A star-type σ *is a* king star-type *if* $\operatorname{tp}_{\boldsymbol{x}} \sigma \in \mathrm{K}(\Pi, \mathrm{T})$ *is a king type. Otherwise the star-type* σ *is called a* peasant star-type*. Note that the size of a star-type is polynomially bounded by the size of the type instance.*

Definition 73. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *and let* $\mathfrak{A}$ *be a model for* (Π, T)*. If* $a \in A$*,* the star-type $\sigma = \operatorname{stp}^{\mathfrak{A}}[a]$ *of* a *is defined by:*

$$\operatorname{stp}^{\mathfrak{A}}[a] = \left\{ \operatorname{tp}^{\mathfrak{A}}[a, b] \;\middle|\; b \in A \setminus \{a\} \right\}.$$

Remark 30. *Then* σ *is indeed a star-type over* (Π, T)*.*

Proof. That σ is nonempty follows from the observation that $\mathfrak{A}$ as a $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure must have cardinality at least 2. We verify the conditions for a star-type:

($\sigma \boldsymbol{x}$) Let $\tau, \tau' \in \sigma$. Then $\tau = \operatorname{tp}^{\mathfrak{A}}[a, b]$ and $\tau' = \operatorname{tp}^{\mathfrak{A}}[a, c]$ for some $b, c \in A \setminus \{a\}$, so $\operatorname{tp}_{\boldsymbol{x}} \tau = \operatorname{tp}^{\mathfrak{A}}[a] = \operatorname{tp}_{\boldsymbol{x}} \tau'$.

Let $\pi = \operatorname{tp}_{\boldsymbol{x}} \sigma = \operatorname{tp}^{\mathfrak{A}}[a]$.

($\sigma \kappa \boldsymbol{x}$) Suppose that $\pi = \kappa \in \mathrm{K}(\Pi, \mathrm{T})$ is a king type, so $\kappa \not\sim^{\mathrm{T}} \kappa$. Since $\sigma \subseteq \mathrm{T}$ and every $\tau \in \sigma$ has $\operatorname{tp}_{\boldsymbol{x}} \tau = \kappa$, we must have that no $\tau \in \sigma$ has $\operatorname{tp}_{\boldsymbol{y}} \tau = \kappa$.

($\sigma \kappa \boldsymbol{y}$) Let $\kappa \in \mathrm{K}(\Pi, \mathrm{T}) \setminus \{\pi\}$. By Condition ($\mathcal{R} \kappa$) for $\mathfrak{A}$, there is a unique $b \in A$ realizing κ. Since $\kappa \neq \pi$, we have $b \neq a$. Hence $\tau = \operatorname{tp}^{\mathfrak{A}}[a, b]$ is the unique $\tau \in \sigma$ that has $\operatorname{tp}_{\boldsymbol{y}} \tau = \kappa$.

($\sigma\boldsymbol{m}$) Let $\boldsymbol{m} \in \bar{\boldsymbol{m}}$. Since $\mathfrak{A}$ is a $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure, it must satisfy the intended interpretation eq. (6.2), so in particular there is some $b \in A \setminus \{a\}$ such that $\tau = \mathrm{tp}^{\mathfrak{A}}[a, b]$ has $\boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y}) \in \tau$.

□

Remark 31. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*. Let* $\mathrm{T}' \subseteq \mathrm{T}$ *be a nonempty subset that is closed under inversion. According to Remark 23,* (Π, T') *is a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*. Suppose that* $\mathrm{K}(\Pi, \mathrm{T}) = \mathrm{K}(\Pi, \mathrm{T}')$*.*

Then if $\sigma \subseteq \mathrm{T}'$ *is a star-type over* (Π, T)*, then it is also a star-type over* (Π, T')*.*

Proof. We verify the conditions for a star-type over (Π, T'):

($\sigma\boldsymbol{x}$) follows from Condition ($\sigma\boldsymbol{x}$) for the star-type σ over (Π, T).

($\sigma\kappa\boldsymbol{x}$) follows from Condition ($\sigma\kappa\boldsymbol{x}$) for the star-type σ over (Π, T) together with the assumption $\mathrm{K}(\Pi, \mathrm{T}) = \mathrm{K}(\Pi, \mathrm{T}')$.

($\sigma\kappa\boldsymbol{y}$) follows from Condition ($\sigma\kappa\boldsymbol{y}$) for the star-type σ over (Π, T) together with the assumption $\mathrm{K}(\Pi, \mathrm{T}) = \mathrm{K}(\Pi, \mathrm{T}')$.

($\sigma\boldsymbol{m}$) follows from Condition ($\sigma\boldsymbol{m}$) for the star-type σ over (Π, T).

□

Remark 32 (Star-type extension)**.** *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*, let* σ *be a star-type over* (Π, T) *and let* $\pi = \mathrm{tp}_{\boldsymbol{x}}\, \sigma$*. Let* $\tau \in \mathrm{T}$ *be a 2-type such that* $\mathrm{tp}_{\boldsymbol{x}}\, \tau = \pi$ *and* $\mathrm{tp}_{\boldsymbol{y}}\, \tau$ *is not a king type. Then* $\sigma' = \sigma \cup \{\tau\}$ *is also a star-type over* (Π, T)*.*

Proof. It is straightforward to verify the conditions for a star-type:

($\sigma\boldsymbol{x}$) follows from the assumption that $\mathrm{tp}_{\boldsymbol{x}}\, \pi = \mathrm{tp}_{\boldsymbol{x}}\, \sigma$.

($\sigma\kappa\boldsymbol{x}$) follows from the assumption that $\mathrm{tp}_{\boldsymbol{y}}\, \pi$ is not a king type.

($\sigma\kappa\boldsymbol{y}$) follows from the assumption that $\mathrm{tp}_{\boldsymbol{y}}\, \pi$ is not a king type.

($\sigma\boldsymbol{m}$) follows since σ' extends σ.

□

Definition 74. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*. A* certificate $\mathcal{S}$ *for the type instance* (Π, T) *is a nonempty set of star-types over* (Π, T) *satisfying the following conditions:*

($\mathcal{S}i$) *If* $\tau \in \sigma$ *for some* $\sigma \in \mathcal{S}$*, then* $\tau^{-1} \in \sigma'$ *for some* $\sigma' \in \mathcal{S}$*, that is there are star-types for the endpoints of every 2-type used in the certificate. Equivalently,* $\mathrm{T}' = \cup\mathcal{S}$ *is closed under inversion. Note that by Remark 23,* (Π, T') *is also a type instance over* Σ*. Call* (Π, T') *the* filtered type instance *of* $\mathcal{S}$*.*

($\mathcal{S}$K) *We require that* $\mathrm{K}(\Pi, \mathrm{T}') = \mathrm{K}(\Pi, \mathrm{T})$, *that is the notion of a king type with respect to the filtered type instance coincides with the notion of a king type with respect to the original type instance. Note that then by Remark 31, we may think of* (Π, T') *as a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *and also that every star-type* $\sigma \in \mathcal{S}$ *is a also a star-type over* (Π, T').

($\mathcal{S}\pi$) *If* $\pi \in \Pi$, *then some (possibly many)* $\sigma \in \mathcal{S}$ *has* $\mathrm{tp}_{\boldsymbol{x}}\, \sigma = \pi$, *that is there is a star-type for every* 1*-type.*

($\mathcal{S}\kappa$) *If* $\kappa \in \mathrm{K}(\Pi, \mathrm{T})$, *then a unique* $\sigma \in \mathcal{S}$ *has* $\mathrm{tp}_{\boldsymbol{x}}\, \sigma = \kappa$, *that is there is a unique star-type for every king-type. Note that the existence is already implied by Condition* ($\mathcal{S}\pi$) .

($\mathcal{S}c$) *Let* $\pi, \pi' \in \Pi$. *If it is not the case that* $\pi = \pi' = \kappa \in \mathrm{K}(\Pi, \mathrm{T})$, *then some* $\tau \in \mathrm{T}'$ *has* $\mathrm{tp}_{\boldsymbol{x}}\, \tau = \pi$ *and* $\mathrm{tp}_{\boldsymbol{y}}\, \tau = \pi'$, *that is if* π *and* π' *are not the same king type, then they are connectable.*

Remark 33. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$, *let* $\mathcal{S}$ *be a certificate for* (Π, T) *and let* $\mathrm{T}' = \cup \mathcal{S}$. *If a structure* $\mathfrak{A}$ *realizes the filtered type instance* (Π, T'), *then it realizes* (Π, T).

Proof. Follows from **??**. □

Note that in general the size of a certificate may be exponential in the size of the type instance. However, polynomial certificates exist:

Lemma 7 (Certificate extraction)**.** *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$, *let* $\mathfrak{A}$ *be a model for* (Π, T) *and let* $(\Pi, \mathrm{T}') = \Pi\mathrm{T}[\mathfrak{A}]$, $\mathrm{T}' \subseteq \mathrm{T}$ *according to* **??** *be the type instance of* $\mathfrak{A}$. *For every* 2*-type* $\tau \in \mathrm{T}'$, *let* $a_\tau \neq b_\tau \in A$ *be two distinct elements realizing* τ: $\mathrm{tp}^{\mathfrak{A}}[a_\tau, b_\tau] = \tau$. *The choice of the elements is made symmetrically, that is* $a_{\tau^{-1}} = b_\tau$ *and* $b_{\tau^{-1}} = a_\tau$ *for every* $\tau \in \mathrm{T}'$. *Let*

$$\mathcal{S} = \left\{ \mathrm{stp}^{\mathfrak{A}}[a_\tau] \;\middle|\; \tau \in \mathrm{T}' \right\}.$$

Then $\mathcal{S}$ *is a certificate for* (Π, T). *Moreover, its size is linearly bounded by* $|\mathrm{T}'|$, *hence also by the size of* (Π, T).

Proof. That $\mathcal{S}$ is nonempty follows from the observation that $\mathfrak{A}$ must have cardinality at least 2. That $\mathcal{S}$ is a set of star-types over (Π, T) follows from Remark 30. We verify the conditions for a certificate:

($\mathcal{S}i$) That $\cup \mathcal{S} = \mathrm{T}'$ is closed under inversion follows from Condition ($\mathcal{T}i$) for the type instance (Π, T').

($\mathcal{S}$K) That $\mathrm{K}(\Pi, \mathrm{T}') = \mathrm{K}(\Pi, \mathrm{T})$ follows from **??**.

($\mathcal{S}\pi$) That every 1-type $\pi \in \Pi$ has a star-type $\sigma \in \mathcal{S}$ such that $\mathrm{tp}_{\boldsymbol{x}}\, \sigma = \pi$ follows from Condition ($\mathcal{R}\pi$) for $\mathfrak{A}$.

($\mathcal{S}\kappa$) That every king-type $\kappa \in \mathrm{K}(\Pi, \mathrm{T})$ has a unique star-type $\sigma \in \mathcal{S}$ such that $\mathrm{tp}_{\boldsymbol{x}}\, \sigma = \kappa$ follows from Condition ($\mathcal{R}\kappa$) for $\mathfrak{A}$.

($\mathcal{S}c$) Let $\pi, \pi' \in \Pi$ be such that they are not the same king type. We will find distinct elements $a \neq b \in A$ such that $\mathrm{tp}^{\mathfrak{A}}[a] = \pi$ and $\mathrm{tp}^{\mathfrak{A}}[b] = \pi'$. Then $\tau = \mathrm{tp}^{\mathfrak{A}}[a, b] \in \mathrm{T}'$ will connect π and π'.

First suppose that π is a peasant type. By Conditions ($\mathcal{R}\Pi$) and ($\mathcal{R}\kappa$) for $\mathfrak{A}$, we must have that π is realized at least 2 times in $\mathfrak{A}$. Let $a_1 \neq a_2 \in A$ be two distinct elements realizing π and let $b \in A$ be an element realizing π'. Then $H = \{a_1, a_2\} \setminus \{b\}$ is nonempty and let $a \in H$. Then $a \neq b$ and $\tau = \mathrm{tp}^{\mathfrak{A}}[a, b] \in \mathrm{T}'$.

Next suppose that $\pi \in \mathrm{K}(\Pi, \mathrm{T})$ is a king type. Then $\pi' \neq \pi$ by assumption, so for any two elements $a, b \in A$ such that $\mathrm{tp}^{\mathfrak{A}}[a] = \pi$ and $\mathrm{tp}^{\mathfrak{A}}[b] = \pi'$ we have that $a \neq b$, so $\tau = \mathrm{tp}^{\mathfrak{A}}[a, b] \in \mathrm{T}'$ is appropriate.

□

Lemma 8 (Certificate expansion)**.** *Let $\mathcal{S}$ be a certificate for the type instance* (Π, T) *over the classified signature* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*. Then* (Π, T) *has a finite model.*

More precisely, let $t \geq |\mathrm{T}|$ be a parameter. TODO: From upstairs!

Proof. We adapt the standard strategy[1] used in the proof of the finite model property for the logic $\mathcal{L}^2$, as presented in [2]. Let $\mathrm{T}' = \cup\mathcal{S}$. We build a model $\mathfrak{A}$ for (Π, T') as follows (clearly, this will also be a model for . The domain A of $\mathfrak{A}$ is the union of the following disjoint sets of elements:

- The singleton set $A^\sigma = \{a^\sigma\}$ for every king star-type $\sigma \in \mathcal{S}$, $\mathrm{tp}_{\boldsymbol{x}}\, \sigma \in \mathrm{K}(\Pi, \mathrm{T}')$. Call the elements a^σ the kings.

- The three disjoint copies of t elements $A^\sigma = A_0^\sigma \cup A_1^\sigma \cup A_2^\sigma$ for every peasant star-type $\sigma \in \mathcal{S}$, $\mathrm{tp}_{\boldsymbol{x}}\, \sigma \in \mathrm{P}(\Pi, \mathrm{T}')$, where $A_i^\sigma = \{a_{i1}^\sigma, a_{i2}^\sigma, \ldots, a_{it}^\sigma\}$ for $i \in \{0, 1, 2\}$. Call the elements a_{ij}^σ the peasants.

Let $\sigma : A \to \mathcal{S}$ denote the intended star-type of the elements: $\sigma(a) = \sigma$ on A^σ. Let $\pi : A \to \Pi$ denote the intended 1-type of the elements: $\pi(a) = \mathrm{tp}_{\boldsymbol{x}}(\sigma(a))$. We proceed to consistently assign 2-types to pairs of distinct elements from the structure on stages.

Realization of kings We first find witnesses for the intended star-types of the kings. Let $\sigma \in \mathcal{S}$, $\kappa = \mathrm{tp}_{\boldsymbol{x}}\, \sigma \in \mathrm{K}(\Pi, \mathrm{T}')$ be a king star-type. Consider the unique element $a = a^\sigma$ having intended star-type σ. For every $\tau \in \sigma$ we will find an element b_τ for the assignment $\mathrm{tp}^{\mathfrak{A}}[a, b_\tau] = \tau$, such that all elements b_τ are distinct and are distinct from a and also $\mathrm{tp}_{\boldsymbol{y}}\, \tau = \pi(b_\tau)$.

[1] with the slight difference that our approach doesn't need *a court*, since the information about it is implicit in the certificate

1. If $\mathrm{tp}_{\boldsymbol{y}}\,\tau = \kappa' \in \mathrm{K}(\Pi, \mathrm{T}')$ is a king type, by ($\sigma 2$) we have $\kappa' \neq \kappa$. By ($\mathcal{S}3$) there is a unique star-type σ' having $\mathrm{tp}_{\boldsymbol{x}}\,\sigma' = \kappa'$ and a unique king $b_\tau = a_{\sigma'} \neq a$ having intended star-type σ'. Note that this assignment is symmetric, that is at the point of considering the type $\tau^{-1} \in \sigma'$ for the king b_τ, we would choose a as the opposite side of the assignment: we claim that $\tau^{-1} \in \sigma'$ and it is the unique 2-type from σ' having $\mathrm{tp}_{\boldsymbol{y}}\,\tau^{-1} = \kappa$. Indeed, $\tau^{-1} \in \mathrm{T}'$, since T' is closed under inversion; By ($\sigma 3$) we have that there is a unique $\tau' \in \sigma'$ connecting κ' and κ. Since σ' is the unique star-type in $\mathcal{S}$ having $\mathrm{tp}_{\boldsymbol{x}}\,\sigma' = \kappa'$, we have that τ' is the unique 2-type from T' connecting κ' and κ. Hence we must have $\tau' = \tau^{-1}$. Note that by ($\sigma 3$) asserting that for the star-type σ there is a unique 2-type $\tau \in \sigma$ that connects it to a king type (distinct from the origin), we have assigned a 2-types between every pair of distinct kings.
2. If $\mathrm{tp}_{\boldsymbol{y}}\,\tau = \pi' \in \mathrm{P}(\Pi, \mathrm{T}')$ is a peasant type (hence $\pi' \neq \kappa$), we simultaneously find distinct elements b_η for all 2-types $\eta \in \sigma$ that are parallel to τ. Let

   $$\mathrm{U} = \{\eta \in \sigma \mid (\eta \parallel \tau)\} = \left\{\eta \in \sigma \;\middle|\; \mathrm{tp}_{\boldsymbol{y}}\,\eta = \pi'\right\}$$

   be the set of all such 2-types η. Since $\mathrm{U} \subseteq \sigma \subseteq \mathrm{T}'$, there are at most t 2-types η in U. Since $\mathrm{T}' = \cup\mathcal{S}$ and T' is closed under inversion, for every $\eta \in \mathrm{U}$ we can find a star-type $\sigma'_\eta \in \mathcal{S}$ containing its inverse: $\eta^{-1} \in \sigma'_\eta$. Since $\mathrm{tp}_{\boldsymbol{x}}\,\sigma'_\eta = \mathrm{tp}_{\boldsymbol{x}}\,\eta^{-1} = \pi'$ is not a king type, there are enough distinct elements $b_\eta \in A_0^{\sigma'_\eta}$ having intended star-type σ'_η. Note that these will be distinct from a, since a is a king. We assign $\mathrm{tp}^{\mathfrak{A}}[a, b_\eta] = \eta$ for all $\eta \in \mathrm{U}$.

Realization of peasants We now find witnesses for the intended star-types of the remaining peasant elements. Let $\sigma \in \mathcal{S}$ be such that $\pi = \mathrm{tp}_{\boldsymbol{x}}\,\sigma \in \mathrm{P}(\Pi, \mathrm{T}')$ is a peasant type. Let $a = a^\sigma_{ij}$, where $i \in \{0, 1, 2\}$ and $j \in [1, t]$ be an arbitrary peasant with intended star-type σ. For every $\tau \in \sigma$ we will find an element b_τ for the assignment $\mathrm{tp}^{\mathfrak{A}}[a, b_\tau] = \tau$, and again we will ensure that all b_τ are distinct and distinct from a and also that $\mathrm{tp}_{\boldsymbol{y}}\,\tau = \pi(b_\tau)$.

1. If $\mathrm{tp}_{\boldsymbol{y}}\,\tau = \kappa'$ is a king type then let c be the element realizing it: $\pi(c) = \kappa'$. We consider two cases. First suppose that $\mathrm{tp}^{\mathfrak{A}}[c, a] = \eta$ has already been assigned during the realization of kings. By that construction we must have that $a = b_\eta$ and so $\sigma = \sigma(a) = \sigma'_\eta$, so that $\eta^{-1} \in \sigma(a)$. Note that $\mathrm{tp}_{\boldsymbol{y}}\,\eta^{-1} = \kappa'$. We claim that $\eta^{-1} = \tau$. This is immediate by Condition ($\sigma 3$), which asserts that there is a unique 2-type $\tau \in \sigma$ having $\mathrm{tp}_{\boldsymbol{y}}\,\tau = \kappa'$ — a king type. Hence in this case the needed 2-types $\tau \in \sigma$ have already been assigned in the opposite direction during the realization of kings.

   Next suppose that $\mathrm{tp}^{\mathfrak{A}}[c, a]$ has not been assigned during the realization of kings. Then just assign $\mathrm{tp}^{\mathfrak{A}}[a, c] = \tau$. Note that this may extend the actual star-type of the king c beyond its intended star-type $\sigma(c)$ by adding the type τ^{-1}, but by Remark 32, this extension is still a star-type. That is, in the end, the structure may realize *more* than the intended star-types, but, importantly, *not less*.

2. If $\mathrm{tp}_{\boldsymbol{y}}\,\tau = \pi'$ is not a king type, we simultaneously find distinct peasants b_η for all $\eta \in \sigma$ that are parallel to τ. Let $\mathrm{U} = \left\{\eta \in \sigma \,\middle|\, \mathrm{tp}_{\boldsymbol{y}}\,\eta = \pi'\right\}$ be the set of all such 2-types η. The key to consistency is to use elements from the *next copy* as witnesses. Let $i' = (i + 1 \bmod 3) \in \{0, 1, 2\}$. Since $\mathrm{U} \subseteq \sigma \subseteq \mathrm{T}'$, there are at most t 2-types η in U. Since $\mathrm{T}' = \cup\mathcal{S}$ and T' is closed under inversion, for every $\eta \in \mathrm{U}$ we can find a star-type $\sigma'_\eta \in \mathcal{S}$ containing its inverse: $\eta^{-1} \in \sigma'_\eta$. Since $\mathrm{tp}_{\boldsymbol{x}}\,\sigma'_\eta = \mathrm{tp}_{\boldsymbol{x}}\,\eta^{-1} = \pi'$ is not a king type, there are enough distinct elements $b_\eta \in A_{i'}^{\sigma'_\eta}$ from the next copy of peasants having intended star-type σ'_η. Since these elements are from the next copy, they are distinct from a. We assign $\mathrm{tp}^{\mathfrak{A}}[a, b_\eta] = \eta$ for all $\eta \in \mathrm{U}$. None of these assignments clash with each other, since they have been made between pairs of elements from consecutive copies.

Completion For any pair of distinct elements $a, b \in A$ that has not yet been assigned a 2-type, assign $\mathrm{tp}^{\mathfrak{A}}[a, b] = \tau$ to arbitrary 2-type τ that connects the 1-types $\pi(a)$ and $\pi(b)$. This is possible because during realization of kings we have assigned a 2-type between every pair of distinct kings and by $(\mathcal{S}4)$.

This structure is a $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$-structure by $(\sigma 4)$ and is a model for (Π, T) by $(\mathcal{S}2)$. □

Proposition 11. *The logic $\mathcal{L}^2$ has the finite model property. The (finite) type realizability problem for $\mathcal{L}^2$ is in* NPTime.

Proof. Let (Π, T) be a type instance for the classified signature $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$. Guess a polynomial certificate for (Π, T). By Lemma 7 and Lemma 8, such a certificate exists iff (Π, T) is realizable. The logic has the finite model property since the model constructed in Lemma 8 is finite. □

Recall Remark 29, stating that a structure is a model for a formula iff it is a model for the type instance consisting of the types consistent with the formula, so as a corollary we get the standard result:

Corollary 3. *The logic $\mathcal{L}^2$ has the finite model property and its (finite) satisfiability problem is in* NExpTime.

6.2 Type realizibility with equivalences

In this section we consider the logic $\mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}}$ featuring $e \geq 2$ equivalence symbols $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_e$ in refinement. Abbreviate the coarsest equivalence symbol $\boldsymbol{e} = \boldsymbol{e}_e$.

As the logic $\mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}}$ relativizes the logic, terms from the previous section relativize. In this setting by convention the message symbols $\boldsymbol{m}_i$ of a classified signature $\langle\Sigma, \bar{\boldsymbol{m}}\rangle$ must be disjoint from the builtin equivalence symbols. The following reductions relativize:

$$\text{(FIN-)SAT-}\mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\mathrm{NPTime}} \text{(FIN-)CL-SAT-}\mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}}$$
$$\text{(FIN-)CL-SAT-}\mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\mathrm{NExpTime}} \text{(FIN-)TP-REALIZ-}\mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}}.$$

We proceed to define new terms. The terminology is loosely based on [4].

Definition 75. *Let Σ be a predicate signature. A 2-type $\tau \in \mathrm{T}[\Sigma]$ is a* galactic type *if $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \in \tau$. Otherwise, if $(\neg \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y})) \in \tau$, the type is a* cosmic type*. The set of galactic 2-types is $\mathrm{T_g}[\Sigma]$. The set of cosmic 2-types is $\mathrm{T_c}[\Sigma]$. These sets partition the set of all 2-types $\mathrm{T}[\Sigma]$.*

Informally, we think of the $\boldsymbol{e}$-classes in a structure as *galaxies*; of the whole structure as the *cosmos*; of the galactic 2-types as characterizing the interactions in the internals of the galaxies, while cosmic 2-types characterize the interactions between different galaxies.

Definition 76. *Let (Π, T) be a type instance over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$. Two 1-types $\pi, \pi' \in \Pi$ are* galactically connectable *(written $\pi \sim_{\mathrm{g}}^{\mathrm{T}} \pi'$), if $\pi = \mathrm{tp}_{\boldsymbol{x}}\, \tau$ and $\pi' = \mathrm{tp}_{\boldsymbol{y}}\, \tau$ for some galactic $\tau \in \mathrm{T} \cap \mathrm{T_g}[\Sigma]$. The types are* cosmically connectable *(written $\pi \sim_{\mathrm{c}}^{\mathrm{T}} \pi'$), if $\pi = \mathrm{tp}_{\boldsymbol{x}}\, \tau$ and $\pi' = \mathrm{tp}_{\boldsymbol{y}}\, \tau$ for some cosmic $\tau \in \mathrm{T} \cap \mathrm{T_c}[\Sigma]$.*

The galactic and cosmic connectability are symmetric since T is closed under inversions. Note that the types π and π' may be both galactically and cosmically connectable. Also note that the general Definition 67 of connectable types is equivalent to:

$$\pi \sim^{\mathrm{T}} \pi' \text{ iff } \pi \sim_{\mathrm{g}}^{\mathrm{T}} \pi' \text{ or } \pi \sim_{\mathrm{c}}^{\mathrm{T}} \pi'.$$

Recall that a 1-type κ is a king type *if $\kappa \not\sim^{\mathrm{T}} \kappa$ and that the set of king types is $\mathrm{K}(\Pi, \mathrm{T})$. A 1-type ν is a* noble type *if $\nu \not\sim_{\mathrm{c}}^{\mathrm{T}} \nu$. The set of noble types is $\mathrm{N}(\Pi, \mathrm{T})$. Note that every king type is a noble type. A 1-type $\pi \in \Pi$ is a peasant type if it is not a noble type. The set of peasant types is $\mathrm{P}(\Pi, \mathrm{T}) = \Pi \setminus \mathrm{N}(\Pi, \mathrm{T})$. Note that this is a different definition of a peasant type from the definition in the case of no builtin equivalence symbols.*

Remark 34. *Let (Π, T) be a type instance over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ and let $\mathrm{T}' \subseteq \mathrm{T}$ be a nonempty subset of 2-types that is closed under inversion. According to Remark 23, (Π, T') is also a type instance over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$. Additionally, $\mathrm{N}(\Pi, \mathrm{T}) \subseteq \mathrm{N}(\Pi, \mathrm{T}')$ and $\mathrm{P}(\Pi, \mathrm{T}') \subseteq \mathrm{P}(\Pi, \mathrm{T}')$. This is the analogue of Remark 23.*

Remark 35. *Let (Π, T) be a type instance over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ and let $\mathfrak{A}$ be a model for (Π, T). Then every noble type $\nu \in \mathrm{N}(\Pi, \mathrm{T})$ is realized (possibly many times) in a unique galaxy of $\mathfrak{A}$. Note that the opposite does not hold: there might be some peasant type that is realized in a unique galaxy. This is a partial analogue of Remark 24.*

Remark 36. *Let $\mathfrak{A}$ be a $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure and let $(\Pi, \mathrm{T}) = \Pi\mathrm{T}[\mathfrak{A}]$ be the type instance of $\mathfrak{A}$. Then $\mathrm{N}(\Pi, \mathrm{T})$ is the set of 1-types that are realized in a unique galaxy of $\mathfrak{A}$ and $\mathrm{P}(\Pi, \mathrm{T})$ is the set of 1-types that are realized in at least 2 galaxies in $\mathfrak{A}$. This is an analogue of Remark 24, but a restricted one.*

Proof. By Remark 35, it suffices to show that any 1-type that is cosmically connectable with itself is realized in at least 2 galaxies: Let $\pi \in \Pi$ be a 1-type and $\tau \in \mathrm{T}$ be a cosmic 2-type such that $\mathrm{tp}_{\boldsymbol{x}}\, \tau = \mathrm{tp}_{\boldsymbol{y}}\, \tau = \pi$. Then $\tau = \mathrm{tp}^{\mathfrak{A}}[a, b]$ for some $a \neq b \in A$. Since τ is cosmic, we must have that a and b are in different galaxies. $\square$

We will define a class of structures, the nobly distinguished structures that will be simler to work with.

Definition 77. *Let $\mathfrak{A}$ be a $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure, let $(\Pi, \mathrm{T}) = \Pi\mathrm{T}[\mathfrak{A}]$, let $E = \boldsymbol{e}^{\mathfrak{A}}$ and let $\mathcal{G} = \mathscr{E}E$ be the set of galaxies of $\mathfrak{A}$. A galaxy $X \in \mathcal{G}$ is a* noble galaxy *if it contains an element $a \in X$ that realizes a noble type: $\mathrm{tp}^{\mathfrak{A}}[a] \in \mathrm{N}(\Pi, \mathrm{T})$. Otherwise, if every element $a \in X$ realizes a peasant type, then the galaxy is a* peasant galaxy*. Note that a noble galaxy may contain elements that realize peasant types. The set of noble galaxies is $\mathcal{G}_{\mathrm{N}}$ and the set of peasant galaxies is $\mathcal{G}_{\mathrm{P}}$.*

Definition 78. *Let $\mathfrak{A}$ be $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure, let $(\Pi, \mathrm{T}) = \Pi\mathrm{T}[\mathfrak{A}]$, let $E = \boldsymbol{e}^{\mathfrak{A}}$ and let $\mathcal{G} = \mathscr{E}E$. The structure $\mathfrak{A}$ is* peasantly united *if whenever $\pi \in \mathrm{P}(\Pi, \mathrm{T})$ is a peasant type that is realized in some peasant galaxy $X \in \mathcal{G}_{\mathrm{P}}$, then π is also realized by some element in some other peasant galaxy $Y \in \mathcal{G}_{\mathrm{P}} \setminus \{X\}$.*

Lemma 9 (Peasant unitedness)**.** *Let $\mathfrak{A}$ be a (finite) $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure and let $(\Pi, \mathrm{T}) = \Pi\mathrm{T}[\mathfrak{A}]$. Then there is a (finite) peasantly united $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure $\mathfrak{A}'$ such that $\Pi\mathrm{T}[\mathfrak{A}'] = (\Pi, \mathrm{T})$.*

Proof. The idea is to copy the peasant galaxies in order to witness the peasant unitedness condition. Let $E = \boldsymbol{e}^{\mathfrak{A}}$ and let $\mathcal{G} = \mathscr{E}E$. We describe $\mathfrak{A}'$ by describing its galaxies $\mathcal{G}'$. The noble galaxies $\mathcal{G}'_{\mathrm{N}}$ of $\mathfrak{A}'$ coincide with the noble galaxies $\mathcal{G}_{\mathrm{N}}$ of $\mathfrak{A}$. The peasant galaxies $\mathcal{G}'_{\mathrm{P}}$ of $\mathfrak{A}'$ consist of two copies X_1, X_2 of each peasant galaxy $X \in \mathcal{G}$. This naturally assigns the 1-type of every element $a \in A'$ and the 2-types between any pair of distinct elements that do not come from the two copies of the same galaxy. Note that already at this point, the partial structure $\mathfrak{A}'$ already satisfies the existential parts eq. (6.2), so it is already a partial $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure. We proceed to complete $\mathfrak{A}'$. Let $X \in \mathcal{G}_{\mathrm{P}}$ be a peasant $\mathfrak{A}$-galaxy and let $a_1 \in X_1$ and $b_2 \in X_2$ be any two elements from the corresponding copies in $\mathfrak{A}'$. Note that $a, b \in X$ and let $\pi = \mathrm{tp}^{\mathfrak{A}'}[a] = \mathrm{tp}^{\mathfrak{A}}[a]$ and $\pi' = \mathrm{tp}^{\mathfrak{A}'}[b] = \mathrm{tp}^{\mathfrak{A}}[b]$. Since X is a peasant galaxy, we must have that $\pi, \pi' \in \mathrm{P}(\Pi, \mathrm{T})$ are peasant 1-types. By Remark 36, π is realized in at least 2 galaxies in $\mathfrak{A}$. So there is some element $c \in A$ realizing π such that $c \notin X$. Now consider the 2-type $\tau = \mathrm{tp}^{\mathfrak{A}}[c, b] \in \mathrm{T}$. Since $b \in X$, we must have that this is a cosmic 2-type connecting π with π'. So we can assign $\mathrm{tp}^{\mathfrak{A}'}[a_1, b_2] = \tau$.

Note that $\mathfrak{A}'$ is peasantly united by construction: any (peasant) type that is realized in some peasant galaxy $X_i \in \mathcal{G}'_{\mathrm{P}}$ is also realized in its copy X_{3-i}. Also note that $\mathfrak{A}'$ is finite iff $\mathfrak{A}$ is finite. $\square$

Definition 79. *A $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure $\mathfrak{A}$ is a* nobly distinguished model *of the type instance (Π, T) if $\mathfrak{A}$ is a model for (Π, T) and the following conditions are satisfied:*

($\mathcal{R}d$) *Let $E = \boldsymbol{e}^{\mathfrak{A}}$. We require that if $a \in A$ has $\mathrm{tp}^{\mathfrak{A}}[a] \in \mathrm{N}(\Pi, \mathrm{T})$, then every $b \in E[a]$ has $\mathrm{tp}^{\mathfrak{A}}[b] \in \mathrm{N}(\Pi, \mathrm{T})$. That is, if some element in a galaxy realizes a noble type, then all elements from that galaxy realize noble types.*

($\mathcal{R}\nu$) *If $\pi \in \Pi$ is realized (possibly many times) in a unique galaxy, then $\pi \in \mathrm{N}(\Pi, \mathrm{T})$ is noble.*

Remark 37. *Let $\mathfrak{A}$ be a nobly distinguished model for* (Π, T). *Then the galaxies of $\mathfrak{A}$ partition into* noble galaxies, *whose elements only realize noble types and* peasant galaxies, *whose elements only realize peasant types.*

Proof. This follows from Condition ($\mathcal{R}d$) for $\mathfrak{A}$. ∎

Remark 38. *Let $\mathfrak{A}$ be a nobly distinguished model for (Π, T) and let $(\Pi, \mathrm{T}') = \Pi\mathrm{T}[\mathfrak{A}]$ be the type instance of $\mathfrak{A}$. Then $\mathrm{N}(\Pi, \mathrm{T}') = \mathrm{N}(\Pi, \mathrm{T})$ and $\mathfrak{A}$ is a nobly distinguished model for (Π, T').*

Proof. By **??** we have that $\mathrm{N}(\Pi, \mathrm{T}) \subseteq \mathrm{N}(\Pi, \mathrm{T}')$. By Condition ($\mathcal{R}\nu$) we have that $\mathrm{N}(\Pi, \mathrm{T}') \subseteq \mathrm{N}(\Pi, \mathrm{T})$. By Remark 26 we have that $\mathfrak{A}$ is a model for (Π, T'). We verify the conditions for $\mathfrak{A}$ to be nobly distinguished for (Π, T'):

($\mathcal{R}d$) follows from Condition ($\mathcal{R}d$) for the nobly distinguished (Π, T)-model $\mathfrak{A}$ and the observation $\mathrm{N}(\Pi, \mathrm{T}') = \mathrm{N}(\Pi, \mathrm{T})$.

($\mathcal{R}\nu$) follows from Condition ($\mathcal{R}\nu$) for the nobly distinguished (Π, T)-model $\mathfrak{A}$ and the observation $\mathrm{N}(\Pi, \mathrm{T}') = \mathrm{N}(\Pi, \mathrm{T})$.

∎

Remark 39. *Let $\mathfrak{A}$ be a nobly distinguished model for (Π, T) and let $(\Pi, \mathrm{T}') = \Pi\mathrm{T}[\mathfrak{A}]$ be the type instance of $\mathfrak{A}$. Then $\mathrm{N}(\Pi, \mathrm{T}') = \mathrm{N}(\Pi, \mathrm{T})$ and so also $\mathrm{P}(\Pi, \mathrm{T}') = \mathrm{P}(\Pi, \mathrm{T})$.*

Proof. This follows from Condition ($\mathcal{R}\nu$) for $\mathfrak{A}$. ∎

Definition 80. *The (finite) nobly distinguished type realizability problem for $\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$ is the following: given a classified signature $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ and a type instance (Π, T) over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$, is there a (finite) $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure $\mathfrak{A}$ that is a nobly distinguished model for (Π, T). Denote the nobly distinguished type realizability problem for the logic $\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$ by* ND-TP-REALIZ-$\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$ *and its finite version by* FIN-ND-TP-REALIZ-$\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$.

We now proceed to reduce the (finite) type realizability problem to the (finite) nobly distinguished type realizability problem.

Definition 81. *Let (Π, T) be a type instance for $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$. For every 1-type $\pi \in \Pi$, let $\boldsymbol{p}^\pi$ be a new unary predicate symbol. Let $\Sigma' = \Sigma + \langle \boldsymbol{p}^\pi \mid \pi \in \Pi \rangle$ be an enrichment of Σ featuring these new symbols. For every $\pi \in \Pi$, define the following set of literals:*

$$\boldsymbol{p}_\pi(\boldsymbol{x}) = \{\boldsymbol{p}^\pi(\boldsymbol{x})\} \cup \left\{ \neg \boldsymbol{p}^{\pi'}(\boldsymbol{x}) \;\middle|\; \pi' \in \Pi \setminus \{\pi\} \right\}.$$

Let $\perp \notin \Pi$ be a special element and define the special set $\boldsymbol{p}_\perp(\boldsymbol{x})$ of literals:

$$\boldsymbol{p}_\perp(\boldsymbol{x}) = \{\neg \boldsymbol{p}^\pi(\boldsymbol{x}) \mid \pi \in \Pi\}.$$

For every $\pi \in \Pi$, $\rho \in \Pi \cup \{\perp\}$, let π_ρ be the following 1-type over Σ':

$$\pi_\rho = \pi \cup \boldsymbol{p}_\rho(\boldsymbol{x}).$$

We refer to π_ρ as the ρ-copy of π. Define $(\Pi_\Pi, \mathrm{T}_\mathrm{T})$ as follows:

$$\Pi_\Pi = \{\pi_\rho \mid \pi \in \Pi, \rho \in \Pi \cup \{\bot\}\}$$
$$\mathrm{T}_\mathrm{T} = \{\tau \cup \boldsymbol{p}_\rho(\boldsymbol{x}) \cup \boldsymbol{p}_{\rho'}(\boldsymbol{x}) \mid \rho, \rho' \in \Pi \cup \{\bot\}\}\,.$$

Clearly, $(\Pi_\Pi, \mathrm{T}_\mathrm{T})$ is a type instance for $\langle \Sigma', \bar{\boldsymbol{m}}\rangle$. Note that the size of $(\Pi_\Pi, \mathrm{T}_\mathrm{T})$ is quadratic, hence polynomially bounded, with respect to the size of the original type instance (Π, T).

Definition 82. *Let (Π, T) be a type instance over $\langle \Sigma, \bar{\boldsymbol{m}}\rangle$ and let $\Sigma' = \Sigma + \langle \boldsymbol{p}^\pi \mid \pi \in \Pi\rangle$. Let $\Pi_\bullet \subseteq \Pi_\Pi$ and $\mathrm{T}_\bullet \subseteq \mathrm{T}_\mathrm{T}$ be such that $(\Pi_\bullet, \mathrm{T}_\bullet)$ is a type instance over $\langle \Sigma', \bar{\boldsymbol{m}}\rangle$. Then $(\Pi_\bullet, \mathrm{T}_\bullet)$ is a* promotion *of (Π, T) if the following conditions are satisfied:*

($\mathcal{P}\pi$) *If $\pi \in \Pi$, then $\pi_\rho \in \Pi_\bullet$ for some (possibly many) $\rho \in \pi \cup \{\bot\}$. That is, $\Pi_\bullet$ contains a copy of every 1-type from Π.*

($\mathcal{P}\nu$) *Let $\pi' \in \Pi_\bullet$. Then $\pi' = \pi_\rho$ for some $\pi, \rho \in \Pi$ iff $\pi' \in \mathrm{N}(\Pi_\bullet, \mathrm{T}_\bullet)$, that is the noble types in the promotion are exactly the ones of the form π_ρ for $\rho \neq \bot$.*

Lemma 10 (Noble distinguishability)**.** *Let (Π, T) be a type instance over $\langle \Sigma, \bar{\boldsymbol{m}}\rangle$ and let $\Sigma' = \Sigma + \langle \boldsymbol{p}^\pi \mid \pi \in \Pi\rangle$. Then (Π, T) has a (finite) model iff there is some promotion $(\Pi_\bullet, \mathrm{T}_\bullet)$ of (Π, T) that has a (finite) nobly distinguished model.*

Proof. First, suppose that $\mathfrak{A}$ is a model for (Π, T) and let $E = \boldsymbol{e}^\mathfrak{A}$. We define a promotion $(\Pi_\bullet, \mathrm{T}_\bullet)$ of (Π, T) and a Σ'-enrichment $\mathfrak{A}'$ that is a nobly distinguished model for $(\Pi_\bullet, \mathrm{T}_\bullet)$. Let $(\Pi, \mathrm{T}') = \Pi\mathrm{T}[\mathfrak{A}]$ be the type instance of $\mathfrak{A}$. Recall that $\mathrm{N}(\Pi, \mathrm{T}) \subseteq \mathrm{N}(\Pi, \mathrm{T}')$ by **??**. For every noble $\nu \in \mathrm{N}(\Pi, \mathrm{T}')$, let $X_\nu \in \mathscr{E}E$ be the unique galaxy realizing it. Note that there might be distinct noble types realized in the same galaxy: $X_\nu = X_{\nu'}$ for $\nu \neq \nu' \in \mathrm{N}(\Pi, \mathrm{T}')$. Let $X_\mathrm{N} = \{X_\nu \mid \nu \in \mathrm{N}(\Pi, \mathrm{T}')\}$ be the set of galaxies realizing some (possibly many) noble type. For every $X \in X_\mathrm{N}$ choose an arbitrary noble type ν realized in it: $X = X_\nu$. Define the enrichment $\mathfrak{A}'$ as follows: for every $a \in A$:

- if $a \in X_\nu$ is an element of some noble galaxy, then let $\mathrm{tp}^{\mathfrak{A}'}[a] = \mathrm{tp}^{\mathfrak{A}}[a]_\nu$
- otherwise, let $\mathrm{tp}^{\mathfrak{A}'}[a] = \mathrm{tp}^{\mathfrak{A}}[a]_\bot$.

Let $(\Pi_\bullet, \mathrm{T}_\bullet) = \Pi\mathrm{T}[\mathfrak{A}']$ be the type instance of $\mathfrak{A}'$, so $\mathfrak{A}'$ is a model for $(\Pi_\bullet, \mathrm{T}_\bullet)$. We first veirfy that $(\Pi_\bullet, \mathrm{T}_\bullet)$ is a promotion of (Π, T):

($\mathcal{P}\pi$) By construction, for every $a \in A$ we have that $\mathrm{tp}^{\mathfrak{A}'}[a] = \mathrm{tp}^{\mathfrak{A}}[a]_\rho$ for some $\rho \in \Pi \cup \{\bot\}$, and by Condition ($\mathcal{R}\Pi$) for $\mathfrak{A}$ as a (Π, T)-model, we have that $\Pi_\bullet$ contains a copy of every 1-type from Π.

($\mathcal{P}\nu$) Let $\pi' \in \Pi_\bullet$. First suppose that $\pi' = \pi_\rho$ for some $\pi, \rho \in \Pi$. By construction, $\rho \in \mathrm{N}(\Pi, \mathrm{T}')$. We claim that $\pi' \in \mathrm{N}(\Pi_\bullet, \mathrm{T}_\bullet)$. Suppose not for the sake of contradiction, that is suppose that $\pi' \sim_\mathrm{c}^{\mathrm{T}_\bullet} \pi'$. Let $\tau \in \mathrm{T}_\bullet$ be a cosmic 2-type connecting π' with itself. By Condition ($\mathcal{R}\mathrm{T}$)for $\mathfrak{A}'$ as a $(\Pi_\bullet, \mathrm{T}_\bullet)$-model, there are some $a \neq b \in A$ such that $\mathrm{tp}^{\mathfrak{A}'}[a,b] = \tau$. But then $\mathrm{tp}^{\mathfrak{A}'}[a] = \mathrm{tp}^{\mathfrak{A}'}[b] = \pi' = \pi_\nu$ and so by construction we must have that $a, b \in X_\nu$ are in the same galaxy, which is a contradiction with τ being cosmic.

Next suppose

By construction, $(\Pi_\bullet, \mathrm{T}_\bullet)$ is a promotion of (Π, T) and $\mathfrak{A}'$ is a model of $(\Pi_\bullet, \mathrm{T}_\bullet)$. We need to check that $\mathfrak{A}'$ is nobly distinguished. We claim that every peasant type $\pi' \in \mathrm{P}(\Pi_\bullet, \mathrm{T}_\bullet)$ has the form $\pi' = \pi_\perp$ for some $\pi \in \Pi$. Suppose not and let $\pi' \in \mathrm{P}(\Pi_\bullet, \mathrm{T}_\bullet)$ be some peasant type having the form $\pi' = \pi_\nu$ for some $\pi \in \Pi$ and $\nu \in \mathrm{N}(\Pi, \mathrm{T}')$. Since π' is a peasant type, there is some cosmic $\tau' \in \mathrm{T}_\bullet$ such that $\mathrm{tp}_{\boldsymbol{x}}\, \tau' = \mathrm{tp}_{\boldsymbol{y}}\, \tau'$. Since $(\Pi_\bullet, \mathrm{T}_\bullet)$ is the type instance of $\mathfrak{A}'$, there are some $a \neq b \in A$ such that $\mathrm{tp}^{\mathfrak{A}'}[a,b] = \tau'$ and since τ' is cosmic, a and b must lie in distinct E-classes. But $\mathrm{tp}^{\mathfrak{A}'}[a] = \mathrm{tp}^{\mathfrak{A}'}[b] = \pi' = \pi_\nu$ and by construction we must have that $a, b \in X_\nu$ — a contradiction. Therefore every peasant type $\pi' \in \mathrm{P}(\Pi_\bullet, \mathrm{T}_\bullet)$ has the form $\pi' = \pi_\perp$ for some $\pi \in \Pi$. So every noble type $\nu' \in \mathrm{N}(\Pi_\bullet, \mathrm{T}_\bullet)$ must have the form $\nu' = \pi_\nu$ for some $\pi \in \Pi$ and $\nu \in \mathrm{N}(\Pi, \mathrm{T}')$. Now by construction it is immediate that $\mathfrak{A}'$ is nobly distinguished.

Next, suppose that $(\Pi_\bullet, \mathrm{T}_\bullet)$ is a promotion of (Π, T) and that $\mathfrak{A}'$ is a model for $(\Pi_\bullet, \mathrm{T}_\bullet)$. Then the reduct of $\mathfrak{A}'$ to a Σ-structure is a model for (Π, T). $\square$

Denote the type realizability problem for $\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$ restricted to the class of nobly distinguished structures by ND-TP-REALIZ-$\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$ and the finite version of that problem by FIN-ND-TP-REALIZ-$\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$. The noble distinguishability lemma shows that the class of nobly distinguished models is a nondeterministic polynomial time reduction class:

$$\text{(FIN-)TP-REALIZ-}\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}} \leq_{\mathrm{m}}^{\mathrm{NPT{\small IME}}} \text{(FIN-)ND-TP-REALIZ-}\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}.$$

Definition 83. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*. A* cosmic spectrum $\sigma \subseteq (\mathrm{T} \cap \mathrm{T_c}[\Sigma])$ *over* (Π, T) *is a nonempty set of cosmic* 2*-types satisfying the following conditions:*

($\sigma\boldsymbol{x}$) *An* internal type *for* σ *is any* 1*-type* $\pi \in (\mathrm{tp}_{\boldsymbol{x}} \restriction \sigma)$*. Denote the set of internal types for* σ *by* $\mathrm{Tp}_{\boldsymbol{x}}\, \sigma = (\mathrm{tp}_{\boldsymbol{x}} \restriction \sigma)$*. We require that either* $\mathrm{Tp}_{\boldsymbol{x}}\, \sigma \subseteq \mathrm{N}(\Pi, \mathrm{T})$ *(in that case we refer to* σ *as a* noble cosmic spectrum*), or* $\mathrm{Tp}_{\boldsymbol{x}}\, \sigma \cap \mathrm{N}(\Pi, \mathrm{T}) = \varnothing$*, equivalently* $\mathrm{Tp}_{\boldsymbol{x}}\, \sigma \subseteq \mathrm{P}(\Pi, \mathrm{T})$ *(and we refer to* σ *as a* peasant cosmic spectrum*). This is the analogue of Condition* ($\sigma\boldsymbol{x}$) *for star-types.*

($\sigma\nu\boldsymbol{x}$) *If* σ *is a noble cosmic spectrum and* $\nu \in \mathrm{Tp}_{\boldsymbol{x}}\, \sigma$ *is any (noble) internal type, then no* $\tau \in \sigma$ *has* $\mathrm{tp}_{\boldsymbol{y}}\, \tau = \nu$*. This is the analogue of Condition* ($\sigma\kappa\boldsymbol{x}$) *for star-types.*

($\sigma\nu\boldsymbol{y}$) *If* $\nu \in \mathrm{N}(\Pi, \mathrm{T}) \setminus \mathrm{Tp}_{\boldsymbol{x}}\, \sigma$ *is any noble type that is not internal and* $\pi \in \mathrm{Tp}_{\boldsymbol{x}}\, \sigma$ *is any internal type, then some (possibly many)* $\tau \in \sigma$ *has* $\mathrm{tp}_{\boldsymbol{x}}\, \tau = \pi$ *and* $\mathrm{tp}_{\boldsymbol{y}}\, \tau = \nu$*. This is the analogue of Condition* ($\sigma\kappa\boldsymbol{y}$) *for star-types.*

$\cancel{(\sigma\boldsymbol{m})}$ *This is not a condition. The notion of local consistency will be the analogue of Condition* ($\sigma\boldsymbol{m}$) *for star-types.*

Remark 40. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *and let* $\mathfrak{A}$ *be a nobly distinguished model for* (Π, T)*. Then the noble types* $\nu \in \mathrm{N}(\Pi, \mathrm{T})$ *are exactly the* 1*-types that are realized in a unique galaxy in* $\mathfrak{A}$*.*

Definition 84. *Let* (Π, T) *be a type instance over* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*. Let* $\mathfrak{A}$ *be a nobly distinguished model for* (Π, T) *and suppose that* $E = \boldsymbol{e}^{\mathfrak{A}} \neq (A \times A)$ *is not full on* A*, equivalently that* E *has at least* 2 *equivalence classes. The* cosmic spectrum $\sigma = \mathrm{csp}^{\mathfrak{A}}[X]$ *of an* E*-class* $X \in \mathscr{E}E$ *is the set of cosmic types realized at* X*:*

$$\mathrm{csp}^{\mathfrak{A}}[X] = \left\{ \mathrm{tp}^{\mathfrak{A}}[a, b] \in \mathrm{T_c}[\Sigma] \;\middle|\; a \in X, b \in A \setminus \{a\} \right\} = \left\{ \mathrm{tp}^{\mathfrak{A}}[a, b] \;\middle|\; a \in X, b \in A \setminus X \right\}.$$

Remark 41. *Then* σ *is indeed a cosmic spectrum over* (Π, T)*.*

Proof. That σ is nonempty follows from the assumption that E is not full on A. We verify the conditions for a cosmic spectrum:

($\sigma\boldsymbol{x}$) Let $\sigma \in \mathcal{S}$, so $\sigma = \mathrm{csp}^{\mathfrak{A}}[X]$ for some galaxy $X \in \mathscr{E}E$. Since $\mathfrak{A}$ is nobly distinguished, either X is a noble galaxy in which case it only realizes noble types, or is a peasant galaxy in which case it only realizes peasant types.

($\sigma\nu\boldsymbol{x}$) follows from Remark 40.

($\sigma\nu\boldsymbol{y}$)

□

We like to think about a cosmic spectrum of a galaxy as the *reason why* that galaxy is possible. For this we need to define the notion of a *locally consistent* cosmic spectrum. For this we extract a type instance $(\Pi^\sigma, \mathrm{T}^\sigma)$ over the *simpler logic* $\mathcal{L}^2(\mathrm{e}-1)\mathrm{E}_{\mathsf{refine}}$ out of the cosmic spectrum σ:

Definition 85. *Let* (Π, T) *be a type instance over the* $\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$*-classified signature* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$*. Let* $\sigma \subseteq (\mathrm{T} \cap \mathrm{T_c}[\Sigma])$ *be a cosmic spectrum over* (Π, T)*. Let* $\boldsymbol{in}$ *(intended to label the inside of the galaxy) and* $\boldsymbol{bh}$ *(intended to label some* black hole *outside the galaxy) be two new unary predicate symbols. Define the following sets of literals:*

$$\begin{aligned} \mathcal{I}(\boldsymbol{x}) &= \{\boldsymbol{in}(\boldsymbol{x}), \neg \boldsymbol{bh}(\boldsymbol{x})\} \\ \mathcal{O}(\boldsymbol{x}) &= \{\neg \boldsymbol{in}(\boldsymbol{x}), \neg \boldsymbol{bh}(\boldsymbol{x})\} \\ \mathcal{B}(\boldsymbol{x}) &= \{\neg \boldsymbol{in}(\boldsymbol{x}), \boldsymbol{bh}(\boldsymbol{x})\}. \end{aligned}$$

For a 1*-type* $\pi \in \Pi$ *or a* 2*-type* $\tau \in \mathrm{T}$*, denote by* π^- *and* τ^- *the reducts of* π *and* τ *to the language* $\Sigma - \{\boldsymbol{e}\}$*. That is,* $\pi^- \subset \pi$ *and* $\tau^- \subset \tau$ *consist of those literals that do not feature the predicate symbol* $\boldsymbol{e}$*.*

Define the spectral type instance $(\Pi^\sigma, \mathrm{T}^\sigma)$ *of the cosmic spectrum* σ *as a type instance over the* $\mathcal{L}^2(\mathrm{e}-1)\mathrm{E}_{\mathsf{refine}}$*-signature* $\langle \Sigma - \langle \boldsymbol{e} \rangle + \langle \boldsymbol{in}, \boldsymbol{bh} \rangle, \bar{\boldsymbol{m}} \rangle$[2] *as follows:*

[2]Note that here we use the condition that no builtin equivalence symbol $\boldsymbol{e}$ occurs as a message symbol.

($\mathcal{I}$) *For every* $\pi \in (\mathrm{tp}_{\boldsymbol{x}} \restriction \sigma)$ *inside* σ, *add the* 1*-type* $(\pi^- \cup \mathcal{I}(\boldsymbol{x}))$ *to* Π^σ.

($\mathcal{O}$) *For every* $\pi \in (\mathrm{tp}_{\boldsymbol{y}} \restriction \sigma)$ *outside* σ, *add the* 1*-type* $(\pi^- \cup \mathcal{O}(\boldsymbol{x}))$ *to* Π^σ. *Note that some* 1*-types might occur both inside and outside of* σ.

($\mathcal{B}$) *Let* β *be an arbitrary* 1*-type extending* $\mathcal{B}(\boldsymbol{x})$. *Call* β *the* black hole type *and add it to* Π^σ.

($\mathcal{II}$) *For every galactic* $\tau \in (\mathrm{T} \cap \mathrm{T}_{\mathrm{g}}[\Sigma])$, *add the* 2*-type* $(\tau^- \cup \mathcal{I}(\boldsymbol{x}) \cup \mathcal{I}(\boldsymbol{y}))$ *to* T^σ.

($\mathcal{IO}$) *For every (cosmic)* $\tau \in \sigma$, *add the* 2*-type* $(\tau^- \cup \mathcal{I}(\boldsymbol{x}) \cup \mathcal{O}(\boldsymbol{y}))$ *and its inverse to* T^σ.

($\mathcal{OO}$) *For every* 2*-type* $\tau \in \mathrm{T}$, *add the* 2*-type* $(\tau^- \cup \mathcal{O}(\boldsymbol{x}) \cup \mathcal{O}(\boldsymbol{y}))$ *to* T^σ.

($\mathcal{IB}$) *For every* 1*-type* $\pi' \in \Pi^\sigma$, $\mathcal{I}(\boldsymbol{x}) \subseteq \pi'$ *that comes from the inside, let* τ *be an arbitrary* 2*-type connecting* π' *and the black hole type* β *that witnesses no message symbols for* π' *and everything for* β, *that is* $\mathrm{tp}_{\boldsymbol{x}}\,\tau = \pi'$, $\mathrm{tp}_{\boldsymbol{y}}\,\tau = \beta$, $(\neg \boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y})) \in \tau$ *and* $\boldsymbol{m}(\boldsymbol{y}, \boldsymbol{x}) \in \tau$ *for every* $\boldsymbol{m} \in \bar{\boldsymbol{m}}$. *Add* τ *and its inverse to* T^σ.

($\mathcal{OB}$) *For every* 1*-type* $\pi' \in \Pi^\sigma$, $\mathcal{O}(\boldsymbol{x}) \subseteq \pi'$ *that comes from the outside, let* τ *be an arbitrary* 2*-type connecting* π' *and the black hole type* β *that witnesses every message symbol for* π' *and for* β, *that is* $\mathrm{tp}_{\boldsymbol{x}}\,\tau = \pi'$, $\mathrm{tp}_{\boldsymbol{y}}\,\tau = \beta$, $\boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y}) \in \tau$ *and* $\boldsymbol{m}(\boldsymbol{y}, \boldsymbol{x}) \in \tau$ *for every* $\boldsymbol{m} \in \bar{\boldsymbol{m}}$. *Add* τ *and its inverse to* T^σ.

This completes the description of the spectral type instance $(\Pi^\sigma, \mathrm{T}^\sigma)$.

The cosmic spectrum σ *is* locally consistent *if its spectral type instance* $(\Pi^\sigma, \mathrm{T}^\sigma)$ *over the* $\mathcal{L}^2(e-1)\mathrm{E}_{\mathsf{refine}}$*-classified signature* $\langle \Sigma - \langle \boldsymbol{e} \rangle + \langle \boldsymbol{in}, \boldsymbol{bh} \rangle, \bar{\boldsymbol{m}} \rangle$ *is realizable.*

Remark 42. *Let* (Π, T) *be a type instance over the* $\mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}}$*-classified signature* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$. *Let* $\mathfrak{A}$ *be a model for* (Π, T) *such that* $E = \boldsymbol{e}^{\mathfrak{A}}$ *is not full on* A *and let* $X \in \mathscr{E}E$ *be any galaxy. Then the cosmic spectrum* $\sigma = \mathrm{csp}^{\mathfrak{A}}[X]$ *is a locally consistent cosmic spectrum over* (Π, T).

Proof. Let $\Sigma' = \Sigma - \langle \boldsymbol{e} \rangle + \langle \boldsymbol{in}, \boldsymbol{bh} \rangle$. We build a Σ'-structure $\mathfrak{A}'$ based on $\mathfrak{A}$ that realizes the spectral type instance $(\Pi^\sigma, \mathrm{T}^\sigma)$. The domain of $\mathfrak{A}'$ is $A' = A \cup \{o\}$, where o is the new *black hole* element. The 1-type of every element $a \in A'$ is defined as follows:

($\mathcal{I}$) If $a \in X$ is inside the galaxy, then let $\mathrm{tp}^{\mathfrak{A}'}[a] = \mathrm{tp}^{\mathfrak{A}}[a]^- \cup \mathcal{I}(\boldsymbol{x})$.

($\mathcal{O}$) If $a \in A \setminus X$ is outside the galaxy, then let $\mathrm{tp}^{\mathfrak{A}'}[a] = \mathrm{tp}^{\mathfrak{A}}[a]^- \cup \mathcal{O}(\boldsymbol{x})$.

($\mathcal{B}$) If $a = o$ is the black hole, then let $\mathrm{tp}^{\mathfrak{A}'}[a] = \beta$.

The 2-type between every pair of distinct elements $a \neq b \in A'$ is defined as follows:

($\mathcal{II}$) If both $a, b \in X$ are inside, then let $\mathrm{tp}^{\mathfrak{A}'}[a, b] = \mathrm{tp}^{\mathfrak{A}}[a, b]^- \cup \mathcal{I}(\boldsymbol{x}) \cup \mathcal{I}(\boldsymbol{y})$. Note that this assignment is symmetric, that is we would assign $\mathrm{tp}^{\mathfrak{A}'}[a, b]^{-1}$ to $\mathrm{tp}^{\mathfrak{A}'}[b, a]$.

($\mathcal{IO}$) If $a \in X$ is inside and $b \in A \setminus X$ is outside, then let $\mathrm{tp}^{\mathfrak{A}'}[a,b] = \mathrm{tp}^{\mathfrak{A}}[a,b]^- \cup \mathcal{I}(\boldsymbol{x}) \cup \mathcal{O}(\boldsymbol{y})$. Note that this case covers the symmetric assignments where $a \in A \setminus X$ and $b \in X$.

($\mathcal{OO}$) If both $a, b \in A \setminus X$ are outside, then let $\mathrm{tp}^{\mathfrak{A}'}[a,b] = \mathrm{tp}^{\mathfrak{A}}[a,b]^- \cup \mathcal{O}(\boldsymbol{x}) \cup \mathcal{O}(\boldsymbol{y})$. Note that this assignment is symmetric.

($\mathcal{IB}$) If $a \in X$ is inside and $b = o$ is the black hole, let $\tau \in \mathrm{T}^\sigma$ be the unique 2-type connecting $\mathrm{tp}^{\mathfrak{A}'}[a]$ and β and assign $\mathrm{tp}^{\mathfrak{A}'}[a,b] = \tau$.

($\mathcal{OB}$) If $a \in A \setminus X$ is outside and $b = o$ is the black hole, let $\tau \in \mathrm{T}^\sigma$ be the unique 2-type connecting $\mathrm{tp}^{\mathfrak{A}'}[a]$ and β and assign $\mathrm{tp}^{\mathfrak{A}'}[a,b] = \tau$.

We have to verify that $\mathfrak{A}'$ is indeed a $\langle \Sigma', \bar{\boldsymbol{m}} \rangle$-structure, that is that the star-type of every element of $a \in A'$ contains $\boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y})$ for every message symbol $\boldsymbol{m} \in \bar{\boldsymbol{m}}$:

($\mathcal{I}$) If $a \in X$ is inside, then every $\boldsymbol{m} \in \bar{\boldsymbol{m}}$ that is included in $\mathrm{stp}^{\mathfrak{A}}[a]$ is also included in $\mathrm{stp}^{\mathfrak{A}'}[a]$, since the 2-type τ^- contains $\boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y})$ iff τ contains $\boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y})$.

($\mathcal{O}$) If $a \in A \setminus X$ is outside, then we just need to recall that at stage ($\mathcal{OB}$), we have used a 2-type $\mathrm{tp}^{\mathfrak{A}'}[a,o] = \tau$ that witnesses every message symbol for a: $\boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y}) \in \tau$ for every $\boldsymbol{m} \in \bar{\boldsymbol{m}}$.

($\mathcal{B}$) If $a = o$ is the black hole, then we just need to recall that X is nonempty and at stage ($\mathcal{IB}$) for every $a \in X$ we have used a 2-type $\tau = \mathrm{tp}^{\mathfrak{A}'}[a,o]$ that witnesses every message symbol for o: $\boldsymbol{m}(\boldsymbol{y}, \boldsymbol{x}) \in \tau$ for every $\boldsymbol{m} \in \bar{\boldsymbol{m}}$.

Since $\mathfrak{A}$ was a model for (Π, T), it is clear that $\mathfrak{A}'$ is a model for $(\Pi^\sigma, \mathrm{T}^\sigma)$. $\square$

Remark 43. *Relation between kings in above.*

Definition 86. *A certificate $\mathcal{S}$ for the type instance (Π, T) over the $\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$-classified signature $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ is a nonempty set of cosmic spectrums over (Π, T) satisfying the following conditions:*

($\mathcal{S}^\star i$) *If $\tau \in \sigma$ for some $\sigma \in \mathcal{S}$, then $\tau^{-1} \in \sigma'$ for some $\sigma' \in \mathcal{S}$, that is there are cosmic spectrums for the endpoints of every (cosmic) 2-type used in the certificate. Equivalently, $\mathrm{T}' = \cup\mathcal{S}$ is closed under inversion. Note that then (Π, T') is also a type instance over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ and also that every cosmic spectrum $\sigma \in \mathcal{S}$ is a cosmic spectum over (Π, T). Call (Π, T') the* filtered type instance *of $\mathcal{S}$.*

($\mathcal{S}^\star$N) *We require that $\mathrm{N}(\Pi, \mathrm{T}') = \mathrm{N}(\Pi, \mathrm{T})$, that is the notion of a noble type with respect to the filtered type instance coincides with the notion of a noble type with respect to the original type instance.*

($\mathcal{S}^\star\pi$) *If $\pi \in \Pi$, then some (possibly many) $\sigma \in \mathcal{S}$ has $\pi \in \mathrm{Tp}_{\boldsymbol{x}}\,\sigma$.*

($\mathcal{S}^\star\nu$) *If $\nu \in \mathrm{N}(\Pi, \mathrm{T})$, then a unique $\sigma \in \mathcal{S}$ has $\nu \in \mathrm{Tp}_{\boldsymbol{x}}\,\sigma$. Note that the existence is already implied by ($\mathcal{S}^\star\pi$) .*

($\mathcal{S}^\star c$) *Let $\pi, \pi' \in \Pi$. If it is not the case that both π and π' are noble and are internal for the same cosmic spectrum $\sigma \in \mathcal{S}$, then some cosmic $\tau \in \mathrm{T}'$ has $\mathrm{tp}_{\boldsymbol{x}}\, \tau = \pi$ and $\mathrm{tp}_{\boldsymbol{y}}\, \tau = \pi'$, that is if π and π' don't come from the same noble galaxy, then they are cosmically connectable.*

Again, in general the size of a certificate is only exponentially bounded in terms of the size of the type instance. But again, polynomial certificates exist:

Lemma 11 (Certificate extraction). *Let $\mathfrak{A}$ be a nobly distinguished model for the type instance (Π, T) such that $E = \boldsymbol{e}^{\mathfrak{A}}$ is not full on A. Let (Π, T') be the type instance of $\mathfrak{A}$. For every cosmic $\tau \in \mathrm{T}'$, let $a_\tau \neq b_\tau \in A$ be two distinct elements realizing τ: $\mathrm{tp}^{\mathfrak{A}}[a_\tau, b_\tau] = \tau$. Note that necessarily the elements a_τ and b_τ are in different galaxies. The choice is made symmetrically, that is $a_{\tau^{-1}} = b_\tau$ and $b_\tau = a_{\tau^{-1}}$. Let*

$$\mathcal{S} = \left\{ \mathrm{csp}^{\mathfrak{A}}[E[a_\tau]] \;\middle|\; \tau \in \mathrm{T}' \cap \mathrm{T_c}[\Sigma] \right\}.$$

Then $\mathcal{S}$ is a certificate for (Π, T). Moreover, its size is linearly bounded by $|\mathrm{T}'|$, hence also by the size of (Π, T).

Proof. We check the conditions for a certificate:

($\mathcal{S}1'$) By construction, (Π, T') is a type instance and every $\sigma \in \mathcal{S}$ is a locally consistent cosmic spectrum over (Π, T').

($\mathcal{S}2'$) Every 1-type $\pi \in \Pi$ is witnessed since $\mathfrak{A}$ is a model for (Π, T').

($\mathcal{S}3'$) Every noble 1-type $\nu \in \mathrm{N}(\Pi, \mathrm{T}')$ is witnessed uniquely, since the noble 1-types are exactly those that are realized at a single galaxy of $\mathfrak{A}$.

($\mathcal{S}4'$) Suppose that $\pi, \pi' \in \Pi$ is a pair of 1-types that are not cosmically connectable. Consider the elements realizing these 1-types in $\mathfrak{A}$: $\pi^{\mathfrak{A}}, \pi'^{\mathfrak{A}} \subseteq A$. These sets are nonempty, since $\mathfrak{A}$ is a model for (Π, T'). Next we claim that both $\pi^{\mathfrak{A}}$ and $\pi'^{\mathfrak{A}}$ are included in the same galaxy of $\mathfrak{A}$. Suppose not and let $a \in \pi^{\mathfrak{A}}$ and $b \in \pi'^{\mathfrak{A}}$ be elements realizing the 1-types from different galaxies. Then $\mathrm{tp}^{\mathfrak{A}}[a, b] \in \mathrm{T}'$ is a cosmic type connecting π and π' — a contradiction. Therefore $\pi^{\mathfrak{A}}, \pi'^{\mathfrak{A}} \subseteq X$ for some $X \in \mathscr{E}E$. Since $\mathfrak{A}$ is nobly distinguished and (Π, T') is the type instance of $\mathfrak{A}$, the 1-types π and π' must be noble over (Π, T'). By ($\mathcal{S}3'$), they are witnessed uniquely in the certificate by the cosmic spectrum $\sigma = \mathrm{csp}^{\mathfrak{A}}[X]$.

□

Lemma 12 (King lemma). *Let $\mathcal{S}$ be a certificate for (Π, T), let $\mathrm{T}' = \cup\mathcal{S}$.*

(Kings are the same since nobles are the same)

Let $\kappa \in \mathrm{K}(\Pi, \mathrm{T})$, $\sigma \in \mathcal{S}$ be the unique such that $\kappa \in \mathrm{Tp}_{\boldsymbol{x}}\, \sigma$ and let $\pi \in \Pi \setminus \mathrm{Tp}_{\boldsymbol{x}}\, \sigma$. Then κ and π are uniquely connecable – NOPE!

Lemma 13 (Certificate expansion)**.** *Let $\mathcal{S}$ be a certificate for the type instance* (Π, T) *over the classified signature* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ *over* $\mathcal{L}^2\mathrm{eE}_{\mathsf{refine}}$*. Then* (Π, T) *has a finite nobly distinguished model.*

More precisely, let $\mathrm{T}' = \cup\mathcal{S} \subseteq \mathrm{T}$ *and let* $t \geq |\mathrm{T}'|$ *be a parameter. Then the filtered type instance* (Π, T') *has a finite nobly distinguished model* $\mathfrak{A}$ *in which every king type* $\kappa \in \mathrm{K}(\Pi, \mathrm{T}')$ *is realized once and every peasant type* $\pi \in \mathrm{P}(\Pi, \mathrm{T}')$ *is realized at least* t *times.*

Proof. Let $\Sigma' = \Sigma - \langle \boldsymbol{e} \rangle + \langle \boldsymbol{in}, \boldsymbol{bh} \rangle$. For every locally consistent cosmic spectrum $\sigma \in \mathcal{S}$, let $\mathfrak{A}^\sigma$ be a finite $\langle \Sigma', \bar{\boldsymbol{m}} \rangle$-structure that is a model for the spectral type instance $(\Pi^\sigma, \mathrm{T}^\sigma)$. Assume that the domains of the structures $\mathfrak{A}^\sigma$ are disjoint. Let $X^\sigma = \boldsymbol{in}^{\mathfrak{A}^\sigma}$ be the set of inside elements in the model $\mathfrak{A}^\sigma$. We describe the domain A of $\mathfrak{A}$ and the interpretation $E = a^{\mathfrak{A}}$ of the coarsest builtin equivalence symbol by specifying the galaxies $X \in \mathscr{E}E$ of $\mathfrak{A}$:

- For every noble $\sigma \in \mathcal{S}$, there is a unique galaxy X^σ that has intended cosmic spectrum σ. That is $\mathcal{X}^\sigma = \{X^\sigma\}$. Call the galaxies X^σ the noble galaxies.
- For every peasant $\sigma \in \mathcal{S}$, there are three disjoint copies of t galaxies: $\mathcal{X}^\sigma = \mathcal{X}^\sigma_0 \cup \mathcal{X}^\sigma_1 \cup \mathcal{X}^\sigma_2$, $\mathcal{X}^\sigma_i = \{X^\sigma_{i1}, X^\sigma_{i2}, \dots, X^\sigma_{it}\}$ for $i \in \{0, 1, 2\}$. Call the galaxies X^σ_{ij} the peasant galaxies.

Let $\sigma : \mathscr{E}E \to \mathcal{S}$ denote the intended cosmic spectrum of the galaxies: $\sigma(X) = \sigma$ on $\mathcal{X}^\sigma$. Assign the 1-types and the galactic 2-types to the corresponding models for the spectral instances:

- $\mathrm{tp}^{\mathfrak{A}}[a] = \mathrm{tp}^{\mathfrak{A}^\sigma}[a] \cup \{\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{x})\}$ and
- $\mathrm{tp}^{\mathfrak{A}}[a, b] = \mathrm{tp}^{\mathfrak{A}^\sigma}[a, b] \cup \{\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{x}), \boldsymbol{e}(\boldsymbol{y}, \boldsymbol{y}), \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{e}(\boldsymbol{y}, \boldsymbol{x})\}$ on $X \in \mathcal{X}^\sigma$.

It remains to assign cosmic types consistently between pairs of elements in distinct galaxies.

Realization of kings TODO: lemma Let $\kappa \in \mathrm{K}(\Pi, \mathrm{T}')$. There is a unique (noble) cosmic spectrum $\sigma \in \mathcal{S}$ such that $\kappa \in (\mathrm{tp}_{\boldsymbol{x}} \restriction \sigma)$. Let $\kappa' \in \mathrm{K}(\Pi, \mathrm{T}') \setminus \{\kappa\}$. Then there is a unique (cosmic) 2-type $\tau \in \cup\mathcal{S}$ connecting κ and κ'.

Realization of noble galaxies We first find witnesses for the intended cosmic spectrums of noble galaxies. Let $\sigma \in \mathcal{S}$, $(\mathrm{tp}_{\boldsymbol{x}} \restriction \sigma) \subseteq \mathrm{N}(\Pi, \mathrm{T}')$ be any noble cosmic spectrum. Its corresponding galaxy is X^σ. Let $\tau \in \sigma$ be any (cosmic) type that needs to be realized. Since σ is noble, we have that $\nu = \mathrm{tp}_{\boldsymbol{x}}\, \tau \in \mathrm{N}(\Pi, \mathrm{T}')$ is noble. Let $Y = X^\sigma \cap \nu^{\mathfrak{A}}$ be the set of elements of $\mathfrak{A}$ that have intended 1-type ν. Let $\sigma' \in \mathcal{S}$ be a cosmic spectrum containing τ^{-1}. Since σ is noble we must have that $\sigma' \neq \sigma$. We consider two cases:

- If $\nu' = \mathrm{tp}_{\boldsymbol{y}}\, \tau \in \mathrm{N}(\Pi, \mathrm{T}')$ is a noble type, then $\sigma' \in \mathcal{S}$ is the unique (noble) cosmic spectrum such that $\nu' \in (\mathrm{tp}_{\boldsymbol{x}} \restriction \sigma)$, so $X^{\sigma'}$ is the unique galaxy having intended cosmic spectrum σ'. Let $Z = X^{\sigma'} \cap \nu'^{\mathfrak{A}}$ be the set of elements of $\mathfrak{A}$ having intended

$\square$

6.3 TODO

Let $\pi, \pi' \in \Pi$ be any unordered pair of 1-types. Let $\Delta_\pi, \Delta_{\pi'}$ be the cosmic spectrums containing π and π' inside. We are going to assign 2-types consistently between every pair or distinct elements $a \neq b \in A$ from distinct galaxies, in such a way that the (π, π')-reduced star-type of them is realized.

($\mathcal{KK}$) Suppose that $\pi = \kappa \in \mathrm{K}(\Pi, \mathrm{T}')$ and $\pi' = \kappa' \in \mathrm{K}(\Pi, \mathrm{T}')$ are king types. Suppose that $\kappa \neq \kappa'$.

($\mathcal{K}$) If $\kappa \in \mathrm{K}(\Pi, \mathrm{T}')$ is a king type, then there is a unique element $a \in A$ having $\pi(a) = \kappa$.

($\mathcal{N}$) If $\nu \in \mathrm{N}(\Pi, \mathrm{T}')$ is any noble type, then there is a unique (noble) cosmic spectrum σ containing it inside; that is there is a unique galaxy X that contains all the elements having intended 1-type ν.

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