

Satisfiability with Agreement and Counting

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Glossary

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| $ A $ the cardinality of A . 1 | \mathbb{S}_n the set of permutations of $[1, n]$. 2 |
| $\wp A$ the powerset of A . 1 | Ω an alphabet. 2 |
| $\wp^+ A$ the set of nonempty subsets of A . 1 | $w = w_1 w_2 \dots w_n$ a word. 2 |
| $\wp^\kappa A$ the set of subsets of A of cardinality κ . 1 | Ω^* the set of words over Ω . 2 |
| $A \times B$ the cartesian product of A and B . 1 | Ω^+ the set of nonempty words over Ω . 2 |
| $\text{dom } R$ the domain of R . 1 | Ω^n the set of words of length n over Ω . 2 |
| $\text{ran } R$ the range of R . 1 | \mathbb{B} the bits. 2 |
| R^{-1} the inverse of R . 1 | \mathbb{B}^+ the bitstrings. 2 |
| $R \upharpoonright S$ the restriction of R to S . 1 | $\ n\ $ the bitsize of n . 2 |
| $R[a]$ the R -successors of a . 1 | \bar{n} the binary encoding of n . 2 |
| $S \circ R$ the composition of S and R . 1 | \underline{b} the number encoded by b . 2 |
| id_A the identity on A . 1 | N_t the largest t -bit number. 2 |
| $f : A \rightarrow B$ a total function from A to B . 1 | \mathbb{B}_t the t -bit numbers. 2 |
| $f : A \hookrightarrow B$ an injective function from A into B . 1 | $\Omega_{\mathcal{C}}$ the symbol alphabet. 2 |
| $f : A \twoheadrightarrow B$ a surjective function from A onto B . 1 | \mathcal{V} the variable symbols. 2 |
| $f : A \leftrightarrow B$ a bijective function between A and B . 1 | \mathbf{x} the first variable symbol. 2 |
| $f : A \rightsquigarrow B$ a partial function from A to B . 1 | \mathbf{y} the second variable symbol. 2 |
| $f(a) \simeq b$ f is defined at a with value b . 1 | \mathbf{z} the third variable symbol. 2 |
| $f(a) \simeq \perp$ f is not defined at a . 1 | Σ a predicate signature. 3 |
| $\ A\ $ the length of A . 1 | \mathbf{p}_i a predicate symbol. 3 |
| $\langle a, b, c \rangle$ a sequence. 1 | ar \mathbf{p}_i the arity of \mathbf{p}_i . 3 |
| ε the empty sequence. 1 | $\text{At}[\Sigma]$ the atomic formulas over Σ . 3 |
| $A + B$ the concatenation of A and B . 1 | $\text{Lit}[\Sigma]$ the literals over Σ . 3 |
| $A - B$ A without the elements of B . 1 | $\mathcal{C}[\Sigma]$ the first-order formulas with counting quantifiers over Σ . 3 |
| \mathbb{N} the natural numbers. 1 | $\mathcal{L}[\Sigma]$ the first-order formulas over Σ . 3 |
| \mathbb{N}^+ the positive natural numbers. 1 | $\text{vr } \varphi$ the variables occurring φ . 3 |
| $[n, m]$ the discrete interval between n and m . 1 | $\text{fvr } \varphi$ the variables freely occurring φ . 3 |
| \log the base-2 logarithm. 2 | $\mathcal{L}^v[\Sigma]$ the v -variable first-order formulas over Σ . 3 |
| $v \prec w$ lexicographically smaller. 2 | $\mathcal{C}^v[\Sigma]$ the v -variable first-order formulas with counting quantifiers over Σ . 3 |
| | $\text{qr } \varphi$ the quantifier rank of φ . 3 |
| | $\mathcal{L}_r[\Sigma]$ the r -rank first-order formulas over |

Glossary

- Σ . 4
- $\mathcal{C}_r[\Sigma]$ the r -rank first-order formulas with counting quantifiers over Σ . 4
- $\mathcal{L}_r^v[\Sigma]$ the r -rank v -variable first-order formulas over Σ . 4
- $\mathcal{C}_r^v[\Sigma]$ the r -rank v -variable first-order formulas with counting quantifiers over Σ . 4
- \mathfrak{A} a structure. 4
- $\varphi^{\mathfrak{A}}$ interpretation of φ in \mathfrak{A} . 5
- SAT \mathcal{K} the satisfiable sentences of \mathcal{K} . 5
- FINSAT \mathcal{K} the finitely satisfiable sentences of \mathcal{K} . 5
- $\varphi \equiv \psi$ logically equivalent formulas. 5
- $\mathfrak{A} \equiv \mathfrak{B}$ elementary equivalent structures. 5
- $\mathfrak{A} \equiv_r \mathfrak{B}$ r -rank equivalent structures. 5
- $\mathfrak{A} \equiv^v \mathfrak{B}$ v -variable equivalent structures. 5
- $\mathfrak{A} \equiv_r^v \mathfrak{B}$ r -rank v -variable equivalent structures. 5
- \mathfrak{p} parital isomorphism. 6
- $G_r(\mathfrak{A}, \mathfrak{B})$ the r -round Ehrenfeucht-Fraïssé game. 6
- $\text{supp } \bar{a}$ the support of \bar{a} . 6
- \bar{a}_i^a substitute of \bar{a} . 6
- $G_r^v(\mathfrak{A}, \mathfrak{B})$ the r -round v -pebble game. 6
- $\Pi[\Sigma]$ the set of 1-types over Σ . 7
- $\mathsf{T}[\Sigma]$ the set of 1-types over Σ . 7
- τ^{-1} the inverse of the type τ . 7
- $\text{tp}_{\mathbf{x}} \tau$ the \mathbf{x} -type of τ . 7
- $\text{tp}_{\mathbf{y}} \tau$ the \mathbf{y} -type of τ . 7
- $\text{tp}^{\mathfrak{A}}[a]$ the 1-type of a in \mathfrak{A} . 7
- $\pi^{\mathfrak{A}}$ the interpretation of the 1-type π in \mathfrak{A} . 7
- $\text{tp}^{\mathfrak{A}}[a, b]$ the 1-type of a in \mathfrak{A} . 7
- $\tau^{\mathfrak{A}}$ the interpretation of the 2-type τ in \mathfrak{A} . 8
- PTime complexity class. 8
- $A \leq^{\text{PTime}} B$ A is polynomial-time reducible to B . 8
- $A =^{\text{PTime}} B$ A and B are polynomial-time equivalent. 8
- B a bit setup. 9
- $[\mathbf{u}:\text{data}]^{\mathfrak{A}}$ \mathbf{u} -data at \mathfrak{A} . 9
- $[\mathbf{u}:\text{eq-d}](\mathbf{x})$ \mathbf{u} -data at \mathbf{x} is d . 9
- $[\mathbf{u}:\text{eq}](\mathbf{x}, \mathbf{y})$ \mathbf{u} -data equal at \mathbf{x} and \mathbf{y} . 9
- $[\mathbf{u}:\text{eq-01}](\mathbf{x}, \mathbf{y})$ \mathbf{u} -data at \mathbf{x} and \mathbf{y} is 0 and 1. 9
- $[\mathbf{u}:\text{eq-10}](\mathbf{x}, \mathbf{y})$ \mathbf{u} -data at \mathbf{x} and \mathbf{y} is 1 and 0. 9
- C a counter setup. 9
- $[\mathsf{C}:\text{data}]^{\mathfrak{A}}$ C -data at \mathfrak{A} . 10
- $[\mathsf{C}:\text{eq-d}](\mathbf{x})$ C -data at \mathbf{x} is d . 10
- $[\mathsf{C}:\text{eq}](\mathbf{x}, \mathbf{y})$ C -data equal at \mathbf{x} and \mathbf{y} . 10
- $[\mathsf{C}:\text{less}]\mathbf{d}(\mathbf{x})$ C -data at \mathbf{x} less than d . 10
- $[\mathsf{C}:\text{betw-d-e}](\mathbf{x})$ C -data at \mathbf{x} between d and e . 10
- $[\mathsf{C}:\text{allbetw-d-e}]$ C -data between d and e . 10
- $[\mathsf{C}:\text{less}](\mathbf{x}, \mathbf{y})$ C -data at \mathbf{x} less than C -data at \mathbf{y} . 10
- $[\mathsf{C}:\text{succ}](\mathbf{x}, \mathbf{y})$ C -data at \mathbf{y} succeeds C -data at \mathbf{x} . 10
- $[\mathsf{V}(p):\text{data}]^{\mathfrak{A}}a$ the value of the p -th counter at a . 11
- $[\mathsf{V}:\text{data}]^{\mathfrak{A}}$ the V -data at a . 11
- $[\mathsf{V}:\text{eq-v}](\mathbf{x})$ the V -data at \mathbf{x} . 11
- $[\mathsf{V}(pq):\text{at-}i\text{-eq}](\mathbf{x})$ equal i -th bits at p and q at \mathbf{x} . 11
- $[\mathsf{V}(pq):\text{at-}i\text{-eq-01}](\mathbf{x})$ equal i -th bits at p and q are 0 and 1. 11
- $[\mathsf{V}(pq):\text{at-}i\text{-eq-10}](\mathbf{x})$ equal i -th bits at p and q are 1 and 0. 11
- $[\mathsf{V}(pq):\text{eq}](\mathbf{x})$ equal p and q V -data at \mathbf{x} . 11
- $[\mathsf{V}(pq):\text{less}](\mathbf{x})$ V -data at p less than at q . 12
- $[\mathsf{V}(pq):\text{succ}](\mathbf{x})$ V -data at q succeeds the data at p . 12
- $[\mathsf{P}:\text{alldiff}]$ P -data at different positions is different. 12
- $[\mathsf{P}:\text{perm}]$ P -data is a permutation. 12
- $\mathcal{E}E$ the set of equivalence classes of E . 13
- $[\mathbf{e}:\text{refl}]$ \mathbf{e} is reflexive. 13
- $[\mathbf{e}:\text{symm}]$ \mathbf{e} is symmetric. 13
- $[\mathbf{e}:\text{trans}]$ \mathbf{e} is transitive. 13
- $[\mathbf{e}:\text{equiv}]$ \mathbf{e} is transitive. 13
- $[\mathbf{d}, \mathbf{e}:\text{refine}]$ refinement. 14

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| $[d, e:\text{global}]$ global agreement. 14 | $[E:\text{chperm}]^{\mathfrak{A}}$ characteristic E-permutation in \mathfrak{A} . 21 |
| $[d, e:\text{local}]$ local agreement. 14 | $[L:\text{fixperm}]$ fixed permutation condition. 22 |
| $[e_1, e_2, \dots, e_e:\text{refine}]$ symbols in refinement. 15 | $[L:\text{locperm}]$ local agreement condition. 23 |
| $[e_1, e_2, \dots, e_e:\text{global}]$ symbols in global agreement. 15 | $[L:\text{el-}i]$ local refinement induced by levels. 23 |
| $[e_1, e_2, \dots, e_e:\text{local}]$ symbols in local agreement. 15 | ltr translation of local agreement to refinement. 24 |
| $\Lambda_p^v e E_a$ logic notation. 17 | g granularity. 25 |
| $[P:\text{alleq}]$ P-data equal everywhere. 18 | G granularity color setup. 26 |
| $[P:\text{globperm}]$ P-data is a global permutation. 18 | $[\Gamma:d]$ finer equivalence granularity formula. 26 |
| $[L:\text{eg-}i](x, y)$ global refinement induced by levels. 19 | grtr granularity translation. 26 |
| gtr translation of global agreement to refinement. 19 | $[\Sigma:\text{cell}](x, y)$ Σ -cell formula. 29 |
| | \mathcal{O} organ-equivalence relation. 31 |
| | O sub-organ-equivalence relation. 31 |

1 Introduction

1.1 Notation

The cardinal number $|A|$ is the *cardinality* of the set A . The set $\wp A$ is the *powerset* of A . The set $\wp^+ A = \wp A \setminus \{\emptyset\}$ is the *set of nonempty subsets* of A . If κ is a cardinal number, the set $\wp^\kappa A = \{S \in \wp A \mid |S| = \kappa\}$ is the κ -*powerset* of A . The *cartesian product* of A and B is $A \times B$. The sets A and B *properly intersect* if $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$.

If R is a binary relation, its *domain* is $\text{dom } R$ and its *range* is $\text{ran } R$. The *inverse* of $R \subseteq A \times B$ is

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}.$$

If S is a set and $R \subseteq A \times B$, the *restriction* of R to S is

$$R \upharpoonright S = \{(a, b) \in R \mid a \in S\}.$$

If $R \subseteq A \times B$ is a binary relation and $a \in A$, the R -*successors* of a are

$$R[a] = \{b \in B \mid (a, b) \in R\}.$$

If $S \subseteq B \times C$ and $R \subseteq A \times B$ are two binary relations, their *composition* is

$$S \circ R = \{(a, c) \in A \times C \mid (\exists b \in B)(a, b) \in R \wedge (b, c) \in S\}.$$

A *function* is formally just a functional relation. The *identity function* on A is id_A . A *total function* from A to B is denoted $f : A \rightarrow B$. A *injective function* from A into B is denoted $f : A \hookrightarrow B$. A *surjective function* from A onto B is denoted $f : A \twoheadrightarrow B$. A *bijective function* between A and B is denoted $f : A \leftrightarrow B$. A *partial function* from A to B is denoted $f : A \rightsquigarrow B$. If $f : A \rightsquigarrow B$ is a partial function and $a \in A$, the notation $f(a) \simeq b$ means that f is defined at a and its value is b ; the notation $f(a) \simeq \perp$ means that f is not defined at a .

A *sequence* is formally just a function with domain an ordinal number. If A is a sequence, its *length* $\|A\|$ is just the domain of A . The sequence consisting of the elements a, b and c in that order is $\langle a, b, c \rangle$. The *empty sequence* is ε . A *finite sequence* is a sequence of finite length. If A and B are two sequences, their *concatenation* is $A + B$, and the sequence obtained from A by dropping all elements of B is $A - B$.

The set of *natural numbers* is $\mathbb{N} = \{0, 1, \dots\}$. The set of *positive natural numbers* is $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. If $n, m \in \mathbb{N}$ are natural numbers, the *discrete interval* $[n, m]$ between n and m is

$$[n, m] = \begin{cases} \{n, n+1, \dots, m\} & \text{if } n \leq m \\ \emptyset & \text{otherwise.} \end{cases}$$

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The function **log** is the *base-2 logarithm*.

An n -vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{N}^n$ is just a tuple of natural numbers. The n -vector \mathbf{v} is *lexicographically smaller* than the n -vector \mathbf{w} (written $\mathbf{v} \prec \mathbf{w}$) if there is a position $p \in [1, n]$ such that $v_p < w_p$ and $v_q = w_q$ for all $q \in [p + 1, n]$.

The set of n -permutations of $[1, n]$ is \mathbb{S}_n . We think of an n -permutation ν as an n -vector $\nu = (\nu(1), \nu(2), \dots, \nu(n))$.

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is *polynomially bounded* if there is a polynomial p and a number $n_0 \in \mathbb{N}$ such that $f(n) \leq p(n)$ for all $n \geq n_0$. The function f is *exponentially bounded* if there is a polynomial p and a number $n_0 \in \mathbb{N}$ such that $f(n) \leq 2^{p(n)}$ for all $n \geq n_0$. We are going to use these terms implicitly with respect to quantities that depend on one another. For example, the cardinality of \mathbb{S}_n is exponentially bounded by n .

An *alphabet* Ω is just a nonempty set. The elements of Ω are *characters*. A *word* $w = w_1 w_2 \dots w_n$ is a finite sequence of characters. The set of words over Ω is Ω^* . The set of nonempty words over Ω is $\Omega^+ = \Omega^* \setminus \{\varepsilon\}$. If $n \in \mathbb{N}$, the set of words of length n over Ω is Ω^n .

The set of *bits* is $\mathbb{B} = \{0, 1\}$. The set of *bitstrings* is \mathbb{B}^+ . The bitstrings are read right-to-left, that is the bitstring $b = 10$ has first character 0. If $t < u \in \mathbb{N}^+$, the t -bit bitstrings \mathbb{B}^t are embedded into the u -bit bitstrings \mathbb{B}^u by appending leading zeroes. If $n \in \mathbb{N}$, the *bitsize* $\|n\|$ of n is:

$$\|n\| = \begin{cases} 1 & \text{if } n = 0 \\ \lfloor \log n \rfloor + 1 & \text{otherwise.} \end{cases}$$

If $n \in \mathbb{N}$, the *binary encoding* of n is $\bar{n} \in \mathbb{B}^{\|n\|}$. If $b \in \mathbb{B}^t$, the *number encoded by* b is \underline{b} . The *largest t -bit number* is $N_t = 2^t - 1$. The set of *t -bit numbers* is $\mathbb{B}_t = [0, N_t]$.

1.2 Syntax

The *symbol alphabet* for the first-order logic with counting quantifiers is

$$\Omega_{\mathcal{C}} = \left\{ \vee, \wedge, \rightarrow, \leftrightarrow, \neg, \exists, \forall, =, (, , ,); \leq, =, \geq, ^0, ^1 \right\}.$$

The propositional connectives are listed in increasing order of precedence. The *negation* \neg is unary; the *disjunction* \vee , *conjunction* \wedge and *equivalence* \leftrightarrow are left-associative; the *implication* \rightarrow is right-associative. Note that we consider logics with *formal equality* $=$.

A *counting quantifier* is a word over $\Omega_{\mathcal{C}}$ of the form $\exists^{\leq \bar{m}}$ or $\exists^{=\bar{m}}$ or $\exists^{\geq \bar{m}}$, where $m \in \mathbb{N}$ and $\bar{m} \in \mathbb{B}^+$ is the binary encoding of m . Note that this encoding of the counting quantifiers is *succinct*. As we note in remark 1, this succinct representation allows for exponentially small counting formulas compared to their pure first-order equivalents. We denote the counting quantifiers by $\exists^{\leq m}$, $\exists^{=m}$ and $\exists^{\geq m}$, that is, we omit the encoding notation for m .

The sequence $\mathcal{V} = \langle \mathbf{v}_1, \mathbf{v}_2, \dots \rangle$ is a countable sequence of distinct *variable symbols*. We pay special attention to $\mathbf{x} = \mathbf{v}_1$, $\mathbf{y} = \mathbf{v}_2$ and $\mathbf{z} = \mathbf{v}_3$, the *first*, *second* and *third* variable symbol, respectively.

A *predicate signature* $\Sigma = \langle \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_s \rangle$ is a finite sequence of distinct *predicate symbols* \mathbf{p}_i together with their *arities* $\text{ar } \mathbf{p}_i \in \mathbb{N}^+$. A predicate signature is *unary* or *monadic* if all of its predicate symbols have arity 1. A predicate signature is *binary* if all of its predicate symbols have arity 1 or 2. For the purposes of this work we will not be considering constant and function symbols—constant symbols can be simulated by a fresh unary predicate symbol having the intended interpretation of being true at a unique element; presence of function symbols on the other hand leads quite easily to undecidable satisfiability problems. By convention $\Omega_{\mathcal{C}}$, \mathcal{V} and Σ are disjoint.

Let Σ be a predicate signature. The set of *atomic formulas* $\mathcal{At}[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\alpha ::= (x = y) \mid p(x_1, x_2, \dots, x_n)$$

for $x, y \in \mathcal{V}$, $p \in \Sigma$, $n = \text{ar } p$ and $x_1, x_2, \dots, x_n \in \mathcal{V}$.

The set of *literals* $\mathcal{Lit}[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\lambda ::= \alpha \mid (\neg \alpha).$$

The set of *first-order formulas with counting quantifiers* $\mathcal{C}[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\begin{aligned} \varphi ::= & \alpha \mid (\neg \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi) \mid (\exists x \varphi) \mid (\forall x \varphi) \mid \\ & \mid (\exists^{\leq m} x \varphi) \mid (\exists^=^m x \varphi) \mid (\exists^{\geq m} x \varphi) \end{aligned}$$

for $x \in \mathcal{V}$ and $m \in \mathbb{N}$.

The set of *first-order formulas* $\mathcal{L}[\Sigma] \subset \mathcal{C}[\Sigma]$ over Σ consists of the formulas that do not feature a counting quantifier.

The set of variables occurring in φ is $\text{vr } \varphi \subset \mathcal{V}$. The set of variables freely occurring in φ is $\text{fvr } \varphi \subset \mathcal{V}$. A formula φ is a *sentence* if $\text{fvr } \varphi = \emptyset$. For $v \in \mathbb{N}$, a formula φ is a *v-variable formula* if $\text{vr } \varphi \subseteq \{v_1, v_2, \dots, v_v\}$. The set of *v-variable first-order formulas* over Σ is $\mathcal{L}^v[\Sigma]$. The set of *v-variable first-order formulas with counting quantifiers* over Σ is $\mathcal{C}^v[\Sigma]$.

If $\varphi \in \mathcal{C}[\Sigma]$, the *quantifier rank* $\text{qr } \varphi \in \mathbb{N}$ of φ is defined as follows. If φ matches:

- $(x = y)$, then $\text{qr } \varphi = 0$
- $p(x_1, x_2, \dots, x_n)$, then $\text{qr } \varphi = 0$
- $(\neg \psi)$, then $\text{qr } \varphi = \text{qr } \psi$
- $\psi_1 \oplus \psi_2$ for $\oplus \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$, then $\text{qr } \varphi = \max(\text{qr } \psi_1, \text{qr } \psi_2)$
- $(\exists x \psi)$ or $(\forall x \psi)$, then $\text{qr } \varphi = 1 + \text{qr } \psi$
- $(\exists^{\leq m} x \psi)$ or $(\exists^=^m x \psi)$, then $\text{qr } \varphi = m + 1 + \text{qr } \psi$
- $(\exists^{\geq m} x \psi)$, then $\text{qr } \varphi = m + \text{qr } \psi$.

1 Introduction

An r -rank formula is a formula having quantifier rank r . The set of r -rank first-order formulas over Σ is $\mathcal{L}_r[\Sigma]$. The set of r -rank first-order formulas with counting quantifiers over Σ is $\mathcal{C}_r[\Sigma]$. The set of r -rank v -variable first-order formulas over Σ is $\mathcal{L}_r^v[\Sigma]$. The set of r -rank v -variable first-order formulas with counting quantifiers over Σ is $\mathcal{C}_r^v[\Sigma]$.

If φ is a formula and $x_1, x_2, \dots, x_n \in \mathcal{V}$ are distinct variables, we use the notation $\varphi(x_1, x_2, \dots, x_n)$, a *focused formula*, to show that we are interested in the free occurrences of the variables x_i in φ . If $\varphi(x_1, x_2, \dots, x_n)$ is a focused formula and $y_1, y_2, \dots, y_n \in \mathcal{V}$, then $\varphi(y_1, y_2, \dots, y_n)$ denotes the formula φ where all free occurrences of x_i are replaced by y_i . The notation $\varphi = \varphi(x_1, x_2, \dots, x_n)$ means that $\text{fvr } \varphi \subseteq \{x_1, x_2, \dots, x_n\}$.

We will omit unnecessary brackets in formulas.

1.3 Semantics

If Σ is a predicate signature, a Σ -structure \mathfrak{A} consists of a nonempty set A (the *domain* of \mathfrak{A}), together with a relation $p^{\mathfrak{A}} \subseteq A^{\text{ar } p}$ (the *interpretation* of p at \mathfrak{A}) for every predicate symbol $p \in \Sigma$. A structure is *finite* if its domain is finite. We omit the standard definition of semantic notions. Seldom it will be useful to consider *structures with possibly empty domain*. We will be explicit when this is the case. If \mathfrak{A} is a structure and $B \subseteq A$ there is a substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with possibly empty domain B . We call it the substructure induced by B and denote it $(\mathfrak{A} \upharpoonright B)$.

Note that the interpretation of the counting quantifiers is clear: $\exists^{\leq m} x \varphi$ means that “at most m elements satisfy φ ”; $\exists^=m x \varphi$ means that “exactly m elements satisfy φ ”; $\exists^{\geq m} x \varphi$ means that “at least m elements satisfy φ ”.

The *standard translation* $\text{st} : \mathcal{C}[\Sigma] \rightarrow \mathcal{L}[\Sigma]$ of first-order formulas with counting quantifiers to logically equivalent first-order formulas is defined as follows. If φ matches:

- $(x = y)$ or $p(x_1, x_2, \dots, x_n)$, then $\text{st } \varphi = \varphi$
- $(\neg \psi)$, then $\text{st } \varphi = (\neg \text{st } \psi)$
- $(\psi_1 \oplus \psi_2)$ for $\oplus \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$, then $\text{st } \varphi = (\text{st } \psi_1 \oplus \text{st } \psi_2)$
- $(Qx \psi)$ for $Q \in \{\exists, \forall\}$, then $\text{st } \varphi = (Qx \text{st } \psi)$
- $\exists^{\leq m} x \psi(x)$ or $\exists^=m x \psi(x)$ or $\exists^{\geq m} x \psi(x)$, then let

$$\begin{aligned} \theta_{\leq} &= \forall y_1 \forall y_2 \dots \forall y_m \forall y_{m+1} \left(\bigwedge_{1 \leq i \leq m+1} \text{st } \psi(y_i) \right) \rightarrow \bigvee_{1 \leq i < j \leq m+1} y_i = y_j \\ \theta_{\geq} &= \exists y_1 \exists y_2 \dots \exists y_m \left(\bigwedge_{1 \leq i \leq m} \text{st } \psi(y_i) \right) \wedge \bigwedge_{1 \leq i < j \leq m} y_i \neq y_j \end{aligned}$$

where y_1, y_2, \dots, y_{m+1} are distinct variable symbols not occurring in φ . The formula θ_{\leq} asserts that there are at most m distinct values satisfying ψ . The formula θ_{\geq} asserts that there are at least m distinct values satisfying ψ . If $\varphi = (\exists^{\leq m} x \psi(x))$,

then $\text{st } \varphi = \theta_{\leq}$. If $\varphi = (\exists^m x \psi(x))$, then $\text{st } \varphi = (\theta_{\leq} \wedge \theta_{\geq})$. If $\varphi = (\exists^{\geq m} x \psi(x))$, then $\text{st } \varphi = \theta_{\geq}$.

Remark 1. The translation of a first-order formula with counting quantifiers φ to a logically equivalent first-order formula $\psi = \text{st } \varphi$ preserves quantifier rank. However, the resulting formula ψ may have exponentially larger length.

A predicate signature with intended interpretations Σ is formally a predicate signature together with an *intended interpretation condition* \mathcal{A} , which is formally a class of Σ -structures. A Σ -structure \mathfrak{A} is then just an element of \mathcal{A} . That is, when we speak about a predicate signature with intended interpretations, we are considering the logics strictly over the class of structures respecting the intended interpretation condition. The semantic concepts are relativised appropriately in this context. For example, if $\Sigma = \langle e \rangle$ is a predicate signature consisting of the single binary predicate symbol e , having intended interpretation as an equivalence, then the Σ -formula $\forall x e(x, x)$ is logically valid. From now on, we will use the term *predicate signature* as *predicate signature with possible intended interpretations*.

The predicate signature Σ' is an *enrichment* of the predicate signature Σ if Σ' contains all predicate symbols of Σ and respects their intended interpretation in Σ . A Σ' -structure \mathfrak{A}' is an enrichment of the Σ -structure \mathfrak{A} if they have the same domain and the same interpretation of the predicate symbols of Σ . The basic semantic significance of enrichment is that if $\varphi(x_1, x_2, \dots, x_n)$ is a Σ -formula and $a_1, a_2, \dots, a_n \in A$, then $\mathfrak{A} \models \varphi(a_1, a_2, \dots, a_n)$ iff $\mathfrak{A}' \models \varphi(a_1, a_2, \dots, a_n)$.

If $\varphi(x_1, x_2, \dots, x_n)$ is a focused formula, the interpretation of φ in \mathfrak{A} is

$$\varphi^{\mathfrak{A}} = \{(a_1, a_2, \dots, a_n) \in A^n \mid \mathfrak{A} \models \varphi(a_1, a_2, \dots, a_n)\}.$$

If Σ is a predicate signature and φ is a Σ -sentence, then φ is *satisfiable* if there is a Σ -structure that is a model for φ ; φ is *finitely satisfiable* if there is a *finite* Σ -structure that is a model for φ . If $\mathcal{K} \subseteq \mathcal{C}[\Sigma]$ is a family of formulas over the predicate signature Σ , the set of *satisfiable sentences* is $\text{SATK} \subseteq \mathcal{K}$ and the set of *finitely satisfiable sentences* is $\text{FINSATK} \subseteq \mathcal{K}$. The family \mathcal{K} has the *finite model property* if $\text{SATK} = \text{FINSATK}$. By the Löwenheim-Skolem theorem, every satisfiable sentence φ has a finite or countable model (assuming the intended interpretation condition of the predicate signature is first-order-definable). In this work the intended interpretation conditions of the predicate signatures will always be first-order-definable formula and we will silently assume that all structures are either finite or countable.

Two Σ -sentences φ and ψ are *logically equivalent* (written $\varphi \equiv \psi$) if they have the same models.

Two Σ -structures \mathfrak{A} and \mathfrak{B} are *elementary equivalent* (written $\mathfrak{A} \equiv \mathfrak{B}$) if they satisfy the same first-order sentences (hence also the same first-order sentences with counting quantifiers). The structures \mathfrak{A} and \mathfrak{B} are *r -rank equivalent* (written $\mathfrak{A} \equiv_r \mathfrak{B}$) if they satisfy the same r -rank first-order sentences. The structures \mathfrak{A} and \mathfrak{B} are *v -variable equivalent* (written $\mathfrak{A} \equiv^v \mathfrak{B}$) if they satisfy the same v -variable first-order sentences. The structures \mathfrak{A} and \mathfrak{B} are *r -rank v -variable equivalent* (written $\mathfrak{A} \equiv_r^v \mathfrak{B}$) if they satisfy the same r -rank v -variable first-order sentences.

1.4 Logic games

Logic games capture structure equivalence. Let Σ be a predicate signature and let \mathfrak{A} and \mathfrak{B} be Σ -structures. A *partial isomorphism* $\mathfrak{p} : A \rightsquigarrow B$ from \mathfrak{A} to \mathfrak{B} is a partial mapping that is an isomorphism between the induced substructures $(\mathfrak{A} \upharpoonright \text{dom } \mathfrak{p})$ and $(\mathfrak{B} \upharpoonright \text{ran } \mathfrak{p})$.

Let $r \in \mathbb{N}^+$. The r -round *Ehrenfeucht-Fraïssé game* $G_r(\mathfrak{A}, \mathfrak{B})$ is a two-player game, played with a pair of pebbles, one for each structure. The two players are Spoiler and Duplicator. Initially the pebbles are off the structures. During each round, Spoiler picks a pebble and places it on some element in its designated structure. Duplicator responds by picking the other pebble and placing it on some element in the other structure. Thus during round i , the players play a pair of elements $a_i \mapsto b_i \in A \times B$. Collect the sequences of played elements $\bar{a} = \langle a_1, a_2, \dots, a_r \rangle$ and $\bar{b} = \langle b_1, b_2, \dots, b_r \rangle$. Duplicator wins the match if the relation $\bar{a} \mapsto \bar{b} = \{a_1 \mapsto b_1, a_2 \mapsto b_2, \dots, a_r \mapsto b_r\} \subseteq A \times B$, built from the pairs of elements in each round, is a partial isomorphism from \mathfrak{A} to \mathfrak{B} . Ehrenfeucht's theorem says that Duplicator has a winning strategy for $G_r(\mathfrak{A}, \mathfrak{B})$ iff $\mathfrak{A} \equiv_r \mathfrak{B}$. Fraïssé's theorem gives a back-and-forth characterization of the winning strategy for Duplicator [1, ch. 2]:

Theorem 1. *Suppose that $(\mathfrak{I}_0, \mathfrak{I}_1, \dots, \mathfrak{I}_r)$ is a sequence of nonempty sets of partial isomorphisms between \mathfrak{A} and \mathfrak{B} with the following properties:*

1. *For every $j < r$, $\mathfrak{p} \in \mathfrak{I}_{j+1}$ and $a \in A$, there is $\mathfrak{q} \in \mathfrak{I}_j$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $a \in \text{dom } \mathfrak{q}$.*
2. *For every $j < r$, $\mathfrak{p} \in \mathfrak{I}_{j+1}$ and $b \in B$, there is $\mathfrak{q} \in \mathfrak{I}_j$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $b \in \text{ran } \mathfrak{q}$.*

Then $\mathfrak{A} \equiv_r \mathfrak{B}$.

Let $v \in \mathbb{N}^+$. By convention \perp is an element not occurring in any structure. If $\bar{a} = a_1 a_2 \dots a_v \in (A \cup \{\perp\})^v$ is a vector, the *support* of \bar{a} is $\text{supp } \bar{a} = \{i \in [1, v] \mid a_i \in A\}$ and an i -substitute of \bar{a} is $\bar{a}_i^a = a_1 a_2 \dots a_{i-1} a a_{i+1} \dots a_v$. If \mathfrak{A} and \mathfrak{B} are Σ -structures, a v -partial isomorphism from \mathfrak{A} to \mathfrak{B} is a pair $\mathfrak{p} = \bar{a} \mapsto \bar{b}$, where $\bar{a} \in (A \cup \{\perp\})^v$, $\bar{b} \in (B \cup \{\perp\})^v$, $\text{supp } \bar{a} = \text{supp } \bar{b}$ and $\bar{a}' \mapsto \bar{b}'$ is a partial isomorphism, where \bar{a}' and \bar{b}' are the subsequences of \bar{a} and \bar{b} with indexes from the support. We also think of \mathfrak{p} as a partial function $\mathfrak{p} : A \rightsquigarrow B$, defined by

$$\mathfrak{p} = \left\{ \bar{a}_i \mapsto \bar{b}_i \mid i \in \text{supp } \bar{a} \right\}.$$

We now consider *pebble games*, which characterise constrained variables equivalence of structures. Let $r \in \mathbb{N}^+$. The r -round v -pebble game $G_r^v(\mathfrak{A}, \mathfrak{B})$ is a two-player game, with v pairs of pebbles. Initially the pebbles are off the board. During each round, Spoiler picks a pair of pebbles and puts one of them on an element in one of the structures. Duplicator responds by putting the other pebble from the pair on an element from the other structure. Thus the state of the game after round i is thus captured by two sequences $\bar{a}_i \in (A \cup \{\perp\})^v$ and $\bar{b}_i \in (B \cup \{\perp\})^v$ having the same support, showing which pairs of pebbles have been placed where. Duplicator wins the match if for each $i \leq r$ we have that $\bar{a}_i \mapsto \bar{b}_i$ is a v -partial isomorphism. Similarly to the Ehrenfeucht-Fraïssé game, we have that Duplicator has a winning strategy for $G_r^v(\mathfrak{A}, \mathfrak{B})$ iff $\mathfrak{A} \equiv_r^v \mathfrak{B}$. The following is a back-and-forth characterization of the winning strategy for Duplicator [1, ch. 2]:

Theorem 2. Suppose that $(\mathfrak{I}_0, \mathfrak{I}_1, \dots, \mathfrak{I}_r)$ is a sequence of nonempty sets of v -partial isomorphisms between \mathfrak{A} and \mathfrak{B} with the following properties:

1. For every $j \in [0, r-1]$, $i \in [1, v]$, $\bar{a} \mapsto \bar{b} \in \mathfrak{I}_{j+1}$ and $a \in A$, there is $b \in B$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathfrak{I}_j$.
2. For every $j \in [0, r-1]$, $i \in [1, v]$, $\bar{a} \mapsto \bar{b} \in \mathfrak{I}_{j+1}$ and $b \in B$, there is $a \in A$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathfrak{I}_j$.

Then $\mathfrak{A} \equiv_r^v \mathfrak{B}$.

1.5 Types

Let $\Sigma = \langle p_1, p_2, \dots, p_s \rangle$ be a predicate signature. A 1-type π over Σ is a maximal consistent set of literals featuring only the variable symbol \mathbf{x} . The set of 1-types over Σ is $\Pi[\Sigma]$. Note that consistency here is relativised by the intended interpretations of the predicate signature. For example if Σ contains the binary predicate symbol e with intended interpretation as an equivalence, then every 1-type over Σ includes the literal $e(\mathbf{x}, \mathbf{x})$. Also note that the cardinality of a 1-type over Σ is exponentially bounded by the length s of Σ and the cardinality of $\Pi[\Sigma]$ is doubly exponentially bounded by s .

A 2-type τ over Σ is a maximal consistent set of literals featuring only the variable symbols \mathbf{x} and \mathbf{y} and including the literal $(\mathbf{x} \neq \mathbf{y})$. The set of 2-types over Σ is $T[\Sigma]$. Again, consistency is relativised by the intended interpretation of the predicate signature. For example if Σ contains the binary predicate symbol e with intended interpretation as an equivalence, then if $e(\mathbf{x}, \mathbf{y}) \in \tau$, then $e(\mathbf{y}, \mathbf{x}) \in \tau$. Again, the cardinality of a 2-type over Σ is exponentially bounded by s and the cardinality of $T[\Sigma]$ is doubly exponentially bounded by s .

If $\tau \in T[\Sigma]$, the *inverse* τ^{-1} of τ is the 2-type obtained from τ by swapping the variables \mathbf{x} and \mathbf{y} in every literal. The \mathbf{x} -type of τ is the 1-type $\text{tp}_{\mathbf{x}} \tau$ consisting of all the literals of τ featuring only the variable symbol \mathbf{x} . Similarly, the \mathbf{y} -type of τ is the 1-type $\text{tp}_{\mathbf{y}} \tau$ consisting of all the literals of τ featuring only the variable symbol \mathbf{y} , that is replaced by \mathbf{x} . For example we have the identity $\text{tp}_{\mathbf{x}} \tau^{-1} = \text{tp}_{\mathbf{y}} \tau$.

If \mathfrak{A} is a Σ -structure and $a \in A$, the 1-type of a in \mathfrak{A} is

$$\text{tp}^{\mathfrak{A}}[a] = \{\lambda(\mathbf{x}) \in \text{Lit}[\Sigma] \mid \mathfrak{A} \models \lambda(a)\}.$$

If $\text{tp}^{\mathfrak{A}}[a] = \pi$, we say that the 1-type π is *realized* by a in \mathfrak{A} . The interpretation of the 1-type π in \mathfrak{A} is the set of elements realizing π :

$$\pi^{\mathfrak{A}} = \{a \in A \mid \text{tp}^{\mathfrak{A}}[a] = \pi\}.$$

If $a \neq b \in A$, the 2-type of (a, b) in \mathfrak{A} is

$$\text{tp}^{\mathfrak{A}}[a, b] = \{\lambda(\mathbf{x}, \mathbf{y}) \in \text{Lit}[\Sigma] \mid \mathfrak{A} \models \lambda(a, b)\}.$$

1 Introduction

We do not define a 2-type in case $a = b$. If $\text{tp}^{\mathfrak{A}}[a, b] = \tau$, we say that the 2-type τ is *realized* by (a, b) in \mathfrak{A} . The interpretation of the 2-type τ in \mathfrak{A} is the set of pairs realizing τ :

$$\tau^{\mathfrak{A}} = \{(a, b) \in A \times A \mid a \neq b \wedge \text{tp}^{\mathfrak{A}}[a, b] = \tau\}.$$

1.6 Complexity

We denote the complexity classes **P**TIME, NPTIME, PSPACE, EXPTIME, NEXPTIME and for $e \in \mathbb{N}^+$, the e -exponential deterministic and nondeterministic time classes e EXPTIME and NeEXPTIME. The complexity class ELEMENTARY is the union of the complexity classes e EXPTIME for $e \in \mathbb{N}^+$. If $A \subseteq \Omega_1^*$ and $B \subseteq \Omega_2^*$ are decision problems, the problem A is *polynomial-time reducible* to B (written $A \leq^{\text{PTIME}} B$) if there is a polynomial-time algorithm $f : \Omega_1^* \rightarrow \Omega_2^*$ such that $a \in A$ iff $f(a) \in B$. Similar reductions are defined analogously. The decision problems A and B are *polynomial-time equivalent* (written $A =^{\text{PTIME}} B$) if $A \leq^{\text{PTIME}} B$ and $B \leq^{\text{PTIME}} A$.

A decision problem is *hard* for a complexity class if any decision problem of that complexity class is polynomial-time reducible to it. A decision problem is *complete* for a complexity class if it is hard for that class and contained in that class. A *Wang tile* t is a square tile with colored sides, formally $t = (e_t, n_t, w_t, s_t) \in \mathbb{N}^4$. A *tile family* T is a finite set of Wang tiles. Given a tile family T , a finite list of tiles from the tile family $I = \langle t_1, t_2, \dots, t_w \rangle$ and a natural number $N \geq w$, the *square tiling problem* is the problem of tiling the $N \times N$ square with tiles from T such that two adjacent tiles share the same color of their adjacent sides and the first w tiles on the first row are exactly the tiles I . Formally, the tiling of the square is a function $f : [1, N] \times [1, N] \rightarrow T$, $f(i, j) = (e(i, j), n(i, j), w(i, j), s(i, j))$ such that $e(i, j) = w(i, j + 1)$, $s(i, j) = n(i + 1, j)$ for $i, j \in [1, N - 1]$ and $f(1, k) = t_k$ for $k \in [1, w]$. Without loss of generality, assume that $e_t, n_t, w_t, s_t \in [1, 4|T|]$ for all $t \in T$. We use the following well-known results on tiling problems for proving hardness of the satisfiability of logics.

The *tiling problem* is defined as follows. Given (T, I, n) , a tile family T , an initial segment I and a natural number $n > |I|$ *coded in unary*, is there a tiling of the $N \times N$ square, where $N = n$. The tiling problem is NPTIME-complete [2].

The *exponential tiling problem* is defined as follows. Given (T, I, n) , a tile family T , an initial segment I and a natural number $n > |I|$ *coded in binary*, is there a tiling of the $N \times N$ square, where $N = n$. The exponential tiling problem is NEXPTIME-complete.

The *doubly exponential tiling problem* is defined as follows. Given (T, I, n) , a tile family T , an initial segment I and a natural number $n > |I|$ *coded in binary*, is there a tiling of the $N \times N$ square, where $N = 2^n$. The doubly exponential tiling problem is N2EXPTIME-complete.

2 Counter setups

2.1 Bits

A *bit setup* $\mathbf{B} = \langle \mathbf{u} \rangle$ is a predicate signature consisting of a single unary predicate symbol \mathbf{u} .

Definition 1. Let \mathfrak{A} be a \mathbf{B} -structure. Define the function $[\mathbf{u}:\text{data}]^{\mathfrak{A}} : A \rightarrow \mathbb{B}$ by:

$$[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = \begin{cases} 1 & \text{if } \mathfrak{A} \models \mathbf{u}(a) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2. Let $d \in \mathbb{B}$. Define the quantifier-free $\mathcal{L}^1[\mathbf{B}]$ -formula $[\mathbf{u}:\text{eq}-d](x)$ by:

$$[\mathbf{u}:\text{eq}-d](x) = \begin{cases} \mathbf{u}(x) & \text{if } d = 1 \\ \neg \mathbf{u}(x) & \text{otherwise.} \end{cases}$$

If \mathfrak{A} is a \mathbf{B} -structure, $a \in A$ and $d \in \mathbb{B}$, then $\mathfrak{A} \models [\mathbf{u}:\text{eq}-d](a)$ iff $[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = d$.

Definition 3. Define the quantifier-free $\mathcal{L}^2[\mathbf{B}]$ -formulas $[\mathbf{u}:\text{eq}](x, y)$, $[\mathbf{u}:\text{eq}-01](x, y)$ and $[\mathbf{u}:\text{eq}-10](x, y)$ by:

$$\begin{aligned} [\mathbf{u}:\text{eq}](x, y) &= \mathbf{u}(x) \leftrightarrow \mathbf{u}(y) \\ [\mathbf{u}:\text{eq}-01](x, y) &= \neg \mathbf{u}(x) \wedge \mathbf{u}(y) \\ [\mathbf{u}:\text{eq}-10](x, y) &= \mathbf{u}(x) \wedge \neg \mathbf{u}(y). \end{aligned}$$

If \mathfrak{A} is a \mathbf{B} -structure and $a, b \in A$, then:

- $\mathfrak{A} \models [\mathbf{u}:\text{eq}](a, b)$ iff $[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = [\mathbf{u}:\text{data}]^{\mathfrak{A}} b$
- $\mathfrak{A} \models [\mathbf{u}:\text{eq}-01](a, b)$ iff $[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = 0$ and $[\mathbf{u}:\text{data}]^{\mathfrak{A}} b = 1$
- $\mathfrak{A} \models [\mathbf{u}:\text{eq}-10](a, b)$ iff $[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = 1$ and $[\mathbf{u}:\text{data}]^{\mathfrak{A}} b = 0$.

2.2 Counters

A *t-bit counter setup* for $t \in \mathbb{N}^+$ is a predicate signature $\mathbf{C} = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \rangle$ consisting of t distinct unary predicate symbols \mathbf{u}_i .

2 Counter setups

Definition 4. Let \mathfrak{A} be a C-structure. Define the function $[C:\text{data}]^{\mathfrak{A}} : A \rightarrow \mathbb{B}_t$ by:

$$[C:\text{data}]^{\mathfrak{A}}a = \sum_{1 \leq i \leq t} 2^{i-1} [u_i:\text{data}]^{\mathfrak{A}}a.$$

Definition 5. Let $d \in \mathbb{B}_t$. Define the quantifier-free $\mathcal{L}^1[C]$ -formula $[C:\text{eq-d}](x)$ by:

$$[C:\text{eq-d}](x) = \bigwedge_{1 \leq i \leq t} [u_i:\text{eq-}\bar{d}_i](x).$$

If \mathfrak{A} is a C-structure, $a \in A$ and $d \in \mathbb{B}_t$, then $\mathfrak{A} \models [C:\text{eq-d}](a)$ iff $[C:\text{data}]^{\mathfrak{A}}a = d$.

If A is a nonempty set and $\text{data} : A \rightarrow \mathbb{B}_t$ is any function, there is a C-structure \mathfrak{A} over A such that $[C:\text{data}]^{\mathfrak{A}} = \text{data}$.

Definition 6. Define the quantifier-free $\mathcal{L}^2[C]$ -formula $[C:\text{eq}](x, y)$ by:

$$[C:\text{eq}](x, y) = \bigwedge_{1 \leq i \leq t} [u_i:\text{eq}](x, y).$$

If \mathfrak{A} is a C-structure and $a, b \in A$, then $\mathfrak{A} \models [C:\text{eq}](a, b)$ iff $[C:\text{data}]^{\mathfrak{A}}a = [C:\text{data}]^{\mathfrak{A}}b$.

Definition 7. Let $d \in \mathbb{B}_t$. Define the quantifier-free $\mathcal{L}^1[C]$ -formula $[C:\text{less-d}](x)$ by:

$$[C:\text{less-d}](x) = \bigvee_{1 \leq j \leq t} \neg u_j(x) \wedge \neg [u_j:\text{eq-}\bar{d}_j](x) \wedge \bigwedge_{j < k \leq t} [u_k:\text{eq-}\bar{d}_k](x).$$

If \mathfrak{A} is a C-structure, $a \in A$ and $d \in \mathbb{B}_t$, then $\mathfrak{A} \models [C:\text{less-d}](a)$ iff $[C:\text{data}]^{\mathfrak{A}}a < d$.

Definition 8. Let $d \leq e \in \mathbb{B}_t$. Define the quantifier-free $\mathcal{L}^1[C]$ -formula $[C:\text{betw-d-e}](x)$ by:

$$[C:\text{betw-d-e}](x) = \neg [C:\text{less-d}](x) \wedge ([C:\text{less-e}](x) \vee [C:\text{eq-e}](x)).$$

If \mathfrak{A} is a C-structure, $a \in A$ and $d \leq e \in \mathbb{B}_t$, then

$$\mathfrak{A} \models [C:\text{betw-d-e}](a) \text{ iff } d \leq [C:\text{data}]^{\mathfrak{A}}a \leq e.$$

Definition 9. Let $d \leq e \in \mathbb{B}_t$. Define the $\mathcal{L}^1[C]$ -sentence $[C:\text{allbetw-d-e}]$ by:

$$[C:\text{allbetw-d-e}] = \forall x [C:\text{betw-d-e}](x).$$

If \mathfrak{A} is a C-structure and $d \leq e \in \mathbb{B}_t$, then $\mathfrak{A} \models [C:\text{betw-d-e}]$ iff $d \leq [C:\text{data}]^{\mathfrak{A}}a \leq e$ for all $a \in A$.

Definition 10. Define the quantifier-free $\mathcal{L}^2[C]$ -formula $[C:\text{less}](x, y)$ by:

$$[C:\text{less}](x, y) = \bigvee_{1 \leq j \leq t} [u_j:\text{eq-01}](x, y) \wedge \bigwedge_{j < k \leq n} [u_k:\text{eq}](x, y).$$

If \mathfrak{A} is a C-structure and $a, b \in A$, then $\mathfrak{A} \models [C:\text{less}](a, b)$ iff $[C:\text{data}]^{\mathfrak{A}}a < [C:\text{data}]^{\mathfrak{A}}b$.

Definition 11. Define the quantifier-free $\mathcal{L}^2[C]$ -formula $[C:\text{succ}](x, y)$ by:

$$[C:\text{succ}](x, y) = \bigvee_{1 \leq j \leq t} \bigwedge_{1 \leq i < j} [u_i:\text{eq-10}](x, y) \wedge [u_j:\text{eq-01}](x, y) \wedge \bigwedge_{j < k \leq n} [u_k:\text{eq}](x, y).$$

If \mathfrak{A} is a C-structure and $a, b \in A$, then:

$$\mathfrak{A} \models [C:\text{succ}](a, b) \text{ iff } [C:\text{data}]^{\mathfrak{A}}b = 1 + [C:\text{data}]^{\mathfrak{A}}a.$$

2.3 Vectors

Let $n, t \in \mathbb{N}^+$. Recall the set of n -dimensional t -bit vectors is \mathbb{B}_t^n . An n -dimensional t -bit vector setup is a predicate signature $V = \langle \mathbf{u}_{11}, \mathbf{u}_{12}, \dots, \mathbf{u}_{nt} \rangle$ of (nt) distinct unary predicate symbols. The counter setup $V(p)$ of V at position $p \in [1, n]$ is $V(p) = \langle \mathbf{u}_{p1}, \mathbf{u}_{p2}, \dots, \mathbf{u}_{pt} \rangle$.

Definition 12. Let \mathfrak{A} be a V -structure and $a \in A$. We refer to $[V(p):\text{data}]^{\mathfrak{A}}a$ as the value of the p -th counter at a . Define the function $[V:\text{data}]^{\mathfrak{A}} : A \rightarrow \mathbb{B}_t^n$ by:

$$[V:\text{data}]^{\mathfrak{A}}a = ([V(1):\text{data}]^{\mathfrak{A}}a, [V(2):\text{data}]^{\mathfrak{A}}a, \dots, [V(n):\text{data}]^{\mathfrak{A}}a).$$

Definition 13. Let $\mathbf{v} = (d_1, d_2, \dots, d_n) \in \mathbb{B}_t^n$ be an n -dimensional t -bit vector. Define the quantifier-free $\mathcal{L}^1[V]$ -formula $[V:\text{eq-v}](\mathbf{x})$ by:

$$[V:\text{eq-v}](\mathbf{x}) = \bigwedge_{1 \leq p \leq n} [V(p):\text{eq-d}_p](\mathbf{x}).$$

If \mathfrak{A} is a V -structure, $a \in A$ and $\mathbf{v} \in \mathbb{B}_t^n$, then $\mathfrak{A} \models [V:\text{eq-v}](a)$ iff $[V:\text{data}]^{\mathfrak{A}}a = \mathbf{v}$.

If \mathfrak{A} is a nonempty set and $\text{data} : A \rightarrow \mathbb{B}_t^n$ is any function, then there is a V -structure \mathfrak{A} over A such that $[V:\text{data}]^{\mathfrak{A}} = \text{data}$.

Definition 14. Let $p, q \in [1, n]$ and let $i \in [1, t]$. Define the quantifier-free $\mathcal{L}^1[V]$ -formulas $[V(pq):\text{at-}i\text{-eq}](\mathbf{x})$, $[V(pq):\text{at-}i\text{-eq-01}](\mathbf{x})$ and $[V(pq):\text{at-}i\text{-eq-10}](\mathbf{x})$ by:

$$\begin{aligned} [V(pq):\text{at-}i\text{-eq}](\mathbf{x}) &= \mathbf{u}_{pi}(\mathbf{x}) \leftrightarrow \mathbf{u}_{qi}(\mathbf{x}) \\ [V(pq):\text{at-}i\text{-eq-01}](\mathbf{x}) &= \neg \mathbf{u}_{pi}(\mathbf{x}) \wedge \mathbf{u}_{qi}(\mathbf{x}) \\ [V(pq):\text{at-}i\text{-eq-10}](\mathbf{x}) &= \mathbf{u}_{pi}(\mathbf{x}) \wedge \neg \mathbf{u}_{qi}(\mathbf{x}). \end{aligned}$$

If \mathfrak{A} is a V -structure and $a \in A$, then:

- $\mathfrak{A} \models [V(pq):\text{at-}i\text{-eq}](a)$ iff $[\mathbf{u}_{pi}:\text{data}]^{\mathfrak{A}}a = [\mathbf{u}_{qi}:\text{data}]^{\mathfrak{A}}a$, that is the values of the i -th bit at positions p and q at a are equal
- $\mathfrak{A} \models [V(pq):\text{at-}i\text{-eq-01}](a)$ iff $[\mathbf{u}_{pi}:\text{data}]^{\mathfrak{A}}a = 0$ and $[\mathbf{u}_{qi}:\text{data}]^{\mathfrak{A}}a = 1$, that is the i -th bit at position p at a is 0 and the i -th bit at position q at a is 1
- $\mathfrak{A} \models [V(pq):\text{at-}i\text{-eq-10}](a)$ iff $[\mathbf{u}_{pi}:\text{data}]^{\mathfrak{A}}a = 1$ and $[\mathbf{u}_{qi}:\text{data}]^{\mathfrak{A}}a = 0$, that is the i -th bit at position p at a is 1 and the i -th bit at position q at a is 0.

Definition 15. Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[V]$ -formula $[V(pq):\text{eq}](\mathbf{x})$ by:

$$[V(pq):\text{eq}](\mathbf{x}) = \bigwedge_{1 \leq i \leq t} [V(pq):\text{at-}i\text{-eq}](\mathbf{x}).$$

If \mathfrak{A} is a V -structure and $a \in A$, then:

$$\mathfrak{A} \models [V(pq):\text{eq}](a) \text{ iff } [V(p):\text{data}]^{\mathfrak{A}}a = [V(q):\text{data}]^{\mathfrak{A}}a.$$

2 Counter setups

Definition 16. Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[V]$ -formula $[V(pq):\text{less}](x)$ by:

$$[V(pq):\text{less}](x) = \bigvee_{1 \leq j \leq t} [V(pq):\text{at-}j\text{-eq-01}](x) \wedge \bigwedge_{j < k \leq t} [V(pq):\text{at-}k\text{-eq}](x).$$

If \mathfrak{A} is a V-structure and $a \in A$, then:

$$\mathfrak{A} \models [V(pq):\text{less}](a) \text{ iff } [V(p):\text{data}]^{\mathfrak{A}}a < [V(q):\text{data}]^{\mathfrak{A}}a.$$

Definition 17. Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[V]$ -formula

$$[V(pq):\text{succ}](x) = \bigvee_{1 \leq j \leq t} \bigwedge_{1 \leq i < j} [V(pq):\text{at-}i\text{-eq-10}](x) \wedge [V(pq):\text{at-}j\text{-eq-01}](x) \wedge \bigwedge_{j < k \leq t} [V(pq):\text{at-}k\text{-eq}](x).$$

If \mathfrak{A} is a V-structure and $a \in A$, then:

$$\mathfrak{A} \models [V(pq):\text{succ}](a) \text{ iff } [V(q):\text{data}]^{\mathfrak{A}}a = 1 + [V(p):\text{data}]^{\mathfrak{A}}a.$$

2.4 Permutations

Let $n \in \mathbb{N}^+$. An n -permutation setup $P = \langle \mathbf{u}_{11}, \mathbf{u}_{12}, \dots, \mathbf{u}_{nt} \rangle$ is just an n -dimensional t -bit vector setup, where $t = \|n\|$ is the bitsize of n . Recall that the set \mathbb{S}_n of permutations of $[1, n]$ is a subset of \mathbb{B}_t^n .

Definition 18. Define the quantifier-free $\mathcal{L}^1[P]$ -sentence $[P:\text{alldiff}]$ by:

$$[P:\text{alldiff}] = \forall x \bigwedge_{1 \leq p < q \leq n} \neg [P(pq):\text{eq}](x).$$

If \mathfrak{A} is a P-structure then $\mathfrak{A} \models [P:\text{alldiff}]$ iff $[P(p):\text{data}]^{\mathfrak{A}}a \neq [P(q):\text{data}]^{\mathfrak{A}}a$ for all $a \in A$ and $p \neq q \in [1, n]$.

Definition 19. Define the quantifier-free $\mathcal{L}^1[P]$ -sentence $[P:\text{perm}]$ by:

$$[P:\text{perm}] = [P:\text{betw-1-n}] \wedge [P:\text{alldiff}].$$

If \mathfrak{A} is a P-structure then $\mathfrak{A} \models [P:\text{perm}]$ iff $[P:\text{data}]^{\mathfrak{A}}a \in \mathbb{S}_n$ for all $a \in A$.

If A is a nonempty set and $\text{data} : A \rightarrow \mathbb{S}_n$ is any function, then there is a P-structure $\mathfrak{A} \models [P:\text{perm}]$ over A such that $[P:\text{data}]^{\mathfrak{A}} = \text{data}$.

3 Equivalence relations

An *equivalence relation* $E \subseteq A \times A$ on A is a relation that is reflexive, symmetric and transitive. The set of *equivalence classes* of E is $\mathcal{E}E = \{E[a] \mid a \in A\}$.

Let $E = \langle e \rangle$ be a predicate signature consisting of a single binary predicate symbol e . Define the $\mathcal{L}^2[E]$ -sentence $[e:\text{refl}]$ by:

$$[e:\text{refl}] = \forall x e(x, x).$$

Define the $\mathcal{L}^2[E]$ -sentence $[e:\text{symm}]$ by:

$$[e:\text{symm}] = \forall x \forall y e(x, y) \rightarrow e(y, x).$$

Define the $\mathcal{L}^3[E]$ -sentence $[e:\text{trans}]$ by:

$$[e:\text{trans}] = \forall x \forall y \forall z (e(x, y) \wedge e(y, z)) \rightarrow e(x, z).$$

Define the $\mathcal{L}^3[E]$ -sentence $[e:\text{equiv}]$ by:

$$[e:\text{equiv}] = [e:\text{refl}] \wedge [e:\text{symm}] \wedge [e:\text{trans}].$$

Let \mathfrak{A} be an E -structure and let $E = e^{\mathfrak{A}}$. Then E is reflexive iff $\mathfrak{A} \models [e:\text{refl}]$; E is symmetric iff $\mathfrak{A} \models [e:\text{symm}]$; E is transitive iff $\mathfrak{A} \models [e:\text{trans}]$; E is an equivalence on A iff $\mathfrak{A} \models [e:\text{equiv}]$. It can be shown that transitivity and equivalence cannot be defined in the two-variable fragment with counting $\mathcal{C}^2[E]$.

3.1 Two equivalence relations in agreement

Definition 20. Let $\langle D, E \rangle \subseteq A \times A$ be a sequence of two equivalence relations on A . The relation D is *finer* than the relation E if every equivalence class of D is a subset of some equivalence class of E . Equivalently, $D \subseteq E$. Equivalently,

$$(\forall a \in A)(\forall b \in A) D(a, b) \rightarrow E(a, b).$$

If D is finer than E , then E is coarser than D . The sequence $\langle D, E \rangle$ is a sequence of equivalence relations on A in *refinement* if D is finer E .

The sequence $\langle D, E \rangle$ is a sequence of equivalence relations in *global agreement* if either D is finer than E or E is finer than D .

The sequence $\langle D, E \rangle$ is a sequence of equivalence relations in *local agreement* if for every $a \in A$, either $D[a] \subseteq E[a]$ or $E[a] \subseteq D[a]$. Equivalently, no two equivalence classes $E[a]$ and $D[b]$ properly intersect. Equivalently

$$(\forall a \in A) [(\forall b \in A) D(a, b) \rightarrow E(a, b)] \vee [(\forall b \in A) E(a, b) \rightarrow D(a, b)].$$

3 Equivalence relations

Let $E = \langle \mathbf{d}, \mathbf{e} \rangle$ be a predicate signature consisting of the two binary predicate symbols \mathbf{d} and \mathbf{e} . Let \mathfrak{A} is an E -structure and suppose that \mathbf{d} and \mathbf{e} are interpreted in \mathfrak{A} as equivalence relations on A . Let $D = \mathbf{d}^{\mathfrak{A}}$ and $E = \mathbf{e}^{\mathfrak{A}}$ be the interpretations of the two symbols.

Definition 21. Define the $\mathcal{L}^2[E]$ -sentence $[\mathbf{d}, \mathbf{e}:\text{refine}]$ by:

$$[\mathbf{d}, \mathbf{e}:\text{refine}] = \forall x \forall y (\mathbf{d}(x, y) \rightarrow \mathbf{e}(x, y)).$$

Then $\langle D, E \rangle$ is in refinement iff $\mathfrak{A} \models [\mathbf{d}, \mathbf{e}:\text{refine}]$.

Definition 22. Define the $\mathcal{L}^2[E]$ -sentence $[\mathbf{d}, \mathbf{e}:\text{global}]$ by:

$$[\mathbf{d}, \mathbf{e}:\text{global}] = [\mathbf{d}, \mathbf{e}:\text{refine}] \vee [\mathbf{e}, \mathbf{d}:\text{refine}].$$

Then $\langle D, E \rangle$ is in global agreement iff $\mathfrak{A} \models [\mathbf{d}, \mathbf{e}:\text{global}]$.

Definition 23. Define the $\mathcal{L}^2[E]$ -sentence $[\mathbf{d}, \mathbf{e}:\text{local}]$ by:

$$[\mathbf{d}, \mathbf{e}:\text{local}] = \forall x ((\forall y \mathbf{d}(x, y) \rightarrow \mathbf{e}(x, y)) \vee (\forall y \mathbf{e}(x, y) \rightarrow \mathbf{d}(x, y))).$$

Then $\langle D, E \rangle$ is in global agreement iff $\mathfrak{A} \models [\mathbf{d}, \mathbf{e}:\text{local}]$.

Lemma 1. If $\langle D, E \rangle \subseteq A \times A$ is a sequence two equivalence relations on A , then it is in local agreement iff $L = D \cup E$ is an equivalence relation on A .

Proof. The union of two equivalence relations on A is a reflexive and symmetric relation.

First suppose that D and E are in local agreement. We claim that L is transitive. Let $a, b, c \in A$ be such that $(a, b) \in L$ and $(b, c) \in L$. Since D and E are in local agreement, without loss of generality $D[b] \subseteq E[b]$. Since $(a, b) \in L$, either $a \in D[b] \subseteq E[b]$ or $a \in E[b]$. Similarly $c \in E[b]$. Therefore $(a, c) \in E \subseteq L$.

Next suppose that L is an equivalence relation, let $b \in A$ and assume towards a contradiction that $D[b] \not\subseteq E[b]$ and $E[b] \not\subseteq D[b]$. There is some $a \in D[b] \setminus E[b]$ and $c \in E[b] \setminus D[b]$. Then $(a, b) \in D \subseteq L$ and $(b, c) \in E \subseteq L$, hence $(a, c) \in L$. Without loss of generality $(a, c) \in E$. Since $c \in E[b]$, we have $a \in E[b]$ — a contradiction. \square

3.2 Many equivalence relations in agreement

Let e be a positive natural number.

Definition 24. Let $\langle E_1, E_2, \dots, E_e \rangle \subseteq A \times A$ be a sequence of equivalence relations on A .

The sequence is in refinement if $E_1 \subseteq E_2 \subseteq \dots \subseteq E_e$.

The sequence is in global agreement if the equivalence relations form a chain under inclusion, that is for all $i, j \in [1, e]$, either $E_i \subseteq E_j$ or $E_j \subseteq E_i$. Equivalently, there is a (not necessarily unique) permutation $\nu \in \mathbb{S}_e$ such that $E_{\nu(1)} \subseteq E_{\nu(2)} \subseteq \dots \subseteq E_{\nu(e)}$.

The sequence is in local agreement if for every element $a \in A$, the equivalence classes $E_1[a], E_2[a], \dots, E_e[a]$ form a chain under inclusion. Equivalently, no two equivalence classes $E_i[a]$ and $E_j[b]$ properly intersect.

3.2 Many equivalence relations in agreement

Let $E = \langle e_1, e_2, \dots, e_e \rangle$ be a predicate signature consisting of e binary predicate symbols. Let \mathfrak{A} be an E -structure and suppose that the symbols e_i are interpreted as equivalence relations on A . Let $E_i = e_i^{\mathfrak{A}}$.

Definition 25. Define the $\mathcal{L}^2[E]$ -sentence $[e_1, e_2, \dots, e_e:\text{refine}]$ by:

$$[e_1, e_2, \dots, e_e:\text{refine}] = \forall x \forall y \bigwedge_{1 \leq i < e} e_i(x, y) \rightarrow e_{i+1}(x, y).$$

Then $\langle E_1, E_2, \dots, E_e \rangle$ is in refinement iff $\mathfrak{A} \models [e_1, e_2, \dots, e_e:\text{refine}]$.

Definition 26. Define the $\mathcal{L}^2[E]$ -sentence $[e_1, e_2, \dots, e_e:\text{global}]$ by:

$$[e_1, e_2, \dots, e_e:\text{global}] = \bigvee_{\nu \in \mathbb{S}_e} [e_{\nu(1)}, e_{\nu(2)}, \dots, e_{\nu(e)}:\text{refine}].$$

Then $\langle E_1, E_2, \dots, E_e \rangle$ is in global agreement iff $\mathfrak{A} \models [e_1, e_2, \dots, e_e:\text{global}]$. Note that the length of the formula $[e_1, e_2, \dots, e_e:\text{global}]$ grows exponentially as e grows.

Definition 27. Define the $\mathcal{L}^2[E]$ -sentence $[e_1, e_2, \dots, e_e:\text{local}]$ by:

$$[e_1, e_2, \dots, e_e:\text{local}] = \forall x \bigvee_{\nu \in \mathbb{S}_e} \forall y \bigwedge_{1 \leq i < e} e_{\nu(i)}(x, y) \rightarrow e_{\nu(i+1)}(x, y).$$

Then $\langle E_1, E_2, \dots, E_e \rangle$ is in local agreement iff $\mathfrak{A} \models [e_1, e_2, \dots, e_e:\text{local}]$. Note that the length of the formula $[e_1, e_2, \dots, e_e:\text{local}]$ grows exponentially as e grows.

Let $E = \langle E_1, E_2, \dots, E_e \rangle \subseteq A \times A$ be a sequence of equivalence relations on A .

Theorem 3. The sequence E is in local agreement iff the union $\cup S$ of any nonempty subset $S \subseteq E$ is an equivalence relation on A .

Proof. First suppose that the equivalence relations E_i are in local agreement. We show that the union $\cup S$ of arbitrary nonempty subset $S = \{E_{i(1)}, E_{i(2)}, \dots, E_{i(s)}\}$ is an equivalence relation by induction on s , the cardinality of S . If $s = 1$, this statement is trivial. Suppose $s > 1$. By the induction hypothesis, $D = \cup \{E_{i(1)}, E_{i(2)}, \dots, E_{i(s-1)}\}$ is an equivalence relation on A . We claim that D and $E_{i(s)}$ are in local agreement. Indeed, let $a \in A$ be arbitrary and consider $D[a] = E_{i(1)}[a] \cup E_{i(2)}[a] \cup \dots \cup E_{i(s-1)}[a]$ and $E_{i(s)}[a]$. Since all equivalences E_k are in local agreement, either $E_{i(s)}[a] \subseteq E_{i(j)}[a]$ for some $j \in [1, s-1]$, or $E_{i(j)}[a] \subseteq E_{i(s)}[a]$ for all $j \in [1, s-1]$. In the first case $E_{i(s)}[a] \subseteq D[a]$; in the second case $D[a] \subseteq E_{i(s)}[a]$. Thus D and $E_{i(s)}$ are in local agreement. By lemma 1, $\cup S = D \cup E_{i(s)}$ is an equivalence relation on A .

Next suppose that the equivalences are not in local agreement. There is an element $a \in A$ such that $\{E_i[a] \mid i \in [1, e]\}$ is not a chain. There are $i, j \in [1, e]$ such that $E_i[a] \not\subseteq E_j[a]$ and $E_j[a] \not\subseteq E_i[a]$. Thus E_i and E_j are not in local agreement. By lemma 1, the union $E_i \cup E_j$ is not an equivalence relation on A . \square

Suppose that the sequence $E = \langle E_1, E_2, \dots, E_e \rangle$ is in local agreement.

3 Equivalence relations

Definition 28. An index set is an element $I \in \wp^+[1, e]$. Define $E[\cdot] : \wp^+[1, e] \rightarrow \wp^+E$ by:

$$E[I] = \{E_i \mid i \in I\}.$$

The level sequence $L = \langle L_1, L_2, \dots, L_e \rangle \subseteq A \times A$ of the sequence E is defined as follows. For $k \in [1, e]$:

$$L_k = \cap \left\{ \cup E[I] \mid I \in \wp^k[1, e] \right\}.$$

Remark 2. All L_k are equivalence relations on A .

Proof. Let $I \in \wp^k[1, e]$ be a k -index set, where $k \geq 1$. By theorem 3, $\cup E[I]$ is an equivalence relation on A . Since intersection of equivalence relations on A is again an equivalence relation on A , the level $L_k = \cap \left\{ \cup E[I] \mid I \in \wp^k[1, e] \right\}$ is an equivalence relation on A . \square

Remark 3. The level sequence L is a sequence of equivalence relations on A in refinement.

Proof. Let $i < j \in [1, e]$. Let $J \in \wp^j[1, e]$ be any j -index set. We claim that $L_i \subseteq \cup E[J]$. Indeed, choose some i -index set $I \subset J$. By the definition of L_i we have $L_i \subseteq \cup E[I] \subseteq \cup E[J]$. Hence $L_i \subseteq \cap \left\{ \cup E[J] \mid J \in \wp^j[1, e] \right\} = L_j$. \square

Let $a \in A$. Since the sequence E is in local agreement, there is a permutation $\nu \in \mathbb{S}_e$ such that:

$$E_{\nu(1)}[a] \subseteq E_{\nu(2)}[a] \subseteq \dots \subseteq E_{\nu(e)}[a]. \quad (3.1)$$

Lemma 2. If $\nu \in \mathbb{S}_e$ is a permutation satisfying eq. (3.1), then $L_{\nu^{-1}(i)}[a] = E_i[a]$ for all $i \in [1, e]$.

Proof. Let $k = \nu^{-1}(i)$, so $\nu(k) = i$. We claim that $L_k[a] = E_i[a]$. First, consider the k -index set $K = \{\nu(1), \nu(2), \dots, \nu(k)\}$. By the definition of L_k , followed by eq. (3.1), we have $L_k[a] \subseteq \cup E[K][a] = E_{\nu(k)}[a] = E_i[a]$. Next, let $K \subseteq \wp^k[1, e]$ be any k -index set. By the pigeonhole principle, there is some $k' \geq k$ such that $k' \in K$. By eq. (3.1) we have:

$$E_i[a] = E_{\nu(k)}[a] \subseteq E_{\nu(k')}[a] \subseteq \cup E[K][a].$$

Hence $E_i[a] \subseteq \cap \left\{ \cup E[K][a] \mid K \in \wp^k[1, e] \right\} = L_k[a]$. \square

4 Reductions

We restrict our attention to binary predicate signatures, consisting of unary and binary predicate symbols only. To denote various logics with builtin equivalence symbols, we use the notation

$$\Lambda_p^v e E_a$$

where:

- $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ is the *ground logic*
- v , if given, bounds the number of variables
- e , if given, bounds the number of builtin equivalence symbols
- $a \in \{\text{refine}, \text{global}, \text{local}\}$, if given, gives the agreement condition between the builtin equivalence symbols
- p , the *signature power*, specifies constraints on the signature:
 - if $p = 0$, the signature consists of only constantly many unary predicate symbols in addition to the builtin equivalence symbols
 - if $p = 1$, the signature consists of unboundedly many unary predicate symbols in addition to the builtin equivalence symbols
 - if p is not given, the signature consists of unboundedly many unary and binary predicate symbols in addition to the builtin equivalence symbols. This is the commonly investigated fragment with respect to satisfiability of the two-variable logics with or without counting quantifiers.

For example \mathcal{L}_1 is the monadic first-order logic, featuring only unary predicate symbols. $\mathcal{L}_0 1E$ is the first-order logic of a single equivalence relation. \mathcal{C}^2 is the two-variable logic with counting quantifiers, featuring unary and binary predicate symbols. $\mathcal{L}^2 2E$ is the two-variable logic, featuring unary, binary predicate symbols and two builtin equivalence symbols. $\mathcal{C}_1^2 2E_{\text{local}}$ is the two-variable logic with counting quantifiers, featuring unary predicate symbols and two builtin equivalence symbols in local agreement. $\mathcal{L}_1 E_{\text{global}}$ is the monadic first-order logic featuring many equivalence symbols in global agreement.

When we working with a concrete logic, for example $\mathcal{C}_2^2 2E_{\text{local}}$, we implicitly assume an appropriate generic predicate signature Σ for it. In this case, there are two builtin equivalence symbols d and e in Σ and in addition Σ contains arbitrary many unary

4 Reductions

and binary predicate symbols. The *intended interpretation* of the builtin equivalence symbols is fixed by an appropriate condition θ . In this case:

$$\theta = [d:\text{equiv}] \wedge [e:\text{equiv}] \wedge [d, e:\text{local}].$$

Note that the interpretation condition might in general be a first-order formula outside the logic in interest, as in this case, since for instance $[d:\text{equiv}]$ uses the variables x, y and z and the logic $\mathcal{C}_2^2\text{E}_{\text{local}}$ is a two-variable logic. Recall that when talking about semantics, we include the intended interpretation condition in the definition of Σ -structures.

4.1 Global agreement to refinement

In this section we demonstrate how (finite) satisfiability in logics featuring builtin equivalence symbols in global agreement reduces to (finite) satisfiability in logics featuring builtin equivalence symbols in refinement. Our strategy is to encode the permutation of the builtin equivalence symbols in global agreement that turns them in refinement into a permutation setup.

Fix an arbitrary ground logic $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ and think of Σ as a predicate signature for the logics $\Lambda\text{E}_{\text{global}}$ and $\Lambda\text{E}_{\text{refine}}$. The e builtin equivalence symbols of Σ are e_1, e_2, \dots, e_e .

Let φ be a $\Lambda[\Sigma]$ -sentence. The class of $\Lambda\text{E}_{\text{refine}}$ -structures satisfying φ coincides with the class of $\Lambda\text{E}_{\text{global}}$ -structures satisfying:

$$\varphi \wedge [e_1, e_2, \dots, e_e:\text{refine}].$$

Hence:

$$(\text{FIN})\text{SATA}\Lambda\text{E}_{\text{refine}} \leq^{\text{PTIME}} (\text{FIN})\text{SATA}\Lambda\text{E}_{\text{global}}.$$

Since the length of the formula $[e_1, e_2, \dots, e_e:\text{refine}]$ grows polynomially as e grows:

$$(\text{FIN})\text{SATA}\Lambda\text{E}_{\text{refine}} \leq^{\text{PTIME}} (\text{FIN})\text{SATA}\Lambda\text{E}_{\text{global}}.$$

Consider the opposite direction. Let $P = \langle u_{11}, u_{12}, \dots, u_{et} \rangle$ be an e -permutation setup (where $t = \|e\|$).

Definition 29. Define the $\mathcal{L}^2[P]$ -sentence $[P:\text{alleq}]$ by:

$$[P:\text{alleq}] = \forall x \forall y \bigwedge_{1 \leq i \leq e} [P:\text{eq-}i](x, y).$$

If \mathfrak{A} is a P -structure, then $\mathfrak{A} \models [P:\text{alleq}]$ iff $[P:\text{data}]^{\mathfrak{A}}a = [P:\text{data}]^{\mathfrak{A}}b$ for all $a, b \in A$. If A is a nonempty set and $v \in \mathbb{B}_t^e$ is any e -dimensional t -vector, there is a P -structure \mathfrak{A} over A such that $\mathfrak{A} \models [P:\text{alleq}]$ and $[P:\text{data}]^{\mathfrak{A}}a = v$ for all $a \in A$.

Definition 30. Define the $\mathcal{L}^2[P]$ -sentence $[P:\text{globperm}]$ by:

$$[P:\text{globperm}] = [P:\text{perm}] \wedge [P:\text{alleq}].$$

4.1 Global agreement to refinement

If \mathfrak{A} be a P-structure then $\mathfrak{A} \models [\text{P:globperm}]$ iff there is a permutation $\nu \in \mathbb{S}_e$ such that $[\text{P:data}]^{\mathfrak{A}}a = \nu$ for all $a \in A$.

If A be a nonempty set and $\nu \in \mathbb{S}_e$ is any permutation, there is a P-structure \mathfrak{A} over A such that $\mathfrak{A} \models [\text{P:globperm}]$ and $[\text{P:data}]^{\mathfrak{A}}a = \nu$ for all $a \in A$.

Let $L = \langle l_1, l_2, \dots, l_e \rangle + P$ be a predicate signature consisting of the binary predicate symbols l_k in addition to the symbols from P .

Definition 31. For $i \in [1, e]$, define the quantifier-free $\mathcal{L}^2[L]$ -formula $[\text{L:eg-}i](x, y)$ by:

$$[\text{L:eg-}i](x, y) = \bigwedge_{1 \leq k \leq e} [\text{P:eq-}k-i](x) \rightarrow l_k(x, y).$$

Remark 4. Let \mathfrak{A} be an L-structure and suppose that $\mathfrak{A} \models [\text{P:globperm}]$ and that the binary symbols l_k are interpreted as equivalence relations on A in refinement. Recall that there is a permutation $\nu \in \mathbb{S}_e$ such that $[\text{P:data}]^{\mathfrak{A}}a = \nu$ for all $a \in A$. Then for all $i \in [1, e]$:

$$[\text{L:eg-}i]^{\mathfrak{A}} = l_{\nu^{-1}(i)}^{\mathfrak{A}}.$$

In particular, $\langle [\text{L:eg-}1]^{\mathfrak{A}}, [\text{L:eg-}2]^{\mathfrak{A}}, \dots, [\text{L:eg-}e]^{\mathfrak{A}} \rangle$ is a sequence of equivalence relations on A in global agreement.

Proof. Let $k = \nu^{-1}(i)$, so $\nu(k) = i$ and $[\text{P}(k):\text{data}]^{\mathfrak{A}}a = i$. Since ν is a permutation, for every $k' \in [1, e]$:

$$\mathfrak{A} \models [\text{P:eq-}k'-i](a) \text{ iff } [\text{P}(k'):\text{data}]^{\mathfrak{A}}a = i \text{ iff } k' = k. \quad (4.1)$$

Let $a, b \in A$. First suppose that $\mathfrak{A} \models [\text{L:eg-}i](a, b)$. By eq. (4.1) we have $\mathfrak{A} \models [\text{P:eq-}k-i](a)$, hence $\mathfrak{A} \models l_k(a, b)$.

Now suppose that $\mathfrak{A} \models \neg[\text{L:eg-}i](a, b)$. There is some $k' \in [1, e]$ such that:

$$\mathfrak{A} \models \neg([\text{P:eq-}k'-i](a) \rightarrow l_{k'}(a, b)) \equiv [\text{P:eq-}k'-i](a) \wedge \neg l_{k'}(a, b).$$

By eq. (4.1) we have $k' = k$, hence $\mathfrak{A} \models \neg l_k(a, b)$. □

Let $E = \langle e_1, e_2, \dots, e_e \rangle$ be a predicate signature consisting of the binary predicate symbols e_i . Let Σ be a predicate signature enriching E and not containing any symbols from L . Let $\Sigma' = \Sigma \cup L$ and $L' = \Sigma' - E$.

Definition 32. Define the syntactic operation $\text{gtr} : \Lambda[\Sigma] \rightarrow \Lambda[L']$ by:

$$\text{gtr } \varphi = \varphi' \wedge [\text{P:globperm}],$$

where φ' is obtained from φ by replacing all occurrences of a subformula of the form $e_i(x, y)$ by the formula $[\text{L:eg-}i](x, y)$, where x and y are (not necessarily distinct) variables and $i \in [1, e]$.

4 Reductions

Remark 5. Let φ be a $\Lambda[\Sigma]$ -formula and let \mathfrak{A} be a Σ -structure. Suppose that $\mathfrak{A} \models \varphi$ and that the symbols e_1, e_2, \dots, e_e are interpreted in \mathfrak{A} as equivalence relations on A in global agreement. Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \text{gtr } \varphi$ and that the symbols l_1, l_2, \dots, l_e are interpreted in \mathfrak{A}' as equivalence relations on A in refinement.

Proof. There is a permutation $\nu \in \mathbb{S}_e$ such that $e_{\nu(1)}^{\mathfrak{A}} \subseteq e_{\nu(2)}^{\mathfrak{A}} \subseteq \dots \subseteq e_{\nu(e)}^{\mathfrak{A}}$. Consider an enrichment \mathfrak{A}' of \mathfrak{A} to a Σ' -structure where $l_k^{\mathfrak{A}'} = e_{\nu(k)}^{\mathfrak{A}}$, so the interpretations of l_k in \mathfrak{A}' are equivalence relations on A in refinement. We can interpret the unary predicate symbols from permutation setup P in \mathfrak{A}' so that $\mathfrak{A}' \models [P:\text{globperm}]$ and $[P:\text{data}]^{\mathfrak{A}'} a = \nu$ for all $a \in A$. By remark 4, for every $i \in [1, e]$:

$$[L:\text{eg-}i]^{\mathfrak{A}'} = l_{\nu^{-1}(i)}^{\mathfrak{A}'} = e_{\nu(\nu^{-1}(i))}^{\mathfrak{A}'} = e_i^{\mathfrak{A}'} = e_i^{\mathfrak{A}}.$$

Hence $\mathfrak{A}' \models \forall x \forall y e_i(x, y) \leftrightarrow [L:\text{eg-}i](x, y)$. Since $\mathfrak{A}' \models \varphi$ we have $\mathfrak{A}' \models \text{gtr } \varphi$. \square

Remark 6. Let φ be a $\Lambda[\Sigma]$ -formula and let \mathfrak{A} be an L' -structure. Suppose that $\mathfrak{A} \models \text{gtr } \varphi$ and that the symbols l_1, l_2, \dots, l_e are interpreted in \mathfrak{A} as equivalence relations on A in refinement. Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \varphi$ and that the symbols e_1, e_2, \dots, e_e are interpreted as equivalence relations on A in global agreement in \mathfrak{A}' .

Proof. Consider an enrichment \mathfrak{A}' of \mathfrak{A} to a Σ' -structure where $e_i^{\mathfrak{A}'} = [L:\text{eg-}i]^{\mathfrak{A}'}$. By remark 4, $\langle e_1^{\mathfrak{A}'}, e_2^{\mathfrak{A}'}, \dots, e_e^{\mathfrak{A}'} \rangle$ is a sequence of equivalence relations on A in global agreement. For every $i \in [1, e]$ we have $\mathfrak{A}' \models \forall x \forall y e_i(x, y) \leftrightarrow [L:\text{eg-}i](x, y)$ by definition. Since $\mathfrak{A}' \models \text{gtr } \varphi$ we have $\mathfrak{A}' \models \varphi$. \square

The last two remarks show that a $\Lambda e E_{\text{global}}$ -formula φ has essentially the same models as the $\Lambda e E_{\text{refine}}$ -formula $\text{gtr } \varphi$, so we have shown:

Proposition 1. The logic $\Lambda e E_{\text{global}}$ has the finite model property iff the logic $\Lambda e E_{\text{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent: $(\text{FIN})\text{SATA} \Lambda e E_{\text{global}} =^{\text{PTIME}} (\text{FIN})\text{SATA} \Lambda e E_{\text{refine}}$.

Since the relative size of $\text{gtr } \varphi$ with respect to φ grows polynomially as e grows, we have shown:

Proposition 2. The logic $\Lambda E_{\text{global}}$ has the finite model property iff the logic $\Lambda E_{\text{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent: $(\text{FIN})\text{SATA} \Lambda E_{\text{global}} =^{\text{PTIME}} (\text{FIN})\text{SATA} \Lambda E_{\text{refine}}$.

The reduction is two-variable first-order and uses additional (et) unary predicate symbols for the permutation setup P , so it is also valid for the two-variable fragments $\Lambda_0^2 e E_a$, $\Lambda_1^2 e E_a$ and $\Lambda_1^2 E_a$ for $a \in \{\text{global}, \text{refine}\}$ respectively.

4.2 Local agreement to refinement

In this section demonstrate how (finite) satisfiability in logics featuring builtin equivalence symbols in local agreement reduces to (finite) satisfiability in logics featuring builtin equivalence symbols in refinement. Our strategy is to start with the level equivalences which form a refinement, and to encode a permutation specifying the local chain structure for every element in the structure.

Fix an arbitrary ground logic $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ and think of Σ as a predicate signature for the logics $\Lambda eE_{\text{local}}$ and $\Lambda eE_{\text{refine}}$. The e builtin equivalence symbols of Σ are e_1, e_2, \dots, e_e .

Let φ be a $\Lambda[\Sigma]$ -sentence. The class of $\Lambda eE_{\text{refine}}$ -structures satisfying φ coincides with the class of $\Lambda eE_{\text{local}}$ -structures satisfying

$$\varphi \wedge [e_1, e_2, \dots, e_e : \text{refine}].$$

Hence:

$$(\text{FIN})\text{SATA} \Lambda eE_{\text{refine}} \leq^{\text{PTIME}} (\text{FIN})\text{SATA} \Lambda eE_{\text{local}}.$$

Since the size of the formula $[e_1, e_2, \dots, e_e : \text{refine}]$ grows polynomially as e grows, we have:

$$(\text{FIN})\text{SATA} \Lambda eE_{\text{refine}} \leq^{\text{PTIME}} (\text{FIN})\text{SATA} \Lambda eE_{\text{local}}.$$

Consider the opposite direction. Let $E = \langle e_1, e_2, \dots, e_e \rangle$ be a predicate signature consisting of the binary predicate symbols e_i (not necessarily interpreted as equivalences). Let \mathfrak{A} be an E -structure and suppose that the symbols e_i are interpreted in \mathfrak{A} as equivalence relations on A in local agreement. Let $E_i = e_i^{\mathfrak{A}}$ for $i \in [1, e]$. Recall that for every $a \in A$ there is a permutation $\nu \in \mathbb{S}_e$ satisfying eq. (3.1):

$$E_{\nu(1)}[a] \subseteq E_{\nu(2)}[a] \subseteq \dots \subseteq E_{\nu(e)}[a]. \quad (4.2)$$

Definition 33. The characteristic E -permutation of a in \mathfrak{A} is the lexicographically smallest permutation $\nu \in \mathbb{S}_e$ satisfying eq. (4.2). Define the function $[\text{E:chperm}]^{\mathfrak{A}} : A \rightarrow \mathbb{S}_e$ so that $[\text{E:chperm}]^{\mathfrak{A}} a$ is the characteristic E -permutation of a in \mathfrak{A} .

Remark 7. Let $a \in A$, $\nu = [\text{E:chperm}]^{\mathfrak{A}} a$ and $i < j \in [1, e]$. Suppose that $E_{\nu(i)}[a] = E_{\nu(j)}[a]$. Then $\nu(i) < \nu(j)$.

Proof. Suppose not. For some $i < j \in [1, e]$ we have $\nu(i) \geq \nu(j)$. Since ν is a permutation and $i \neq j$, we have $\nu(i) > \nu(j)$. Since $E_{\nu(i)}[a] = E_{\nu(j)}[a]$, by eq. (4.2) we have $E_{\nu(k)} = E_{\nu(i)}$ for all $k \in [i, j]$. Consider the permutation $\mu \in \mathbb{S}_e$ defined by:

$$\mu(k) = \begin{cases} \nu(j) & \text{if } k = i \\ \nu(i) & \text{if } k = j \\ \nu(k) & \text{otherwise.} \end{cases}$$

Clearly, μ is a permutation satisfying eq. (4.2) that is lexicographically smaller than ν — a contradiction. \square

4 Reductions

Remark 8. Let $a, b \in A$ and let $\alpha = [\text{E:chperm}]^{\mathfrak{A}} a$ and $\beta = [\text{E:chperm}]^{\mathfrak{A}} b$. Let $i \in [1, e]$ and suppose that $(a, b) \in E_i$. Then $\alpha^{-1}(i) = \beta^{-1}(i)$.

Proof. Suppose not, so $\alpha^{-1}(i) \neq \beta^{-1}(i)$. Let $p = \alpha^{-1}(i)$ and $q = \beta^{-1}(i)$. Without loss of generality, suppose that $p < q$. Thus p is the position of i in the permutation α and $q > p$ is the position of i in the permutation β . By the pigeonhole principle, there is $k \in [1, e]$ that occurs after i in α and before j in β : $p < \alpha^{-1}(k)$ and $\beta^{-1}(k) < q$. Since β is the characteristic E-permutation of b in \mathfrak{A} , by eq. (4.2) we have $E_k[b] \subseteq E_i[b]$. Since $(a, b) \in E_i$, we have $E_k[b] \subseteq E_i[a]$. Since $E_k[b] \subseteq E_i[a]$ are equivalence classes, $E_k[a] \subseteq E_i[a]$. Since k occurs after i in α , which is the characteristic E-permutation of a in \mathfrak{A} , by eq. (4.2) we have $E_k[a] = E_i[a]$. By remark 7, $i < k$. By the contrapositive of remark 7, $E_k[b] = E_i[b]$ is impossible. Since k occurs before i in β , by eq. (4.2) we have $E_k[b] \subset E_i[b]$. Hence

$$E_k[b] \subset E_i[b] = E_i[a] = E_k[a]$$

— a contradiction — since the equivalence classes $E_k[b]$ and $E_k[a]$ are either equal or disjoint. \square

Let $L = \langle L_1, L_2, \dots, L_e \rangle \subseteq A \times A$ be the levels of $E = \langle E_1, E_2, \dots, E_e \rangle$. Recall that by remark 3, the levels are equivalence relations on A in refinement.

Remark 9. Let $a \in A$, $\alpha = [\text{E:chperm}]^{\mathfrak{A}} a$ and let $k \in [1, e]$. Then $L_k[a] = E_{\alpha(k)}[a]$.

Proof. Since α satisfies eq. (4.2), by lemma 2:

$$L_k[a] = L_{\alpha^{-1}(\alpha(k))}[a] = E_{\alpha(k)}[a].$$

\square

Remark 10. Let $a, b \in A$, $\alpha = [\text{E:chperm}]^{\mathfrak{A}} a$, $\beta = [\text{E:chperm}]^{\mathfrak{A}} b$ and $k \in [1, e]$. Suppose that $(a, b) \in L_k$. Then $\alpha(k) = \beta(k)$. That is, the elements connected at level k agree at position k in their characteristic permutations.

Proof. By remark 9, $L_k[a] = E_{\alpha(k)}[a]$, thus $(a, b) \in E_{\alpha(k)}$. By remark 7,

$$k = \alpha^{-1}(\alpha(k)) = \beta^{-1}(\alpha(k)).$$

Hence $\beta(k) = \alpha(k)$. \square

Let $P = \langle \mathbf{u}_{11}, \mathbf{u}_{12}, \dots, \mathbf{u}_{et} \rangle$ be an e -permutation setup. Let $L = \langle \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_e \rangle + P$ be a predicate signature containing the binary predicate symbols \mathbf{l}_k (not necessarily interpreted as equivalence relations) together with the symbols from P .

Definition 34. Define the $\mathcal{L}^2[L]$ -sentence $[\text{L:fixperm}]$ by:

$$[\text{L:fixperm}] = \forall \mathbf{x} \forall \mathbf{y} \bigwedge_{1 \leq k \leq e} \mathbf{l}_k(\mathbf{x}, \mathbf{y}) \rightarrow [P(k):\text{eq}](\mathbf{x}, \mathbf{y}).$$

Definition 35. Define the $\mathcal{L}^2[\mathbf{L}]$ -sentence $[\mathbf{L}:\text{locperm}]$ by:

$$[\mathbf{L}:\text{locperm}] = [\mathbf{P}:\text{perm}] \wedge [\mathbf{L}:\text{fixperm}].$$

Remark 11. Let \mathfrak{A} be an \mathbf{L} -structure and suppose that $\mathfrak{A} \models [\mathbf{L}:\text{locperm}]$. Let $a, b \in A$, $k \in [1, e]$ and suppose that $\mathfrak{A} \models \mathbf{l}_k(a, b)$. Let $\alpha = [\mathbf{P}:\text{data}]^{\mathfrak{A}}a$ and $\beta = [\mathbf{P}:\text{data}]^{\mathfrak{A}}b$ be the e -permutations at a and b , encoded by the permutation setup \mathbf{P} . Then $\alpha(k) = \beta(k)$.

Proof. Since $\mathfrak{A} \models [\mathbf{L}:\text{fixperm}]$ and $\mathfrak{A} \models \mathbf{l}_k(a, b)$, we have $\mathfrak{A} \models [\mathbf{P}:\text{eq-}k](a, b)$, which means $\alpha(k) = \beta(k)$. \square

Definition 36. For $i \in [1, e]$, define the quantifier-free $\mathcal{L}^2[\mathbf{L}]$ -formula $[\mathbf{L}:\text{el-}i]$ by:

$$[\mathbf{L}:\text{el-}i](\mathbf{x}, \mathbf{y}) = \bigwedge_{1 \leq k \leq n} [\mathbf{L}:\text{eq-}k-i](\mathbf{x}) \rightarrow \mathbf{l}_k(\mathbf{x}, \mathbf{y}).$$

Remark 12. Let \mathfrak{A} be an \mathbf{L} -structure and suppose that $\mathfrak{A} \models [\mathbf{L}:\text{locperm}]$ and that the binary symbols \mathbf{l}_k are interpreted in \mathfrak{A} as equivalence relations on A in refinement. Define $\nu : A \rightarrow \mathbb{S}_e$ by $\nu(a) = [\mathbf{P}:\text{data}]^{\mathfrak{A}}a$ for $a \in A$. Let $a \in A$ be arbitrary. Then for all $i \in [1, e]$:

$$[\mathbf{L}:\text{el-}i]^{\mathfrak{A}}[a] = \mathbf{l}_{\nu(a)^{-1}(i)}^{\mathfrak{A}}[a].$$

Proof. Let $E_i = [\mathbf{L}:\text{el-}i]^{\mathfrak{A}}$ and $L_i = \mathbf{l}_i^{\mathfrak{A}}$ for every $i \in [1, e]$. Let $i \in [1, e]$ be arbitrary. Let $\alpha = \nu(a)$ and $k = \alpha^{-1}(i)$, so $\alpha = [\mathbf{P}:\text{data}]^{\mathfrak{A}}a$ and $\alpha(k) = i$. We have to show that $E_i[a] = L_k[a]$. Since α is a permutation, for every $k' \in [1, e]$ we have:

$$\mathfrak{A} \models [\mathbf{P}:\text{eq-}k'-i](a) \text{ iff } \alpha(k') = i \text{ iff } k' = k. \quad (4.3)$$

First, suppose $b \in E_i[a]$. Then $\mathfrak{A} \models [\mathbf{L}:\text{el-}i](a, b)$ and by eq. (4.3) we have $\mathfrak{A} \models \mathbf{l}_k(a, b)$, hence $b \in L_k[a]$.

Next, suppose $b \notin E_i[a]$. Then $\mathfrak{A} \models \neg[\mathbf{L}:\text{el-}i](a, b)$, so there is some $k' \in [1, e]$ such that $\mathfrak{A} \models \neg([\mathbf{L}:\text{eq-}k'-i](a) \rightarrow \mathbf{l}_{k'}(a, b)) \equiv [\mathbf{L}:\text{eq-}k'-i](a) \wedge \neg\mathbf{l}_{k'}(a, b)$. By eq. (4.3) we have $k' = k$. Hence $\mathfrak{A} \models \neg\mathbf{l}_k(a, b)$, so $b \notin L_k[a]$. \square

Remark 13. Let \mathfrak{A} and ν are declared as in remark 12. Then the interpretations $\langle [\mathbf{L}:\text{el-}1]^{\mathfrak{A}}, [\mathbf{L}:\text{el-}2]^{\mathfrak{A}}, \dots, [\mathbf{L}:\text{el-}e]^{\mathfrak{A}} \rangle$ are equivalence relations on A in local agreement.

Proof. Let $E_i = [\mathbf{L}:\text{el-}i]^{\mathfrak{A}}$ and $L_i = \mathbf{l}_i^{\mathfrak{A}}$ for every $i \in [1, e]$. Let $i \in [1, e]$ be arbitrary. We check that E_i is reflexive, symmetric and transitive.

- For reflexivity, let $a \in A$. By remark 12, $E_i[a] = L_k[a]$ for $k = \nu(a)^{-1}(i)$. But $L_k[a]$ is an equivalence class, hence $a \in L_k[a]$, so $(a, a) \in E_i$.
- For symmetry, let $a, b \in A$ and $(a, b) \in E_i$. Let $k = \nu(a)^{-1}(i)$ so that $i = \nu(k)$. By remark 12, $E_i[a] = L_k[a]$. Thus $\mathfrak{A} \models \mathbf{l}_k(a, b)$ and by remark 11, $i = \nu(a)(k) = \nu(b)(k)$. By remark 12:

$$E_i[b] = [\mathbf{L}:\text{el-}i]^{\mathfrak{A}}[b] = \mathbf{l}_{\nu(b)^{-1}(i)}^{\mathfrak{A}}[b] = L_k[b] = L_k[a].$$

Since $a \in L_k[a] = E_i[b]$, we have $(b, a) \in E_i$.

4 Reductions

- For transitivity, continue the argument for symmetry. Let $c \in E_i[b]$. Then $c \in E_i[b] = L_k[a] = E_i[a]$, thus $(a, c) \in E_i$.

By remark 12, since the relations L_k are in refinement, we have that E_1, E_2, \dots, E_e are in local agreement. \square

Let $E = \langle e_1, e_2, \dots, e_e \rangle$ be a predicate signature consisting of binary predicate symbols. Let Σ be a predicate signature enriching E and not containing any symbols from L . Let $\Sigma' = \Sigma + L$ and $L' = \Sigma' - E$.

Definition 37. Define the syntactic operation $\text{ltr} : \Lambda[\Sigma] \rightarrow \Lambda[L']$ by:

$$\text{ltr } \varphi = \varphi' \wedge [L:\text{locperm}],$$

where φ' is obtained from φ by replacing all occurrences of a subformula of the form $e_i(x, y)$ by the formula $[L:\text{el-}i](x, y)$, where x and y are (not necessarily distinct) variable symbols and $i \in [1, e]$.

Remark 14. Let φ be a $\Lambda[\Sigma]$ -formula and let \mathfrak{A} be a Σ -structure. Suppose that $\mathfrak{A} \models \varphi$ and that the symbols e_1, e_2, \dots, e_e are interpreted in \mathfrak{A} as equivalence relations on A in local agreement. Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \text{ltr } \varphi$ and that the symbols l_1, l_2, \dots, l_e are interpreted in \mathfrak{A}' as equivalence relations on A in refinement.

Proof. Since the binary symbols e_1, e_2, \dots, e_e are interpreted as equivalence relations on A in local agreement in \mathfrak{A} , we may define the levels $L_1, L_2, \dots, L_e \subseteq A \times A$ and the characteristic E-permutation mapping $\nu = [E:\text{chperm}]^{\mathfrak{A}} : A \rightarrow \mathbb{S}_e$. Consider an enrichment \mathfrak{A}' of \mathfrak{A} where $l_i^{\mathfrak{A}'} = L_i$. By remark 3, L_i are equivalences on A in refinement. We interpret the unary symbols from the permutation setup P so that $[P:\text{data}]^{\mathfrak{A}'} a = \nu(a)$ for all $a \in A$. By remark 10, $\mathfrak{A}' \models [L:\text{fixperm}]$. By remark 12, followed by lemma 2, for every $i \in [1, e]$ and $a \in A$ we have:

$$[L:\text{el-}i]^{\mathfrak{A}'}[a] = l_{\nu(a)^{-1}(i)}^{\mathfrak{A}'}[a] = l_{\nu(a)(\nu(a)^{-1}(i))}^{\mathfrak{A}'}[a] = e_i^{\mathfrak{A}'}[a].$$

By remark 13, the interpretations $[L:\text{el-}i]$ are equivalence relations and since they have the same classes as the interpretations of e_i , we have $\mathfrak{A}' \models \forall x \forall y e_i(x, y) \leftrightarrow [L:\text{el-}i](x, y)$ and since $\mathfrak{A}' \models \varphi$, we have $\mathfrak{A}' \models \text{ltr } \varphi$. \square

Remark 15. Let φ be a $\Lambda[\Sigma]$ -formula and let \mathfrak{A} be an L' -structure. Suppose that $\mathfrak{A} \models \text{ltr } \varphi$ and that the symbols l_1, l_2, \dots, l_e are interpreted as equivalence relations on A in refinement in \mathfrak{A} . Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \varphi$ and that the binary symbols e_1, e_2, \dots, e_e are interpreted as equivalence relations on A in global agreement in \mathfrak{A}' .

Proof. Consider an enrichment \mathfrak{A}' of \mathfrak{A} to a Σ' -structure where $e_i^{\mathfrak{A}'} = [L:\text{el-}i]^{\mathfrak{A}}$. By remark 13, $e_i^{\mathfrak{A}'}$ are equivalence relations on A in local agreement. For every $i \in [1, e]$ we have $\mathfrak{A}' \models \forall x \forall y e_i(x, y) \leftrightarrow [L:\text{el-}i](x, y)$ by definition. Since $\mathfrak{A}' \models \text{ltr } \varphi$ we have $\mathfrak{A}' \models \varphi$. \square

The last two remarks show that a $\Lambda eE_{\text{local}}$ -formula φ has essentially the same models as the $\Lambda eE_{\text{refine}}$ -formula $\text{ltr } \varphi$, so we have shown:

Proposition 3. *The logic $\Lambda eE_{\text{local}}$ has the finite model property iff the logic $\Lambda eE_{\text{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent: $(\text{FIN})\text{SAT}\Lambda eE_{\text{local}} =^{\text{PTIME}} (\text{FIN})\text{SAT}\Lambda eE_{\text{refine}}$.*

Since the relative size of $\text{ltr } \varphi$ with respect to φ grows polynomially as e grows, we have shown:

Proposition 4. *The logic ΛE_{local} has the finite model property iff the logic $\Lambda E_{\text{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent: $(\text{FIN})\text{SAT}\Lambda E_{\text{local}} =^{\text{PTIME}} (\text{FIN})\text{SAT}\Lambda E_{\text{refine}}$.*

The reduction is two-variable first-order and uses additional (*et*) unary predicate symbols for the permutation setup P , so it is also valid for the two-variable fragments $\Lambda_0^2 eE_a$, $\Lambda_1^2 eE_a$ and $\Lambda_1^2 E_a$ for $a \in \{\text{local}, \text{refine}\}$ respectively.

4.3 Granularity

In this section we demonstrate how to replace the finest equivalence from a sequence of equivalences in refinement with a counter setup. This works if the structures are granular, that is, if the finest equivalence doesn't have many classes within a single bigger equivalence class.

Definition 38. *Let $\langle D, E \rangle \subseteq A \times A$ be a sequence of two equivalence relations on A in refinement. Let $g \in \mathbb{N}^+$. The sequence is g -granular if every E -equivalence class includes at most g D -equivalence classes.*

Definition 39. *Let $g \in \mathbb{N}^+$ and let $\langle D, E \rangle \subseteq A \times A$ be g -granular. The function $c : A \rightarrow [1, g]$ is a g -granular coloring for the sequence, if two E -equivalent elements have the same color iff they are D -equivalent. That is, for every $(a, b) \in E$ we have $c(a) = c(b)$ iff $(a, b) \in D$.*

Remark 16. *Let $g \in \mathbb{N}^+$ and let $\langle D, E \rangle \subseteq A \times A$ be g -granular. Then there is a g -granular coloring for the sequence.*

Proof. Let X be an E -class. Since $D \subseteq E$ is g -granular, the set $S = \{D[a] \mid a \in X\}$ has cardinality at most g . Let $\iota : S \hookrightarrow [1, g]$ be any injective function. Define the color c on X as $c(a) = \iota D[a]$. \square

Remark 17. *Let $E \subseteq A \times A$ be an equivalence relation on A , $g \in \mathbb{N}^+$ and $c : A \rightarrow [1, g]$. Then there is an equivalence relation $D \subseteq E$ on A such that $\langle D, E \rangle$ is g -granular, having c as a g -granular coloring.*

Proof. Take $D = \{(a, b) \in E \mid c(a) = c(b)\}$. \square

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Definition 40. Let $g \in \mathbb{N}^+$ and let $t = \|g\|$ be the bitsize of g . A g -color setup $\mathbf{G} = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \rangle$ is just a t -bit counter setup.

Let $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ be a ground logic, $g \in \mathbb{N}^+$ and $\mathbf{G} = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \rangle$ be a g -color setup. Let Σ be a predicate signature containing the binary symbols \mathbf{d} and \mathbf{e} and not containing any symbols from \mathbf{G} . Let $\Sigma' = \Sigma + \mathbf{G}$ and $\Gamma = \Sigma' - \{\mathbf{d}\}$.

Definition 41. Define the quantifier-free $\mathcal{L}^2[\Gamma]$ -formula $[\Gamma:\mathbf{d}](\mathbf{x}, \mathbf{y})$ by:

$$[\Gamma:\mathbf{d}](\mathbf{x}, \mathbf{y}) = \mathbf{e}(\mathbf{x}, \mathbf{y}) \wedge [\mathbf{G}:\mathbf{eq}](\mathbf{x}, \mathbf{y}).$$

Definition 42. Define the syntactic operation $\text{grtr} : \Lambda[\Sigma] \rightarrow \Lambda[\Gamma]$ by:

$$\text{grtr } \varphi = \varphi' \wedge [\mathbf{G}:\text{betw-1-}g],$$

where φ' is obtained from the formula φ by replacing all subformulas of the form $\mathbf{d}(x, y)$ by $[\Gamma:\mathbf{d}](x, y)$, where x and y are (not necessarily distinct) variable symbols.

Lemma 3. Let \mathfrak{A} be a Σ -structure and suppose that the sequence of symbols $\langle \mathbf{d}, \mathbf{e} \rangle$ is interpreted in \mathfrak{A} as a g -granular sequence $\langle D, E \rangle \subseteq A \times A$. Suppose that $\mathfrak{A} \models \varphi$. Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \text{grtr } \varphi$.

Proof. By remark 16, there exists a g -granular coloring $c : A \rightarrow [1, g]$. We interpret the unary symbols in \mathbf{G} so that $[\mathbf{G}:\text{data}]^{\mathfrak{A}'} = c$. Since \mathfrak{A}' is an enrichment of \mathfrak{A} , we have $\mathfrak{A}' \models \varphi$. Let $a, b \in A$. Then $\mathfrak{A}' \models [\Gamma:\mathbf{d}](a, b)$ is equivalent to:

$$\mathfrak{A}' \models \mathbf{e}(a, b) \text{ and } \mathfrak{A}' \models [\mathbf{G}:\mathbf{eq}](a, b),$$

which is equivalent to:

$$(a, b) \in E \text{ and } [\mathbf{G}:\text{data}]^{\mathfrak{A}'} a = [\mathbf{G}:\text{data}]^{\mathfrak{A}'} b,$$

which, since $[\mathbf{G}:\text{data}]^{\mathfrak{A}'} = c$ is a g -granular coloring, is equivalent to:

$$(a, b) \in D.$$

Hence $\mathfrak{A}' \models \forall \mathbf{x} \forall \mathbf{y} \mathbf{d}(\mathbf{x}, \mathbf{y}) \leftrightarrow [\Gamma:\mathbf{d}](\mathbf{x}, \mathbf{y})$ and since $\mathfrak{A}' \models \varphi$, we have $\mathfrak{A}' \models \text{grtr } \varphi$. \square

Lemma 4. Let \mathfrak{A} be a Γ -structure and suppose that the binary symbol \mathbf{e} is interpreted in \mathfrak{A} as an equivalence relation on A . Suppose that $\mathfrak{A} \models \text{grtr } \varphi$. Then there is a Σ' -structure \mathfrak{A}' enriching \mathfrak{A} such that $\mathfrak{A}' \models \varphi$ and the sequence of binary symbols $\langle \mathbf{d}, \mathbf{e} \rangle$ is interpreted in \mathfrak{A}' as a g -granular sequence $\langle D, E \rangle \subseteq A \times A$.

Proof. Since $\mathfrak{A} \models [\mathbf{G}:\text{betw-1-}g]$, we have $[\mathbf{G}:\text{data}]^{\mathfrak{A}} a \in [1, g]$ for all $a \in A$. Define $c : A \rightarrow [1, g]$ by $c(a) = [\mathbf{G}:\text{data}]^{\mathfrak{A}} a$. By remark 16, we can find $D \subseteq E$ such that the sequence $\langle D, E \rangle$ is g -granular, having c as a g -granular coloring. Consider the Σ' -structure \mathfrak{A}' , where $\mathbf{d}^{\mathfrak{A}'} = D$. Since \mathfrak{A}' is an enrichment of \mathfrak{A} $\text{grtr } \varphi$, we have $\mathfrak{A}' \models \varphi$. Let $a, b \in A$. Then $\mathfrak{A}' \models [\Gamma:\mathbf{d}](a, b)$ is equivalent to:

$$\mathfrak{A}' \models \mathbf{e}(a, b) \text{ and } \mathfrak{A}' \models [\mathbf{G}:\mathbf{eq}](a, b),$$

which is equivalent to:

$$(a, b) \in E \text{ and } c(a) = c(b),$$

which, since c is a g -granular coloring, is equivalent to:

$$(a, b) \in D.$$

Hence $\mathfrak{A}' \models \forall \mathbf{x} \forall \mathbf{y} e(\mathbf{x}, \mathbf{y}) \leftrightarrow [\Gamma : \mathbf{d}](\mathbf{x}, \mathbf{y})$ and since $\mathfrak{A}' \models \text{grtr } \varphi$, we have $\mathfrak{A}' \models \varphi$. \square

5 Monadic two-variable fragments

In this chapter we investigate questions about (finite) satisfiability of two-variable first-order sentences featuring unary predicate symbols and builtin equivalence symbols in refinement. We will employ the game-theoretic characterisations of equivalence of structures as described in section 1.4. For the general case of several builtin equivalence relations without agreement, it is known that:

- the full binary two-variable first-order logic of a single builtin equivalence symbol \mathcal{L}^21E has the finite model property and its (finite) satisfiability problem is NEXPTIME-complete [3]
- the full binary two-variable first-order logic of two builtin equivalence symbols \mathcal{L}^22E lacks the finite model property and its satisfiability and finite satisfiability problems are both N2EXPTIME-complete [4]
- both the satisfiability and finite satisfiability problems for \mathcal{L}^2eE , where $e \geq 3$ are undecidable [3].

5.1 Cells

Let $U = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_u \rangle$ be a monadic predicate signature. Let $E = \langle \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_e \rangle$ be a predicate signature consisting of builtin equivalence symbols in refinement, where $e \geq 1$. Let $\Sigma = U + E$. Abbreviate the finest equivalence symbol as $\mathbf{d} = \mathbf{e}_1$.

Definition 43. Define quantifier-free $\mathcal{L}^2[\Sigma]$ -formula $[\Sigma:\text{cell}](\mathbf{x}, \mathbf{y})$ by:

$$[\Sigma:\text{cell}](\mathbf{x}, \mathbf{y}) = \mathbf{d}(\mathbf{x}, \mathbf{y}) \wedge \bigwedge_{1 \leq i \leq u} \mathbf{u}_i(\mathbf{x}) \leftrightarrow \mathbf{u}_i(\mathbf{y}).$$

If \mathfrak{A} is a Σ -structure and $D = \mathbf{d}^{\mathfrak{A}}$, then the interpretation $C = [\Sigma:\text{cell}]^{\mathfrak{A}} \subseteq A \times A$ is an equivalence relation on A that refines D . The cells of \mathfrak{A} are the equivalence classes of C . That is, a cell is a maximal set of D -equivalent elements satisfying the same \mathbf{u} -predicates.

Remark 18. If \mathfrak{A} is a Σ -structure and the elements $a, b \in A$ are in the same cell, then $\text{tp}^{\mathfrak{A}}[a] = \text{tp}^{\mathfrak{A}}[b]$.

Proof. Since \mathbf{e}_j is a builtin equivalence symbol, $\mathfrak{A} \models \mathbf{e}_j(a, a)$ and $\mathfrak{A} \models \mathbf{e}_j(b, b)$ for every $j \in [1, e]$. Since a and b are in the same cell, $\mathfrak{A} \models [\Sigma:\text{cell}](a, b)$. In particular, $\mathfrak{A} \models \mathbf{u}_i(a) \leftrightarrow \mathbf{u}_i(b)$ for every $i \in [1, u]$. \square

Remark 19. If \mathfrak{A} is a Σ -structure, $a, b \in A$ are in the same cell $C[a]$, $a', b' \in A$ are in the same cell $C[a']$ and $C[a] \neq C[a']$, then $\text{tp}^{\mathfrak{A}}[a, a'] = \text{tp}^{\mathfrak{A}}[b, b']$.

Proof. By remark 18, it is sufficient to check that $\mathfrak{A} \models e_j(a, a')$ iff $\mathfrak{A} \models e_j(b, b')$ for all $j \in [1, e]$. This is clear, since C refines the finest equivalence relation D and the equivalence symbols e_j are in refinement. \square

Lemma 5. Let \mathfrak{A} be a Σ -structure. There is some $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{A} \equiv^2 \mathfrak{B}$ and every \mathfrak{B} -cell has cardinality at most 2.

Proof. Let $C \subseteq A \times A$ be the cell equivalence relation of \mathfrak{A} . Execute the following process: for every \mathfrak{A} -cell, if it has cardinality 1, select the element from that cell; otherwise select two elements from that cell. Let $B \subseteq A$ be the set of selected elements and let $\mathfrak{B} = \mathfrak{A} \upharpoonright B$. By construction, every \mathfrak{B} -cell has cardinality at most 2. We claim that $\mathfrak{A} \equiv^2 \mathfrak{B}$. Let $h = C \cap (A \times B)$ relates elements from A with elements from B in the same cell. Note that for all $a \in A$:

$$|C[a]| \geq 2 \text{ iff } |h[a]| = 2. \quad (5.1)$$

Recall that a 2-partial isomorphism \mathbf{p} from \mathfrak{A} to \mathfrak{B} is an object $\bar{a} \mapsto \bar{b}$, where $\bar{a} = a_1 a_2 \in (A \cup \{\perp\})^2$, $\bar{b} = b_1 b_2 \in (B \cup \{\perp\})^2$, $\text{supp } \bar{a} = \text{supp } \bar{b}$ and we think of $\mathbf{p} : A \rightsquigarrow B$ as a partial function from A to B . Consider the set \mathcal{I} of 2-partial isomorphisms from \mathfrak{A} to \mathfrak{B} that are included in h . This set is nonempty, since $\langle \perp, \perp \rangle \mapsto \langle \perp, \perp \rangle \in \mathcal{I}$. Let $r \in \mathbb{N}^+$. We claim that the sequence $\mathcal{I}_0 = \mathcal{I}_1 = \dots = \mathcal{I}_r = \mathcal{I}$ satisfies the back-and-forth conditions of theorem 2. Let $i \in [1, 2]$ and let $\bar{a} \mapsto \bar{b} \in \mathcal{I}$. Without loss of generality, suppose $i = 1$. Let $\bar{a} = a_1 a_2$ and $\bar{b} = b_1 b_2$.

1. For the forth condition, let $a \in A$. We have to find some $b \in B$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathcal{I}$. If $a_2 = \perp$ then $b_2 = \perp$ and $b = a$ works. Suppose $a_2 \in A$. Then $(a_2, b_2) \in h \subseteq C$, so a_2 and b_2 are in the same \mathfrak{A} -cell $C[a_2]$. Let $s = |C[a_2]|$ be the cardinality of that cell.

If $a = a_2$ then $b = b_2$ works.

If $a \neq a_2$ and $a \in C[a_2]$, then $s \geq 2$. By eq. (5.1) there is $b \in h[a_2]$ such that $b \neq b_2$. This b works.

If $a \notin C[a_2]$, let $b \in h[a]$ be any. Then $a_2, b_2 \in C[a_2]$, $a, b \in C[a]$, $C[a_2] \neq C[a]$ and by remark 19, this b works.

2. For the back condition, let $b \in B$. We have to find some $a \in A$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathcal{I}$. If $b_2 = \perp$ then $a_2 = \perp$ and $a = b$ works. Suppose $b_2 \in B$. Then $(a_2, b_2) \in h \subseteq C$, so a_2 and b_2 are in the same \mathfrak{A} -cell $C[b_2]$. Let $s = |C[b_2]|$ be the cardinality of that cell.

If $b = b_2$ then $a = a_2$ works.

If $b \neq b_2$ and $b \in C[b_2]$, then $s \geq 2$. Thus there is some $a \in C[b_2]$ such that $a \neq a_2$. This a works.

If $b \notin C[b_2]$, then $a_2 \neq b$. By remark 19, $a = b$ works.

By theorem 2, $\mathfrak{A} \equiv_r^2 \mathfrak{B}$. Since $r \in \mathbb{N}^+$ was arbitrary, $\mathfrak{A} \equiv^2 \mathfrak{B}$. \square

5.2 Organs

Let $U = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_u \rangle$ be a monadic predicate signature. Let $E = \langle \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_e \rangle$ be a predicate signature consisting of builtin equivalence symbols in refinement, where $e \geq 2$. Let $\Sigma = U + E$. Abbreviate the finest two equivalence symbols as $\mathbf{d} = \mathbf{e}_1$ and $\mathbf{e} = \mathbf{e}_2$.

Definition 44. Let \mathfrak{A} be a Σ -structure and let $D = \mathbf{d}^{\mathfrak{A}}$ and $E = \mathbf{e}^{\mathfrak{A}}$. Recall that the set of D -classes is $\mathcal{E}D$. Two D -classes $X, Y \in \mathcal{E}D$ are organ-equivalent, if $X \times Y \subseteq E$ and the induced substructures $(\mathfrak{A} \upharpoonright X)$ and $(\mathfrak{A} \upharpoonright Y)$ are isomorphic. The organ-equivalence relation is $\mathcal{O} \subseteq \mathcal{E}D \times \mathcal{E}D$. Since D refines E , the organ-equivalence is an equivalence relation on $\mathcal{E}D$. An organ is an organ-equivalence-class. That is, an organ is a maximal set of isomorphic D -classes, included in the same E -class. For any two organ-equivalent D -classes $(X, Y) \in \mathcal{O}$, fix an isomorphism $\mathfrak{h}_{XY} : (\mathfrak{A} \upharpoonright X) \rightarrow (\mathfrak{A} \upharpoonright Y)$ consistently, so that $\mathfrak{h}_{XX} = \text{id}_X$, $\mathfrak{h}_{YX} = \mathfrak{h}_{XY}^{-1}$ and if $(Y, Z) \in \mathcal{O}$ then $\mathfrak{h}_{XZ} = \mathfrak{h}_{YZ} \circ \mathfrak{h}_{XY}$. Two elements $a, b \in A$ are sub-organ-equivalent if $(D[a], D[b]) \in \mathcal{O}$ and $\mathfrak{h}_{D[a]D[b]}(a) = b$. Since the isomorphisms \mathfrak{h}_{XY} are chosen consistently, the sub-organ-equivalence relation $\mathcal{O} \subseteq A \times A$ is an equivalence relation on A that refines E .

Remark 20. If \mathfrak{A} is a Σ -structure and $a, b, a', b' \in A$ be such that a and b are sub-organ-equivalent, a' and b' are sub-organ-equivalent and $D[a]$ and $D[a']$ are not organ-equivalent, then $\text{tp}^{\mathfrak{A}}[a, a'] = \text{tp}^{\mathfrak{A}}[b, b']$.

Proof. Since a and b are sub-organ-equivalent, we have $\mathfrak{h}_{D[a]D[b]}(a) = b$, hence $\text{tp}^{\mathfrak{A}}[a] = \text{tp}^{\mathfrak{A}}[b]$. Similarly, $\text{tp}^{\mathfrak{A}}[a'] = \text{tp}^{\mathfrak{A}}[b']$. Since $D[a]$ and $D[a']$ are not organ-equivalent, in particular $D[a] \neq D[a']$, so $\mathfrak{A} \models \neg \mathbf{d}(a, a')$. Since $D[b]$ is organ-equivalent to $D[a]$, $D[b']$ is organ-equivalent to $D[a']$ and organ-equivalence is equivalence relation, we have $\mathfrak{A} \models \neg \mathbf{d}(b, b')$. Since the sub-organ-equivalence \mathcal{O} is an equivalence relation on A that refines E and the equivalence symbols \mathbf{e}_j are in refinement, $\mathfrak{A} \models \mathbf{e}_j(a, a')$ iff $\mathfrak{A} \models \mathbf{e}_j(b, b')$ for all $j \in [2, e]$. \square

Lemma 6. Let \mathfrak{A} be a Σ -structure. There is some $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{A} \equiv^2 \mathfrak{B}$ and every \mathfrak{B} -organ has cardinality at most 2.

Proof. Let $D = \mathbf{d}^{\mathfrak{A}}$, $E = \mathbf{e}^{\mathfrak{A}}$ and let $\mathcal{A} = \mathcal{E}D$ be the set of D -classes. Let $\mathcal{O} \subseteq \mathcal{A} \times \mathcal{A}$ be the \mathfrak{A} -organ-equivalence relation on \mathcal{A} . Execute the following process: for every \mathfrak{A} -organ, if it has cardinality 1, select the D -class from that organ; otherwise select two D -classes from that organ (note that these will be isomorphic). Let $\mathcal{B} \subseteq \mathcal{A}$ be the set of selected D -classes. Let $B = \cup \mathcal{B} \subseteq A$ be the set of elements in the selected classes and let $\mathfrak{B} = (\mathfrak{A} \upharpoonright B)$. By construction, every \mathfrak{B} -organ has cardinality at most 2. We claim that $\mathfrak{A} \equiv^2 \mathfrak{B}$. Let $\mathcal{H} = \mathcal{O} \cap \mathcal{A} \times \mathcal{B}$ relates the D -classes with the D -classes from \mathcal{B} in the same organ. Let $h = \cup \{\mathfrak{h}_{XY} \mid (X, Y) \in \mathcal{H}\}$, so h relates the elements of A with their isomorphic elements from B . Note that for all D -classes $D[a] \in \mathcal{A}$:

$$|\mathcal{O}[D[a]]| \geq 2 \text{ iff } |\mathcal{H}[D[a]]| = 2 \text{ iff } |h[a]| = 2. \quad (5.2)$$

Recall that a 2-partial isomorphism \mathbf{p} from \mathfrak{A} to \mathfrak{B} is an object $\bar{a} \mapsto \bar{b}$, where $\bar{a} = a_1 a_2 \in (A \cup \{\perp\})^2$, $\bar{b} = b_1 b_2 \in (B \cup \{\perp\})^2$, $\text{supp } \bar{a} = \text{supp } \bar{b}$ and we think of \mathbf{p} as a relation

between A and B . Consider the set \mathfrak{I} of 2-partial isomorphisms from \mathfrak{A} to \mathfrak{B} that are included in h . This set is nonempty, since $\langle \perp, \perp \rangle \mapsto \langle \perp, \perp \rangle \in \mathfrak{I}$.

Let $r \in \mathbb{N}^+$. We claim that the sequence $\mathfrak{I}_0 = \mathfrak{I}_0 = \dots = \mathfrak{I}_r = \mathfrak{I}$ satisfies the back-and-forth conditions of theorem 2. Let $i \in [1, 2]$ and $\bar{a} \mapsto \bar{b} \in \mathfrak{I}$. Without loss of generality, suppose $i = 1$. Let $\bar{a} = a_1 a_2$ and $\bar{b} = b_1 b_2$.

1. For the forth condition, let $a \in A$. We have to find some $b \in B$ such that $\bar{a}_i^a \mapsto \bar{a}_i^b \in \mathfrak{I}$. If $a_2 = \perp$ then $b_2 = \perp$ and $b = a$ works. Suppose $a_2 \in A$. Then $(a_2, b_2) \in h$, so $D[a_2]$ and $D[b_2]$ are in the same \mathfrak{A} -organ $\mathcal{O}[D[a_2]]$. Let $s = |\mathcal{O}[D[a_2]]|$ be the cardinality of that organ.

If $a = a_2$ then $b = b_2$ works.

If $a \neq a_2$ and $D[a] \in \mathcal{O}[D[a_2]]$, then $s \geq 2$. By eq. (5.2) there is $b \in h[a_2]$ such that $b \neq b_2$. This b works.

If $D[a] \notin \mathcal{O}[D[a_2]]$, let $b \in h[a]$ be any. Then $D[b_2] \in \mathcal{O}[D[a_2]]$, $D[b] \in \mathcal{O}[D[a]]$, $\mathcal{O}[D[a_2]] \neq \mathcal{O}[D[a]]$ and by remark 20, this b works.

2. For the back condition, let $b \in B$. We have to find some $a \in A$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathfrak{I}$. If $b_2 = \perp$ then $a_2 = \perp$ and $a = b$ works. Suppose $b_2 \in B$. Then $D[a_2]$ and $D[b_2]$ are in the same \mathfrak{A} -organ $\mathcal{O}[D[b_2]]$. Let $s = |\mathcal{O}[D[b_2]]|$ be the cardinality of that organ.

If $b = b_2$ then $a = a_2$ works.

If $b \neq b_2$ and $D[b] \in \mathcal{O}[D[b_2]]$, then $s \geq 2$. Thus there is $a \in A$ such that $b \in h[a]$ and $a \neq a_2$. This a works.

If $D[b] \notin \mathcal{O}[D[b_2]]$, then $a_2 \neq b$. By remark 20, $a = b$ works.

By theorem 2, $\mathfrak{A} \equiv_r^2 \mathfrak{B}$. Since $r \in \mathbb{N}^+$ was arbitrary, $\mathfrak{A} \equiv^2 \mathfrak{B}$. □

5.3 Satisfiability

Let $U(u) = \langle u_1, u_2, \dots, u_u \rangle$ be an unary predicate signature having length u . Let $E(e) = \langle e_1, e_2, \dots, e_e \rangle$ be a binary predicate consisting of e builtin equivalence symbols in refinement. Let $\Sigma(u, e) = U(u) + E(e)$, so $\Sigma(u, e)$ is a generic predicate signature for the logics $\mathcal{L}_0^2 e E_{\text{refine}}$ and $\mathcal{L}_1^2 e E_{\text{refine}}$.

Remark 21. Let $u, e \in \mathbb{N}$ such that $e \geq 2$ and consider $\Sigma = \Sigma(u, e)$. Abbreviate $\mathbf{d} = e_1$ and $\mathbf{e} = e_2$. Let \mathfrak{A} be a Σ -structure. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{A} \equiv^2 \mathfrak{B}$ and $\langle \mathbf{d}^{\mathfrak{B}}, \mathbf{e}^{\mathfrak{B}} \rangle$ is g -granular for $g = 3 \cdot (3^{2^u} - 1)$.

Proof. By lemma 5, there is $\mathfrak{B}' \subseteq \mathfrak{A}$ such that $\mathfrak{A} \equiv^2 \mathfrak{B}'$ and the cells of \mathfrak{B}' have cardinality at most 2. By lemma 6, there is $\mathfrak{B} \subseteq \mathfrak{B}'$ such that $\mathfrak{B} \equiv^2 \mathfrak{B}'$ and the organs of \mathfrak{B} have cardinality at most 2. Let $D = \mathbf{d}^{\mathfrak{B}}$ and $E = \mathbf{e}^{\mathfrak{B}}$. Let X be any E -class. Since every D -class includes at most 2^u cells and is nonempty, every D -class has cardinality at most $3 \cdot 2^u$ and there are at most $(3^{2^u} - 1)$ nonisomorphic D -classes included in X .

Since every \mathfrak{B} -organ has cardinality at most 2, there are at most $3 \cdot (3^{2^u} - 1)$ D -classes included in X . \square

Recall from section 4.3 that the finest equivalence relation in a g -granular structure can be simulated by introducing $\|g\|$ fresh unary predicate symbols. By induction on e we get:

Corollary 1. *Let $u, e \in \mathbb{N}$ such that $e \geq 2$. Then $\mathcal{L}^2[\Sigma(u, e)]$ has the finite model property and*

$$(\text{FIN})\text{SAT}\mathcal{L}^2[\Sigma(u, e)] \leq^{\text{PTIME}} (\text{FIN})\text{SAT}\mathcal{L}^2[\Sigma(u', e')].$$

where $e' = e - 1$, $u' = u + \|g\|$ and $g = 3 \cdot (3^{2^u} - 1)$.

Remark 22. *Consider $\Sigma = \Sigma(u, 1)$ be a predicate signature consisting of the unary predicate symbols \mathbf{u}_i and the single builtin equivalence symbol \mathbf{d} . Let \mathfrak{A} be a Σ -structure. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{A} \equiv^2 \mathfrak{B}$ and the cardinality of \mathfrak{B} is at most $9 \cdot 2^u \cdot (3^{2^u} - 1)$.*

Proof. Let $\Sigma' = \Sigma + \langle \mathbf{e} \rangle$ and let \mathfrak{A}' be a Σ' -enrichment of \mathfrak{A} where $\mathbf{e}^{\mathfrak{A}'} = A \times A$ is interpreted as the full relation on A , so the symbols \mathbf{d}, \mathbf{e} are interpreted in \mathfrak{A}' as equivalence relations on A in refinement. By lemma 5, there is $\mathfrak{B}' \subseteq \mathfrak{A}'$ such that $\mathfrak{A} \equiv^2 \mathfrak{B}'$ and the cells of \mathfrak{B}' have cardinality at most 2. By lemma 6, there is $\mathfrak{B} \subseteq \mathfrak{B}'$ such that $\mathfrak{B} \equiv^2 \mathfrak{B}'$ and the organs of \mathfrak{B} have cardinality at most 2. Let $D = \mathbf{d}^{\mathfrak{B}}$ and $E = \mathbf{e}^{\mathfrak{B}}$. Clearly $E = B \times B$, so there is a single E -class and the \mathfrak{B} -organs partition the D -classes into groups of isomorphic classes. Since every D -class includes at most 2^u cells and is nonempty, every D -class has cardinality at most $3 \cdot 2^u$. Since \mathfrak{B} is g -granular for $g = 3 \cdot (3^{2^u} - 1)$ and there is a single E -class, the cardinality of \mathfrak{B} is at most $9 \cdot 2^u \cdot (3^{2^u} - 1)$. \square

When the number of unary predicate symbols is constant u , independent of the length of the formulas, we get:

Proposition 5. *The logic $\mathcal{L}_0^2 \mathbf{eE}_{\text{refine}}$ has the finite model property and its (finite) satisfiability problem is in PTIME.*

By proposition 1 and proposition 3, the same holds for the logics $\mathcal{L}_0^2 \mathbf{eE}_{\text{global}}$ and $\mathcal{L}_0^2 \mathbf{eE}_{\text{local}}$.

For $u \in \mathbb{N}$ and $e \in \mathbb{N}^+$, let $g = 3 \cdot (3^{2^u} - 1)$ and $u' = u + \|g\|$. Since

$$\|g\| = \lfloor \log 3 \cdot (3^{2^u} - 1) \rfloor + 1 < 2 + 2 \cdot 2^u + 1 < 8^u,$$

we have $u' < 9^u$. By induction on e , the logic $\mathcal{L}_1^2 \mathbf{eE}_{\text{refine}}$ has the finite model property and for $e' = e - 1$

$$(\text{FIN})\text{SAT}\mathcal{L}_1^2 \mathbf{eE}_{\text{refine}} \leq^{\text{EXPTIME}} (\text{FIN})\text{SAT}\mathcal{L}_1^2 \mathbf{e'E}_{\text{refine}}.$$

Since the even stronger logic $\mathcal{L}^2 \mathbf{1E}_{\text{refine}}$ has the finite model property and its (finite) satisfiability problem is NEXPTIME-complete [3], we get:

Proposition 6. *The logic $\mathcal{L}_1^2eE_{\text{refine}}$ has the finite model property and its (finite) satisfiability problem is in NeEXPTIME.*

By proposition 1 and proposition 3, the same holds for the logics $\mathcal{L}_1^2eE_{\text{global}}$ and $\mathcal{L}_1^2eE_{\text{local}}$.

If we allow unboundedly many unary predicate symbols, u is on the order of the length of the formula, and we get:

Proposition 7. *The logic $\mathcal{L}_1^2E_{\text{refine}}$ has the finite model property and its (finite) satisfiability problem is in ELEMENTARY.*

By proposition 2 and proposition 4, this also holds for the logics $\mathcal{L}_1^2E_{\text{global}}$ and $\mathcal{L}_1^2E_{\text{local}}$. Trivially, this also holds for the logics $\mathcal{L}_0^2E_{\text{refine}}$, $\mathcal{L}_0^2E_{\text{global}}$ and $\mathcal{L}_0^2E_{\text{local}}$.

Using finer methods, we will give much better estimates for the complexity of the (finite) satisfiability problems for $\mathcal{L}_1^2eE_{\text{refine}}$, $\mathcal{L}_1^2E_{\text{refine}}$ and $\mathcal{L}_0^2E_{\text{refine}}$.

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