Satisfiability with Equivalences in Agreement, Part 2

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Overview

Two-variable Logic

- The two-variable logic $\mathcal{L}^2$ is the fragment of first-order logic featuring only the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ (with formal equality and restricted to just unary and binary predicate symbols).
- It is known that $\mathcal{L}^2$ has the finite model property [Mortimer, 1975] and its (finite) satisfiability problem is $\mathrm{NE{\scriptsize XP}T{\scriptsize IME}}$-complete [Grädel, Kolaitis, Vardi, 1997].
- We develop a technique that allows us to show that the two-variable logic with equivalences in refinement $\mathcal{L}^2 e\mathrm{E}_{\text{refine}}$ has the finite model property and its (finite) satisfiability problem is in $\mathrm{NE{\scriptsize XP}T{\scriptsize IME}}$.

Scott Normal Form

Theorem (Scott, 1962)

There is a polynomial-time reduction $\mathrm{sctr} : \mathcal{L}^2 \to \mathcal{L}^2$ *which reduces every sentence* φ *to a sentence* $\mathrm{sctr}\,\varphi$ *in Scott normal form:*

$$\forall \boldsymbol{x} \forall \boldsymbol{y} (\alpha_0(\boldsymbol{x}, \boldsymbol{y}) \vee \boldsymbol{x} = \boldsymbol{y}) \wedge \bigwedge_{1 \leq i \leq m} \forall \boldsymbol{x} \exists \boldsymbol{y} (\alpha_i(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{x} \neq \boldsymbol{y}),$$

where $m \geq 1$*, the formulas* α_i *are quantifier-free and use at most linearly many new unary predicate symbols. The sentences* φ *and* $\mathrm{sctr}\,\varphi$ *are satisfiable over the same domains of cardinality at least* 2*.*

Scott Normal Form

$$\forall \boldsymbol{x} \forall \boldsymbol{y}(\alpha_0(\boldsymbol{x}, \boldsymbol{y}) \vee \boldsymbol{x} = \boldsymbol{y}) \wedge \bigwedge_{1 \leq i \leq m} \forall \boldsymbol{x} \exists \boldsymbol{y}(\alpha_i(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{x} \neq \boldsymbol{y})$$

Strategy: if ψ is a subformula of φ of the form $Qx\alpha(x, y)$, where α is quantifier-free, then φ and $\varphi' \wedge \forall y(\boldsymbol{p}(y) \leftrightarrow Qx\alpha(x, y))$ are equisatisfiable, where $\boldsymbol{p}(y)$ is a new unary symbol and φ' is obtained from φ by replacing the subformula ψ by $\boldsymbol{p}(y)$.

Classified Signatures

- we can replace the existential parts $\alpha_i, i \geq 1$ with fresh binary predicate symbols $\boldsymbol{m}_i$ with interpretation $\forall \boldsymbol{x} \forall \boldsymbol{y}(\boldsymbol{m}_i(\boldsymbol{x}, \boldsymbol{y}) \leftrightarrow \alpha_i(\boldsymbol{x}, \boldsymbol{y}))$
- A *classified signature* $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ consists of a signature Σ together with a sequence of distinct binary predicate symbols $\bar{\boldsymbol{m}} = \boldsymbol{m}_1 \boldsymbol{m}_2 \ldots \boldsymbol{m}_m$ from Σ.
- A *structure* $\mathfrak{A}$ for $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ is a structure for Σ satisfying the existential parts:

$$\bigwedge_{1 \leq i \leq m} \forall \boldsymbol{x} \exists \boldsymbol{y}(\boldsymbol{m}_i(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{x} \neq \boldsymbol{y})$$

Classified Signatures

- The (finite) classified satisfiability problem is: given a classified signature $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ and a quantifier-free formula $\alpha(\boldsymbol{x}, \boldsymbol{y})$, is there a $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \forall \boldsymbol{x} \forall \boldsymbol{y} \alpha(\boldsymbol{x}, \boldsymbol{y})$.
- Scott tells us how (finite) satisfiability reduces to (finite) classified satisfiability.

Types

Let $\Sigma = \langle \boldsymbol{p}^1, \boldsymbol{p}^2, \ldots, \boldsymbol{p}^n \rangle$ be a predicate signature.

- A 1-*type* π over Σ is a maximal consistent set of literals featuring only the variable $\boldsymbol{x}$ (in model theory known as atomic type).
- A 2-*type* τ over Σ is maximal consistent set of literals featuring the variable symbols $\boldsymbol{x}$ and $\boldsymbol{y}$ and including $(\boldsymbol{x} \neq \boldsymbol{y})$.
- If τ is a 2-type, the $\boldsymbol{x}$-*type* $\mathrm{tp}_{\boldsymbol{x}}\tau$ is the 1-type consisting of those literals featuring only the variable $\boldsymbol{x}$,
- the *inverse* τ^{-1} is the 2-type obtained from swapping $\boldsymbol{x}$ and $\boldsymbol{y}$ in the literals of τ
- and $\mathrm{tp}_{\boldsymbol{y}}\tau = \mathrm{tp}_{\boldsymbol{x}}(\tau^{-1})$.

Type Instances

- A *type instance* T over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ is a nonempty set of 2-types that is closed under inversion.
- The type instance $\mathrm{T}[\mathfrak{A}]$ of a $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure $\mathfrak{A}$ is:

$$\mathrm{T}[\mathfrak{A}] = \left\{ \mathrm{tp}^{\mathfrak{A}}[a, b] \;\middle|\; a \in A, b \in A \setminus \{a\} \right\},$$

  where $\mathrm{tp}^{\mathfrak{A}}[a, b]$ is the 2-type realized by (a, b) in $\mathfrak{A}$.
- Type instances are typically exponentially bigger than the classified signature.

Type Realizability

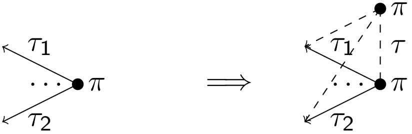
- The *(finite) type realizability problem* is: given a classified signature $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$ and a type instance T over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$, is there a (finite) $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$-structure $\mathfrak{A}$ such that $\mathrm{T}[\mathfrak{A}] = \mathrm{T}$.
- We aim to show that the type realizability problem for $\mathcal{L}^2$ is in $\mathrm{NP}\textsc{Time}$.
- (Finite) classified satisfiability reduces in nondeterministic exponential time to (finite) type realizability: just guess a type instance consisting of 2-types consistent with $\alpha(x, y)$.

Kings and Workers

- Let T be a type instance over $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$. The 1-types of T are $\Pi_{\mathrm{T}} = (\mathrm{tp}_{\boldsymbol{x}} \restriction \mathrm{T})$.
- A 1-type $\kappa \in \Pi_{\mathrm{T}}$ is a *king type* if no $\tau \in \mathrm{T}$ has $\mathrm{tp}_{\boldsymbol{x}}\tau = \mathrm{tp}_{\boldsymbol{y}}\tau = \kappa$. The set of king types is K_{T}.
- The remaining 1-types are the *worker types*.
- In models of T, king types are realized uniquely, while worker types are realized by at least two elements.

Worker Copies

If $\mathfrak{A}$ is a model for T, any worker element can be copied:
$\pi \in \mathrm{W}_{\mathrm{T}}$, $\tau \in \mathrm{T}$, $\mathrm{tp}_{\boldsymbol{x}}\tau = \mathrm{tp}_{\boldsymbol{y}}\tau = \pi$

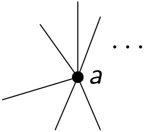


This doesn't work for kings!

$$\mathrm{T}[\pi] = \begin{cases} \Pi_{\mathrm{T}} \text{ if } \pi \text{ is a worker type} \\ \Pi_{\mathrm{T}} \setminus \{\pi\} \text{ otherwise, if it is a king type} \end{cases}$$

Star-types

The *star-type* of $a \in A$ is: $\mathrm{stp}^{\mathfrak{A}}[a] = \left\{ \mathrm{tp}^{\mathfrak{A}}[a, b] \;\middle|\; b \in A \setminus \{a\} \right\}$



Gives reason why the existential condition is satisfied locally at a.

Star-types

A *star-type* σ over T is a nonempty subset $\sigma \subseteq \mathrm{T}$ satisfying:

($\sigma\boldsymbol{x}$) If $\tau, \tau' \in \sigma$, then $\mathrm{tp}_{\boldsymbol{x}}\tau = \mathrm{tp}_{\boldsymbol{x}}\tau'$. Denote $\mathrm{tp}_{\boldsymbol{x}}\tau$ for any $\tau \in \sigma$ by $\pi = \mathrm{tp}_{\boldsymbol{x}}\sigma$.

($\sigma\pi\boldsymbol{y}$) If $\pi' \in \mathrm{T}[\pi]$, then some $\tau \in \sigma$ has $\mathrm{tp}_{\boldsymbol{y}}\tau = \pi'$.

($\sigma\kappa\boldsymbol{y}$) If $\kappa' \in \mathrm{T}[\pi] \cap \mathrm{K}_{\mathrm{T}}$ and if $\tau, \tau' \in \sigma$ have $\mathrm{tp}_{\boldsymbol{y}}\tau = \mathrm{tp}_{\boldsymbol{y}}\tau' = \kappa'$, then $\tau = \tau'$.

($\sigma\boldsymbol{m}$) If $\boldsymbol{m} \in \bar{\boldsymbol{m}}$, then some $\tau \in \sigma$ has $\boldsymbol{m}(\boldsymbol{x}, \boldsymbol{y}) \in \tau$.

Certificates

A *certificate* $\mathcal{S}$ for T is a nonempty set of star-types satisfying:

($\mathcal{S}\tau$) If $\tau \in \mathrm{T}$, then some $\sigma \in \mathcal{S}$ has $\tau \in \sigma$, that is there is a star-type containing each 2-type.

($\mathcal{S}\kappa$) If $\kappa \in \mathrm{K}_{\mathrm{T}}$ and if $\sigma, \sigma' \in \mathcal{S}$ have $\mathrm{tp}_{\boldsymbol{x}}\sigma = \mathrm{tp}_{\boldsymbol{x}}\sigma' = \kappa$, then $\sigma = \sigma'$.

Then:

($\mathcal{S}\pi$) If $\pi \in \Pi_{\mathrm{T}}$, then some $\sigma \in \mathcal{S}$ has $\mathrm{tp}_{\boldsymbol{x}}\sigma = \pi$.

($\mathcal{S}\kappa'$) If $\kappa \in \mathrm{K}_{\mathrm{T}}$, then a unique $\sigma \in \mathcal{S}$ has $\mathrm{tp}_{\boldsymbol{x}}\sigma = \pi$.

Certificate Extraction

A star-type for every $\tau \in \mathrm{T}$ is sufficient!

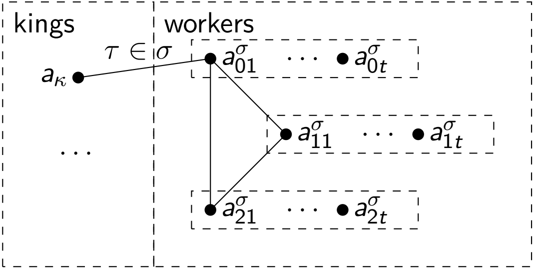
- Let $\mathfrak{A}$ be a model for T
- For every $\tau \in \mathrm{T}$ choose $a_\tau, b_\tau \in A$ such that $\mathrm{tp}^{\mathfrak{A}}[a_\tau, b_\tau] = \tau$
- $\mathcal{S} = \left\{ \mathrm{stp}^{\mathfrak{A}}[a_\tau] \;\middle|\; \tau \in \mathrm{T} \right\}$ is a *polynomial* certificate

Certificate Expansion

Theorem

Let $\mathcal{S}$ be a certificate for the type instance T and let $t \geq |\mathrm{T}|$. Then T has a finite model in which every worker type is realized at least t times.

Construction



- single element for each king
- 3 blocks of t elements for each worker star-type
- king–to–element determined by the star-type of the element
- worker–to–worker between consecutive blocks
- completion to a full structure

Summary

- Type realizability for $\mathcal{L}^2$ is in $\mathrm{NPT{\scriptsize IME}}$
- $\mathcal{L}^2$ has the finite model property and its satisfiability problem is in $\mathrm{NE{\scriptsize XP}T{\scriptsize IME}}$

Strategy

- Consider the two-variable logic with a single builtin equivalence symbol $\mathcal{L}^2 1\mathrm{E}$
- Equivalence classes are structures for the *simpler* $\mathcal{L}^2$
- Exploit the previous result to ensure *classes are "consistent"* and figure out how to *glue them together*
- Make sure the argument is suitable for induction to get to $\mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}}$

Galaxies and Cosmos

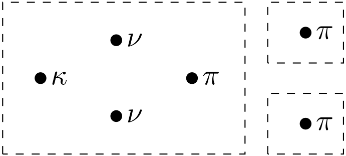
Let $\mathfrak{A}$ be a $\mathcal{L}^2 1\mathrm{E}$-structure for the type instance T.

- classes of $\mathfrak{A}$ are the *galaxies* of $\mathfrak{A}$
- $\mathfrak{A}$ is *the cosmos*
- $\tau \in \mathrm{T}$ is *galactic* if $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) \in \tau$, $\mathrm{T}^{\mathrm{g}} \subseteq \mathrm{T}$
- otherwise τ is *cosmic*, $\mathrm{T}^{\mathrm{c}} \subseteq \mathrm{T}$

Noble and Peasant Types

- $\nu \in \mathsf{\Pi}_{\mathrm{T}}$ is *noble* if no cosmic τ has $\mathrm{tp}_{\boldsymbol{x}}\tau = \mathrm{tp}_{\boldsymbol{y}}\tau = \nu$; the set of noble types is N_{T}
- $\pi \in \mathsf{\Pi}_{\mathrm{T}}$ is *peasant* if it is not noble; the set of peasant types is $\mathsf{\Pi}_{\mathrm{T}}$
- kings are noble
- peasants are workers
- a galaxy is *noble* if it realizes a noble type
- a galaxy is *peasant* if it realizes only peasant types

Example



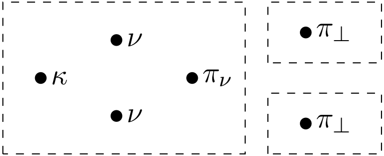
Noble galaxies may contain peasants!

Noble Distinguishability

- A structure is *nobly distinguished* if every noble galaxy realizes only noble types

Noble Distinguishability

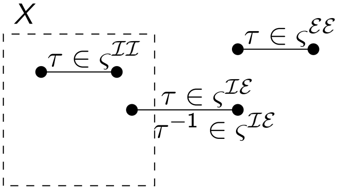
- A structure is *nobly distinguished* if every noble galaxy realizes only noble types
- Reduction: tag peasants in a noble galaxy with some noble type from that galaxy



Cosmic Spectrums

- Let $\mathfrak{A}$ be a nobly distinguished model for T having at least 2 galaxies
- The cosmic spectrum of a galaxy $X \subset A$ is

$$\mathrm{csp}^{\mathfrak{A}}[X] = (\varsigma^{\mathcal{II}}, \varsigma^{\mathcal{IE}}, \varsigma^{\mathcal{EI}}, \varsigma^{\mathcal{EE}})$$



Cosmic Spectrums

A cosmic spectrum $\varsigma = (\varsigma^{\mathcal{II}}, \varsigma^{\mathcal{IE}}, \varsigma^{\mathcal{EI}}, \varsigma^{\mathcal{EE}})$ over T is a tuple satisfying:

($\varsigma\mathcal{II}$) The set of *internal types* $\varsigma^{\mathcal{II}} \subseteq \mathrm{T}^{\mathrm{g}}$ is a set of galactic types that is closed under inversion.

($\varsigma\mathcal{IE}$) The set of *boundary types* $\varsigma^{\mathcal{IE}} \subseteq \mathrm{T}^{\mathrm{c}}$ is a nonempty set of cosmic types.

($\varsigma\mathcal{EI}$) The set of *inverted boundary types* is: $\varsigma^{\mathcal{EI}} = \left\{\tau^{-1} \;\middle|\; \tau \in \varsigma^{\mathcal{IE}}\right\}$.

($\varsigma\mathcal{EE}$) The set of *external types* $\varsigma^{\mathcal{EE}} \subseteq \mathrm{T}$ is a set of 2-types that is closed under inversion.

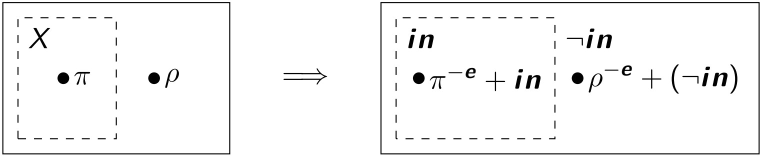
($\varsigma\mathrm{T}$) We require that $\mathrm{T} = \varsigma^{\mathcal{II}} \cup \varsigma^{\mathcal{IE}} \cup \varsigma^{\mathcal{EI}} \cup \varsigma^{\mathcal{EE}}$.

($\varsigma\mathrm{NP}$) The (nonempty) set $\mathsf{Tp}_{\boldsymbol{x}}\,\varsigma = (\mathrm{tp}_{\boldsymbol{x}} \restriction \varsigma^{\mathcal{IE}})$ is the set of *internal 1-types* of ς. We require that $\mathsf{Tp}_{\boldsymbol{x}}\,\varsigma \subseteq \mathrm{N}_{\mathrm{T}}$ or $\mathsf{Tp}_{\boldsymbol{x}}\,\varsigma \subseteq \mathrm{P}_{\mathrm{T}}$.

Locally consistent cosmic spectrums

- The *spectral type instance* T^{ς} of the $\mathcal{L}^2 e\mathrm{E}_{\text{refine}}$-cosmic spectrum ς is $\mathrm{T}^{\varsigma} = \mathrm{T}^{\varsigma}_{\mathcal{I}\mathcal{I}} \cup \mathrm{T}^{\varsigma}_{\mathcal{I}\mathcal{E}} \cup \mathrm{T}^{\varsigma}_{\mathcal{E}\mathcal{I}} \cup \mathrm{T}^{\varsigma}_{\mathcal{E}\mathcal{E}}$, where $\mathrm{T}^{\varsigma}_{\mathcal{X}\mathcal{Y}} = \left\{ \tau_{\mathcal{X}\mathcal{Y}} \;\middle|\; \tau \in \varsigma^{\mathcal{X}\mathcal{Y}} \right\}$, where $\tau_{\mathcal{X}\mathcal{Y}}$ works by "removing $\boldsymbol{e}$ and tagging the ends with a new predicate symbol $\boldsymbol{in}$".
- ς is *locally consistent* if T^{ς} is realizable over *the simpler* $\mathcal{L}^2(e-1)\mathrm{E}_{\text{refine}}$

$\mathfrak{A} : \mathcal{L}^2 e\mathrm{E}_{\text{refine}}$ $\qquad$ $\mathfrak{A}' : \mathcal{L}^2(e-1)\mathrm{E}_{\text{refine}}$



Certificates

A *certificate* $\mathcal{S}$ for the type instance T is a nonempty set of locally consistent cosmic spectrums satisfying:

($\mathcal{S}\mathrm{T}^{\mathrm{c}}$) If $\tau \in \mathrm{T}^{\mathrm{c}}$ then some $\varsigma \in \mathcal{S}$ has $\tau \in \varsigma^{\mathcal{I}\mathcal{E}}$.

($\mathcal{S}\mathrm{T}^{\mathrm{g}}$) If $\tau \in \mathrm{T}^{\mathrm{g}}$ then some $\varsigma \in \mathcal{S}$ has $\tau \in \varsigma^{\mathcal{I}\mathcal{I}}$.

($\mathcal{S}\nu$) If $\nu \in \mathrm{N}_{\mathrm{T}}$ and $\varsigma, \varsigma' \in \mathcal{S}$ have $\nu \in \mathsf{Tp}_{\boldsymbol{x}}\,\varsigma$ and $\nu \in \mathsf{Tp}_{\boldsymbol{x}}\,\varsigma'$, then $\varsigma' = \varsigma$.

Analogues

$$
\begin{aligned}
\mathcal{L}^2 &\Longleftrightarrow \mathcal{L}^2 e\mathrm{E}_{\mathsf{refine}} \\
\text{models} &\Longleftrightarrow \text{nobly distinguished models} \\
\text{elements} &\Longleftrightarrow \text{galaxies} \\
\text{kings} &\Longleftrightarrow \text{nobles} \\
\text{workers} &\Longleftrightarrow \text{peasants} \\
\text{star-types} &\Longleftrightarrow \text{locally consistent cosmic spectrums} \\
\text{certificates} &\Longleftrightarrow \text{certificates}
\end{aligned}
$$

Certificate Expansion

Theorem

Let $\mathcal{S}$ be a certificate for the type instance T over the $\mathcal{L}^2\mathrm{eE}_{\text{refine}}$-classified signature $\langle \Sigma, \bar{\boldsymbol{m}} \rangle$. Then T has a finite model in which each worker type is realized at least t times.

- Type realizability for $\mathcal{L}^2\mathrm{eE}_{\text{refine}}$ is in $\mathrm{NPT{\scriptsize IME}}$
- Satisfiability for $\mathcal{L}^2\mathrm{eE}_{\text{refine}}$ is in $\mathrm{NE{\scriptsize XP}T{\scriptsize IME}}$