

Satisfiability with Agreement and Counting

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Contents

1	Introduction	1
1.1	Notation	1
1.2	Syntax	2
1.3	Semantics	4
1.4	Logic games	6
1.5	Types	7
1.6	Normal forms	8
1.7	Complexity	8
2	Counter setups	11
2.1	Bits	11
2.2	Counters	11
2.3	Vectors	13
2.4	Permutations	15
3	Equivalence relations	17
3.1	Two equivalence relations in agreement	17
3.2	Many equivalence relations in agreement	18
4	Reductions	21
4.1	Global agreement to refinement	22
4.2	Local agreement to refinement	25
4.3	Granularity	29
5	Monadic logics	33
5.1	Cells	33
5.2	Organs	34
5.3	Satisfiability	36
5.4	Hardness with a single equivalence	37
5.5	Hardness with many equivalences in refinement	43
6	Two-variable logics	47
6.1	Type realizability of the two-variable first-order logic	47
6.2	Type realizability of the two-variable first-order logic with equivalences in refinement	52
6.3	NOTES	56

Glossary

$ A $ the cardinality of A . 1	$v \prec w$ lexicographically smaller. 2
$\wp A$ the powerset of A . 1	\mathbb{S}_n the set of permutations of $[1, n]$. 2
$\wp^+ A$ the set of nonempty subsets of A . 1	$\exp_a^e(x)$ tetration. 2
$\wp^\kappa A$ the set of subsets of A of cardinality κ . 1	Ω an alphabet. 2
$A \times B$ the cartesian product of A and B . 1	$w = w_1 w_2 \dots w_n$ a word. 2
$\text{dom } R$ the domain of R . 1	Ω^* the set of words over Ω . 2
$\text{ran } R$ the range of R . 1	Ω^+ the set of nonempty words over Ω . 2
R^{-1} the inverse of R . 1	Ω^n the set of words of length n over Ω . 2
$R \upharpoonright S$ the restriction of R to S . 1	\mathbb{B} the bits. 2
$R[a]$ the R -successors of a . 1	\mathbb{B}^+ the bitstrings. 2
$S \circ R$ the composition of S and R . 1	$\ n\ $ the bitsize of n . 2
id_A the identity on A . 1	\bar{n} the binary encoding of n . 2
$f : A \rightarrow B$ a total function from A to B . 1	\underline{b} the number encoded by b . 2
$f : A \hookrightarrow B$ an injective function from A into B . 1	N_t the largest t -bit number. 2
$f : A \twoheadrightarrow B$ a surjective function from A onto B . 1	\mathbb{B}_t the t -bit numbers. 2
$f : A \leftrightarrow B$ a bijective function between A and B . 1	$\Omega_{\mathcal{C}}$ the symbol alphabet. 2
$f : A \rightsquigarrow B$ a partial function from A to B . 1	\mathcal{V} the variable symbols. 3
$f(a) \simeq b$ f is defined at a with value b . 1	\mathbf{x} the first variable symbol. 3
$f(a) \simeq \perp$ f is not defined at a . 1	\mathbf{y} the second variable symbol. 3
ch_S^A characteristic function. 1	\mathbf{z} the third variable symbol. 3
$\ A\ $ the length of A . 1	Σ a predicate signature. 3
$\langle a, b, c \rangle$ a sequence. 1	\mathbf{p}_i a predicate symbol. 3
ε the empty sequence. 1	$\text{ar } \mathbf{p}_i$ the arity of \mathbf{p}_i . 3
$A + B$ the concatenation of A and B . 1	$\mathcal{A}t[\Sigma]$ the atomic formulas over Σ . 3
$A - B$ A without the elements of B . 1	$\mathcal{L}it[\Sigma]$ the literals over Σ . 3
\mathbb{N} the natural numbers. 1	$\mathcal{C}[\Sigma]$ the first-order formulas with counting quantifiers over Σ . 3
\mathbb{N}^+ the positive natural numbers. 1	$\mathcal{L}[\Sigma]$ the first-order formulas over Σ . 3
$[n, m]$ the discrete interval between n and m . 1	$\text{vr } \varphi$ the variables occurring φ . 3
\log the base-2 logarithm. 2	$\text{fvr } \varphi$ the variables freely occurring φ . 3
	$\mathcal{L}^v[\Sigma]$ the v -variable first-order formulas over Σ . 3
	$\mathcal{C}^v[\Sigma]$ the v -variable first-order formulas with counting quantifiers over Σ . 3

Glossary

$\text{qr } \varphi$ the quantifier rank of φ . 3	$[\mathbf{u}:\text{eq}](\mathbf{x}, \mathbf{y})$ \mathbf{u} -data equal at \mathbf{x} and \mathbf{y} . 11
$\mathcal{L}_r[\Sigma]$ the r -rank first-order formulas over Σ . 4	$[\mathbf{u}:\text{eq-01}](\mathbf{x}, \mathbf{y})$ \mathbf{u} -data at \mathbf{x} and \mathbf{y} is 0 and 1. 11
$\mathcal{C}_r[\Sigma]$ the r -rank first-order formulas with counting quantifiers over Σ . 4	$[\mathbf{u}:\text{eq-10}](\mathbf{x}, \mathbf{y})$ \mathbf{u} -data at \mathbf{x} and \mathbf{y} is 1 and 0. 11
$\mathcal{L}_r^v[\Sigma]$ the r -rank v -variable first-order formulas over Σ . 4	C a counter setup. 11
$\mathcal{C}_r^v[\Sigma]$ the r -rank v -variable first-order formulas with counting quantifiers over Σ . 4	$[\mathbf{C}:\text{data}]^{\mathfrak{A}}$ C-data at \mathfrak{A} . 12
\mathfrak{A} a structure. 4	$[\mathbf{C}:\text{eq-}d](\mathbf{x})$ C-data at \mathbf{x} is d . 12
$\varphi^{\mathfrak{A}}$ interpretation of φ in \mathfrak{A} . 5	$[\mathbf{C}:\text{eq}](\mathbf{x}, \mathbf{y})$ C-data equal at \mathbf{x} and \mathbf{y} . 12
SAT \mathcal{K} the satisfiable sentences of \mathcal{K} . 5	$[\mathbf{C}:\text{less}]d(\mathbf{x})$ C-data at \mathbf{x} less than d . 12
FINSAT \mathcal{K} the finitely satisfiable sentences of \mathcal{K} . 6	$[\mathbf{C}:\text{betw-}d\text{-}e](\mathbf{x})$ C-data at \mathbf{x} between d and e . 12
$\varphi \equiv \psi$ logically equivalent formulas. 6	$[\mathbf{C}:\text{allbetw-}d\text{-}e]$ C-data between d and e . 12
$\mathfrak{A} \equiv \mathfrak{B}$ elementary equivalent structures. 6	$[\mathbf{C}:\text{less}](\mathbf{x}, \mathbf{y})$ C-data at \mathbf{x} less than C-data at \mathbf{y} . 13
$\mathfrak{A} \equiv_r \mathfrak{B}$ r -rank equivalent structures. 6	$[\mathbf{C}:\text{succ}](\mathbf{x}, \mathbf{y})$ C-data at \mathbf{y} succeeds C-data at \mathbf{x} . 13
$\mathfrak{A} \equiv^v \mathfrak{B}$ v -variable equivalent structures. 6	$[\mathbf{V}(p):\text{data}]^{\mathfrak{A}}a$ the value of the p -th counter at a . 13
$\mathfrak{A} \equiv_r^v \mathfrak{B}$ r -rank v -variable equivalent structures. 6	$[\mathbf{V}:\text{data}]^{\mathfrak{A}}$ the V-data at a . 13
\mathfrak{p} parital isomorphism. 6	$[\mathbf{V}:\text{eq-v}](\mathbf{x})$ the V-data at \mathbf{x} . 13
$G_r(\mathfrak{A}, \mathfrak{B})$ the r -round Ehrenfeucht-Fraïssé game. 6	$[\mathbf{V}(pq):\text{at-}i\text{-eq}](\mathbf{x})$ equal i -th bits at p and q at \mathbf{x} . 14
$\Pi[\Sigma]$ the set of 1-types over Σ . 7	$[\mathbf{V}(pq):\text{at-}i\text{-eq-01}](\mathbf{x})$ equal i -th bits at p and q are 0 and 1. 14
$\mathbf{T}[\Sigma]$ the set of 1-types over Σ . 7	$[\mathbf{V}(pq):\text{at-}i\text{-eq-10}](\mathbf{x})$ equal i -th bits at p and q are 1 and 0. 14
τ^{-1} the inverse of the type τ . 7	$[\mathbf{V}(pq):\text{eq}](\mathbf{x})$ equal p and q V-data at \mathbf{x} . 14
$\text{tp}_{\mathbf{x}} \tau$ the \mathbf{x} -type of τ . 7	$[\mathbf{V}(pq):\text{less}](\mathbf{x})$ V-data at p less than at q . 14
$\text{tp}_{\mathbf{y}} \tau$ the \mathbf{y} -type of τ . 7	$[\mathbf{V}(pq):\text{succ}](\mathbf{x})$ V-data at q succeeds the data at p . 14
$\text{tp}^{\mathfrak{A}}[a]$ the 1-type of a in \mathfrak{A} . 7	$[\mathbf{P}:\text{alldiff}]$ P-data at different positions is different. 15
$\pi^{\mathfrak{A}}$ the interpretation of the 1-type π in \mathfrak{A} . 7	$[\mathbf{P}:\text{perm}]$ P-data is a permutation. 15
$\text{tp}^{\mathfrak{A}}[a, b]$ the 1-type of a in \mathfrak{A} . 7	$\mathcal{E}E$ the set of equivalence classes of E . 17
$\tau^{\mathfrak{A}}$ the interpretation of the 2-type τ in \mathfrak{A} . 7	$[\mathbf{e}:\text{refl}]$ e is reflexive. 17
PTime complexity class. 8	$[\mathbf{e}:\text{symm}]$ e is symmetric. 17
$A \leq_m^{\text{PTime}} B$ A is polynomial-time reducible to B . 9	$[\mathbf{e}:\text{trans}]$ e is transitive. 17
$A =_m^{\text{PTime}} B$ A and B are polynomial-time equivalent. 9	$[\mathbf{e}:\text{equiv}]$ e is transitive. 17
B a bit setup. 11	$[\mathbf{d}, \mathbf{e}:\text{refine}]$ refinement. 18
$[\mathbf{u}:\text{data}]^{\mathfrak{A}}$ \mathbf{u} -data at \mathfrak{A} . 11	$[\mathbf{d}, \mathbf{e}:\text{global}]$ global agreement. 18
$[\mathbf{u}:\text{eq-}d](\mathbf{x})$ \mathbf{u} -data at \mathbf{x} is d . 11	

- $[d, e:\text{local}]$ local agreement. 18
 $[e_1, e_2, \dots, e_e:\text{refine}]$ symbols in refinement. 19
 $[e_1, e_2, \dots, e_e:\text{global}]$ symbols in global agreement. 19
 $[e_1, e_2, \dots, e_e:\text{local}]$ symbols in local agreement. 19
 $\Lambda_p^v e E_a$ logic notation. 21
 $[P:\text{alleq}]$ P-data equal everywhere. 22
 $[P:\text{globperm}]$ P-data is a global permutation. 22
 $[L:\text{eg-}i](x, y)$ global refinement induced by levels. 23
 grtr translation of global agreement to refinement. 23
 $[E:\text{chperm}]^{\mathfrak{A}}$ characteristic E-permutation in \mathfrak{A} . 25
 $[L:\text{fixperm}]$ fixed permutation condition. 26
 $[L:\text{locperm}]$ local agreement condition. 27
 $[L:\text{el-}i]$ local refinement induced by levels. 27
 ltr translation of local agreement to refinement. 28
 g granularity. 29
 G granularity color setup. 29
 $[\Gamma:d]$ finer equivalence granularity formula. 30
 grtr granularity translation. 30
 $[\Sigma:\text{cell}](x, y)$ Σ -cell formula. 33
 \mathcal{O} organ-equivalence relation. 34
 \mathcal{O} sub-organ-equivalence relation. 35
 $[D:\text{Data}]^{\mathfrak{A}}$ D-Data. 39
 $[D:\text{Zero}](x)$ zero D-Data at x . 39
 $[D:\text{Largest}](x)$ maximum D-Data at x . 40
 m_i message symbols. 47
 $\langle \Sigma, \bar{m} \rangle$ classified signature. 47
 (Π, T) type instance. 48
 $\pi \sim \pi'$ connectable 1-types. 49

1 Introduction

1.1 Notation

The cardinal number $|A|$ is the *cardinality* of the set A . The set $\wp A$ is the *powerset* of A . The set $\wp^+ A = \wp A \setminus \{\emptyset\}$ is the *set of nonempty subsets* of A . If κ is a cardinal number, the set $\wp^\kappa A = \{S \in \wp A \mid |S| = \kappa\}$ is the κ -*powerset* of A . The *cartesian product* of A and B is $A \times B$. The sets A and B *properly intersect* if $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$.

If R is a binary relation, its *domain* is $\text{dom } R$ and its *range* is $\text{ran } R$. The *inverse* of $R \subseteq A \times B$ is

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}.$$

If S is a set and $R \subseteq A \times B$, the *restriction* of R to S is

$$R \upharpoonright S = \{(a, b) \in R \mid a \in S\}.$$

If $R \subseteq A \times B$ is a binary relation and $a \in A$, the R -*successors* of a are

$$R[a] = \{b \in B \mid (a, b) \in R\}.$$

If $S \subseteq B \times C$ and $R \subseteq A \times B$ are two binary relations, their *composition* is

$$S \circ R = \{(a, c) \in A \times C \mid (\exists b \in B)(a, b) \in R \wedge (b, c) \in S\}.$$

A *function* is formally just a functional relation. The *identity function* on A is id_A . A *total function* from A to B is denoted $f : A \rightarrow B$. A *injective function* from A into B is denoted $f : A \hookrightarrow B$. A *surjective function* from A onto B is denoted $f : A \twoheadrightarrow B$. A *bijective function* between A and B is denoted $f : A \leftrightarrow B$. A *partial function* from A to B is denoted $f : A \rightsquigarrow B$. If $f : A \rightsquigarrow B$ is a partial function and $a \in A$, the notation $f(a) \simeq b$ means that f is defined at a and its value is b ; the notation $f(a) \simeq \perp$ means that f is not defined at a . If $S \subseteq A$, the *characteristic function* of S in A is $\text{ch}_S^A : A \rightarrow \{0, 1\}$.

A *sequence* is formally just a function with domain an ordinal number. If A is a sequence, its *length* $\|A\|$ is just the domain of A . The sequence consisting of the elements a, b and c in that order is $\langle a, b, c \rangle$. The *empty sequence* is ε . A *finite sequence* is a sequence of finite length. If A and B are two sequences, their *concatenation* is $A + B$, and the sequence obtained from A by dropping all elements of B is $A - B$.

The set of *natural numbers* is $\mathbb{N} = \{0, 1, \dots\}$. The set of *positive natural numbers* is $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. If $n, m \in \mathbb{N}$ are natural numbers, the *discrete interval* $[n, m]$ between n

1 Introduction

and m is

$$[n, m] = \begin{cases} \{n, n+1, \dots, m\} & \text{if } n \leq m \\ \emptyset & \text{otherwise.} \end{cases}$$

The function **log** is the *base-2 logarithm*.

An n -vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{N}^n$ is just a tuple of natural numbers. The n -vector \mathbf{v} is *lexicographically smaller*¹ than the n -vector \mathbf{w} (written $\mathbf{v} \prec \mathbf{w}$) if there is a position $p \in [1, n]$ such that $v_p < w_p$ and $v_q = w_q$ for all $q \in [p+1, n]$.

The set of n -permutations of $[1, n]$ is \mathbb{S}_n . We think of an n -permutation ν as an n -vector $\nu = (\nu(1), \nu(2), \dots, \nu(n))$.

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is *polynomially bounded* if there is a polynomial p and a number $n_0 \in \mathbb{N}$ such that $f(n) \leq p(n)$ for all $n \geq n_0$. The function f is *exponentially bounded* if there is a polynomial p and a number $n_0 \in \mathbb{N}$ such that $f(n) \leq 2^{p(n)}$ for all $n \geq n_0$. We are going to use these terms implicitly with respect to quantities that depend on one another. For example, the cardinality of \mathbb{S}_n is exponentially bounded by n .

Define the *tetration* operation $\text{exp}_a^e(x)$ by $\text{exp}_a^0(x) = x$ and $\text{exp}_a^{e+1}(x) = a^{\text{exp}_a^e(x)}$, so $\text{exp}_a^e(x) = a^{a^{\dots^{a^x}}}$ is a tower of e exponentiations.

An *alphabet* Ω is just a nonempty set. The elements of Ω are *characters*. A *word* $w = w_1 w_2 \dots w_n$ is a finite sequence of characters. The set of words over Ω is Ω^* . The set of nonempty words over Ω is $\Omega^+ = \Omega^* \setminus \{\varepsilon\}$. If $n \in \mathbb{N}$, the set of words of length n over Ω is Ω^n .

The set of *bits* is $\mathbb{B} = \{0, 1\}$. The set of *bitstrings* is \mathbb{B}^+ . The bitstrings are read right-to-left, that is the bitstring $b = 10$ has first character 0. If $t < u \in \mathbb{N}^+$, the t -bit bitstrings \mathbb{B}^t are embedded into the u -bit bitstrings \mathbb{B}^u by appending leading zeroes. If $n \in \mathbb{N}$, the *bitsize* $\|n\|$ of n is:

$$\|n\| = \begin{cases} 1 & \text{if } n = 0 \\ \lfloor \log n \rfloor + 1 & \text{otherwise.} \end{cases}$$

If $n \in \mathbb{N}$, the *binary encoding* of n is $\overline{n} \in \mathbb{B}^{\|n\|}$. If $b \in \mathbb{B}^t$, the *number encoded by* b is \underline{b} . The *largest t -bit number* is $N_t = 2^t - 1$. The set of *t -bit numbers* is $\mathbb{B}_t = [0, N_t]$.

1.2 Syntax

The *symbol alphabet* for the first-order logic with counting quantifiers is

$$\Omega_C = \left\{ \neg, \wedge, \vee, \rightarrow, \leftrightarrow; \exists, \forall; =; (, , ,); \leq, =, \geq, {}^0, {}^1 \right\}.$$

The propositional connectives are listed in decreasing order of precedence. The *negation* \neg is unary; the *disjunction* \vee , *conjunction* \wedge and *equivalence* \leftrightarrow are left-associative; the

¹the higher positions to the right are more significant; it may *look like* this ordering is the anti-lexicographic one, for example $(1, 1, 0) \prec (0, 0, 1)$.

implication \rightarrow is right-associative. The quantifiers bind as strong as the negation. Note that we consider logics with *formal equality* $=$.

A *counting quantifier* is a word over $\Omega_{\mathcal{C}}$ of the form $\exists^{\leq \bar{m}}$ or $\exists^{=\bar{m}}$ or $\exists^{\geq \bar{m}}$, where $m \in \mathbb{N}$ and $\bar{m} \in \mathbb{B}^+$ is the binary encoding of m . Note that this encoding of the counting quantifiers is *succinct*. As we note in Remark 1, this succinct representation allows for exponentially small counting formulas compared to their pure first-order equivalents. We denote the counting quantifiers by $\exists^{\leq m}$, $\exists^{=m}$ and $\exists^{\geq m}$, that is, we omit the encoding notation for m .

The sequence $\mathcal{V} = \langle v_1, v_2, \dots \rangle$ is a countable sequence of distinct *variable symbols*. We pay special attention to $\mathbf{x} = v_1$, $\mathbf{y} = v_2$ and $\mathbf{z} = v_3$, the *first*, *second* and *third* variable symbol, respectively.

A *predicate signature* $\Sigma = \langle p_1, p_2, \dots, p_s \rangle$ is a finite sequence of distinct *predicate symbols* p_i together with their *arities* $\text{ar } p_i \in \mathbb{N}^+$. A predicate signature is *unary* or *monadic* if all of its predicate symbols have arity 1. A predicate signature is *binary* if all of its predicate symbols have arity 1 or 2. For the purposes of this work we will not be considering constant and function symbols—constant symbols can be simulated by a fresh unary predicate symbol having the intended interpretation of being true at a unique element; presence of function symbols on the other hand leads quite easily to undecidable satisfiability problems. By convention $\Omega_{\mathcal{C}}$, \mathcal{V} and Σ are disjoint.

Let Σ be a predicate signature. The set of *atomic formulas* $\text{At}[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\alpha ::= (x = y) \mid p(x_1, x_2, \dots, x_n)$$

for $x, y \in \mathcal{V}$, $p \in \Sigma$, $n = \text{ar } p$ and $x_1, x_2, \dots, x_n \in \mathcal{V}$.

The set of *literals* $\text{Lit}[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\lambda ::= \alpha \mid (\neg \alpha).$$

The set of *first-order formulas with counting quantifiers* $\mathcal{C}[\Sigma] \subset (\Omega_{\mathcal{C}} \cup \mathcal{V} \cup \Sigma)^*$ over Σ is generated by the grammar:

$$\begin{aligned} \varphi ::= & \alpha \mid (\neg \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi) \mid (\exists x \varphi) \mid (\forall x \varphi) \mid \\ & \mid (\exists^{\leq m} x \varphi) \mid (\exists^{=m} x \varphi) \mid (\exists^{\geq m} x \varphi) \end{aligned}$$

for $x \in \mathcal{V}$ and $m \in \mathbb{N}$.

The set of *first-order formulas* $\mathcal{L}[\Sigma] \subset \mathcal{C}[\Sigma]$ over Σ consists of the formulas that do not feature a counting quantifier.

The set of variables occurring in φ is $\text{vr } \varphi \subset \mathcal{V}$. The set of variables freely occurring in φ is $\text{fvr } \varphi \subset \mathcal{V}$. A formula φ is a *sentence* if $\text{fvr } \varphi = \emptyset$. For $v \in \mathbb{N}$, a formula φ is a *v-variable formula* if $\text{vr } \varphi \subseteq \{v_1, v_2, \dots, v_v\}$. The set of *v-variable first-order formulas* over Σ is $\mathcal{L}^v[\Sigma]$. The set of *v-variable first-order formulas with counting quantifiers* over Σ is $\mathcal{C}^v[\Sigma]$.

If $\varphi \in \mathcal{C}[\Sigma]$, the *quantifier rank* $\text{qr } \varphi \in \mathbb{N}$ of φ is defined as follows. If φ matches:

1 Introduction

- $(x = y)$, then $\text{qr } \varphi = 0$
- $p(x_1, x_2, \dots, x_n)$, then $\text{qr } \varphi = 0$
- $(\neg\psi)$, then $\text{qr } \varphi = \text{qr } \psi$
- $\psi_1 \oplus \psi_2$ for $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\text{qr } \varphi = \max(\text{qr } \psi_1, \text{qr } \psi_2)$
- $(\exists x\psi)$ or $(\forall x\psi)$, then $\text{qr } \varphi = 1 + \text{qr } \psi$
- $(\exists^{\leq m}x\psi)$ or $(\exists^=m x\psi)$, then $\text{qr } \varphi = m + 1 + \text{qr } \psi$
- $(\exists^{\geq m}x\psi)$, then $\text{qr } \varphi = m + \text{qr } \psi$.

An r -rank formula is a formula having quantifier rank r . The set of r -rank first-order formulas over Σ is $\mathcal{L}_r[\Sigma]$. The set of r -rank first-order formulas with counting quantifiers over Σ is $\mathcal{C}_r[\Sigma]$. The set of r -rank v -variable first-order formulas over Σ is $\mathcal{L}_r^v[\Sigma]$. The set of r -rank v -variable first-order formulas with counting quantifiers over Σ is $\mathcal{C}_r^v[\Sigma]$.

If φ is a formula and $x_1, x_2, \dots, x_n \in \mathcal{V}$ are distinct variables, we use the notation $\varphi(x_1, x_2, \dots, x_n)$, a *focused formula*, to show that we are interested in the free occurrences of the variables x_i in φ . If $\varphi(x_1, x_2, \dots, x_n)$ is a focused formula and $y_1, y_2, \dots, y_n \in \mathcal{V}$, then $\varphi(y_1, y_2, \dots, y_n)$ denotes the formula φ where all free occurrences of x_i are replaced by y_i . The notation $\varphi = \varphi(x_1, x_2, \dots, x_n)$ means that $\text{fvr } \varphi \subseteq \{x_1, x_2, \dots, x_n\}$.

We will omit unnecessary brackets in formulas.

1.3 Semantics

If Σ is a predicate signature, a Σ -structure \mathfrak{A} consists of a nonempty set A (the *domain* of \mathfrak{A}), together with a relation $p^{\mathfrak{A}} \subseteq A^{\text{ar } p}$ (the *interpretation* of p at \mathfrak{A}) for every predicate symbol $p \in \Sigma$. A structure is *finite* if its domain is finite. We omit the standard definition of semantic notions. Seldom it will be useful to consider *structures with possibly empty domain*. We will be explicit when this is the case. If \mathfrak{A} is a structure and $B \subseteq A$ there is a substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with possibly empty domain B . We call it the substructure induced by B and denote it $(\mathfrak{A} \upharpoonright B)$.

Note that the interpretation of the counting quantifiers is clear: $\exists^{\leq m}x\varphi$ means that “at most m elements satisfy φ ”; $\exists^=m x\varphi$ means that “exactly m elements satisfy φ ”; $\exists^{\geq m}x\varphi$ means that “at least m elements satisfy φ ”.

The *standard translation* $\text{st} : \mathcal{C}[\Sigma] \rightarrow \mathcal{L}[\Sigma]$ of first-order formulas with counting quantifiers to logically equivalent first-order formulas is defined as follows. If φ matches:

- $(x = y)$ or $p(x_1, x_2, \dots, x_n)$, then $\text{st } \varphi = \varphi$
- $(\neg\psi)$, then $\text{st } \varphi = (\neg \text{st } \psi)$
- $(\psi_1 \oplus \psi_2)$ for $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\text{st } \varphi = (\text{st } \psi_1 \oplus \text{st } \psi_2)$
- $(Qx\psi)$ for $Q \in \{\exists, \forall\}$, then $\text{st } \varphi = (Qx \text{st } \psi)$

- $(\exists^{\leq m} x \psi(x))$ or $(\exists^m x \psi(x))$ or $(\exists^{\geq m} x \psi(x))$, then let

$$\theta_{\leq} = \forall y_1 \forall y_2 \dots \forall y_m \forall y_{m+1} \left(\bigwedge_{1 \leq i \leq m+1} \text{st } \psi(y_i) \rightarrow \bigvee_{1 \leq i < j \leq m+1} y_i = y_j \right)$$

$$\theta_{\geq} = \exists y_1 \exists y_2 \dots \exists y_m \left(\bigwedge_{1 \leq i \leq m} \text{st } \psi(y_i) \wedge \bigwedge_{1 \leq i < j \leq m} y_i \neq y_j \right)$$

where y_1, y_2, \dots, y_{m+1} are distinct variable symbols not occurring in φ . The formula θ_{\leq} asserts that there are at most m distinct values satisfying ψ . The formula θ_{\geq} asserts that there are at least m distinct values satisfying ψ . If $\varphi = (\exists^{\leq m} x \psi(x))$, then $\text{st } \varphi = \theta_{\leq}$. If $\varphi = (\exists^m x \psi(x))$, then $\text{st } \varphi = (\theta_{\leq} \wedge \theta_{\geq})$. If $\varphi = (\exists^{\geq m} x \psi(x))$, then $\text{st } \varphi = \theta_{\geq}$.

Remark 1. The translation of a first-order formula with counting quantifiers φ to a logically equivalent first-order formula $\psi = \text{st } \varphi$ preserves quantifier rank. However, the resulting formula ψ may have exponentially larger length.

A predicate signature with intended interpretations Σ is formally a predicate signature together with an *intended interpretation condition* \mathcal{A} , which is formally a class of Σ -structures. A Σ -structure \mathfrak{A} is then just an element of \mathcal{A} . That is, when we speak about a predicate signature with intended interpretations, we are considering the logics strictly over the class of structures respecting the intended interpretation condition. The semantic concepts are relativised appropriately in this context. For example, if $\Sigma = \langle e \rangle$ is a predicate signature consisting of the single binary predicate symbol e , having intended interpretation as an equivalence, then the Σ -formula $\forall x e(x, x)$ is logically valid. From now on, we will use the term *predicate signature* as *predicate signature with possible intended interpretations*.

The predicate signature Σ' is an *enrichment* of the predicate signature Σ if Σ' contains all predicate symbols of Σ and respects their intended interpretation in Σ . A Σ' -structure \mathfrak{A}' is an enrichment of the Σ -structure \mathfrak{A} if they have the same domain and the same interpretation of the predicate symbols of Σ . The basic semantic significance of enrichment is that if $\varphi(x_1, x_2, \dots, x_n)$ is a Σ -formula and $a_1, a_2, \dots, a_n \in A$, then $\mathfrak{A} \models \varphi(a_1, a_2, \dots, a_n)$ iff $\mathfrak{A}' \models \varphi(a_1, a_2, \dots, a_n)$. If \mathfrak{A}' is an enrichment of \mathfrak{A} then \mathfrak{A} is a *reduct*² of \mathfrak{A}' .

If $\varphi(x_1, x_2, \dots, x_n)$ is a focused formula, the interpretation of φ in \mathfrak{A} is

$$\varphi^{\mathfrak{A}} = \{(a_1, a_2, \dots, a_n) \in A^n \mid \mathfrak{A} \models \varphi(a_1, a_2, \dots, a_n)\}.$$

If Σ is a predicate signature and φ is a Σ -sentence, then φ is *satisfiable* if there is a Σ -structure that is a model for φ ; φ is *finitely satisfiable* if there is a *finite* Σ -structure that is a model for φ . If $\mathcal{K} \subseteq \mathcal{C}[\Sigma]$ is a family of formulas over the predicate signature Σ , the set of *satisfiable sentences* is $\text{SATK} \subseteq \mathcal{K}$ and the set of *finitely satisfiable sentences* is

²or why not *empoverishment*?

1 Introduction

$\text{FINSAT}\mathcal{K} \subseteq \mathcal{K}$. The family \mathcal{K} has the *finite model property* if $\text{SAT}\mathcal{K} = \text{FINSAT}\mathcal{K}$. By the Löwenheim-Skolem theorem, every satisfiable sentence φ has a finite or countable model (assuming the intended interpretation condition of the predicate signature is first-order-definable). In this work the intended interpretation conditions of the predicate signatures will always be first-order-definable formula and we will silently assume that all structures are either finite or countable.

Two Σ -sentences φ and ψ are *logically equivalent* (written $\varphi \equiv \psi$) if they have the same models.

Two Σ -structures \mathfrak{A} and \mathfrak{B} are *elementary equivalent* (written $\mathfrak{A} \equiv \mathfrak{B}$) if they satisfy the same first-order sentences (hence also the same first-order sentences with counting quantifiers). The structures \mathfrak{A} and \mathfrak{B} are *r -rank equivalent* (written $\mathfrak{A} \equiv_r \mathfrak{B}$) if they satisfy the same r -rank first-order sentences. The structures \mathfrak{A} and \mathfrak{B} are *v -variable equivalent* (written $\mathfrak{A} \equiv^v \mathfrak{B}$) if they satisfy the same v -variable first-order sentences. The structures \mathfrak{A} and \mathfrak{B} are *r -rank v -variable equivalent* (written $\mathfrak{A} \equiv_r^v \mathfrak{B}$) if they satisfy the same r -rank v -variable first-order sentences.

1.4 Logic games

Logic games capture structure equivalence. Let Σ be a predicate signature and let \mathfrak{A} and \mathfrak{B} be Σ -structures. A *partial isomorphism* $\mathbf{p} : A \rightsquigarrow B$ from \mathfrak{A} to \mathfrak{B} is a partial mapping that is an isomorphism between the induced substructures $(\mathfrak{A} \upharpoonright \text{dom } \mathbf{p})$ and $(\mathfrak{B} \upharpoonright \text{ran } \mathbf{p})$.

Let $r \in \mathbb{N}^+$. The *r -round Ehrenfeucht-Fraïssé game* $G_r(\mathfrak{A}, \mathfrak{B})$ is a two-player game, played with a pair of pebbles, one for each structure. The two players are Spoiler and Duplicator. Initially the pebbles are off the structures. During each round, Spoiler picks a pebble and a structure and places it on some element in that structure. Duplicator responds by picking the other pebble and placing it on some element in the other structure. Thus during round i , the players play a pair of elements $a_i \mapsto b_i \in A \times B$. Collect the sequences of played elements $\bar{a} = \langle a_1, a_2, \dots, a_r \rangle$ and $\bar{b} = \langle b_1, b_2, \dots, b_r \rangle$. Duplicator wins the match if the relation $\bar{a} \mapsto \bar{b} = \{a_1 \mapsto b_1, a_2 \mapsto b_2, \dots, a_r \mapsto b_r\} \subseteq A \times B$, built from the pairs of elements in each round, is a partial isomorphism from \mathfrak{A} to \mathfrak{B} . Ehrenfeucht's theorem says that Duplicator has a winning strategy for $G_r(\mathfrak{A}, \mathfrak{B})$ iff $\mathfrak{A} \equiv_r \mathfrak{B}$. Fraïssé's theorem gives a back-and-forth characterization of the winning strategy for Duplicator [1, ch. 2]:

Theorem 1. *Suppose that $(\mathfrak{I}_0, \mathfrak{I}_1, \dots, \mathfrak{I}_r)$ is a sequence of nonempty sets of partial isomorphisms between \mathfrak{A} and \mathfrak{B} with the following properties:*

1. *For every $j < r$, $\mathbf{p} \in \mathfrak{I}_{j+1}$ and $a \in A$, there is $\mathbf{q} \in \mathfrak{I}_j$ such that $\mathbf{p} \subseteq \mathbf{q}$ and $a \in \text{dom } \mathbf{q}$.*
2. *For every $j < r$, $\mathbf{p} \in \mathfrak{I}_{j+1}$ and $b \in B$, there is $\mathbf{q} \in \mathfrak{I}_j$ such that $\mathbf{p} \subseteq \mathbf{q}$ and $b \in \text{ran } \mathbf{q}$.*

Then $\mathfrak{A} \equiv_r \mathfrak{B}$.

1.5 Types

Let $\Sigma = \langle p_1, p_2, \dots, p_s \rangle$ be a predicate signature. A 1-type π over Σ is a maximal consistent set of literals featuring only the variable symbol \mathbf{x} ³. The set of 1-types over Σ is $\Pi[\Sigma]$. Note that consistency here is relativised by the intended interpretations of the predicate signature. For example if Σ contains the binary predicate symbol e with intended interpretation as an equivalence, then every 1-type over Σ includes the literal $e(\mathbf{x}, \mathbf{x})$. Also note that the cardinality of a 1-type over Σ is polynomially bounded by the length s of Σ and the cardinality of $\Pi[\Sigma]$ is exponentially bounded by s .

A 2-type τ over Σ is a maximal consistent set of literals featuring only the variable symbols \mathbf{x} and \mathbf{y} and including the literal $(\mathbf{x} \neq \mathbf{y})$. The set of 2-types over Σ is $T[\Sigma]$. Again, consistency is relativised by the intended interpretation of the predicate signature. For example, if Σ contains the binary predicate symbol e with intended interpretation as an equivalence, then if $e(\mathbf{x}, \mathbf{y}) \in \tau$, then $e(\mathbf{y}, \mathbf{x}) \in \tau$. Again, the cardinality of a 2-type over Σ is polynomially bounded by s and the cardinality of $T[\Sigma]$ is exponentially bounded by s .

If $\tau \in T[\Sigma]$, the *inverse* τ^{-1} of τ is the 2-type obtained from τ by swapping the variables \mathbf{x} and \mathbf{y} in every literal. The \mathbf{x} -type of τ is the 1-type $\text{tp}_{\mathbf{x}} \tau$ consisting of all the literals of τ featuring only the variable symbol \mathbf{x} . Similarly, the \mathbf{y} -type of τ is the 1-type $\text{tp}_{\mathbf{y}} \tau$ consisting of all the literals of τ featuring only the variable symbol \mathbf{y} , that is replaced by \mathbf{x} . For instance we have the identity $\text{tp}_{\mathbf{x}} \tau^{-1} = \text{tp}_{\mathbf{y}} \tau$. We say that τ *connects* the 1-types $\text{tp}_{\mathbf{x}} \tau$ and $\text{tp}_{\mathbf{y}} \tau$.

If \mathfrak{A} is a Σ -structure and $a \in A$, the 1-type of a in \mathfrak{A} is

$$\text{tp}^{\mathfrak{A}}[a] = \{\lambda(\mathbf{x}) \in \text{Lit}[\Sigma] \mid \mathfrak{A} \models \lambda(a)\}.$$

If $\text{tp}^{\mathfrak{A}}[a] = \pi$, we say that the 1-type π is *realized* by a in \mathfrak{A} . The interpretation of the 1-type π in \mathfrak{A} is the set of elements realizing π :

$$\pi^{\mathfrak{A}} = \{a \in A \mid \text{tp}^{\mathfrak{A}}[a] = \pi\}.$$

If $a \neq b \in A$, the 2-type of (a, b) in \mathfrak{A} is

$$\text{tp}^{\mathfrak{A}}[a, b] = \{\lambda(\mathbf{x}, \mathbf{y}) \in \text{Lit}[\Sigma] \mid \mathfrak{A} \models \lambda(a, b)\}.$$

We do not define a 2-type in case $a = b$. If $\text{tp}^{\mathfrak{A}}[a, b] = \tau$, we say that the 2-type τ is *realized* by (a, b) in \mathfrak{A} . The interpretation of the 2-type τ in \mathfrak{A} is the set of pairs realizing τ :

$$\tau^{\mathfrak{A}} = \{(a, b) \in A \times A \mid a \neq b \wedge \text{tp}^{\mathfrak{A}}[a, b] = \tau\}.$$

³this is different than the commonly used notion of type in model theory, where types are sets of general formulas, not just literals

1.6 Normal forms

In two-variable logics, a common technique of reducing formula quantifier rank while preserving satisfiability is Skolemization [2]: Let φ be a \mathcal{L}^2 -sentence. By replacing universally quantified subformulas $\forall x\psi$ by twofold existential negations $\neg\exists x\neg\psi$, without loss of generality assume that only existential quantifiers occur in φ . Consider a subformula ψ of φ that has the lowest possible nontrivial quantifier rank 1. Then $\psi = \psi(y) = \exists x\alpha(x, y)$, where the formula α is quantifier-free, $\{x, y\} = \{\mathbf{x}, \mathbf{y}\}$ and y may or may not necessarily occur freely in α . Introduce a new unary predicate symbol \mathbf{u}_ψ with the intended interpretation $\forall y(\mathbf{u}_\psi(y) \leftrightarrow \exists x\alpha(x, y))$ and let φ' be the formula obtained from φ by replacing the subformula ψ by $\mathbf{u}_\psi(y)$. The original formula φ is equisatisfiable with $\varphi_1 = \forall y(\mathbf{u}_\psi(y) \leftrightarrow \exists x\alpha(x, y)) \wedge \varphi'$ in a strict sense, that is any model for φ can be \mathbf{u}_ψ -enriched into a model for φ_1 and any model for φ_1 is a model for φ . By repeating this process linearly many times, we can bring the formula to a form where the quantifier rank is at most 2 [3, 2]:

Theorem 2 (Scott). *There is a polynomial-time reduction $\text{sctr} : \mathcal{L}^2 \rightarrow \mathcal{L}^2$ which reduces every sentence φ to a sentence $\text{sctr } \varphi$ in Scott normal form:*

$$\forall \mathbf{x} \forall \mathbf{y} (\alpha_0(\mathbf{x}, \mathbf{y}) \vee \mathbf{x} = \mathbf{y}) \wedge \bigwedge_{1 \leq i \leq m} \forall \mathbf{x} \exists \mathbf{y} (\alpha_i(\mathbf{x}, \mathbf{y}) \wedge \mathbf{x} \neq \mathbf{y}),$$

where the formulas α_i are quantifier-free and use at most linearly many new unary predicate symbols. The sentences φ and $\text{sctr } \varphi$ are satisfiable over the same domains. Moreover the length $\text{sctr } \varphi$ is linear in the length of φ .

A completely analogous normal form can be described for the two-variable fragment with counting quantifiers [4]:

Theorem 3 (Pratt-Hartmann). *There is a polynomial-time reduction $\text{prtr} : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ with reduces every sentence φ to a sentence $\text{prtr } \varphi$ in the form:*

$$\forall \mathbf{x} \forall \mathbf{y} (\alpha_0(\mathbf{x}, \mathbf{y}) \vee \mathbf{x} = \mathbf{y}) \wedge \bigwedge_{1 \leq i \leq m} \forall \mathbf{x} \exists^{=M_i} \mathbf{y} (\alpha_i(\mathbf{x}, \mathbf{y}) \wedge \mathbf{x} \neq \mathbf{y}),$$

where the formulas α_i are quantifier-free and may use linearly many new unary and binary predicate symbols. Let $M = \max\{M_1, M_2, \dots, M_m\}$. Then φ and $\text{prtr } \varphi$ are satisfiable over the same domains of cardinality greater than M . Moreover the length $\text{prtr } \varphi$ is linear in the length of φ .

1.7 Complexity

We denote the complexity classes $\mathbf{PTIME} = \text{TIME}[\text{poly}(n)] = \bigcup_{c \in \mathbb{N}^+} \text{TIME}[n^c]$, \mathbf{NPTIME} , \mathbf{PSPACE} , $\mathbf{EXPTIME}$ and $\mathbf{NEXPTIME}$. For $e \in \mathbb{N}^+$, the e -exponential deterministic and nondeterministic time classes are $e\mathbf{EXPTIME} = \text{TIME}[\exp_e^e(\text{poly}(n))]$ and $\mathbf{NeEXPTIME}$. The complexity class $\mathbf{ELEMENTARY}$ is the union of the complexity classes $e\mathbf{EXPTIME}$ for $e \in \mathbb{N}^+$.

The Grzegorzcyk hierarchy \mathcal{E}^i for $i \in \mathbb{N}$ orders the primitive recursive functions by means of the power of recursion needed. The *basic functions* are the zero function $\text{zero}(n) = 0$, the successor function $\text{succ}(n) = n + 1$ and the projection functions $\text{proj}_i^u(n_1, n_2, \dots, n_u) = n_i$. If $u, v \in \mathbb{N}$, $f : \mathbb{N}^u \rightarrow \mathbb{N}$ and $g_1, g_2, \dots, g_u : \mathbb{N}^v \rightarrow \mathbb{N}$ are functions, their *superposition* is the function $h : \mathbb{N}^v \rightarrow \mathbb{N}$ defined by $h(\bar{n}) = f(g_1(\bar{n}), g_2(\bar{n}), \dots, g_u(\bar{n}))$ for $\bar{n} \in \mathbb{N}^v$. If $u \in \mathbb{N}$, $f : \mathbb{N}^u \rightarrow \mathbb{N}$ and $g : \mathbb{N}^{u+2} \rightarrow \mathbb{N}$, their *primitive recursion* is the function $h : \mathbb{N}^{u+1} \rightarrow \mathbb{N}$ defined by:

$$\begin{aligned} h(\bar{n}, 0) &= f(\bar{n}) \\ h(\bar{n}, i + 1) &= g(\bar{n}, i, h(\bar{n}, i)) \end{aligned}$$

for $\bar{n} \in \mathbb{N}^u$. For $i \in \mathbb{N}$, define the function E_i by $E_0(n) = n + 1$ and

$$E_{i+1}(n) = E_i^n(2) = \underbrace{E_i(E_i(\dots E_i(2)))}_n.$$

For $i \in \mathbb{N}$, the i -th level of the Grzegorzcyk hierarchy \mathcal{E}^i as the least set of functions containing the basic functions, the functions E_k for $k \in [0, i]$ and closed under superposition and *limited primitive recursion*, that is a primitive recursion $h : \mathbb{N}^{u+1} \rightarrow \mathbb{N}$ of the functions $f : \mathbb{N}^u \rightarrow \mathbb{N}$, $g : \mathbb{N}^{u+2} \rightarrow \mathbb{N}$, $f, g \in \mathcal{E}^i$, such that there is a function $b : \mathbb{N}^{u+1} \rightarrow \mathbb{N}$, $b \in \mathcal{E}^i$ bounding h : $h(\bar{n}) \leq b(\bar{n})$ for all $n \in \mathbb{N}^{u+1}$. A decision problem $A \subseteq \Omega^*$ is in some level of the Grzegorzcyk hierarchy just in case its characteristic function occurs at that level. The primitive recursive functions are partitioned by the Grzegorzcyk hierarchy. The complexity class ELEMENTARY coincides with the third level of the Grzegorzcyk hierarchy \mathcal{E}^3 .

If $A \subseteq \Omega_1^*$ and $B \subseteq \Omega_2^*$ are decision problems, the problem A is *many-one polynomial-time reducible* to B (written $A \leq_m^{\text{PTIME}} B$) if there is a polynomial-time algorithm $f : \Omega_1^* \rightarrow \Omega_2^*$ such that $a \in A$ iff $f(a) \in B$. Similar reductions are defined analogously. The decision problems A and B are *many-one polynomial-time equivalent* (written $A =_m^{\text{PTIME}} B$) if $A \leq_m^{\text{PTIME}} B$ and $B \leq_m^{\text{PTIME}} A$.

A decision problem is *hard* for a complexity class if any decision problem of that complexity class is polynomial-time reducible to it. A decision problem is *complete* for a complexity class if it is hard for that class and contained in that class.

We will need the following standard domino tiling problem [5, p. 403]: A *domino system* is a triple $D = (T, H, V)$, where $T = [1, k]$ is a finite set of *tiles* and $H, V \subseteq T \times T$ are *horizontal* and *vertical matching relations*. A *tiling* of $m \times m$ for a domino system D with *initial condition* $c^0 = \langle t_1^0, t_2^0, \dots, t_n^0 \rangle$, where $n \leq m$, is a mapping $t : [1, m] \times [1, m] \rightarrow T$ such that:

- $(t(i, j), t(i + 1, j)) \in H$ for all $i \in [1, m - 1]$ and $j \in [1, m]$
- $(t(i, j), t(i, j + 1)) \in V$ for all $i \in [1, m]$ and $j \in [1, m - 1]$
- $t(i, 1) = t_i^0$ for all $i \in [1, n]$.

It is well-known [6, 7] that there exists a domino system D_0 for which:

1 Introduction

- the problem asking whether there exists a tiling of $m \times m$ with initial condition c^0 of length n , where $m = n$, is NPTIME-complete.
- the problem asking whether there exists a tiling of $m \times m$ with initial condition c^0 of length n , where $m = 2^n$, is NEXPTIME-complete.
- the problem asking whether there exists a tiling of $m \times m$ with initial condition c^0 of length n , where $m = 2^{2^n}$, is N2EXPTIME-complete.
- the argument extends to arbitrary exponential towers: the problem asking whether there exists a tiling of $m \times m$ with initial condition c^0 of length n , where $m = \exp_2^e(n)$ is NeEXPTIME-complete.

2 Counter setups

2.1 Bits

A *bit setup* $\mathbf{B} = \langle \mathbf{u} \rangle$ is a predicate signature consisting of a single unary predicate symbol \mathbf{u} .

Definition 1. Let \mathfrak{A} be a \mathbf{B} -structure. Define the function $[\mathbf{u}:\text{data}]^{\mathfrak{A}} : A \rightarrow \mathbb{B}$ by:

$$[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = \begin{cases} 1 & \text{if } \mathfrak{A} \models \mathbf{u}(a) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2. Let $d \in \mathbb{B}$. Define the quantifier-free $\mathcal{L}^1[\mathbf{B}]$ -formula $[\mathbf{u}:\text{eq}-d](x)$ by:

$$[\mathbf{u}:\text{eq}-d](x) = \begin{cases} \mathbf{u}(x) & \text{if } d = 1 \\ \neg \mathbf{u}(x) & \text{otherwise.} \end{cases}$$

If \mathfrak{A} is a \mathbf{B} -structure, $a \in A$ and $d \in \mathbb{B}$, then $\mathfrak{A} \models [\mathbf{u}:\text{eq}-d](a)$ iff $[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = d$.

Definition 3. Define the quantifier-free $\mathcal{L}^2[\mathbf{B}]$ -formulas $[\mathbf{u}:\text{eq}](x, y)$, $[\mathbf{u}:\text{eq}-01](x, y)$ and $[\mathbf{u}:\text{eq}-10](x, y)$ by:

$$\begin{aligned} [\mathbf{u}:\text{eq}](x, y) &= \mathbf{u}(x) \leftrightarrow \mathbf{u}(y) \\ [\mathbf{u}:\text{eq}-01](x, y) &= \neg \mathbf{u}(x) \wedge \mathbf{u}(y) \\ [\mathbf{u}:\text{eq}-10](x, y) &= \mathbf{u}(x) \wedge \neg \mathbf{u}(y). \end{aligned}$$

If \mathfrak{A} is a \mathbf{B} -structure and $a, b \in A$, then:

- $\mathfrak{A} \models [\mathbf{u}:\text{eq}](a, b)$ iff $[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = [\mathbf{u}:\text{data}]^{\mathfrak{A}} b$
- $\mathfrak{A} \models [\mathbf{u}:\text{eq}-01](a, b)$ iff $[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = 0$ and $[\mathbf{u}:\text{data}]^{\mathfrak{A}} b = 1$
- $\mathfrak{A} \models [\mathbf{u}:\text{eq}-10](a, b)$ iff $[\mathbf{u}:\text{data}]^{\mathfrak{A}} a = 1$ and $[\mathbf{u}:\text{data}]^{\mathfrak{A}} b = 0$.

2.2 Counters

A *t-bit counter setup* for $t \in \mathbb{N}^+$ is a predicate signature $\mathbf{C} = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \rangle$ consisting of t distinct unary predicate symbols \mathbf{u}_i .

2 Counter setups

Definition 4. Let \mathfrak{A} be a C-structure. Define the function $[\mathbf{C:data}]^{\mathfrak{A}} : A \rightarrow \mathbb{B}_t$ by:

$$[\mathbf{C:data}]^{\mathfrak{A}} a = \sum_{1 \leq i \leq t} 2^{i-1} [\mathbf{u}_i:\text{data}]^{\mathfrak{A}} a.$$

Definition 5. Let $d \in \mathbb{B}_t$ be a t -bit number. Define the quantifier-free $\mathcal{L}^1[\mathbf{C}]$ -formula $[\mathbf{C:eq-d}](\mathbf{x})$ by:

$$[\mathbf{C:eq-d}](\mathbf{x}) = \bigwedge_{1 \leq i \leq t} [\mathbf{u}_i:\text{eq-}\bar{d}_i](\mathbf{x}).$$

If \mathfrak{A} is a C-structure, $a \in A$ and $d \in \mathbb{B}_t$, then $\mathfrak{A} \models [\mathbf{C:eq-d}](a)$ iff $[\mathbf{C:data}]^{\mathfrak{A}} a = d$.

If A is a nonempty set and $\text{data} : A \rightarrow \mathbb{B}_t$ is any function, there is a C-structure \mathfrak{A} over A such that $[\mathbf{C:data}]^{\mathfrak{A}} = \text{data}$.

Definition 6. Define the quantifier-free $\mathcal{L}^2[\mathbf{C}]$ -formula $[\mathbf{C:eq}](\mathbf{x}, \mathbf{y})$ by:

$$[\mathbf{C:eq}](\mathbf{x}, \mathbf{y}) = \bigwedge_{1 \leq i \leq t} [\mathbf{u}_i:\text{eq}](\mathbf{x}, \mathbf{y}).$$

If \mathfrak{A} is a C-structure and $a, b \in A$, then $\mathfrak{A} \models [\mathbf{C:eq}](a, b)$ iff $[\mathbf{C:data}]^{\mathfrak{A}} a = [\mathbf{C:data}]^{\mathfrak{A}} b$.

The bitstring $a \in \mathbb{B}^t$ encodes a number less than the number encoded by the bitstring $b \in \mathbb{B}^t$, if they differ and at least position where they are different $j \in [1, t]$ the bitstring a has value 0 and the bitstring b has value 1, that is, iff there is a position $j \in [1, t]$ such that the following two conditions hold:

$$a_j = 0 \text{ and } b_j = 1 \quad (\text{Less1})$$

$$a_k = b_k \text{ for all } k \in [j+1, t]. \quad (\text{Less2})$$

Definition 7. Define the quantifier-free $\mathcal{L}^2[\mathbf{C}]$ -formula $[\mathbf{C:less}](\mathbf{x}, \mathbf{y})$ by:

$$[\mathbf{C:less}](\mathbf{x}, \mathbf{y}) = \bigvee_{1 \leq j \leq t} [\mathbf{u}_j:\text{eq-01}](\mathbf{x}, \mathbf{y}) \wedge \bigwedge_{j < k \leq t} [\mathbf{u}_k:\text{eq}](\mathbf{x}, \mathbf{y}).$$

If \mathfrak{A} is a C-structure and $a, b \in A$, then $\mathfrak{A} \models [\mathbf{C:less}](a, b)$ iff $[\mathbf{C:data}]^{\mathfrak{A}} a < [\mathbf{C:data}]^{\mathfrak{A}} b$.

The bitstring $b \in \mathbb{B}^t$ encodes the successor of the number encoded by the bitstring a if there is a position $j \in [1, t]$ such that the following four conditions hold:

$$a_j = 0 \text{ and } b_j = 1 \quad (\text{Succ1})$$

$$a_i = 1 \text{ for all } i \in [1, j-1] \quad (\text{Succ2})$$

$$b_i = 0 \text{ for all } i \in [1, j-1] \quad (\text{Succ3})$$

$$a_k = b_k \text{ for all } k \in [j+1, t]. \quad (\text{Succ4})$$

Definition 8. Define the quantifier-free $\mathcal{L}^2[\mathbf{C}]$ -formula $[\mathbf{C:succ}](\mathbf{x}, \mathbf{y})$ by:

$$[\mathbf{C:succ}](\mathbf{x}, \mathbf{y}) = \bigvee_{1 \leq j \leq t} [\mathbf{u}_j:\text{eq-01}](\mathbf{x}, \mathbf{y}) \wedge \bigwedge_{1 \leq i < j} [\mathbf{u}_i:\text{eq-10}](\mathbf{x}, \mathbf{y}) \wedge \bigwedge_{j < k \leq t} [\mathbf{u}_k:\text{eq}](\mathbf{x}, \mathbf{y}).$$

If \mathfrak{A} is a C-structure and $a, b \in A$, then:

$$\mathfrak{A} \models [\text{C:succ}](a, b) \text{ iff } [\text{C:data}]^{\mathfrak{A}}b = 1 + [\text{C:data}]^{\mathfrak{A}}a.$$

Definition 9. Let $d \in \mathbb{B}_t$. Define the quantifier-free $\mathcal{L}^1[\text{C}]$ -formula $[\text{C:less}]d(\mathbf{x})$ by:

$$[\text{C:less-}d](\mathbf{x}) = \bigvee_{1 \leq j \leq t} \neg \mathbf{u}_j(\mathbf{x}) \wedge \neg [\mathbf{u}_j:\text{eq-}\bar{d}_j](\mathbf{x}) \wedge \bigwedge_{j < k \leq t} [\mathbf{u}_k:\text{eq-}\bar{d}_k](\mathbf{x}).$$

If \mathfrak{A} is a C-structure, $a \in A$ and $d \in \mathbb{B}_t$, then $\mathfrak{A} \models [\text{C:less-}d](a)$ iff $[\text{C:data}]^{\mathfrak{A}}a < d$.

Definition 10. Let $d \leq e \in \mathbb{B}_t$. Define the quantifier-free $\mathcal{L}^1[\text{C}]$ -formula $[\text{C:betw-}d-e](\mathbf{x})$ by:

$$[\text{C:betw-}d-e](\mathbf{x}) = \neg [\text{C:less-}d](\mathbf{x}) \wedge ([\text{C:less-}e](\mathbf{x}) \vee [\text{C:eq-}e](\mathbf{x})).$$

If \mathfrak{A} is a C-structure, $a \in A$ and $d \leq e \in \mathbb{B}_t$, then

$$\mathfrak{A} \models [\text{C:betw-}d-e](a) \text{ iff } d \leq [\text{C:data}]^{\mathfrak{A}}a \leq e.$$

Definition 11. Let $d \leq e \in \mathbb{B}_t$. Define the $\mathcal{L}^1[\text{C}]$ -sentence $[\text{C:allbetw-}d-e]$ by:

$$[\text{C:allbetw-}d-e] = \forall \mathbf{x} [\text{C:betw-}d-e](\mathbf{x}).$$

If \mathfrak{A} is a C-structure and $d \leq e \in \mathbb{B}_t$, then $\mathfrak{A} \models [\text{C:betw-}d-e]$ iff $d \leq [\text{C:data}]^{\mathfrak{A}}a \leq e$ for all $a \in A$.

2.3 Vectors

Let $n, t \in \mathbb{N}^+$. Recall the set of n -dimensional t -bit vectors is \mathbb{B}_t^n . An n -dimensional t -bit vector setup is a predicate signature $V = \langle \mathbf{u}_{11}, \mathbf{u}_{12}, \dots, \mathbf{u}_{nt} \rangle$ of (nt) distinct unary predicate symbols. The counter setup $V(p)$ of V at position $p \in [1, n]$ is $V(p) = \langle \mathbf{u}_{p1}, \mathbf{u}_{p2}, \dots, \mathbf{u}_{pt} \rangle$.

Definition 12. Let \mathfrak{A} be a V -structure and $a \in A$. We refer to $[\text{V}(p):\text{data}]^{\mathfrak{A}}a$ as the value of the p -th counter at a . Define the function $[\text{V:data}]^{\mathfrak{A}} : A \rightarrow \mathbb{B}_t^n$ by:

$$[\text{V:data}]^{\mathfrak{A}}a = ([\text{V}(1):\text{data}]^{\mathfrak{A}}a, [\text{V}(2):\text{data}]^{\mathfrak{A}}a, \dots, [\text{V}(n):\text{data}]^{\mathfrak{A}}a).$$

Definition 13. Let $\mathbf{v} = (d_1, d_2, \dots, d_n) \in \mathbb{B}_t^n$ be an n -dimensional t -bit vector. Define the quantifier-free $\mathcal{L}^1[V]$ -formula $[\text{V:eq-v}](\mathbf{x})$ by:

$$[\text{V:eq-v}](\mathbf{x}) = \bigwedge_{1 \leq p \leq n} [\text{V}(p):\text{eq-}d_p](\mathbf{x}).$$

If \mathfrak{A} is a V -structure, $a \in A$ and $\mathbf{v} \in \mathbb{B}_t^n$, then $\mathfrak{A} \models [\text{V:eq-v}](a)$ iff $[\text{V:data}]^{\mathfrak{A}}a = \mathbf{v}$.

If \mathfrak{A} is a nonempty set and $\text{data} : A \rightarrow \mathbb{B}_t^n$ is any function, then there is a V -structure \mathfrak{A} over A such that $[\text{V:data}]^{\mathfrak{A}} = \text{data}$.

2 Counter setups

Definition 14. Let $p, q \in [1, n]$ and let $i \in [1, t]$. Define the quantifier-free $\mathcal{L}^1[V]$ -formulas $[V(pq):\text{at-}i\text{-eq}](\mathbf{x})$, $[V(pq):\text{at-}i\text{-eq-01}](\mathbf{x})$ and $[V(pq):\text{at-}i\text{-eq-10}](\mathbf{x})$ by:

$$\begin{aligned} [V(pq):\text{at-}i\text{-eq}](\mathbf{x}) &= \mathbf{u}_{pi}(\mathbf{x}) \leftrightarrow \mathbf{u}_{qi}(\mathbf{x}) \\ [V(pq):\text{at-}i\text{-eq-01}](\mathbf{x}) &= \neg \mathbf{u}_{pi}(\mathbf{x}) \wedge \mathbf{u}_{qi}(\mathbf{x}) \\ [V(pq):\text{at-}i\text{-eq-10}](\mathbf{x}) &= \mathbf{u}_{pi}(\mathbf{x}) \wedge \neg \mathbf{u}_{qi}(\mathbf{x}). \end{aligned}$$

If \mathfrak{A} is a V-structure and $a \in A$, then:

- $\mathfrak{A} \models [V(pq):\text{at-}i\text{-eq}](a)$ iff $[\mathbf{u}_{pi}:\text{data}]^{\mathfrak{A}}a = [\mathbf{u}_{qi}:\text{data}]^{\mathfrak{A}}a$, that is the values of the i -th bit at positions p and q at a are equal
- $\mathfrak{A} \models [V(pq):\text{at-}i\text{-eq-01}](a)$ iff $[\mathbf{u}_{pi}:\text{data}]^{\mathfrak{A}}a = 0$ and $[\mathbf{u}_{qi}:\text{data}]^{\mathfrak{A}}a = 1$, that is the i -th bit at position p at a is 0 and the i -th bit at position q at a is 1
- $\mathfrak{A} \models [V(pq):\text{at-}i\text{-eq-10}](a)$ iff $[\mathbf{u}_{pi}:\text{data}]^{\mathfrak{A}}a = 1$ and $[\mathbf{u}_{qi}:\text{data}]^{\mathfrak{A}}a = 0$, that is the i -th bit at position p at a is 1 and the i -th bit at position q at a is 0.

Definition 15. Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[V]$ -formula $[V(pq):\text{eq}](\mathbf{x})$ by:

$$[V(pq):\text{eq}](\mathbf{x}) = \bigwedge_{1 \leq i \leq t} [V(pq):\text{at-}i\text{-eq}](\mathbf{x}).$$

If \mathfrak{A} is a V-structure and $a \in A$, then:

$$\mathfrak{A} \models [V(pq):\text{eq}](a) \text{ iff } [V(p):\text{data}]^{\mathfrak{A}}a = [V(q):\text{data}]^{\mathfrak{A}}a.$$

Definition 16. Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[V]$ -formula $[V(pq):\text{less}](\mathbf{x})$ by:

$$[V(pq):\text{less}](\mathbf{x}) = \bigvee_{1 \leq j \leq t} [V(pq):\text{at-}j\text{-eq-01}](\mathbf{x}) \wedge \bigwedge_{j < k \leq t} [V(pq):\text{at-}k\text{-eq}](\mathbf{x}).$$

If \mathfrak{A} is a V-structure and $a \in A$, then:

$$\mathfrak{A} \models [V(pq):\text{less}](a) \text{ iff } [V(p):\text{data}]^{\mathfrak{A}}a < [V(q):\text{data}]^{\mathfrak{A}}a.$$

Definition 17. Let $p, q \in [1, n]$. Define the quantifier-free $\mathcal{L}^1[V]$ -formula

$$\begin{aligned} [V(pq):\text{succ}](\mathbf{x}) &= \bigvee_{1 \leq j \leq t} \bigwedge_{1 \leq i < j} [V(pq):\text{at-}i\text{-eq-10}](\mathbf{x}) \wedge [V(pq):\text{at-}j\text{-eq-01}](\mathbf{x}) \wedge \\ &\quad \bigwedge_{j < k \leq t} [V(pq):\text{at-}k\text{-eq}](\mathbf{x}). \end{aligned}$$

If \mathfrak{A} is a V-structure and $a \in A$, then:

$$\mathfrak{A} \models [V(pq):\text{succ}](a) \text{ iff } [V(q):\text{data}]^{\mathfrak{A}}a = 1 + [V(p):\text{data}]^{\mathfrak{A}}a.$$

2.4 Permutations

Let $n \in \mathbb{N}^+$. An n -permutation setup $P = \langle \mathbf{u}_{11}, \mathbf{u}_{12}, \dots, \mathbf{u}_{nt} \rangle$ is just an n -dimensional t -bit vector setup, where $t = \|n\|$ is the bitsize of n . Recall that the set \mathbb{S}_n of all permutations of $[1, n]$ is a subset of \mathbb{B}_t^n .

Definition 18. Define the quantifier-free $\mathcal{L}^1[P]$ -sentence $[P:\text{alldiff}]$ by:

$$[P:\text{alldiff}] = \forall \mathbf{x} \bigwedge_{1 \leq p < q \leq n} \neg [P(pq):\text{eq}](\mathbf{x}).$$

If \mathfrak{A} is a P -structure then $\mathfrak{A} \models [P:\text{alldiff}]$ iff $[P(p):\text{data}]^{\mathfrak{A}} a \neq [P(q):\text{data}]^{\mathfrak{A}} a$ for all $a \in A$ and $p \neq q \in [1, n]$.

Definition 19. Define the quantifier-free $\mathcal{L}^1[P]$ -sentence $[P:\text{perm}]$ by:

$$[P:\text{perm}] = [P:\text{betw-1-n}] \wedge [P:\text{alldiff}].$$

If \mathfrak{A} is a P -structure then $\mathfrak{A} \models [P:\text{perm}]$ iff $[P:\text{data}]^{\mathfrak{A}} a \in \mathbb{S}_n$ for all $a \in A$.

If A is a nonempty set and $\text{data} : A \rightarrow \mathbb{S}_n$ is any function, then there is a P -structure $\mathfrak{A} \models [P:\text{perm}]$ over A such that $[P:\text{data}]^{\mathfrak{A}} = \text{data}$.

3 Equivalence relations

An *equivalence relation* $E \subseteq A \times A$ on A is a relation that is reflexive, symmetric and transitive. The set of *equivalence classes* of E is $\mathcal{E}E = \{E[a] \mid a \in A\}$.

Let $E = \langle e \rangle$ be a predicate signature consisting of a single binary predicate symbol e . Define the $\mathcal{L}^2[E]$ -sentence $[e:\text{refl}]$ by:

$$[e:\text{refl}] = \forall x e(x, x).$$

Define the $\mathcal{L}^2[E]$ -sentence $[e:\text{symm}]$ by:

$$[e:\text{symm}] = \forall x \forall y (e(x, y) \rightarrow e(y, x)).$$

Define the $\mathcal{L}^3[E]$ -sentence $[e:\text{trans}]$ by:

$$[e:\text{trans}] = \forall x \forall y \forall z (e(x, y) \wedge e(y, z) \rightarrow e(x, z)).$$

Define the $\mathcal{L}^3[E]$ -sentence $[e:\text{equiv}]$ by:

$$[e:\text{equiv}] = [e:\text{refl}] \wedge [e:\text{symm}] \wedge [e:\text{trans}].$$

Let \mathfrak{A} be an E -structure and let $E = e^{\mathfrak{A}}$. Then E is reflexive iff $\mathfrak{A} \models [e:\text{refl}]$; E is symmetric iff $\mathfrak{A} \models [e:\text{symm}]$; E is transitive iff $\mathfrak{A} \models [e:\text{trans}]$; E is an equivalence on A iff $\mathfrak{A} \models [e:\text{equiv}]$. It can be shown that transitivity and equivalence cannot be defined in the two-variable fragment with counting $\mathcal{C}^2[E]$.

3.1 Two equivalence relations in agreement

Definition 20. Let $\langle D, E \rangle$ be a sequence of two equivalence relations on A . The relation D is *finer* than the relation E if every equivalence class of D is a subset of some equivalence class of E . Equivalently, $D \subseteq E$. Equivalently,

$$(\forall a \in A)(\forall b \in A) (D(a, b) \rightarrow E(a, b)).$$

If D is finer than E , then E is coarser than D . The sequence $\langle D, E \rangle$ is a sequence of equivalence relations on A in *refinement* if D is finer E .

The sequence $\langle D, E \rangle$ is a sequence of equivalence relations in *global agreement* if either D is finer than E or E is finer than D .

The sequence $\langle D, E \rangle$ is a sequence of equivalence relations in *local agreement* if for every $a \in A$, either $D[a] \subseteq E[a]$ or $E[a] \subseteq D[a]$. Equivalently, no two equivalence classes $E[a]$ and $D[b]$ properly intersect. Equivalently,

$$(\forall a \in A) ((\forall b \in A) (D(a, b) \rightarrow E(a, b)) \vee (\forall b \in A) (E(a, b) \rightarrow D(a, b))).$$

3 Equivalence relations

Let $E = \langle \mathbf{d}, \mathbf{e} \rangle$ be a predicate signature consisting of the two binary predicate symbols \mathbf{d} and \mathbf{e} . Let \mathfrak{A} is an E -structure and suppose that \mathbf{d} and \mathbf{e} are interpreted in \mathfrak{A} as equivalence relations on A . Let $D = \mathbf{d}^{\mathfrak{A}}$ and $E = \mathbf{e}^{\mathfrak{A}}$ be the interpretations of the two symbols.

Definition 21. Define the $\mathcal{L}^2[E]$ -sentence $[\mathbf{d}, \mathbf{e}:\text{refine}]$ by:

$$[\mathbf{d}, \mathbf{e}:\text{refine}] = \forall x \forall y (\mathbf{d}(x, y) \rightarrow \mathbf{e}(x, y)).$$

Then $\langle D, E \rangle$ is in refinement iff $\mathfrak{A} \models [\mathbf{d}, \mathbf{e}:\text{refine}]$.

Definition 22. Define the $\mathcal{L}^2[E]$ -sentence $[\mathbf{d}, \mathbf{e}:\text{global}]$ by:

$$[\mathbf{d}, \mathbf{e}:\text{global}] = [\mathbf{d}, \mathbf{e}:\text{refine}] \vee [\mathbf{e}, \mathbf{d}:\text{refine}].$$

Then $\langle D, E \rangle$ is in global agreement iff $\mathfrak{A} \models [\mathbf{d}, \mathbf{e}:\text{global}]$.

Definition 23. Define the $\mathcal{L}^2[E]$ -sentence $[\mathbf{d}, \mathbf{e}:\text{local}]$ by:

$$[\mathbf{d}, \mathbf{e}:\text{local}] = \forall x (\forall y (\mathbf{d}(x, y) \rightarrow \mathbf{e}(x, y)) \vee \forall y (\mathbf{e}(x, y) \rightarrow \mathbf{d}(x, y))).$$

Then $\langle D, E \rangle$ is in global agreement iff $\mathfrak{A} \models [\mathbf{d}, \mathbf{e}:\text{local}]$.

Lemma 1. If $\langle D, E \rangle$ is a sequence two equivalence relations on A , then it is in local agreement iff $L = D \cup E$ is an equivalence relation on A .

Proof. The union of two equivalence relations on A is a reflexive and symmetric relation.

First suppose that D and E are in local agreement. We claim that L is transitive. Let $a, b, c \in A$ be such that $(a, b) \in L$ and $(b, c) \in L$. Since D and E are in local agreement, without loss of generality $D[b] \subseteq E[b]$. Since $(a, b) \in L$, either $a \in D[b] \subseteq E[b]$ or $a \in E[b]$. Similarly $c \in E[b]$. Therefore $(a, c) \in E \subseteq L$.

Next suppose that L is an equivalence relation, let $b \in A$ and assume towards a contradiction that $D[b] \not\subseteq E[b]$ and $E[b] \not\subseteq D[b]$. There is some $a \in D[b] \setminus E[b]$ and $c \in E[b] \setminus D[b]$. Then $(a, b) \in D \subseteq L$ and $(b, c) \in E \subseteq L$, hence $(a, c) \in L$. Without loss of generality $(a, c) \in E$. Since $c \in E[b]$, we have $a \in E[b]$ — a contradiction. \square

3.2 Many equivalence relations in agreement

Let e be a positive natural number.

Definition 24. Let $\langle E_1, E_2, \dots, E_e \rangle$ be a sequence of equivalence relations on A .

The sequence is in refinement if $E_1 \subseteq E_2 \subseteq \dots \subseteq E_e$.

The sequence is in global agreement if the equivalence relations form a chain under inclusion, that is for all $i, j \in [1, e]$, either $E_i \subseteq E_j$ or $E_j \subseteq E_i$. Equivalently, there is a (not necessarily unique) permutation $\nu \in \mathbb{S}_e$ such that $E_{\nu(1)} \subseteq E_{\nu(2)} \subseteq \dots \subseteq E_{\nu(e)}$.

The sequence is in local agreement if for every element $a \in A$ the equivalence classes $E_1[a], E_2[a], \dots, E_e[a]$ form a chain under inclusion. Equivalently, no two equivalence classes $E_i[a]$ and $E_j[b]$ properly intersect.

3.2 Many equivalence relations in agreement

Let $E = \langle e_1, e_2, \dots, e_e \rangle$ be a predicate signature consisting of e binary predicate symbols. Let \mathfrak{A} be an E -structure and suppose that the symbols e_i are interpreted as equivalence relations on A . Let $E_i = e_i^{\mathfrak{A}}$ for $i \in [1, e]$.

Definition 25. Define the $\mathcal{L}^2[E]$ -sentence $[e_1, e_2, \dots, e_e:\text{refine}]$ by:

$$[e_1, e_2, \dots, e_e:\text{refine}] = \forall \mathbf{x} \forall \mathbf{y} \bigwedge_{1 \leq i < e} (e_i(\mathbf{x}, \mathbf{y}) \rightarrow e_{i+1}(\mathbf{x}, \mathbf{y})).$$

Then $\langle E_1, E_2, \dots, E_e \rangle$ is in refinement iff $\mathfrak{A} \models [e_1, e_2, \dots, e_e:\text{refine}]$.

Definition 26. Define the $\mathcal{L}^2[E]$ -sentence $[e_1, e_2, \dots, e_e:\text{global}]$ by:

$$[e_1, e_2, \dots, e_e:\text{global}] = \bigvee_{\nu \in \mathbb{S}_e} [e_{\nu(1)}, e_{\nu(2)}, \dots, e_{\nu(e)}:\text{refine}].$$

Then $\langle E_1, E_2, \dots, E_e \rangle$ is in global agreement iff $\mathfrak{A} \models [e_1, e_2, \dots, e_e:\text{global}]$. Note that the length of the formula $[e_1, e_2, \dots, e_e:\text{global}]$ grows exponentially as e grows.

Definition 27. Define the $\mathcal{L}^2[E]$ -sentence $[e_1, e_2, \dots, e_e:\text{local}]$ by:

$$[e_1, e_2, \dots, e_e:\text{local}] = \forall \mathbf{x} \bigvee_{\nu \in \mathbb{S}_e} \forall \mathbf{y} \bigwedge_{1 \leq i < e} (e_{\nu(i)}(\mathbf{x}, \mathbf{y}) \rightarrow e_{\nu(i+1)}(\mathbf{x}, \mathbf{y})).$$

Then $\langle E_1, E_2, \dots, E_e \rangle$ is in local agreement iff $\mathfrak{A} \models [e_1, e_2, \dots, e_e:\text{local}]$. Note that the length of the formula $[e_1, e_2, \dots, e_e:\text{local}]$ grows exponentially as e grows.

Let $E = \langle E_1, E_2, \dots, E_e \rangle$ be a sequence of equivalence relations on A .

Theorem 4. The sequence E is in local agreement iff the union $\cup S$ of any nonempty subsequence $S \subseteq E$ is an equivalence relation on A .

Proof. First suppose that the equivalence relations E_i are in local agreement. We show that the union $\cup S$ of arbitrary nonempty subsequence $S = \{E_{i(1)}, E_{i(2)}, \dots, E_{i(s)}\}$, where $1 \leq i(1) < i(2) < \dots < i(s) \leq e$, is an equivalence relation by induction on s , the length of S . If $s = 1$ this claim is trivial. Suppose $s > 1$. By the induction hypothesis, $D = \cup\{E_{i(1)}, E_{i(2)}, \dots, E_{i(s-1)}\}$ is an equivalence relation on A . We claim that D and $E_{i(s)}$ are in local agreement. Indeed, let $a \in A$ be arbitrary and consider $D[a] = E_{i(1)}[a] \cup E_{i(2)}[a] \cup \dots \cup E_{i(s-1)}[a]$ and $E_{i(s)}[a]$. Since all equivalences E_k are in local agreement, either $E_{i(s)}[a] \subseteq E_{i(j)}[a]$ for some $j \in [1, s-1]$, or $E_{i(j)}[a] \subseteq E_{i(s)}[a]$ for all $j \in [1, s-1]$. In the first case $E_{i(s)}[a] \subseteq D[a]$; in the second case $D[a] \subseteq E_{i(s)}[a]$. Thus D and $E_{i(s)}$ are in local agreement. By Lemma 1, $\cup S = D \cup E_{i(s)}$ is an equivalence relation on A .

Next suppose that the equivalences are not in local agreement. There is an element $a \in A$ such that $\{E_i[a] \mid i \in [1, e]\}$ is not a chain. There are $i, j \in [1, e]$ such that $E_i[a] \not\subseteq E_j[a]$ and $E_j[a] \not\subseteq E_i[a]$. Thus E_i and E_j are not in local agreement. By Lemma 1, the union $E_i \cup E_j$ is not an equivalence relation on A . \square

Suppose that the sequence $E = \langle E_1, E_2, \dots, E_e \rangle$ is in local agreement.

3 Equivalence relations

Definition 28. An index set is an element $I \in \wp^+[1, e]$. Define $(E \upharpoonright \cdot) : \wp^+[1, e] \rightarrow \wp^+E$ by:

$$(E \upharpoonright I) = \{E_i \mid i \in I\}.$$

That is, $(E \upharpoonright I)$ just collects the equivalences having indices from I .

The level sequence $L = \langle L_1, L_2, \dots, L_e \rangle$ of the sequence E is defined as follows. For $k \in [1, e]$:

$$L_k = \cap \left\{ \cup(E \upharpoonright I) \mid I \in \wp^k[1, e] \right\}.$$

Remark 2. All L_k are equivalence relations on A .

Proof. Let $k \in [1, e]$ and let $K \in \wp^k[1, e]$ be any k -index set. By Theorem 4, $\cup(E \upharpoonright K)$ is an equivalence relation on A . Since intersection of equivalence relations on A is again an equivalence relation on A , the level $L_k = \cap \left\{ \cup(E \upharpoonright K) \mid K \in \wp^k[1, e] \right\}$ is an equivalence relation on A . \square

Remark 3. The level sequence $L = \langle L_1, L_2, \dots, L_e \rangle$ is a sequence of equivalence relations on A in refinement.

Proof. Let $i < j \in [1, e]$. Let $J \in \wp^j[1, e]$ be any j -index set. We claim that $L_i \subseteq \cup(E \upharpoonright J)$. Indeed, choose some i -index set $I \subset J$. By the definition of L_i we have $L_i \subseteq \cup(E \upharpoonright I) \subseteq \cup(E \upharpoonright J)$. Hence $L_i \subseteq \cap \left\{ \cup(E \upharpoonright J) \mid J \in \wp^j[1, e] \right\} = L_j$. \square

Let $a \in A$. Since the sequence $E = \langle E_1, E_2, \dots, E_e \rangle$ is in local agreement, there is a permutation $\nu \in \mathbb{S}_e$ such that:

$$E_{\nu(1)}[a] \subseteq E_{\nu(2)}[a] \subseteq \dots \subseteq E_{\nu(e)}[a]. \quad (3.1)$$

Lemma 2. If $\nu \in \mathbb{S}_e$ is a permutation satisfying eq. (3.1), then $L_{\nu^{-1}(i)}[a] = E_i[a]$ for all $i \in [1, e]$.

Proof. Let $k = \nu^{-1}(i)$, so $\nu(k) = i$. We claim that $L_k[a] = E_i[a]$. First, consider the k -index set $K = \{\nu(1), \nu(2), \dots, \nu(k)\}$. By the definition of L_k , followed by eq. (3.1), we have $L_k[a] \subseteq \cup(E \upharpoonright K)[a] = E_{\nu(k)}[a] = E_i[a]$. Next, let $K \subseteq \wp^k[1, e]$ be any k -index set. By the pigeonhole principle, there is some $k' \geq k$ such that $k' \in K$. By eq. (3.1) we have:

$$E_i[a] = E_{\nu(k)}[a] \subseteq E_{\nu(k')}[a] \subseteq \cup(E \upharpoonright K)[a].$$

Hence $E_i[a] \subseteq \cap \left\{ \cup(E \upharpoonright K)[a] \mid K \in \wp^k[1, e] \right\} = L_k[a]$. \square

4 Reductions

We restrict our attention to binary predicate signatures, consisting of unary and binary predicate symbols only. To denote various logics with builtin equivalence symbols, we use the notation

$$\Lambda_p^v e E_a$$

where:

- $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ is the *ground logic*
- v , if given, bounds the number of variables
- e , if given, bounds the number of builtin equivalence symbols
- $a \in \{\text{refine}, \text{global}, \text{local}\}$, if given, gives the agreement condition between the builtin equivalence symbols
- p , the *signature power*, specifies constraints on the signature:
 - if $p = 0$, the signature consists of only constantly many unary predicate symbols in addition to the builtin equivalence symbols
 - if $p = 1$, the signature consists of unboundedly many unary predicate symbols in addition to the builtin equivalence symbols
 - if p is not given, the signature consists of unboundedly many unary and binary predicate symbols in addition to the builtin equivalence symbols. This is the commonly investigated fragment with respect to satisfiability of the two-variable logics with or without counting quantifiers.

For example \mathcal{L}_1 is the monadic first-order logic, featuring only unary predicate symbols. $\mathcal{L}_0 1E$ is the first-order logic of a single equivalence relation. \mathcal{C}^2 is the two-variable logic with counting quantifiers, featuring unary and binary predicate symbols. $\mathcal{L}^2 2E$ is the two-variable logic, featuring unary, binary predicate symbols and two builtin equivalence symbols. $\mathcal{C}_1^2 2E_{\text{local}}$ is the two-variable logic with counting quantifiers, featuring unary predicate symbols and two builtin equivalence symbols in local agreement. $\mathcal{L}_1 E_{\text{global}}$ is the monadic first-order logic featuring many equivalence symbols in global agreement.

When we working with a concrete logic, for example $\mathcal{C}^2 2E_{\text{local}}$, we implicitly assume an appropriate generic predicate signature Σ for it. In this case, there are two builtin equivalence symbols d and e in Σ and in addition Σ contains arbitrary many unary

4 Reductions

and binary predicate symbols. The *intended interpretation* of the builtin equivalence symbols is fixed by an appropriate condition θ . In this case:

$$\theta = [d:\text{equiv}] \wedge [e:\text{equiv}] \wedge [d, e:\text{local}].$$

Note that the interpretation condition might in general be a first-order formula outside the logic in interest, as in this case, since for instance $[d:\text{equiv}]$ uses the variables x, y and z and the logic $\mathcal{C}^2\mathcal{E}_{\text{local}}$ is a two-variable logic. Recall that when talking about semantics, we include the intended interpretation condition in the definition of Σ -structures.

4.1 Global agreement to refinement

In this section we demonstrate how (finite) satisfiability in logics featuring builtin equivalence symbols in global agreement reduces to (finite) satisfiability in logics featuring builtin equivalence symbols in refinement. Our strategy is to encode the permutation of the builtin equivalence symbols in global agreement that turns them in refinement into a permutation setup.

Fix an arbitrary ground logic $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ and think of Σ as a predicate signature for the logics $\Lambda\mathcal{E}_{\text{global}}$ and $\Lambda\mathcal{E}_{\text{refine}}$. The e builtin equivalence symbols of Σ are e_1, e_2, \dots, e_e .

Let φ be a $\Lambda[\Sigma]$ -sentence. The class of $\Lambda\mathcal{E}_{\text{refine}}$ -structures satisfying φ coincides with the class of $\Lambda\mathcal{E}_{\text{global}}$ -structures satisfying:

$$\varphi \wedge [e_1, e_2, \dots, e_e:\text{refine}].$$

Hence:

$$(\text{FIN})\text{SAT}_{\Lambda\mathcal{E}_{\text{refine}}} \leq_m^{\text{PTIME}} (\text{FIN})\text{SAT}_{\Lambda\mathcal{E}_{\text{global}}}.$$

Since the length of the formula $[e_1, e_2, \dots, e_e:\text{refine}]$ grows polynomially as e grows:

$$(\text{FIN})\text{SAT}_{\Lambda\mathcal{E}_{\text{refine}}} \leq_m^{\text{PTIME}} (\text{FIN})\text{SAT}_{\Lambda\mathcal{E}_{\text{global}}}.$$

Consider the opposite direction. Let $P = \langle u_{11}, u_{12}, \dots, u_{et} \rangle$ be an e -permutation setup (where $t = \|e\|$).

Definition 29. Define the $\mathcal{L}^2[P]$ -sentence $[P:\text{alleq}]$ by:

$$[P:\text{alleq}] = \forall x \forall y \bigwedge_{1 \leq i \leq e} [P(i):\text{eq}](x, y).$$

If \mathfrak{A} is a P -structure, then $\mathfrak{A} \models [P:\text{alleq}]$ iff $[P:\text{data}]^{\mathfrak{A}}a = [P:\text{data}]^{\mathfrak{A}}b$ for all $a, b \in A$. If A is a nonempty set and $v \in \mathbb{B}_t^e$ is any e -dimensional t -vector, there is a P -structure \mathfrak{A} over A such that $\mathfrak{A} \models [P:\text{alleq}]$ and $[P:\text{data}]^{\mathfrak{A}}a = v$ for all $a \in A$.

Definition 30. Define the $\mathcal{L}^2[P]$ -sentence $[P:\text{globperm}]$ by:

$$[P:\text{globperm}] = [P:\text{perm}] \wedge [P:\text{alleq}].$$

4.1 Global agreement to refinement

If \mathfrak{A} be a P-structure then $\mathfrak{A} \models [\text{P:globperm}]$ iff there is a permutation $\nu \in \mathbb{S}_e$ such that $[\text{P:data}]^{\mathfrak{A}}a = \nu$ for all $a \in A$.

If A be a nonempty set and $\nu \in \mathbb{S}_e$ is any permutation, there is a P-structure \mathfrak{A} over A such that $\mathfrak{A} \models [\text{P:globperm}]$ and $[\text{P:data}]^{\mathfrak{A}}a = \nu$ for all $a \in A$.

Let $L = \langle l_1, l_2, \dots, l_e \rangle + P$ be a predicate signature consisting of the binary predicate symbols l_k in addition to the symbols from P .

Definition 31. For $i \in [1, e]$, define the quantifier-free $\mathcal{L}^2[L]$ -formula $[\text{L:eg-}i](x, y)$ by:

$$[\text{L:eg-}i](x, y) = \bigwedge_{1 \leq k \leq e} ([P(k):\text{eq-}i](x) \rightarrow l_k(x, y)).$$

Remark 4. Let \mathfrak{A} be an L-structure and suppose that $\mathfrak{A} \models [\text{P:globperm}]$ and that the binary symbols l_k are interpreted as equivalence relations on A in refinement. Recall that there is a permutation $\nu \in \mathbb{S}_e$ such that $[\text{P:data}]^{\mathfrak{A}}a = \nu$ for all $a \in A$. Then for all $i \in [1, e]$:

$$[\text{L:eg-}i]^{\mathfrak{A}} = l_{\nu^{-1}(i)}^{\mathfrak{A}}.$$

In particular, $\langle [\text{L:eg-}1]^{\mathfrak{A}}, [\text{L:eg-}2]^{\mathfrak{A}}, \dots, [\text{L:eg-}e]^{\mathfrak{A}} \rangle$ is a sequence of equivalence relations on A in global agreement.

Proof. Let $k = \nu^{-1}(i)$, so $\nu(k) = i$ and $[P(k):\text{data}]^{\mathfrak{A}}a = i$. Since ν is a permutation, for every $k' \in [1, e]$:

$$\mathfrak{A} \models [P(k'):\text{eq-}i](a) \text{ iff } [P(k'):\text{data}]^{\mathfrak{A}}a = i \text{ iff } k' = k. \quad (4.1)$$

Let $a, b \in A$. First suppose that $\mathfrak{A} \models [\text{L:eg-}i](a, b)$. By eq. (4.1) we have $\mathfrak{A} \models [P(k):\text{eq-}i](a)$, hence $\mathfrak{A} \models l_k(a, b)$.

Now suppose that $\mathfrak{A} \models \neg[\text{L:eg-}i](a, b)$. There is some $k' \in [1, e]$ such that:

$$\mathfrak{A} \models \neg([P(k'):\text{eq-}i](a) \rightarrow l_{k'}(a, b)) \equiv [P(k'):\text{eq-}i](a) \wedge \neg l_{k'}(a, b).$$

By eq. (4.1) we have $k' = k$, hence $\mathfrak{A} \models \neg l_k(a, b)$. □

Let $E = \langle e_1, e_2, \dots, e_e \rangle$ be a predicate signature consisting of the binary predicate symbols e_i . Let Σ be a predicate signature enriching E and not containing any symbols from L . Let $\Sigma' = \Sigma \cup L$ and $L' = \Sigma' - E$.

Definition 32. Define the syntactic operation $\text{gtr} : \Lambda[\Sigma] \rightarrow \Lambda[L']$ by:

$$\text{gtr } \varphi = \varphi' \wedge [\text{P:globperm}],$$

where φ' is obtained from φ by replacing all occurrences of a subformula of the form $e_i(x, y)$ by the formula $[\text{L:eg-}i](x, y)$, where x and y are (not necessarily distinct) variables and $i \in [1, e]$.

4 Reductions

Remark 5. Let φ be a $\Lambda[\Sigma]$ -formula and let \mathfrak{A} be a Σ -structure. Suppose that $\mathfrak{A} \models \varphi$ and that the symbols e_1, e_2, \dots, e_e are interpreted in \mathfrak{A} as equivalence relations on A in global agreement. Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \text{gtr } \varphi$ and that the symbols l_1, l_2, \dots, l_e are interpreted in \mathfrak{A}' as equivalence relations on A in refinement.

Proof. There is a permutation $\nu \in \mathbb{S}_e$ such that $e_{\nu(1)}^{\mathfrak{A}} \subseteq e_{\nu(2)}^{\mathfrak{A}} \subseteq \dots \subseteq e_{\nu(e)}^{\mathfrak{A}}$. Consider an enrichment \mathfrak{A}' of \mathfrak{A} to a Σ' -structure where $l_k^{\mathfrak{A}'} = e_{\nu(k)}^{\mathfrak{A}}$, so the interpretations of l_k in \mathfrak{A}' are equivalence relations on A in refinement. We can interpret the unary predicate symbols from permutation setup P in \mathfrak{A}' so that $\mathfrak{A}' \models [P:\text{globperm}]$ and $[P:\text{data}]^{\mathfrak{A}'} a = \nu$ for all $a \in A$. By Remark 4, for every $i \in [1, e]$:

$$[L:\text{eg-}i]^{\mathfrak{A}'} = l_{\nu^{-1}(i)}^{\mathfrak{A}'} = e_{\nu(\nu^{-1}(i))}^{\mathfrak{A}'} = e_i^{\mathfrak{A}'} = e_i^{\mathfrak{A}}.$$

Hence $\mathfrak{A}' \models \forall x \forall y (e_i(x, y) \leftrightarrow [L:\text{eg-}i](x, y))$. Since $\mathfrak{A}' \models \varphi$ we have $\mathfrak{A}' \models \text{gtr } \varphi$. \square

Remark 6. Let φ be a $\Lambda[\Sigma]$ -formula and let \mathfrak{A} be an L' -structure. Suppose that $\mathfrak{A} \models \text{gtr } \varphi$ and that the symbols l_1, l_2, \dots, l_e are interpreted in \mathfrak{A} as equivalence relations on A in refinement. Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \varphi$ and that the symbols e_1, e_2, \dots, e_e are interpreted as equivalence relations on A in global agreement in \mathfrak{A}' .

Proof. Consider an enrichment \mathfrak{A}' of \mathfrak{A} to a Σ' -structure where $e_i^{\mathfrak{A}'} = [L:\text{eg-}i]^{\mathfrak{A}'}$. By Remark 4, $\langle e_1^{\mathfrak{A}'}, e_2^{\mathfrak{A}'}, \dots, e_e^{\mathfrak{A}'} \rangle$ is a sequence of equivalence relations on A in global agreement. For every $i \in [1, e]$ we have $\mathfrak{A}' \models \forall x \forall y (e_i(x, y) \leftrightarrow [L:\text{eg-}i](x, y))$ by definition. Since $\mathfrak{A}' \models \text{gtr } \varphi$ we have $\mathfrak{A}' \models \varphi$. \square

The last two remarks show that a $\Lambda e E_{\text{global}}$ -formula φ has essentially the same models as the $\Lambda e E_{\text{refine}}$ -formula $\text{gtr } \varphi$, so we have shown:

Proposition 1. The logic $\Lambda e E_{\text{global}}$ has the finite model property iff the logic $\Lambda e E_{\text{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent: $(\text{FIN})\text{SAT}\Lambda e E_{\text{global}} =_{\text{m}}^{\text{PTIME}} (\text{FIN})\text{SAT}\Lambda e E_{\text{refine}}$.

Since the relative size of $\text{gtr } \varphi$ with respect to φ grows polynomially as e grows, we have shown:

Proposition 2. The logic $\Lambda E_{\text{global}}$ has the finite model property iff the logic $\Lambda E_{\text{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent: $(\text{FIN})\text{SAT}\Lambda E_{\text{global}} =_{\text{m}}^{\text{PTIME}} (\text{FIN})\text{SAT}\Lambda E_{\text{refine}}$.

The reduction is two-variable first-order and uses additional (et) unary predicate symbols for the permutation setup P , so it is also valid for the two-variable fragments $\Lambda_0^2 e E_a$, $\Lambda_1^2 e E_a$ and $\Lambda_1^2 E_a$ for $a \in \{\text{global}, \text{refine}\}$ (but not for the fragment $\Lambda_0^2 E_a$).

4.2 Local agreement to refinement

In this section demonstrate how (finite) satisfiability in logics featuring builtin equivalence symbols in local agreement reduces to (finite) satisfiability in logics featuring builtin equivalence symbols in refinement. Our strategy is to start with the level equivalences which form a refinement, and to encode a permutation specifying the local chain structure for every element in the structure.

Fix an arbitrary ground logic $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ and think of Σ as a predicate signature for the logics $\Lambda eE_{\text{local}}$ and $\Lambda eE_{\text{refine}}$. The e builtin equivalence symbols of Σ are e_1, e_2, \dots, e_e .

Let φ be a $\Lambda[\Sigma]$ -sentence. The class of $\Lambda eE_{\text{refine}}$ -structures satisfying φ coincides with the class of $\Lambda eE_{\text{local}}$ -structures satisfying

$$\varphi \wedge [e_1, e_2, \dots, e_e : \text{refine}].$$

Hence:

$$(\text{FIN})\text{SAT}\Lambda eE_{\text{refine}} \leq_m^{\text{PTIME}} (\text{FIN})\text{SAT}\Lambda eE_{\text{local}}.$$

Since the size of the formula $[e_1, e_2, \dots, e_e : \text{refine}]$ grows polynomially as e grows, we have:

$$(\text{FIN})\text{SAT}\Lambda E_{\text{refine}} \leq_m^{\text{PTIME}} (\text{FIN})\text{SAT}\Lambda E_{\text{local}}.$$

Consider the opposite direction. Let $E = \langle e_1, e_2, \dots, e_e \rangle$ be a predicate signature consisting of the binary predicate symbols e_i (later, we will need these to be not necessarily interpreted as equivalences, but for now we will interpret them as such). Let \mathfrak{A} be an E -structure and suppose that the symbols e_i are interpreted in \mathfrak{A} as equivalence relations on A in local agreement. Let $E_i = e_i^{\mathfrak{A}}$ for $i \in [1, e]$. Recall that for every $a \in A$ there is a permutation $\nu \in \mathbb{S}_e$ satisfying eq. (3.1):

$$E_{\nu(1)}[a] \subseteq E_{\nu(2)}[a] \subseteq \dots \subseteq E_{\nu(e)}[a]. \quad (4.2)$$

Definition 33. *The characteristic E -permutation of a in \mathfrak{A} is the lexicographically smallest permutation $\nu \in \mathbb{S}_e$ satisfying eq. (4.2). Define the function $[\text{E:chperm}]^{\mathfrak{A}} : A \rightarrow \mathbb{S}_e$ so that $[\text{E:chperm}]^{\mathfrak{A}} a$ is the characteristic E -permutation of a in \mathfrak{A} .*

Remark 7. *Let $a \in A$, $\nu = [\text{E:chperm}]^{\mathfrak{A}} a$ and $i < j \in [1, e]$. Suppose that $E_{\nu(i)}[a] = E_{\nu(j)}[a]$. Then $\nu(i) < \nu(j)$.*

Proof. Suppose not. For some $i < j \in [1, e]$ we have $\nu(i) \geq \nu(j)$. Since ν is a permutation and $i \neq j$, we have $\nu(i) > \nu(j)$. Since $E_{\nu(i)}[a] = E_{\nu(j)}[a]$, by eq. (4.2) we have $E_{\nu(k)} = E_{\nu(i)}$ for all $k \in [i, j]$. Consider the permutation $\mu \in \mathbb{S}_e$ defined by:

$$\mu(k) = \begin{cases} \nu(j) & \text{if } k = i \\ \nu(i) & \text{if } k = j \\ \nu(k) & \text{otherwise.} \end{cases}$$

Clearly, μ is a permutation satisfying eq. (4.2) that is lexicographically smaller than ν — a contradiction. \square

4 Reductions

Remark 8. Let $a, b \in A$ and let $\alpha = [\text{E:chperm}]^{\mathfrak{A}}a$ and $\beta = [\text{E:chperm}]^{\mathfrak{A}}b$. Let $i \in [1, e]$ and suppose that $(a, b) \in E_i$. Then $\alpha^{-1}(i) = \beta^{-1}(i)$.

Proof. Suppose not, so $\alpha^{-1}(i) \neq \beta^{-1}(i)$. Let $p = \alpha^{-1}(i)$ and $q = \beta^{-1}(i)$. Without loss of generality, suppose that $p < q$. Thus p is the position of i in the permutation α and $q > p$ is the position of i in the permutation β . By the pigeonhole principle, there is $k \in [1, e]$ that occurs after i in α and before j in β : $p < \alpha^{-1}(k)$ and $\beta^{-1}(k) < q$. Since β is the characteristic E-permutation of b in \mathfrak{A} , by eq. (4.2) we have $E_k[b] \subseteq E_i[b]$. Since $(a, b) \in E_i$, we have $E_k[b] \subseteq E_i[a]$. Since $E_k[b] \subseteq E_i[a]$ are equivalence classes, $E_k[a] \subseteq E_i[a]$. Since k occurs after i in α , which is the characteristic E-permutation of a in \mathfrak{A} , by eq. (4.2) we have $E_k[a] = E_i[a]$. By Remark 7, $i < k$. By the contrapositive of Remark 7, $E_k[b] = E_i[b]$ is impossible. Since k occurs before i in β , by eq. (4.2) we have $E_k[b] \subset E_i[b]$. Hence

$$E_k[b] \subset E_i[b] = E_i[a] = E_k[a]$$

— a contradiction — since the equivalence classes $E_k[b]$ and $E_k[a]$ are either equal or disjoint. \square

Let $L = \langle L_1, L_2, \dots, L_e \rangle$ be the levels of $E = \langle E_1, E_2, \dots, E_e \rangle$. Recall that by Remark 3, the levels are equivalence relations on A in refinement.

Remark 9. Let $a \in A$, $\alpha = [\text{E:chperm}]^{\mathfrak{A}}a$ and let $k \in [1, e]$. Then $L_k[a] = E_{\alpha(k)}[a]$.

Proof. Since α satisfies eq. (4.2), by Lemma 2:

$$L_k[a] = L_{\alpha^{-1}(\alpha(k))}[a] = E_{\alpha(k)}[a].$$

\square

Remark 10. Let $a, b \in A$, $\alpha = [\text{E:chperm}]^{\mathfrak{A}}a$, $\beta = [\text{E:chperm}]^{\mathfrak{A}}b$ and $k \in [1, e]$. Suppose that $(a, b) \in L_k$. Then $\alpha(k) = \beta(k)$. That is, the elements connected at level k agree at position k in their characteristic permutations.

Proof. By Remark 9, $L_k[a] = E_{\alpha(k)}[a]$, thus $(a, b) \in E_{\alpha(k)}$. By Remark 7,

$$k = \alpha^{-1}(\alpha(k)) = \beta^{-1}(\alpha(k)).$$

Hence $\beta(k) = \alpha(k)$. \square

Let $P = \langle \mathbf{u}_{11}, \mathbf{u}_{12}, \dots, \mathbf{u}_{et} \rangle$ be an e -permutation setup. Let $L = \langle \mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_e \rangle + P$ be a predicate signature containing the binary predicate symbols \mathbf{l}_k (not necessarily interpreted as equivalence relations) together with the symbols from P .

Definition 34. Define the $\mathcal{L}^2[L]$ -sentence $[\text{L:fixperm}]$ by:

$$[\text{L:fixperm}] = \forall \mathbf{x} \forall \mathbf{y} \bigwedge_{1 \leq k \leq e} (\mathbf{l}_k(\mathbf{x}, \mathbf{y}) \rightarrow [P(k):\text{eq}](\mathbf{x}, \mathbf{y})).$$

Definition 35. Define the $\mathcal{L}^2[\mathbf{L}]$ -sentence $[\mathbf{L}:\text{locperm}]$ by:

$$[\mathbf{L}:\text{locperm}] = [\mathbf{P}:\text{perm}] \wedge [\mathbf{L}:\text{fixperm}].$$

Remark 11. Let \mathfrak{A} be an \mathbf{L} -structure and suppose that $\mathfrak{A} \models [\mathbf{L}:\text{locperm}]$. Let $a, b \in A$, $k \in [1, e]$ and suppose that $\mathfrak{A} \models \mathbf{l}_k(a, b)$. Let $\alpha = [\mathbf{P}:\text{data}]^{\mathfrak{A}}a$ and $\beta = [\mathbf{P}:\text{data}]^{\mathfrak{A}}b$ be the e -permutations at a and b , encoded by the permutation setup \mathbf{P} . Then $\alpha(k) = \beta(k)$.

Proof. Since $\mathfrak{A} \models [\mathbf{L}:\text{fixperm}]$ and $\mathfrak{A} \models \mathbf{l}_k(a, b)$, we have $\mathfrak{A} \models [\mathbf{P}(k):\text{eq}](a, b)$, which means $\alpha(k) = \beta(k)$. \square

Definition 36. For $i \in [1, e]$, define the quantifier-free $\mathcal{L}^2[\mathbf{L}]$ -formula $[\mathbf{L}:\text{el-}i]$ by:

$$[\mathbf{L}:\text{el-}i](\mathbf{x}, \mathbf{y}) = \bigwedge_{1 \leq k \leq n} ([\mathbf{P}(k):\text{eq-}i](\mathbf{x}) \rightarrow \mathbf{l}_k(\mathbf{x}, \mathbf{y})).$$

Remark 12. Let \mathfrak{A} be an \mathbf{L} -structure and suppose that $\mathfrak{A} \models [\mathbf{L}:\text{locperm}]$ and that the binary symbols \mathbf{l}_k are interpreted in \mathfrak{A} as equivalence relations on A in refinement. Define $\nu : A \rightarrow \mathbb{S}_e$ by $\nu(a) = [\mathbf{P}:\text{data}]^{\mathfrak{A}}a$ for $a \in A$. Let $a \in A$ be arbitrary. Then for all $i \in [1, e]$:

$$[\mathbf{L}:\text{el-}i]^{\mathfrak{A}}[a] = \mathbf{l}_{\nu(a)^{-1}(i)}^{\mathfrak{A}}[a].$$

Proof. Let $E_i = [\mathbf{L}:\text{el-}i]^{\mathfrak{A}}$ and $L_i = \mathbf{l}_i^{\mathfrak{A}}$ for every $i \in [1, e]$. Let $i \in [1, e]$ be arbitrary. Let $\alpha = \nu(a)$ and $k = \alpha^{-1}(i)$, so $\alpha = [\mathbf{P}:\text{data}]^{\mathfrak{A}}a$ and $\alpha(k) = i$. We have to show that $E_i[a] = L_k[a]$. Since α is a permutation, for every $k' \in [1, e]$ we have:

$$\mathfrak{A} \models [\mathbf{P}(k'):\text{eq-}i](a) \text{ iff } \alpha(k') = i \text{ iff } k' = k. \quad (4.3)$$

First, suppose $b \in E_i[a]$. Then $\mathfrak{A} \models [\mathbf{L}:\text{el-}i](a, b)$ and by eq. (4.3) we have $\mathfrak{A} \models \mathbf{l}_k(a, b)$, hence $b \in L_k[a]$.

Next, suppose $b \notin E_i[a]$. Then $\mathfrak{A} \models \neg[\mathbf{L}:\text{el-}i](a, b)$, so there is some $k' \in [1, e]$ such that $\mathfrak{A} \models \neg([\mathbf{P}(k'):\text{eq-}i](a) \rightarrow \mathbf{l}_{k'}(a, b)) \equiv [\mathbf{P}(k'):\text{eq-}i](a) \wedge \neg\mathbf{l}_{k'}(a, b)$. By eq. (4.3) we have $k' = k$. Hence $\mathfrak{A} \models \neg\mathbf{l}_k(a, b)$, so $b \notin L_k[a]$. \square

Remark 13. Let \mathfrak{A} and ν be declared as in Remark 12. Then the sequence of interpretations $\langle [\mathbf{L}:\text{el-}1]^{\mathfrak{A}}, [\mathbf{L}:\text{el-}2]^{\mathfrak{A}}, \dots, [\mathbf{L}:\text{el-}e]^{\mathfrak{A}} \rangle$ is a sequence of equivalence relations on A in local agreement.

Proof. Let $E_i = [\mathbf{L}:\text{el-}i]^{\mathfrak{A}}$ and $L_i = \mathbf{l}_i^{\mathfrak{A}}$ for every $i \in [1, e]$. Let $i \in [1, e]$ be arbitrary. We check that E_i is reflexive, symmetric and transitive.

- For reflexivity, let $a \in A$. By Remark 12, $E_i[a] = L_k[a]$ for $k = \nu(a)^{-1}(i)$. But $L_k[a]$ is an equivalence class, hence $a \in L_k[a]$, so $(a, a) \in E_i$.
- For symmetry, let $a, b \in A$ and $(a, b) \in E_i$. Let $k = \nu(a)^{-1}(i)$ so that $i = \nu(k)$. By Remark 12, $E_i[a] = L_k[a]$. Thus $\mathfrak{A} \models \mathbf{l}_k(a, b)$ and by Remark 11, $i = \nu(a)(k) = \nu(b)(k)$. By Remark 12:

$$E_i[b] = [\mathbf{L}:\text{el-}i]^{\mathfrak{A}}[b] = \mathbf{l}_{\nu(b)^{-1}(i)}^{\mathfrak{A}}[b] = L_k[b] = L_k[a].$$

Since $a \in L_k[a] = E_i[b]$, we have $(b, a) \in E_i$.

4 Reductions

- For transitivity, continue the argument for symmetry. Let $c \in E_i[b]$. Then $c \in E_i[b] = L_k[a] = E_i[a]$, thus $(a, c) \in E_i$.

By Remark 12, since the relations L_k are in refinement, we have that E_1, E_2, \dots, E_e are in local agreement. \square

Let $E = \langle e_1, e_2, \dots, e_e \rangle$ be a predicate signature consisting of binary predicate symbols. Let Σ be a predicate signature enriching E and not containing any symbols from L . Let $\Sigma' = \Sigma + L$ and $L' = \Sigma' - E$.

Definition 37. Define the syntactic operation $\text{ltr} : \Lambda[\Sigma] \rightarrow \Lambda[L']$ by:

$$\text{ltr } \varphi = \varphi' \wedge [L:\text{locperm}],$$

where φ' is obtained from φ by replacing all occurrences of a subformula of the form $e_i(x, y)$ by the formula $[L:\text{el-}i](x, y)$, where x and y are (not necessarily distinct) variable symbols and $i \in [1, e]$.

Remark 14. Let φ be a $\Lambda[\Sigma]$ -formula and let \mathfrak{A} be a Σ -structure. Suppose that $\mathfrak{A} \models \varphi$ and that the symbols e_1, e_2, \dots, e_e are interpreted in \mathfrak{A} as equivalence relations on A in local agreement. Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \text{ltr } \varphi$ and that the symbols l_1, l_2, \dots, l_e are interpreted in \mathfrak{A}' as equivalence relations on A in refinement.

Proof. Since the binary symbols e_1, e_2, \dots, e_e are interpreted as equivalence relations on A in local agreement in \mathfrak{A} , we may define the levels $L_1, L_2, \dots, L_e \subseteq A \times A$ and the characteristic E-permutation mapping $\nu = [E:\text{chperm}]^{\mathfrak{A}} : A \rightarrow \mathbb{S}_e$. Consider an enrichment \mathfrak{A}' of \mathfrak{A} where $l_i^{\mathfrak{A}'} = L_i$. By Remark 3, L_i are equivalences on A in refinement. We interpret the unary symbols from the permutation setup P so that $[P:\text{data}]^{\mathfrak{A}'} a = \nu(a)$ for all $a \in A$. By Remark 10, $\mathfrak{A}' \models [L:\text{fixperm}]$. By Remark 12, followed by Lemma 2, for every $i \in [1, e]$ and $a \in A$ we have:

$$[L:\text{el-}i]^{\mathfrak{A}'}[a] = l_{\nu(a)^{-1}(i)}^{\mathfrak{A}'}[a] = e_{\nu(a)(\nu(a)^{-1}(i))}^{\mathfrak{A}'}[a] = e_i^{\mathfrak{A}'}[a].$$

By Remark 13, the interpretations $[L:\text{el-}i]^{\mathfrak{A}'}$ are equivalence relations. Since the interpretation of the formula $[L:\text{el-}i]$ has the same classes as the interpretation of the symbol e_i , we have $\mathfrak{A}' \models \forall x \forall y (e_i(x, y) \leftrightarrow [L:\text{el-}i](x, y))$. Since $\mathfrak{A}' \models \varphi$ we have $\mathfrak{A}' \models \text{ltr } \varphi$. \square

Remark 15. Let φ be a $\Lambda[\Sigma]$ -formula and let \mathfrak{A} be an L' -structure. Suppose that $\mathfrak{A} \models \text{ltr } \varphi$ and that the symbols l_1, l_2, \dots, l_e are interpreted as equivalence relations on A in refinement in \mathfrak{A} . Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \varphi$ and that the binary symbols e_1, e_2, \dots, e_e are interpreted as equivalence relations on A in global agreement in \mathfrak{A}' .

Proof. Consider an enrichment \mathfrak{A}' of \mathfrak{A} to a Σ' -structure where $e_i^{\mathfrak{A}'} = [L:\text{el-}i]^{\mathfrak{A}}$. By Remark 13, $e_i^{\mathfrak{A}'}$ are equivalence relations on A in local agreement. For every $i \in [1, e]$ we have $\mathfrak{A}' \models \forall x \forall y (e_i(x, y) \leftrightarrow [L:\text{el-}i](x, y))$ by definition. Since $\mathfrak{A}' \models \text{ltr } \varphi$ we have $\mathfrak{A}' \models \varphi$. \square

The last two remarks show that a $\Lambda eE_{\text{local}}$ -formula φ has essentially the same models as the $\Lambda eE_{\text{refine}}$ -formula $\text{ltr } \varphi$, so we have shown:

Proposition 3. *The logic $\Lambda eE_{\text{local}}$ has the finite model property iff the logic $\Lambda eE_{\text{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent: $(\text{FIN})\text{SAT}\Lambda eE_{\text{local}} =_{\text{m}}^{\text{PTIME}} (\text{FIN})\text{SAT}\Lambda eE_{\text{refine}}$.*

Since the relative size of $\text{ltr } \varphi$ with respect to φ grows polynomially as e grows, we have shown:

Proposition 4. *The logic ΛE_{local} has the finite model property iff the logic $\Lambda E_{\text{refine}}$ has the finite model property. The corresponding satisfiability problems are polynomial-time equivalent: $(\text{FIN})\text{SAT}\Lambda E_{\text{local}} =_{\text{m}}^{\text{PTIME}} (\text{FIN})\text{SAT}\Lambda E_{\text{refine}}$.*

The reduction is two-variable first-order and uses additional (*et*) unary predicate symbols for the permutation setup P , so it is also valid for the two-variable fragments $\Lambda_0^2 eE_a$, $\Lambda_1^2 eE_a$ and $\Lambda_1^2 E_a$ for $a \in \{\text{local}, \text{refine}\}$ respectively.

4.3 Granularity

In this section we demonstrate how to replace the finest equivalence from a sequence of equivalences in refinement with a counter setup. This works if the structures are granular, that is, if the finest equivalence doesn't have many classes within a single bigger equivalence class.

Definition 38. *Let $\langle D, E \rangle$ be a sequence of two equivalence relations on A in refinement. Let $g \in \mathbb{N}^+$. The sequence is g -granular if every E -equivalence class includes at most g D -equivalence classes.*

Definition 39. *Let $g \in \mathbb{N}^+$ and let $\langle D, E \rangle$ be g -granular. The function $c : A \rightarrow [1, g]$ is a g -granular coloring for the sequence, if two E -equivalent elements have the same color iff they are D -equivalent. That is, for every $(a, b) \in E$ we have $c(a) = c(b)$ iff $(a, b) \in D$.*

Remark 16. *Let $g \in \mathbb{N}^+$ and let $\langle D, E \rangle$ be g -granular. Then there is a g -granular coloring for the sequence.*

Proof. Let X be an E -class. Since $D \subseteq E$ is g -granular, the set $S = \{D[a] \mid a \in X\}$ has cardinality at most g . Let $\iota : S \hookrightarrow [1, g]$ be any injective function. Define the color c on X as $c(a) = \iota(D[a])$. \square

Remark 17. *Let $E \subseteq A \times A$ be an equivalence relation on A , $g \in \mathbb{N}^+$ and $c : A \rightarrow [1, g]$. Then there is an equivalence relation $D \subseteq E$ on A such that $\langle D, E \rangle$ is g -granular, having c as a g -granular coloring.*

Proof. Take $D = \{(a, b) \in E \mid c(a) = c(b)\}$. \square

Definition 40. *Let $g \in \mathbb{N}^+$ and let $t = \|g\|$ be the bitsize of g . A g -color setup $\mathbf{G} = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \rangle$ is just a t -bit counter setup.*

4 Reductions

Let $\Lambda \in \{\mathcal{L}, \mathcal{C}\}$ be a ground logic, $g \in \mathbb{N}^+$ and $G = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \rangle$ be a g -color setup. Let Σ be a predicate signature containing the binary symbols \mathbf{d} and \mathbf{e} and not containing any symbols from G . Let $\Sigma' = \Sigma + G$ and $\Gamma = \Sigma' - \{\mathbf{d}\}$.

Definition 41. Define the quantifier-free $\mathcal{L}^2[\Gamma]$ -formula $[\Gamma:\mathbf{d}](\mathbf{x}, \mathbf{y})$ by:

$$[\Gamma:\mathbf{d}](\mathbf{x}, \mathbf{y}) = \mathbf{e}(\mathbf{x}, \mathbf{y}) \wedge [\mathbf{G}:\mathbf{eq}](\mathbf{x}, \mathbf{y}).$$

Definition 42. Define the syntactic operation $\text{grtr} : \Lambda[\Sigma] \rightarrow \Lambda[\Gamma]$ by:

$$\text{grtr } \varphi = \varphi' \wedge [\mathbf{G}:\mathbf{betw-1-g}],$$

where φ' is obtained from the formula φ by replacing all subformulas of the form $\mathbf{d}(x, y)$ by $[\Gamma:\mathbf{d}](x, y)$, where x and y are (not necessarily distinct) variable symbols.

Lemma 3. Let \mathfrak{A} be a Σ -structure and suppose that the sequence of symbols $\langle \mathbf{d}, \mathbf{e} \rangle$ is interpreted in \mathfrak{A} as a g -granular sequence $\langle D, E \rangle$. Suppose that $\mathfrak{A} \models \varphi$. Then there is a Σ' -enrichment \mathfrak{A}' of \mathfrak{A} such that $\mathfrak{A}' \models \text{grtr } \varphi$.

Proof. By Remark 16, there exists a g -granular coloring $c : A \rightarrow [1, g]$. We interpret the unary symbols in G so that $[\mathbf{G}:\mathbf{data}]^{\mathfrak{A}'} = c$. Since \mathfrak{A}' is an enrichment of \mathfrak{A} , we have $\mathfrak{A}' \models \varphi$. Let $a, b \in A$. Then $\mathfrak{A}' \models [\Gamma:\mathbf{d}](a, b)$ is equivalent to:

$$\mathfrak{A}' \models \mathbf{e}(a, b) \text{ and } \mathfrak{A}' \models [\mathbf{G}:\mathbf{eq}](a, b),$$

which is equivalent to:

$$(a, b) \in E \text{ and } [\mathbf{G}:\mathbf{data}]^{\mathfrak{A}'} a = [\mathbf{G}:\mathbf{data}]^{\mathfrak{A}'} b,$$

which, since $[\mathbf{G}:\mathbf{data}]^{\mathfrak{A}'} = c$ is a g -granular coloring, is equivalent to:

$$(a, b) \in D.$$

Hence $\mathfrak{A}' \models \forall \mathbf{x} \forall \mathbf{y} (\mathbf{d}(\mathbf{x}, \mathbf{y}) \leftrightarrow [\Gamma:\mathbf{d}](\mathbf{x}, \mathbf{y}))$ and since $\mathfrak{A}' \models \varphi$, we have $\mathfrak{A}' \models \text{grtr } \varphi$. \square

Lemma 4. Let \mathfrak{A} be a Γ -structure and suppose that the binary symbol \mathbf{e} is interpreted in \mathfrak{A} as an equivalence relation on A . Suppose that $\mathfrak{A} \models \text{grtr } \varphi$. Then there is a Σ' -structure \mathfrak{A}' enriching \mathfrak{A} such that $\mathfrak{A}' \models \varphi$ and the sequence of binary symbols $\langle \mathbf{d}, \mathbf{e} \rangle$ is interpreted in \mathfrak{A}' as a g -granular sequence $\langle D, E \rangle$.

Proof. Since $\mathfrak{A} \models [\mathbf{G}:\mathbf{betw-1-g}]$, we have $[\mathbf{G}:\mathbf{data}]^{\mathfrak{A}} a \in [1, g]$ for all $a \in A$. Define $c : A \rightarrow [1, g]$ by $c(a) = [\mathbf{G}:\mathbf{data}]^{\mathfrak{A}} a$. By Remark 16, we can find $D \subseteq E$ such that the sequence $\langle D, E \rangle$ is g -granular, having c as a g -granular coloring. Consider the Σ' -structure \mathfrak{A}' , where $\mathbf{d}^{\mathfrak{A}'} = D$. Since \mathfrak{A}' is an enrichment of $\mathfrak{A} \models \text{grtr } \varphi$, we have $\mathfrak{A}' \models \text{grtr } \varphi$. Let $a, b \in A$. Then $\mathfrak{A}' \models [\Gamma:\mathbf{d}](a, b)$ is equivalent to:

$$\mathfrak{A}' \models \mathbf{e}(a, b) \text{ and } \mathfrak{A}' \models [\mathbf{G}:\mathbf{eq}](a, b),$$

which is equivalent to:

$$(a, b) \in E \text{ and } c(a) = c(b),$$

which, since c is a g -granular coloring, is equivalent to:

$$(a, b) \in D.$$

Hence $\mathfrak{A}' \models \forall \mathbf{x} \forall \mathbf{y} (e(\mathbf{x}, \mathbf{y}) \leftrightarrow [\Gamma:\mathbf{d}](\mathbf{x}, \mathbf{y}))$ and since $\mathfrak{A}' \models \text{grtr } \varphi$, we have $\mathfrak{A}' \models \varphi$. \square

5 Monadic logics

In this chapter we investigate questions about (finite) satisfiability of first-order sentences featuring unary predicate symbols and builtin equivalence symbols in refinement. Our strategy is to extract small substructures of structures and analyse them using Ehrenfeucht-Fraïssé games. It is known that:

- The monadic first-order logic \mathcal{L}_1 has the finite model property and its (finite) satisfiability problem is NEXPTIME-complete [8]
- The first-order logic of a single equivalence relation \mathcal{L}_01E has the finite model property and its (finite) satisfiability problem is PSPACE-complete [6]
- The first-order logic of two equivalence relations \mathcal{L}_02E lacks the finite model property and both the satisfiability and finite satisfiability problems are undecidable [9].

Let $U(u) = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_u \rangle$ be an unary predicate signature consisting of the unary predicate symbols \mathbf{u}_i . Let $E(e) = \langle \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_e \rangle$ be a binary predicate signature consisting of the builtin equivalence symbols \mathbf{e}_j in refinement. Let $\Sigma(u, e) = U(u) + E(e)$, so $\Sigma(u, e)$ is a generic predicate signature for the monadic first-order logic $\mathcal{L}_1eE_{\text{refine}}$.

5.1 Cells

Let $u, e \in \mathbb{N}$, $e \geq 1$ and $\Sigma = \Sigma(u, e) = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_u, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_e \rangle$ be a predicate signature. Abbreviate the finest equivalence symbol $\mathbf{d} = \mathbf{e}_1$.

Definition 43. Define the quantifier-free $\mathcal{L}^2[\Sigma]$ -formula $[\Sigma:\text{cell}](\mathbf{x}, \mathbf{y})$ by:

$$[\Sigma:\text{cell}](\mathbf{x}, \mathbf{y}) = \mathbf{d}(\mathbf{x}, \mathbf{y}) \wedge \bigwedge_{1 \leq i \leq u} (\mathbf{u}_i(\mathbf{x}) \leftrightarrow \mathbf{u}_i(\mathbf{y})).$$

If \mathfrak{A} is a Σ -structure and $D = \mathbf{d}^{\mathfrak{A}}$, then the interpretation $C = [\Sigma:\text{cell}]^{\mathfrak{A}} \subseteq A \times A$ is an equivalence relation on A that refines D . The cells of \mathfrak{A} are the equivalence classes of C . That is, a cell is a maximal set of D -equivalent elements satisfying the same \mathbf{u} -predicates.

Remark 18. Let \mathfrak{A} be a Σ -structure, $r \in \mathbb{N}$, $\bar{a} = a_1 a_2 \dots a_r \in A^r$, $\bar{b} = b_1 b_2 \dots b_r \in A^r$ and a_i and b_i are in the same \mathfrak{A} -cell for all $i \in [1, r]$. Suppose that $a_i = a_j$ iff $b_i = b_j$ for all $i, j \in [1, r]$. Then $\bar{a} \mapsto \bar{b}$ is a partial isomorphism.

Proof. Direct consequence of the fact that the cell equivalence relation refines the finest equivalence relation D and that the elements in the same cell satisfy the same \mathbf{u} -predicates. The equality condition ensures that the mapping is a bijection. \square

Lemma 5. *Let \mathfrak{A} be a Σ -structure and $r \in \mathbb{N}^+$. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \equiv_r \mathfrak{A}$ and every \mathfrak{B} -cell has cardinality at most r .*

Proof. Let $C \subseteq A \times A$ be the \mathfrak{A} -cell equivalence relation. Execute the following process: for every \mathfrak{A} -cell, if it has cardinality less than r , select all elements from that cell; otherwise select r distinct elements from that cell. Let $B \subseteq A$ be the set of selected elements and let $\mathfrak{B} = \mathfrak{A} \upharpoonright B$. By construction, every \mathfrak{B} -cell has cardinality at most r . We claim that $\mathfrak{A} \equiv_r \mathfrak{B}$. Let $h = C \cap (A \times B)$ relates elements from A with elements from B in the same cell. Note that for all $a \in A$:

$$|h[a]| = \min(|C[a]|, r). \quad (5.1)$$

Consider the set \mathfrak{I} of partial isomorphisms from \mathfrak{A} to \mathfrak{B} that have cardinality at most r and that are included in h . This set is nonempty since it contains the empty partial isomorphism. We claim that the sequence $\mathfrak{I}_0 = \mathfrak{I}_1 = \dots = \mathfrak{I}_r = \mathfrak{I}$ satisfies the back-and-forth conditions of Theorem 1. Let $i \in [1, r]$ and let

$$\mathfrak{p} = \bar{a} \mapsto \bar{b} = a_1 a_2 \dots a_r \mapsto b_1 b_2 \dots b_r \in \mathfrak{I}$$

be any partial isomorphism. Without loss of generality, suppose $i = 1$.

1. For the forth condition, let $a \in A$. We have to find some $b \in B$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathfrak{I}$. If $a = a_k$ for some $k \in [2, r]$, then $b = b_k$ is appropriate.

Suppose that $a \neq a_k$ for all $k \in [2, r]$. Let $S \subseteq C[a]$ be the set of \bar{a} -elements in the same \mathfrak{A} -cell as a :

$$S = \{a_k \in C[a] \mid k \in [2, r]\}.$$

Note that $|S| \leq r - 1$ and $|C[a]| \geq |S| + 1$. By eq. (5.1), $|h[a]| \geq |S| + 1$. Hence there is an element $b \in h[a]$ that is distinct from b_k for all $k \in [2, r]$.

2. For the back condition, let $b \in B$. We have to find some $a \in A$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathfrak{I}$. If $b = b_k$ for some $k \in [2, r]$, then $a = a_k$ is appropriate.

Suppose that $b \neq b_k$ for all $k \in [2, r]$. Since $b \in h[b]$, $a = b$ is appropriate.

By Theorem 1, $\mathfrak{A} \equiv_r \mathfrak{B}$. □

5.2 Organs

Let $u, e \in \mathbb{N}$, $e \geq 2$ and $\Sigma = \Sigma(u, e) = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_u, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_e \rangle$ be a predicate signature. Abbreviate the finest two equivalence symbols $\mathbf{d} = \mathbf{e}_1$ and $\mathbf{e} = \mathbf{e}_2$.

Definition 44. *Let \mathfrak{A} be a Σ -structure and let $D = \mathbf{d}^{\mathfrak{A}}$ and $E = \mathbf{e}^{\mathfrak{A}}$. Recall that the set of D -classes is $\mathcal{E}D$. Two D -classes $X, Y \in \mathcal{E}D$ are organ-equivalent if they are included in the same E -class (equivalently $X \times Y \subseteq E$), and the induced substructures $(\mathfrak{A} \upharpoonright X)$ and $(\mathfrak{A} \upharpoonright Y)$ are isomorphic. The organ-equivalence relation is $\mathcal{O} \subseteq \mathcal{E}D \times \mathcal{E}D$. Since D refines E , organ-equivalence is an equivalence relation on $\mathcal{E}D$. An organ is an*

organ-equivalence-class. That is, an organ is a maximal set of isomorphic D -classes, included in the same E -class.

For any two organ-equivalent D -classes $(X, Y) \in \mathcal{O}$, fix an isomorphism

$$\mathfrak{h}_{XY} : (\mathfrak{A} \upharpoonright X) \leftrightarrow (\mathfrak{A} \upharpoonright Y)$$

consistently, so that $\mathfrak{h}_{XX} = \text{id}_X$, $\mathfrak{h}_{YX} = \mathfrak{h}_{XY}^{-1}$ and if $(Y, Z) \in \mathcal{O}$ then $\mathfrak{h}_{XZ} = \mathfrak{h}_{YZ} \circ \mathfrak{h}_{XY}$. Two elements $a, b \in A$ are sub-organ-equivalent if $(D[a], D[b]) \in \mathcal{O}$ and $\mathfrak{h}_{D[a]D[b]}(a) = b$. Since the isomorphisms \mathfrak{h}_{XY} are chosen consistently, sub-organ-equivalence $\mathcal{O} \subseteq A \times A$ is an equivalence relation on A that refines E .

Remark 19. Let \mathfrak{A} be a Σ -structure, $r \in \mathbb{N}$, $\bar{a} = a_1 a_2 \dots a_r \in A^r$, $\bar{b} = b_1 b_2 \dots b_r \in A^r$, a_i and b_i are sub-organ-equivalent for all $i \in [1, r]$. Suppose that $\mathfrak{A} \models \mathbf{d}(a_i, a_j)$ iff $\mathfrak{A} \models \mathbf{d}(b_i, b_j)$ for all $i, j \in [1, r]$. Then $\bar{a} \mapsto \bar{b}$ is a partial isomorphism.

Proof. The condition about the finest equivalence symbol \mathbf{d} ensures that the interpretation of \mathbf{d} is preserved. Since sub-organ-equivalence relates isomorphic elements, the interpretation of the unary symbols and the formal equality is preserved. Since the sub-organ-equivalence $\mathcal{O} \subseteq A \times A$ refines the second finest equivalence relation E , the interpretation of all remaining equivalence symbols \mathbf{e}_j is preserved. \square

Lemma 6. Let \mathfrak{A} be a Σ -structure and $r \in \mathbb{N}^+$. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \equiv_r \mathfrak{A}$ and every \mathfrak{B} -organ has cardinality at most r .

Proof. Let $D = \mathbf{d}^{\mathfrak{A}}$, $E = \mathbf{e}^{\mathfrak{A}}$ and let $\mathcal{A} = \mathcal{E}D$ be the set of D -classes. Let $\mathcal{O} \subseteq \mathcal{A} \times \mathcal{A}$ be the \mathfrak{A} -organ-equivalence relation on \mathcal{A} . Execute the following process: for every \mathfrak{A} -organ, if it has cardinality at most r , select all D -classes from that organ; otherwise select r distinct D -classes from that organ (note that these will be isomorphic). Let $\mathcal{B} \subseteq \mathcal{A}$ be the set of selected D -classes. Let $B = \cup \mathcal{B} \subseteq A$ be the set of elements in the selected classes and let $\mathfrak{B} = (\mathfrak{A} \upharpoonright B)$. By construction, every \mathfrak{B} -organ has cardinality at most r . We claim that $\mathfrak{A} \equiv_r \mathfrak{B}$. Let $\mathcal{H} = \mathcal{O} \cap (\mathcal{A} \times \mathcal{B})$ relates the D -classes with the isomorphic D -classes from \mathcal{B} in the same organ. Let h relates the elements of A with their isomorphic elements from B . Note that for all elements $a \in A$:

$$|h[a]| = \min(|\mathcal{O}[D[a]]|, r). \quad (5.2)$$

Consider the set \mathfrak{I} of partial isomorphisms from \mathfrak{A} to \mathfrak{B} that have cardinality at most r and that are included in h . This set is nonempty since it contains the empty partial isomorphism. We claim that the sequence $\mathfrak{I}_0 = \mathfrak{I}_1 = \dots = \mathfrak{I}_r = \mathfrak{I}$ satisfies the back-and-forth conditions of Theorem 1. Let $i \in [1, r]$ and let

$$\mathfrak{p} = \bar{a} \mapsto \bar{b} = a_1 a_2 \dots a_r \mapsto b_1 b_2 \dots b_r \in \mathfrak{I}$$

be any partial isomorphism. Without loss of generality, suppose $i = 1$.

1. For the forth condition, let $a \in A$. We have to find some $b \in B$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathfrak{I}$. If $a \in D[a_k]$ for some $k \in [2, r]$, then $b = \mathfrak{h}_{D[a_k]D[b_k]}(a)$ is appropriate.

Suppose $a \notin D[a_k]$ for all $k \in [2, r]$. Let $\mathcal{S} \subseteq \mathcal{O}[D[a]]$ be the set of D -classes of \bar{a} -elements in the same \mathfrak{A} -organ as $D[a]$:

$$\mathcal{S} = \{D[a_k] \in \mathcal{O}[D[a]] \mid k \in [2, r]\}.$$

Note that $|\mathcal{S}| \leq r - 1$ and $|\mathcal{O}[D[a]]| \geq |\mathcal{S}| + 1$. By eq. (5.2), $|h[a]| \geq |\mathcal{S}| + 1$. Hence there is some $b \in h[a]$ such that $b \notin D[b_k]$ for all $k \in [2, r]$. This b is appropriate.

2. For the back condition, let $b \in B$. We have to find some $a \in A$ such that $\bar{a}_i^a \mapsto \bar{b}_i^b \in \mathfrak{I}$. If $b \in D[b_k]$ for some $k \in [2, r]$, then $a = \mathfrak{h}_{D[b_k]D[a_k]}(b)$ is appropriate.

Suppose that $b \notin D[b_k]$ for all $k \in [2, r]$. Since $b \in h[b]$, $a = b$ is appropriate.

By Theorem 1, $\mathfrak{A} \equiv_r \mathfrak{B}$. □

5.3 Satisfiability

In this section we will employ the results on cells and organs to bound the size of a small substructure of a general structure.

Remark 20. Let $u, e \in \mathbb{N}$, $e \geq 2$ and consider the predicate signature $\Sigma = \Sigma(u, e) = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_u, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_e \rangle$. Abbreviate $\mathbf{d} = \mathbf{e}_1$ and $\mathbf{e} = \mathbf{e}_2$. Let $r \in \mathbb{N}^+$. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \equiv_r \mathfrak{A}$ and $\langle \mathbf{d}^{\mathfrak{B}}, \mathbf{e}^{\mathfrak{B}} \rangle$ is g -granular for $g = g(u, r) = r \cdot ((r + 1)^{2^u} - 1)$. Furthermore, this \mathfrak{B} has the property that every \mathfrak{B} -cell has cardinality at most r .

Proof. By Lemma 5, there is $\mathfrak{B}' \subseteq \mathfrak{A}$ such that $\mathfrak{B}' \equiv_r \mathfrak{A}$ and every \mathfrak{B}' -cell has cardinality at most r . By Lemma 6, there is $\mathfrak{B} \subseteq \mathfrak{B}'$ such that $\mathfrak{B} \equiv_r \mathfrak{B}'$ and the \mathfrak{B} -organs have cardinality at most r . Let $D = \mathbf{d}^{\mathfrak{B}}$ and $E = \mathbf{e}^{\mathfrak{B}}$. Since every D -class includes at most 2^u cells and is nonempty and every cell has cardinality at most r , there are at most $((r + 1)^{2^u} - 1)$ nonisomorphic D -classes in \mathfrak{B} . Since every E -class includes at most r isomorphic D -classes, we get that $\langle D, E \rangle$ is g -granular. □

Corollary 1. Let $u, e \in \mathbb{N}$, $e \geq 2$ and consider $\Sigma = \Sigma(u, e)$. Let φ be a $\mathcal{L}[\Sigma]$ -sentence having quantifier rank r . By Lemma 3 and Lemma 4 about granularity, the formula φ is essentially equisatisfiable with the formula $\text{grtr } \varphi$, which is a $\Sigma(u + \|g(u, r)\|, e - 1)$ -sentence. Note that $\|g(u, r)\|$ is exponentially bounded by the length $\|\varphi\|$ of the formula. So we have a reduction:

$$(\text{FIN})\text{SAT}_{\mathcal{L}_1} e E_{\text{refine}} \leq_m^{\text{EXPTIME}} (\text{FIN})\text{SAT}_{\mathcal{L}_1} (e - 1) E_{\text{refine}}.$$

If u is a constant independent of φ , then $\|g(u, r)\|$ is polynomially bounded by $\|\varphi\|$. So we have a reduction:

$$(\text{FIN})\text{SAT}_{\mathcal{L}_0} e E_{\text{refine}} \leq_m^{\text{PTIME}} (\text{FIN})\text{SAT}_{\mathcal{L}_1} (e - 1) E_{\text{refine}}.$$

Remark 21. Let $u \in \mathbb{N}$ and consider $\Sigma = \Sigma(u, 1) = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_u, \mathbf{d} \rangle$. Let $r \in \mathbb{N}^+$. There is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{A} \equiv_r \mathfrak{B}$ and $|B| \leq g \cdot r \cdot 2^u$ for $g = g(u, r) = r \cdot ((r+1)^{2^u} - 1)$.

Proof. Let $\Sigma' = \Sigma + \langle \mathbf{e} \rangle$ be an enrichment of Σ with the builtin equivalence symbols \mathbf{e} . Consider an enrichment \mathfrak{A}' of \mathfrak{A} to a Σ' -structure, where $\mathbf{e}^{\mathfrak{A}'} = A \times A$ is interpreted as the full relation on A . Then $\langle \mathbf{d}^{\mathfrak{A}'}, \mathbf{e}^{\mathfrak{A}'} \rangle$ is a sequence of equivalence relations on A in refinement. By Remark 20, there is $\mathfrak{B}' \subseteq \mathfrak{A}'$ such that $\mathfrak{B}' \equiv_r \mathfrak{A}'$ and $\langle \mathbf{d}^{\mathfrak{B}'}, \mathbf{e}^{\mathfrak{B}'} \rangle$ is g -granular. Consider the reduct \mathfrak{B} of \mathfrak{B}' to a Σ -structure. Let $D = \mathbf{d}^{\mathfrak{B}}$ and $E = \mathbf{e}^{\mathfrak{B}}$. Since every \mathfrak{B} -cell has cardinality at most r and every D -class includes at most 2^u cells, we have that every D -class has cardinality at most $r \cdot 2^u$. Since \mathbf{e} was interpreted in \mathfrak{A} as the full relation, it is also interpreted in \mathfrak{B} as the full relation, so there is a single E -class—the whole domain B . Since the sequence $\langle D, E \rangle$ is g -granular, there are at most g D -classes, so $|B| \leq g \cdot r \cdot 2^u$. \square

Corollary 2. The logic $\mathcal{L}_1 1E$ has the finite model property and its (finite) satisfiability problem is in N2EXPTIME .

Combining Corollary 2 with Corollary 1, we get by induction on e :

Proposition 5. For $e \in \mathbb{N}^+$, the logic $\mathcal{L}_1 e E_{\text{refine}}$ has the finite model property and its (finite) satisfiability problem is in $\text{N}(e+1)\text{EXPTIME}$.

By Proposition 1 and Proposition 3, the same holds for $\mathcal{L}_1 e E_{\text{global}}$ and $\mathcal{L}_1 e E_{\text{local}}$.

Proposition 6. The logic $\mathcal{L}_1 E_{\text{refine}}$ has the finite model property and its (finite) satisfiability problem is in the forth level of the Grzegorzcyk hierarchy \mathcal{E}^4 .

By Proposition 2 and Proposition 4, the same holds for $\mathcal{L}_1 E_{\text{global}}$ and $\mathcal{L}_1 E_{\text{local}}$.

Proposition 7. For $e \geq 2$, the logic $\mathcal{L}_0 e E_{\text{refine}}$ has the finite model property and its (finite) satisfiability problem is in NeEXPTIME .

By Proposition 1 and Proposition 3, the same holds for $\mathcal{L}_0 e E_{\text{global}}$ and $\mathcal{L}_0 e E_{\text{local}}$.

5.4 Hardness with a single equivalence

In this section we show that the (finite) satisfiability of monadic first-order logic with a single equivalence symbol $\mathcal{L}_1 1E$ is N2EXPTIME -hard by reducing the doubly exponential tiling problem to such satisfiability. Our strategy is to employ a counter setup of u unary predicate symbols to encode the exponentially many positions of a binary encoding of a doubly exponentially bounded quantity, encoding the coordinates of a cell of the doubly exponential tiling square.

Consider the counter setup $C(u) = \langle \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_u \rangle$ for $u \in \mathbb{N}^+$. Recall that the intention of a counter setup is to encode an arbitrary exponentially bounded value at every element of a structure. Let $D(u) = C(u) + \langle \mathbf{d} \rangle$ be a predicate signature enriching $C(u)$ with the builtin equivalence symbol \mathbf{d} . We will define a system where every \mathbf{d} -equivalence class includes exponentially many cells. These cells will correspond to the exponentially many positions of the binary encoding of a doubly exponential value for

the \mathbf{d} -class. The bit values at each cell position will be encoded by the cardinality of that cell: bit value 0 if the cardinality of the cell is 1 and bit value 1 if the cardinality is greater than 1. This will allow us to encode a doubly exponential value at each \mathbf{d} -class. Call the data $[C:\text{data}]^{\mathfrak{A}}a$, encoded by the counter setup at a the *position* of a .

Let \mathfrak{A} be a $D = D(u)$ -structure.

Definition 45. Define the quantifier-free $\mathcal{L}^2[D]$ -formula $[D:\text{pos-eq}](x, y)$ by:

$$[D:\text{pos-eq}](x, y) = [C:\text{eq}](x, y).$$

Then $\mathfrak{A} \models [D:\text{pos-eq}](a, b)$ iff a and b are at the same positions (in possibly distinct \mathbf{d} -classes): $[C:\text{data}]^{\mathfrak{A}}a = [C:\text{data}]^{\mathfrak{A}}b$.

Definition 46. Define the quantifier-rank-1 $\mathcal{L}^2[D]$ -formula $[D:\text{bit-0}](x)$ by:

$$[D:\text{bit-0}](x) = \forall y (d(y, x) \wedge [D:\text{pos-eq}](y, x) \rightarrow y = x).$$

Then $\mathfrak{A} \models [D:\text{bit-0}](a)$ iff the cell of a has cardinality 1.

Definition 47. Define the quantifier-rank-1 $\mathcal{L}^2[D]$ -formula $[D:\text{bit-1}](x)$ by:

$$[D:\text{bit-1}](x) = \exists y (d(y, x) \wedge [D:\text{pos-eq}](y, x) \wedge y \neq x).$$

Then $\mathfrak{A} \models [D:\text{bit-1}](a)$ iff the cell of a has cardinality greater than 1.

Definition 48. Define the quantifier-free $\mathcal{L}^2[D]$ -formula $[D:\text{pos-zero}](x)$ by:

$$[D:\text{pos-zero}](x) = \bigwedge_{1 \leq i \leq u} \neg u_i(x).$$

Then $\mathfrak{A} \models [D:\text{pos-zero}](a)$ iff the position of a is 0.

Definition 49. Define the quantifier-free $\mathcal{L}^2[D]$ -formula $[D:\text{pos-largest}](x)$ by:

$$[D:\text{pos-largest}](x) = \bigwedge_{1 \leq i \leq u} u_i(x).$$

Then $\mathfrak{A} \models [D:\text{pos-largest}](a)$ iff the position of a is the largest u -bit number N_u .

Definition 50. Define the quantifier-free $\mathcal{L}^2[D]$ -formula $[D:\text{pos-less}](x, y)$ by:

$$[D:\text{pos-less}](x, y) = d(x, y) \wedge [C:\text{less}](x, y).$$

Then $\mathfrak{A} \models [D:\text{pos-less}](a, b)$ iff a and b are in the same \mathbf{d} -class and the position of a is less than the position of b .

Definition 51. Define the quantifier-free $\mathcal{L}^2[D]$ -formula $[D:\text{pos-succ}](x, y)$ by:

$$[D:\text{pos-succ}](x, y) = d(x, y) \wedge [C:\text{succ}](x, y).$$

Then $\mathfrak{A} \models [\mathbf{D:pos-succ}](a, b)$ iff a and b are in the same \mathbf{d} -class and the position of b is the successor of the position of a .

Definition 52. Define the closed $\mathcal{L}^2[\mathbf{D}]$ -sentence $[\mathbf{D:pos-full}]$ by:

$$[\mathbf{D:pos-full}] = \forall \mathbf{x} \exists \mathbf{y} \left(\mathbf{d}(\mathbf{y}, \mathbf{x}) \wedge [\mathbf{D:pos-zero}](\mathbf{y}) \right) \wedge \forall \mathbf{x} \left(\neg [\mathbf{D:pos-largest}](\mathbf{x}) \rightarrow \exists \mathbf{y} [\mathbf{D:pos-succ}](\mathbf{x}, \mathbf{y}) \right).$$

The first part of this formula asserts that every \mathbf{d} -class has an element at position 0. The second part asserts that if a is an element at position p , that is not the largest possible, there exists an element b in the same \mathbf{d} -class at position $p+1$. Therefore in any model of $[\mathbf{D:pos-full}]$, every \mathbf{d} -class has 2^u cells. For example, in particular, every \mathbf{d} -class has cardinality at least 2^u . For the rest of the section, suppose that $\mathfrak{A} \models [\mathbf{D:pos-full}]$.

Definition 53. For every u -bit number $p \in \mathbb{B}_u$, define the $\mathcal{L}^2[\mathbf{D}]$ -formula $[\mathbf{D:pos-p}](\mathbf{x})$ recursively by:

$$[\mathbf{D:pos-0}](\mathbf{x}) = [\mathbf{D:pos-zero}](\mathbf{x})$$

and for $p \in [0, N_u - 1]$:

$$[\mathbf{D:pos-(p+1)}](\mathbf{x}) = \exists \mathbf{y} \left([\mathbf{D:pos-p}](\mathbf{y}) \wedge [\mathbf{D:pos-succ}](\mathbf{y}, \mathbf{x}) \right).$$

In this case, for the formula to be a two-variable formula, the formula $[\mathbf{D:pos-p}](\mathbf{y})$ is obtained from $[\mathbf{D:pos-p}](\mathbf{x})$ by swapping all occurrences (not only the unbounded ones) of the variables \mathbf{x} and \mathbf{y} ¹. Note that the length of the formula $[\mathbf{D:pos-p}](\mathbf{x})$ grows linearly as p grows.

Then $\mathfrak{A} \models [\mathbf{D:pos-p}](a)$ iff p is the position of a .

Definition 54. Let \mathfrak{A} be a \mathbf{D} -structure. Let $D = \mathbf{d}^{\mathfrak{A}}$. Define the function $[\mathbf{D:Data}]^{\mathfrak{A}} : \mathcal{E}D \rightarrow \mathbb{B}^{2^u}$, assigning a 2^u -bit bitstring to any D -class X by:

$$[\mathbf{D:Data}]_p^{\mathfrak{A}} X = \begin{cases} 1 & \text{if } [\mathbf{C:data}]^{\mathfrak{A}}(a) = (p-1) \text{ implies } \mathfrak{A} \models [\mathbf{D:bit-1}](a) \text{ for all } a \in X \\ 0 & \text{otherwise} \end{cases}$$

for $p \in [1, 2^u]$.

Definition 55. Define the quantifier-rank-1 $\mathcal{L}^2[\mathbf{D}]$ -formula $[\mathbf{D:Zero}](\mathbf{x})$ by:

$$[\mathbf{D:Zero}](\mathbf{x}) = \forall \mathbf{y} \left(\mathbf{d}(\mathbf{y}, \mathbf{x}) \rightarrow [\mathbf{D:bit-0}](\mathbf{y}) \right).$$

Then $\mathfrak{A} \models [\mathbf{D:Zero}](a)$ iff the data at the D -class of a encodes 0: $[\mathbf{D:Data}]^{\mathfrak{A}} D[a] = 0$.

¹this is reminiscent to the process of defining a standard translation of modal logic to the two-variable first-order fragment

Definition 56. Define the quantifier-rank-1 $\mathcal{L}^2[D]$ -formula $[D:\text{Largest}](x)$ by:

$$[D:\text{Largest}](x) = \forall y \left(d(y, x) \rightarrow [D:\text{bit-1}](y) \right).$$

Then $\mathfrak{A} \models [D:\text{Largest}](a)$ iff the data at the D -class of a encodes the largest 2^u -bit number: $[D:\text{Data}]^{\mathfrak{A}} D[a] = N_{2^u}$.

Definition 57. Let $M \in \mathbb{B}_{2^u}$ be a t -bit number (where $t \leq 2^u$). Define the $\mathcal{L}^2[D]$ -formula $[D:\text{Eq-}M](x)$ by:

$$\begin{aligned} [D:\text{Eq-}M](x) = & \forall y \left(d(y, x) \rightarrow \bigwedge_{0 \leq p < t} \left([D:\text{pos-}p](y) \rightarrow [D:\text{bit-}(\overline{M}_{p+1})](y) \right) \right) \wedge \\ & \forall x \left([D:\text{pos-}(t-1)](y) \wedge [D:\text{pos-less}](y, x) \rightarrow [D:\text{bit-0}](x) \right). \end{aligned}$$

The first part of this formula asserts that the bits at the first t positions of the d -class of x encode the number M . The second part asserts that all the remaining bits at larger positions are zeroes. Note that the length of this formula is polynomially bounded by t , the bitsize of M . We have $\mathfrak{A} \models [D:\text{Eq-}M](a)$ iff the data at the D -class of a encodes M : $[D:\text{Data}]^{\mathfrak{A}} D[a] = M$.

Definition 58. Define the $\mathcal{L}^6[D]$ -formula $[D:\text{Less}](x, y)$ by:

$$\begin{aligned} [D:\text{Less}](x, y) = & \exists x' \exists y' \left(d(x', x) \wedge d(y', y) \wedge \right. \\ & \left([D:\text{pos-eq}](x', y') \wedge [D:\text{bit-0}](x') \wedge [D:\text{bit-1}](y') \right) \wedge \quad (\text{Less1}) \\ & \forall x'' \left([D:\text{pos-less}](x', x'') \rightarrow \exists y'' \left(d(y'', y') \wedge \right. \right. \\ & \left. \left. [D:\text{pos-eq}](y'', x'') \wedge ([D:\text{bit-0}](y'') \leftrightarrow [D:\text{bit-0}](x'')) \right) \right) \Big). \quad (\text{Less2}) \end{aligned}$$

Then $\mathfrak{A} \models [D:\text{Less}](a, b)$ iff $[D:\text{Data}]^{\mathfrak{A}} D[a] < [D:\text{Data}]^{\mathfrak{A}} D[b]$. By rearrangement and reusing variables, this can be also written using just three variables (but not using just two variables). Indeed, $[D:\text{Less}](x, y)$ is logically equivalent to:

$$\begin{aligned} & \exists z \left(d(z, x) \wedge \exists x \left(x = z \wedge \exists z \left(d(z, y) \wedge \exists y \left(y = z \wedge \right. \right. \right. \right. \\ & \quad \left. \left([D:\text{pos-eq}](x, y) \wedge [D:\text{bit-0}](x) \wedge [D:\text{bit-1}](y) \right) \wedge \quad (\text{Less1}) \\ & \quad \forall z \left([D:\text{pos-less}](x, z) \rightarrow \exists x \left(x = z \wedge \exists z \left(d(z, y) \wedge \exists y \left(y = z \wedge \right. \right. \right. \right. \\ & \quad \left. \left. \left. [D:\text{pos-eq}](y, x) \wedge ([D:\text{bit-0}](y) \leftrightarrow [D:\text{bit-0}](x)) \right) \right) \right) \right) \Big). \quad (\text{Less2}) \end{aligned}$$

Definition 59. Define the $\mathcal{L}^6[D]$ -formula $[D:\text{Succ}](x, y)$ by:

$$\begin{aligned}
 [D:\text{Succ}](x, y) = & \exists x' \exists y' \left(d(x', x) \wedge d(y', y) \wedge \right. \\
 & ([D:\text{pos-eq}](x', y') \wedge [D:\text{bit-0}](x') \wedge [D:\text{bit-1}](y')) \wedge \quad (\text{Succ1}) \\
 & \forall x'' ([D:\text{pos-less}](x'', x') \rightarrow [D:\text{bit-1}](x'')) \wedge \quad (\text{Succ2}) \\
 & \forall y'' ([D:\text{pos-less}](y'', y') \rightarrow [D:\text{bit-0}](y'')) \wedge \quad (\text{Succ3}) \\
 & \left. \forall x'' ([D:\text{pos-less}](x', x'') \rightarrow \exists y'' (d(y'', y') \wedge \right. \\
 & [D:\text{pos-eq}](y'', x'') \wedge ([D:\text{bit-0}](y'') \leftrightarrow [D:\text{bit-0}](x'')))) \left. \right). \quad (\text{Succ4})
 \end{aligned}$$

By rearrangement and reusing variables, this can be also written using just three variables (but not using just two variables).

Then $\mathfrak{A} \models [D:\text{Succ}](a, b)$ iff $[D:\text{Data}]^{\mathfrak{A}} D[b] = 1 + [D:\text{Data}]^{\mathfrak{A}} D[a]$.

Definition 60. Define the $\mathcal{L}^3[D]$ -sentence $[D:\text{Full}]$ by:

$$[D:\text{Full}] = \exists x [D:\text{Zero}](x) \wedge \forall x (\neg [D:\text{Largest}](x) \rightarrow \exists y [D:\text{Succ}](x, y)).$$

If \mathfrak{A} satisfies $[D:\text{Full}]$ then \mathfrak{A} contains a d -class of encoding any possible data: for every $M \in [0, N_{2^u}]$, there is a d -class X such that $[D:\text{Data}]^{\mathfrak{A}} X = M$.

Definition 61. Define the $\mathcal{L}^4[D]$ -formula $[D:\text{Eq}](x, y)$ by:

$$\begin{aligned}
 [D:\text{Eq}](x, y) = & \forall x' \forall y' (d(x', x) \wedge d(y', y) \wedge \\
 & [D:\text{pos-eq}](x', y') \rightarrow ([D:\text{bit-0}](x') \leftrightarrow [D:\text{bit-0}](y'))).
 \end{aligned}$$

By rearrangement and reusing variables, this can be also written using just three variables (but not using just two variables).

Then $\mathfrak{A} \models [D:\text{Eq}](x, y)$ iff $[D:\text{Data}]^{\mathfrak{A}} D[a] = [D:\text{Data}]^{\mathfrak{A}} D[b]$.

Definition 62. Define the $\mathcal{L}^4[D]$ -sentence $[D:\text{Alldiff}]$ by:

$$\begin{aligned}
 [D:\text{Alldiff}] = & \forall x \forall y (\neg d(x, y) \rightarrow \exists x' \exists y' (d(x', x) \wedge d(y', y) \wedge \\
 & [D:\text{pos-eq}](x', y') \wedge \neg ([D:\text{bit-0}](x') \leftrightarrow [D:\text{bit-0}](y')))).
 \end{aligned}$$

By rearrangement and reusing variables, this can be also written using just three variables (but not using just two variables).

If \mathfrak{A} satisfies $[D:\text{Alldiff}]$ then all D -classes in \mathfrak{A} encode different data.

Recall from Section 1.7 that an instance of the *doubly exponential tiling problem* is an initial condition $c^0 = \langle t_1^0, t_2^0, \dots, t_n^0 \rangle \subseteq T = [1, k]$ of tiles from the domino system $D_0 = (T, H, V)$, where $H, V \subseteq T \times T$ are the horizontal and vertical matching relations. We need to define a predicate signature capable enough to express a doubly exponential grid of tiles. Consider the predicate signature

$$D = \langle \mathbf{u}_1^H, \mathbf{u}_2^H, \dots, \mathbf{u}_n^H; \mathbf{u}_1^V, \mathbf{u}_2^V, \dots, \mathbf{u}_n^V; \mathbf{u}_1^T, \mathbf{u}_2^T, \dots, \mathbf{u}_k^T; \mathbf{d} \rangle.$$

It has the following relevant subsignatures:

- $D^H = \langle \mathbf{u}_1^H, \mathbf{u}_2^H, \dots, \mathbf{u}_n^H, \mathbf{d} \rangle$ encodes the horizontal index of a tile
- $D^V = \langle \mathbf{u}_1^V, \mathbf{u}_2^V, \dots, \mathbf{u}_n^V, \mathbf{d} \rangle$ encodes the vertical index of a tile
- $D^{HV} = \langle \mathbf{u}_1^H, \mathbf{u}_2^H, \dots, \mathbf{u}_n^H, \mathbf{u}_1^V, \mathbf{u}_2^V, \dots, \mathbf{u}_n^V, \mathbf{d} \rangle$ encodes the combined horizontal and vertical index of a tile; we need this to define the full grid
- $D^T = \langle \mathbf{u}_1^T, \mathbf{u}_2^T, \dots, \mathbf{u}_k^T \rangle$ encodes the type of a tile.

Let \mathfrak{A} be a D -structure satisfying $[D^{HV}:\text{pos-full}]$ and let $D = \mathbf{d}^{\mathfrak{A}}$. The sentence

$$[D^{HV}:\text{Full}] \wedge [D^{HV}:\text{Alldiff}] \quad (5.3)$$

asserts that the D -classes form a doubly exponential grid. The sentence

$$\forall \mathbf{x} \left(\bigwedge_{1 \leq i \leq k} \mathbf{u}_i^T(\mathbf{x}) \rightarrow \bigwedge_{i < j \leq k} \neg \mathbf{u}_j^T(\mathbf{x}) \right) \quad (5.4)$$

asserts that every element has a unique type. The sentence

$$\forall \mathbf{x} \forall \mathbf{y} \left(\mathbf{d}(\mathbf{x}, \mathbf{y}) \rightarrow \bigwedge_{1 \leq i \leq k} (\mathbf{u}_i^T(\mathbf{x}) \leftrightarrow \mathbf{u}_i^T(\mathbf{y})) \right) \quad (5.5)$$

asserts that all elements in a D -class have the same type—the type of the tile corresponding to that D -class. For $j \in [1, n]$, the sentence

$$\forall \mathbf{x} \left([D^H:\text{Eq}-(j-1)](\mathbf{x}) \wedge [D^V:\text{Zero}](\mathbf{x}) \rightarrow \mathbf{u}_{t_j^0}^T(\mathbf{x}) \right) \quad (5.6)$$

encodes the initial segment in the first row of the square. The sentence

$$\forall \mathbf{x} \forall \mathbf{y} \left([D^H:\text{Succ}](\mathbf{x}, \mathbf{y}) \wedge [D^V:\text{Eq}](\mathbf{x}, \mathbf{y}) \rightarrow \bigvee_{(i,j) \in H} \mathbf{u}_i^T(\mathbf{x}) \wedge \mathbf{u}_j^T(\mathbf{y}) \right) \quad (5.7)$$

encodes the horizontal matching condition. The sentence

$$\forall \mathbf{x} \forall \mathbf{y} \left([D^V:\text{Succ}](\mathbf{x}, \mathbf{y}) \wedge [D^H:\text{Eq}](\mathbf{x}, \mathbf{y}) \rightarrow \bigvee_{(i,j) \in V} \mathbf{u}_i^T(\mathbf{x}) \wedge \mathbf{u}_j^T(\mathbf{y}) \right) \quad (5.8)$$

encodes the vertical matching condition.

Combining $[D^{HV}:\text{pos-full}]$ with the formulas 5.3–5.8, we may encode an instance of the doubly exponential tiling problem as a (finite) satisfiability of a formula, so we have:

Proposition 8. *The (finite) satisfiability problem for the monadic first-order logic with a single equivalence symbol $\mathcal{L}_1 1E$ is N2EXPTIME-hard. More precisely, even the three-variable fragment $\mathcal{L}_1^3 1E$ has this property.*

5.5 Hardness with many equivalences in refinement

The argument from the previous section can be iterated to yield the hardness of the (finite) satisfiability of the monadic first-order logic with several builtin equivalence symbols in refinement $\mathcal{L}_1 eE_{\text{refine}}$. Our strategy is to encode $(e+1)$ -exponential numbers at every equivalence class of the coarsest relation by thinking of the e -exponential numbers at the classes of the second-to-coarsest relation as bit positions.

For $e \in \mathbb{N}^+$, consider the predicate signature $E(e) = \langle e_1, e_2, \dots, e_e \rangle$ consisting of the builtin equivalence symbols e_i in refinement. Abbreviate the *coarsest* equivalence symbol $\mathbf{d} = e_e$.

Definition 63. *Let $e \in \mathbb{N}^+$. An e -exponential setup is a uniform effective polynomial-time process for creating the following data structure. For every $u \in \mathbb{N}^+$, there is a predicate signature $D(e, u)$ having length polynomial in u , consisting of unary predicate symbols and containing $E(e)$. The following data is effectively defined:*

E1 There is a $\mathcal{L}^3[D(e, u)]$ -sentence $[D(e, u):\text{pos-full}]$, whose length grows polynomially as u grows.

E2 If \mathfrak{A} is a $D(e, u)$ -structure, $\mathfrak{A} \models [D(e, u):\text{pos-full}]$ and $D = \mathbf{d}^{\mathfrak{A}}$, then there is a function $[D(e, u):\text{Data}]^{\mathfrak{A}} : \mathcal{E}D \rightarrow \mathbb{B}^{\exp_2^e(u)}$ that assigns an e -exponential bitstring to every D -class.

E3 There is a $\mathcal{L}^3[D(e, u)]$ -formula $[D(e, u):\text{Eq}](\mathbf{x}, \mathbf{y})$ whose length grows polynomially as u grows, such that for all $a, b \in A$:

$$\mathfrak{A} \models [D(e, u):\text{Eq}](a, b) \text{ iff } \underline{[D(e, u):\text{Data}]^{\mathfrak{A}} D[a]} = \underline{[D(e, u):\text{Data}]^{\mathfrak{A}} D[b]}.$$

E4 There is a $\mathcal{L}^3[D(e, u)]$ -formula $[D(e, u):\text{Zero}](\mathbf{x})$, whose length grows polynomially as u grows, such that for all $a \in A$:

$$\mathfrak{A} \models [D(e, u):\text{Zero}](a) \text{ iff } \underline{[D(e, u):\text{Data}]^{\mathfrak{A}} D[a]} = 0.$$

E5 There is a $\mathcal{L}^3[D(e, u)]$ -formula $[D(e, u):\text{Largest}](\mathbf{x})$, whose length grows polynomially as u grows, such that for all $a \in A$:

$$\mathfrak{A} \models [D(e, u):\text{Largest}](a) \text{ iff } \underline{[D(e, u):\text{Data}]^{\mathfrak{A}} D[a]} = N_{\exp_2^e(u)} = \exp_2^{e+1}(u) - 1.$$

E6 There is a $\mathcal{L}^3[D(e, u)]$ -formula $[D(e, u):\text{Less}](\mathbf{x}, \mathbf{y})$, whose length grows polynomially as u grows, such that for all $a, b \in A$:

$$\mathfrak{A} \models [D(e, u):\text{Less}](a, b) \text{ iff } \underline{[D(e, u):\text{Data}]^{\mathfrak{A}} D[a]} < \underline{[D(e, u):\text{Data}]^{\mathfrak{A}} D[b]}.$$

5 Monadic logics

E7 There is a $\mathcal{L}^3[D(e, u)]$ -formula $[D(e, u):Succ](x, y)$, whose length grows polynomially as u grows, such that for all $a, b \in A$:

$$\mathfrak{A} \models [D(e, u):Succ](a, b) \text{ iff } \underline{[D(e, u):Data]^{\mathfrak{A}} D[b]} = \underline{[D(e, u):Data]^{\mathfrak{A}} D[a]} + 1.$$

E8 For every $\exp_2^e(u)$ -bit number M , there is a $\mathcal{L}^3[D(e, u)]$ -formula $[D(e, u):Eq-M](x)$, whose length grows polynomially as u and M grow, such that for all $a \in A$:

$$\mathfrak{A} \models [D(e, u):Eq-M](a) \text{ iff } \underline{[D(e, u):Data]^{\mathfrak{A}} D[a]} = M.$$

The previous section defines a 1-exponential setup. Suppose that we have an e -exponential setup having predicate signature $D = D(e, u)$. Analogously to the previous section, we will describe an $(e+1)$ -exponential setup $D' = D(e+1, u) = D + \langle e \rangle$ which is based on D , where $e = e_{e+1}$ is the new coarsest builtin equivalence symbol in D' . Define the following formulas:

$$\begin{aligned} [D':pos-eq](x, y) &= [D:Eq](x, y) \\ [D':bit-0](x) &= \forall y (e(y, x) \wedge [D':pos-eq](y, x) \rightarrow d(y, x)) \\ [D':bit-1](x) &= \exists y (e(y, x) \wedge [D':pos-eq](y, x) \wedge \neg d(y, x)) \\ [D':pos-zero](x) &= [D:Zero](x) \\ [D':pos-largest](x) &= [D:Largest](x) \\ [D':pos-less](x, y) &= e(x, y) \wedge [D:Less](x, y) \\ [D':pos-succ](x, y) &= e(x, y) \wedge [D:Succ](x, y) \\ [D':pos-full] &= \forall x \exists y (e(y, x) \wedge [D':pos-zero](y)) \wedge \\ &\quad \forall x (\neg [D':pos-largest](x) \rightarrow \exists y [D':pos-succ](x, y)) \\ [D':pos-0](x) &= [D':pos-zero](x) \\ [D':pos-(p+1)](x) &= \exists y ([D':pos-p](y) \wedge [D':pos-succ](y, x)) \\ &\quad \text{for } p \in [0, N_{\exp_2^e(u)} - 1]. \end{aligned} \tag{E1}$$

Let \mathfrak{A} be a D' -structure, $\mathfrak{A} \models [D':pos-full]$ and let $E = e^{\mathfrak{A}}$. Define the function $[D':Data]^{\mathfrak{A}} : \mathcal{E}E \rightarrow \mathbb{B}^{\exp_2^{e+1}(u)}$ assigning a $\exp_2^{e+1}(u)$ -bit bitstring to any E -class X by:

$$[D':Data]_p^{\mathfrak{A}} X = \begin{cases} 1 & \text{if } \mathfrak{A} \models [D':pos-(p-1)](a) \text{ implies } \mathfrak{A} \models [D':bit-1](a) \text{ for all } a \in X \\ 0 & \text{otherwise} \end{cases} \tag{E2}$$

for $p \in [1, \exp_2^{e+1}(u)]$.

5.5 Hardness with many equivalences in refinement

Define the following formulas:

$$[D':\text{Eq}](x, y) = \forall x' \forall y' (e(x', x) \wedge e(y', y) \wedge \quad (\text{E3})$$

$$[D':\text{pos-eq}](x', y') \rightarrow ([D':\text{bit-0}](x') \leftrightarrow [D':\text{bit-0}](y'))$$

$$[D':\text{Zero}](x) = \forall y (e(y, x) \rightarrow [D':\text{bit-0}](y)) \quad (\text{E4})$$

$$[D':\text{Largest}](x) = \forall y (e(y, x) \rightarrow [1:\text{bit-D'}](y)) \quad (\text{E5})$$

$$[D':\text{Less}](x, y) = \exists x' \exists y' (e(x', x) \wedge e(y', y) \wedge \quad (\text{E6})$$

$$([D':\text{pos-eq}](x', y') \wedge [D':\text{bit-0}](x') \wedge [D':\text{bit-1}](y')) \wedge$$

$$\forall x'' ([D':\text{pos-less}](x', x'') \rightarrow \exists y'' (e(y'', y') \wedge$$

$$[D':\text{pos-eq}](y'', x'') \wedge ([D':\text{bit-0}](y'') \leftrightarrow [D':\text{bit-0}](x''))))$$

$$[D':\text{Succ}](x, y) = \exists x' \exists y' (e(x', x) \wedge e(y', y) \wedge \quad (\text{E7})$$

$$([D':\text{pos-eq}](x', y') \wedge [D':\text{bit-0}](x') \wedge [D':\text{bit-1}](y')) \wedge$$

$$\forall x'' ([D':\text{pos-less}](x'', x') \rightarrow [D':\text{bit-1}](x'')) \wedge$$

$$\forall y'' ([D':\text{pos-less}](y'', y') \rightarrow [D':\text{bit-0}](y'')) \wedge$$

$$\forall x'' ([D':\text{pos-less}](x', x'') \rightarrow \exists y'' (e(y'', y') \wedge$$

$$[D':\text{pos-eq}](y'', x'') \wedge ([D':\text{bit-0}](y'') \leftrightarrow [D':\text{bit-0}](x''))))$$

If $M \in \mathbb{B}_{\exp_2^{e+1}(u)}$ is an $\exp_2^{e+1}(u)$ -bit number, let $t = \|M\|$ and define the formula:

$$[D':\text{Eq-M}](x) = \forall y (e(y, x) \rightarrow \bigwedge_{0 \leq p < t} ([D':\text{pos-p}](y) \rightarrow [D':\text{bit-}\overline{M}_{p+1}](y)) \wedge \quad (\text{E8})$$

$$\forall x ([D':\text{pos-(t-1)}](y) \wedge [D':\text{pos-less}](y, x) \rightarrow [D':\text{bit-0}](x))$$

This completes the definition of the $(e+1)$ -exponential setup.

We can encode an instance of the $(e+1)$ -exponential tiling problem into a (finite) satisfiability D-formula completely analogously to the previous section. Thus we have:

Proposition 9. *The (finite) satisfiability problem for the monadic first-order logic with e equivalence symbols in refinement $\mathcal{L}_1 e\text{E}_{\text{refine}}$ is $N(e+1)\text{EXP TIME}$ -hard. Even the three-variable fragment $\mathcal{L}_1^3 e\text{E}_{\text{refine}}$ has this property.*

5 Monadic logics

By Proposition 1 and Proposition 3, the same holds for $\mathcal{L}_1^{(3)}eE_{\text{global}}$ and $\mathcal{L}_1^{(3)}eE_{\text{local}}$.

Proposition 10. *The (finite) satisfiability problem for the monadic first-order logic with many equivalence symbols in refinement $\mathcal{L}_1E_{\text{refine}}$ is ELEMENTARY-hard. Even the three-variable fragment $\mathcal{L}_1^3E_{\text{refine}}$ has this property.*

By Proposition 2 and Proposition 4, the same holds for $\mathcal{L}_1^{(3)}E_{\text{global}}$ and $\mathcal{L}_1^{(3)}E_{\text{local}}$.

6 Two-variable logics

TODO: Introduction

6.1 Type realizability of the two-variable first-order logic

Recall from Section 1.6 about normal forms that every \mathcal{L}^2 -sentence φ can be reduced in polynomial time to a sentence $\text{sctr } \varphi$ in Scott normal form:

$$\forall x \forall y (\alpha_0(x, y) \vee x = y) \wedge \bigwedge_{1 \leq i \leq m} \forall x \exists y (\alpha_i(x, y) \wedge x \neq y),$$

where the formulas α_i are quantifier-free and use at most linearly many new unary predicate symbols. We refer to α_0 as the *universal part*, or the $\forall\forall$ -part of the formula $\text{sctr } \varphi$ and to α_i as the *existential parts*, or the $\forall\exists$ -parts of the formula, for $i \in [1, m]$. For any formula in Scott normal form, we may replace its existential parts by fresh binary predicate symbols: for $i \in [1, m]$, let \mathbf{m}_i be a fresh binary predicate symbol with the intended interpretation $\forall x \forall y (\mathbf{m}_i(x, y) \leftrightarrow \alpha_i(x, y))$. Since this is a universal sentence, it can be incorporated into the $\forall\forall$ -part α_0 of the formula. We refer to the symbols \mathbf{m}_i as the *message symbols*. Hence the formula can be transformed in polynomial time to the form:

$$\forall x \forall y (\alpha(x, y) \vee x = y) \wedge \bigwedge_{1 \leq i \leq m} \forall x \exists y (\mathbf{m}_i(x, y) \wedge x \neq y), \quad (6.1)$$

where the $\forall\forall$ -part α is quantifier-free and over an extended signature. For convenience, we make the existential parts of the formula part of the signature, so we can focus only on the universal part. The following is a term similar to the one defined in [4]:

Definition 64. A classified signature $\langle \Sigma, \bar{\mathbf{m}} \rangle$ for the two-variable first-order logic \mathcal{L}^2 is a predicate signature Σ together with a sequence $\bar{\mathbf{m}} = \mathbf{m}_1 \mathbf{m}_2 \dots \mathbf{m}_m$ of distinct binary predicate symbols from Σ having intended interpretation

$$\bigwedge_{1 \leq i \leq m} \forall x \exists y (\mathbf{m}_i(x, y) \wedge x \neq y). \quad (6.2)$$

That is, a classified signature *automatically includes* the $\forall\exists$ -parts of formulas and $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structures *automatically satisfy* the $\forall\exists$ -parts.

Definition 65. The (finite) classified satisfiability problem for two-variable first-order logic is: given a classified signature $\langle \Sigma, \bar{\mathbf{m}} \rangle$ and a quantifier-free $\mathcal{L}^2[\Sigma]$ -formula $\alpha(x, y)$, is there a (finite) $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structure satisfying eq. (6.1). Denote the classified satisfiability problem for two-variable first-order logic by $\text{CLSAT}\mathcal{L}^2$ and its finite version by $\text{FINCLSAT}\mathcal{L}^2$.

6 Two-variable logics

Scott normal form shows that (finite) satisfiability reduces in polynomial time to (finite) classified satisfiability:

$$(\text{FIN})\text{SAT}\mathcal{L}^2 \leq_m^{\text{PTIME}} (\text{FIN})\text{CLSAT}\mathcal{L}^2.$$

Let $\langle \Sigma, \bar{\mathbf{m}} \rangle$ be a classified signature for \mathcal{L}^2 .

Definition 66. A type instance (Π, \mathbf{T}) over $\langle \Sigma, \bar{\mathbf{m}} \rangle$ is a pair of a set of 1-types $\Pi \subseteq \Pi[\Sigma]$ and a set of 2-types $\mathbf{T} \subseteq \mathbf{T}[\Sigma]$ such that $\text{tp}_{\mathbf{x}} \tau \in \Pi$ and $\text{tp}_{\mathbf{y}} \tau \in \Pi$ for all $\tau \in \mathbf{T}$.

A $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structure \mathfrak{A} realizes (or is a model of) (Π, \mathbf{T}) if it contains at least 2 elements, $\text{tp}^{\mathfrak{A}}[a] \in \Pi$ for all $a \in A$ and $\text{tp}^{\mathfrak{A}}[a, b] \in \mathbf{T}$ for all $a \in A$ and $b \in A \setminus \{a\}$. The structure fully realizes (Π, \mathbf{T}) if additionally for every 1-type $\pi \in \Pi$ there is $a \in A$ such that $\text{tp}^{\mathfrak{A}}[a] = \pi$ (we don't require an analogous condition for the 2-types). The characteristic type instance of \mathfrak{A} is the type instance (Π, \mathbf{T}) defined by:

$$\begin{aligned} \Pi &= \left\{ \text{tp}^{\mathfrak{A}}[a] \mid a \in A \right\} \\ \mathbf{T} &= \left\{ \text{tp}^{\mathfrak{A}}[a, b] \mid a \neq b \in A \right\}. \end{aligned}$$

The (finite) (full) type realizability problem for \mathcal{L}^2 is the following: given a classified signature $\langle \Sigma, \bar{\mathbf{m}} \rangle$ and a type instance (Π, \mathbf{T}) over $\langle \Sigma, \bar{\mathbf{m}} \rangle$, is there a (finite) $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structure that (fully) realizes (Π, \mathbf{T}) . Denote the type realizability problem for \mathcal{L}^2 by $\text{REAL}\mathcal{L}^2$ and its finite version by $\text{FINREAL}\mathcal{L}^2$. Denote the full type realizability problem for \mathcal{L}^2 by $\text{FULLREAL}\mathcal{L}^2$ and its finite version by $\text{FINFULLREAL}\mathcal{L}^2$.

Remark 22. Since the set of 1-types realized in any model of (Π, \mathbf{T}) is a subset of Π , by guessing this subset we can reduce the (finite) type satisfiability problem to the (finite) full type satisfiability problem in nondeterministic polynomial time:

$$(\text{FIN})\text{REAL}\mathcal{L}^2 \leq_m^{\text{NPTIME}} (\text{FIN})\text{FULLREAL}\mathcal{L}^2.$$

Remark 23. Let $\alpha(\mathbf{x}, \mathbf{y})$ be a quantifier-free $\mathcal{L}^2[\Sigma]$ -formula. Let Π be the set of those 1-types over $\langle \Sigma, \bar{\mathbf{m}} \rangle$ that are consistent with α and \mathbf{T} be the set of those 2-types over $\langle \Sigma, \bar{\mathbf{m}} \rangle$ that are consistent with α and the intended interpretation eq. (6.2). Then a $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structure \mathfrak{A} is a model for α iff it is a model for (Π, \mathbf{T}) .

Recall that the number of possible 1-types or 2-types over Σ is exponentially bounded by the length s of Σ and that the cardinality of a 1-type or a 2-type over Σ is polynomially bounded by s . Hence we can reduce the (finite) classified satisfiability problem to the (finite) type realizability problem in exponential time:

$$(\text{FIN})\text{CLSAT}\mathcal{L}^2 \leq_m^{\text{EXPTIME}} (\text{FIN})\text{REAL}\mathcal{L}^2.$$

We study the type realizability problem instead of directly studying the satisfiability problem because it is flexible enough to allow us to employ its solutions in simpler logics to construct a solution in logics with more equivalences.

Let (Π, \mathbf{T}) be a type instance over $\langle \Sigma, \bar{\mathbf{m}} \rangle$.

6.1 Type realizability of the two-variable first-order logic

Definition 67. Two 1-types $\pi, \pi' \in \Pi$ are connectable (written $\pi \sim \pi'$), if $\pi = \text{tp}_x \tau$ and $\pi' = \text{tp}_y \tau$ for some $\tau \in T$. A 1-type $\kappa \in \Pi$ is a king type if $\kappa \not\sim \kappa$.

Remark 24. If two distinct 1-types $\pi, \pi' \in \Pi$ are not connectable, then (Π, T) is not fully realizable.

If \mathfrak{A} is a full model for (Π, T) then every two distinct elements $a \neq b \in A$ realize connectable 1-types. Hence every king type $\kappa \in \Pi$ is realized once in \mathfrak{A} .

A type instance (Π, T) is connected if every two distinct 1-types $\pi, \pi' \in \Pi$ are connectable.

The next definition characterizes the local structure of the ray of 2-types realized around an element of a structure.

Definition 68. A star-type $\sigma \subseteq T$ for the type instance (Π, T) is a nonempty set of 2-types satisfying the following conditions:

1. If $\tau, \tau' \in \sigma$ then $\text{tp}_x \tau = \text{tp}_x \tau'$, that is the x -type of every element of σ is the same. Denote the x -type of every element of σ by $\text{tp}_x \sigma$.
2. If $\kappa = \text{tp}_x \sigma$ is a king type, then no $\tau \in \sigma$ has $\text{tp}_y \tau = \kappa$.
3. If $\kappa \neq \text{tp}_x \sigma$ is any king type, then one $\tau \in \sigma$ has $\text{tp}_y \tau = \kappa$.
4. If $\mathbf{m} \in \bar{\mathbf{m}}$, then $\mathbf{m} \in \tau$ for some (possibly many) $\tau \in \sigma$.

Note that the size of a star-type is polynomially bounded by the size of the type instance.

If \mathfrak{A} is a (Π, T) -structure and $a \in A$, the star-type realized by a is:

$$\text{stp}^{\mathfrak{A}}[a] = \left\{ \text{tp}^{\mathfrak{A}}[a, b] \mid b \in A \setminus \{a\} \right\}.$$

It is straightforward to check that this indeed defines a star-type.

Remark 25 (Star-type extension). Let σ be a star-type, with x -type $\pi = \text{tp}_x \sigma$. Let $\tau \in T$ be a 2-type and suppose that $\text{tp}_x \tau = \pi$ and that $\text{tp}_y \tau$ is not a king type. Then $\sigma \cup \{\tau\}$ is also a star-type.

Definition 69. A certificate \mathcal{S} for the type instance (Π, T) is a nonempty set of star-types satisfying the following conditions:

1. If $\pi \in \Pi$ then some (possibly many) $\sigma \in \mathcal{S}$ has $\text{tp}_x \sigma = \pi$, that is every 1-type is witnessed.
2. If $\kappa \in \Pi$ is a king type, then one $\sigma \in \mathcal{S}$ has $\text{tp}_x \sigma = \kappa$, that is every king type is witnessed once.
3. If $\tau \in \cup \mathcal{S}$, then $\tau^{-1} \in \cup \mathcal{S}$, that is there are witnesses for the endpoints of every 2-type used in the certificate.

Note that in general the size of a certificate may be exponential in terms of the size of the type instance. However, polynomial certificates exist:

Lemma 7 (Certificate extraction). *Let \mathfrak{A} be a full model for the type instance (Π, T) . Recall that by definition the cardinality of \mathfrak{A} is at least 2. Let $T' \subseteq T$ be the set of 2-types realized in \mathfrak{A} . For every 2-type $\tau \in T'$ realized in \mathfrak{A} , let $(a_\tau, b_\tau) \in A \times A$ be a pair of distinct elements realizing τ . Let*

$$\mathcal{S} = \left\{ \text{stp}^{\mathfrak{A}}[a_\tau] \mid \tau \in T' \right\} \cup \left\{ \text{stp}^{\mathfrak{A}}[b_\tau] \mid \tau \in T' \right\}.$$

Then \mathcal{S} is a certificate for the type instance. Moreover, its size is linearly bounded by $|T|$, hence also by the size of the type instance.

Proof. We check the conditions for a certificate:

1. Let $\pi \in \Pi$ be a 1-type. Since \mathfrak{A} is a full model for (Π, T) and A has cardinality at least 2, we may choose a pair of distinct elements such that the first element realizes π .
2. If $\kappa \in \Pi$ is a king type, by Remark 24 it is realized once in \mathfrak{A} . Let $c \in A$ be the unique element realizing κ . If $\sigma \in \mathcal{S}$ has $\text{tp}_x \sigma = \kappa$, then $\sigma = \text{stp}^{\mathfrak{A}}[c]$ so such σ is unique.
3. That $\tau \in \cup \mathcal{S}$ implies $\tau^{-1} \in \cup \mathcal{S}$ follows immediately from the definition of \mathcal{S} .

□

Lemma 8 (Certificate expansion). *Let \mathcal{S} be a certificate for the connected type instance (Π, T) . Then (Π, T) has a finite full model. More precisely, let $t \geq |T|$. Then (Π, T) has a finite model \mathfrak{A} in which for every star-type $\sigma \in \mathcal{S}$ such that $\text{tp}_x \sigma$ is not a king type, there are at least t elements $a \in A$ such that $\sigma \subseteq \text{stp}^{\mathfrak{A}}[a]$.*

Proof. We adapt the standard strategy used in the proof of the finite model property for the logic \mathcal{L}^2 , as presented in [2]¹. We build a model \mathfrak{A} for (Π, T) as follows. The domain A of \mathfrak{A} is the union of the disjoint sets of elements A^σ for all $\sigma \in \mathcal{S}$, that have intended star-type σ as follows. If $\text{tp}_x \sigma$ is a king type, $A^\sigma = \{a^\sigma\}$ is a singleton. Call the element a^σ a *king*. Otherwise, if $\text{tp}_x \sigma$ is not a king type, $A^\sigma = A_0^\sigma \cup A_1^\sigma \cup A_2^\sigma$ consists of three disjoint copies of t elements: $A_i^\sigma = \{a_{i1}^\sigma, a_{i2}^\sigma, \dots, a_{it}^\sigma\}$, where $i \in \{0, 1, 2\}$. Call these elements *peasants*. Let $\sigma : A \rightarrow \mathcal{S}$ denote the intended star-type of the elements: $\sigma(a) = \sigma$ for all $a \in A^\sigma$. Let $\pi : A \rightarrow \Pi$ denote the intended 1-type of the elements: $\pi(a) = \text{tp}_x(\sigma(a))$. We proceed to consistently assign 2-types to pairs of distinct elements from the structure on stages.

Realization of kings We first find witnesses for the intended star-types of the kings. Let $\sigma \in \mathcal{S}$ be such that $\kappa = \text{tp}_x \sigma$ is a king type. Consider the unique element $a = a^\sigma$ having intended star-type σ . For every $\tau \in \sigma$ we will find an element b_τ for the assignment $\text{tp}^{\mathfrak{A}}[a, b_\tau] = \tau$.

¹with the slight difference that our approach doesn't need a *court*, since the information about it is implicit in the certificate

6.1 Type realizability of the two-variable first-order logic

1. If $\text{tp}_y \tau = \kappa'$ is a king type, by Condition 2 for star-types we have $\kappa' \neq \kappa$, so there is a unique star-type σ' having $\text{tp}_x \sigma' = \kappa'$ and a unique king $b_\tau = a^{\sigma'} \neq a$ having intended star-type κ' . Note that this assignment is symmetric, that is at the point of considering the type τ^{-1} for the king b_τ , we would choose a as the opposite side of the assignment.
2. If $\text{tp}_y \tau = \pi'$ is not a king type (hence $\pi' \neq \kappa$), we simultaneously find distinct elements $b_{\tau'}$ for all $\tau' \in \sigma$ having $\text{tp}_y \tau' = \pi'$. Let $T' = \{\tau' \in \sigma \mid \text{tp}_y \tau' = \pi'\}$ be the set of all such τ' . Since $T' \subseteq T$, there are at most t such 2-types τ' . By Condition 3 for certificate, we can find a star-type $\sigma_{\tau'} \in \mathcal{S}$ containing $(\tau')^{-1}$ for every $\tau' \in T'$. Since $\text{tp}_x \sigma_{\tau'} = \pi'$ is not a king type, there are enough elements $b_{\tau'} \in A_0^{\sigma_{\tau'}}$ having intended star-type $\sigma_{\tau'}$. We assign $\text{tp}^{\mathfrak{A}}[a, b_{\tau'}] = \tau'$.

Realization of peasants We now find witnesses for the remaining elements that realize 1-types which are not king types. Let $\sigma \in \mathcal{S}$ be such that $\pi = \text{tp}_x \sigma$ is not a king type. Let $a = a_{ij}^\sigma$ be an arbitrary element with intended star-type σ . For every $\tau \in \sigma$ we will find an element b_τ for the assignment $\text{tp}^{\mathfrak{A}}[a, b_\tau] = \tau$.

1. If $\text{tp}_y \tau = \kappa'$ is a king type then let c be the element realizing it: $\pi(c) = \kappa'$. We consider two cases. First suppose that $\text{tp}^{\mathfrak{A}}[c, a] = \tau'$ has already been assigned during the realization of kings. By construction we must have that $(\tau')^{-1} \in \sigma(a) = \sigma$. Since $\text{tp}_y (\tau')^{-1} = \kappa'$ is a king type, by Condition 2 for certificate, the king type κ' is witnessed once in the star-type σ . Hence $\tau = (\tau')^{-1}$, and so τ is already witnessed by $b_\tau = c$.

Next suppose that $\text{tp}^{\mathfrak{A}}[c, a]$ has not been assigned during the realization of kings. Then just assign $\text{tp}^{\mathfrak{A}}[a, c] = \tau$. Note that this may extend the actual star-type of the king c beyond its intended star-type $\sigma(c)$ by adding the type τ^{-1} , but by Remark 25, this extension is still a star-type. That is, in the end, the structure may realize *more* than the intended star-types, but, importantly, *not less*.

2. If $\text{tp}_y \tau = \pi'$ is not a king type, we simultaneously find distinct peasants $b_{\tau'}$ for all $\tau' \in \sigma$ having $\text{tp}_y \tau' = \pi'$. Let $T' = \{\tau' \in \sigma \mid \text{tp}_y \tau' = \pi'\}$ be the set of all such τ' . By Condition 3 for certificate, we can find a star-type $\sigma_{\tau'} \in \mathcal{S}$ containing $(\tau')^{-1}$ for every $\tau' \in T'$. The key to consistency is to use elements from the *next copy* of peasants: let $i' = (i + 1 \bmod 3) \in \{0, 1, 2\}$. Since $\text{tp}_x \sigma_{\tau'} = \pi'$ is not a king type, there are enough peasants $b_{\tau'} \in A_{i'}^{\sigma_{\tau'}}$ having intended star-type $\sigma_{\tau'}$. We assign $\text{tp}^{\mathfrak{A}}[a, b_{\tau'}] = \tau'$. None of these assignments clash with each other.

Completion For any pair of distinct elements $a, b \in A$ that has not yet been assigned a 2-type, assign $\text{tp}^{\mathfrak{A}}[a, b] = \tau$ to arbitrary 2-type τ that connects the 1-types $\pi(a)$ and $\pi(b)$. This is possible because the type instance (Π, T) is connected.

□

Proposition 11. *The logic \mathcal{L}^2 has the finite model property. The (finite) full type realizability problem for \mathcal{L}^2 is in NPTIME.*

Proof. Let (Π, T) be a type instance. If it is not connected, then it has no model by Remark 24. It is trivial to check satisfiability in the class of structures of cardinality 1. If this fails to yield a model, guess a certificate of polynomial size (and verify in polynomial time that this is indeed a certificate). By Lemma 7 and Lemma 8 such a certificate exists iff (Π, T) is fully satisfiable. \square

Combining this with the reductions given in Remark 22 and Remark 23 gives us another proof of a standard result:

Corollary 3. *The logic \mathcal{L}^2 has the finite model property and its (finite) satisfiability problem is in NEXPTIME.*

6.2 Type realizability of the two-variable first-order logic with equivalences in refinement

In this section we consider the logic $\mathcal{L}^2 eE_{\text{refine}}$ featuring $e \geq 2$ equivalence symbols e_1, e_2, \dots, e_e in refinement. Abbreviate the coarsest equivalence symbol $e = e_e$. Recall from the previous section that every \mathcal{L}^2 -sentence φ can be reduced in polynomial time to an equisatisfiable sentence in the form:

$$\forall x \forall y (\alpha(x, y) \vee x = y) \wedge \bigwedge_{1 \leq i \leq m} \forall x \exists y (\mathbf{m}_i(x, y) \wedge x \neq y),$$

that may use additional unary predicate symbols and the new message symbols \mathbf{m}_i , where the $\forall \forall$ -part α is quantifier-free.

Definition 70. *A classified signature for $\mathcal{L}^2 eE_{\text{refine}}$ is a predicate signature Σ together with a sequence $\bar{\mathbf{m}} = \mathbf{m}_1 \mathbf{m}_2 \dots \mathbf{m}_m$ of distinct binary predicate symbols distinct from the equivalence symbols from Σ having the intended interpretation*

$$\bigwedge_{1 \leq i \leq m} \forall x \exists y (\mathbf{m}_i(x, y) \wedge x \neq y).$$

Again, note that a classified signature *automatically includes* the $\forall \exists$ -part of formulas and $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structures *automatically satisfy* the $\forall \exists$ -part.

The (finite) classified satisfiability problem for $\mathcal{L}^2 eE_{\text{refine}}$ is defined as in Definition 65. Again we have the reduction:

$$(\text{FIN})\text{SAT}\mathcal{L}^2 \leq_m^{\text{P}^{\text{TIME}}} (\text{FIN})\text{CLSAT}\mathcal{L}^2.$$

Let $\langle \Sigma, \bar{\mathbf{m}} \rangle$ be a classified signature for $\mathcal{L}^2 eE_{\text{refine}}$. A type instance (Π, T) over $\langle \Sigma, \bar{\mathbf{m}} \rangle$ is defined as in Definition 66; also the notions of a model for a type instance, full realization, characteristic type instance of a model and the (finite) (full) type realizability problem for $\mathcal{L}^2 eE_{\text{refine}}$ are defined. Again, the reductions from Remark 22 and Remark 23 hold:

$$\begin{aligned} (\text{FIN})\text{REAL}\mathcal{L}^2 eE_{\text{refine}} &\leq_m^{\text{NP}^{\text{TIME}}} (\text{FIN})\text{FULLREAL}\mathcal{L}^2 eE_{\text{refine}} \\ (\text{FIN})\text{CLSAT}\mathcal{L}^2 eE_{\text{refine}} &\leq_m^{\text{EXP}^{\text{TIME}}} (\text{FIN})\text{REAL}\mathcal{L}^2 eE_{\text{refine}}. \end{aligned}$$

6.2 Type realizability of the two-variable first-order logic with equivalences in refinement

We proceed to define new terms, specific to the case of many equivalence symbols. The terminology is loosely based on [4].

Definition 71. A 2-type $\tau \in T[\Sigma]$ is a *galactic type* if $e(\mathbf{x}, \mathbf{y}) \in \tau$. Otherwise, if $(\neg e(\mathbf{x}, \mathbf{y})) \in \tau$, the type is a *cosmic type*. The set of galactic 2-types is $T_g[\Sigma]$. The set of cosmic 2-types is $T_c[\Sigma]$.

Informally, we think of the e -classes in a structure as galaxies, the whole structure as the cosmos, of the galactic 2-types as characterizing the interactions in the internals of the e -classes of a structure, while cosmic 2-types characterize the interactions between elements in different e -classes.

Let (Π, T) be a type instance over $\langle \Sigma, \bar{\mathbf{m}} \rangle$.

Definition 72. Two 1-types $\pi, \pi' \in \Pi$ are *galactically connectable* (written $\pi \sim_g \pi'$), if $\pi = \text{tp}_{\mathbf{x}} \tau$ and $\pi' = \text{tp}_{\mathbf{y}} \tau$ for some galactic $\tau \in T$. The types are *cosmically connectable* (written $\pi \sim_c \pi'$), if $\pi = \text{tp}_{\mathbf{x}} \tau$ and $\pi' = \text{tp}_{\mathbf{y}} \tau$ for some cosmic $\tau \in T$.

Note that the types may be both galactically and cosmically connectable and the (unqualified) definition of connectable types from Definition 67 is equivalent to:

$$\pi \sim \pi' \text{ iff } \pi \sim_g \pi' \text{ or } \pi \sim_c \pi'.$$

Recall that a 1-type π is a *king type* if $\pi \not\sim \pi$. A 1-type π is a *noble type* if $\pi \not\sim_c \pi$. We now generalize Remark 24:

Remark 26. If \mathfrak{A} is a full model for (Π, T) then every king type $\kappa \in \Pi$ is realized once in \mathfrak{A} and every noble type $\nu \in \Pi$ is realized (possibly many times) in a single e -class of \mathfrak{A} .

Recall that (Π, T) is connected if every two distinct 1-types are connectable. Again, if (Π, T) is not connected then (Π, T) is not fully realizable.

Our strategy to resolve the type realizability problem would be to encode the galactic structure of enough e -classes of into instances of the type realizability problem for the logic featuring one less equivalence symbol.

Definition 73. A cosmic spectrum $\sigma \subseteq (T \cap T_c[\Sigma])$ for the type instance (Π, T) is a nonempty set of cosmic types satisfying the following conditions:

1. If $\nu \in (\text{tp}_{\mathbf{x}} \upharpoonright \sigma)$ is a noble type, then no $\tau \in \sigma$ has $\text{tp}_{\mathbf{y}} \tau = \nu$.
2. If $\nu \notin (\text{tp}_{\mathbf{x}} \upharpoonright \sigma)$ is any noble type, then some (possibly many) $\tau \in \sigma$ has $\text{tp}_{\mathbf{y}} \tau = \nu$.
3. If $\kappa \notin (\text{tp}_{\mathbf{x}} \upharpoonright \sigma)$ is any king type, then one $\tau \in \sigma$ has $\text{tp}_{\mathbf{y}} \tau = \kappa$. Note that the existence follows anyway from Condition 2.

Definition 74. Let \mathfrak{A} be a $\langle \Sigma, \bar{\mathbf{m}} \rangle$ -structure and let $E = e^{\mathfrak{A}}$. The cosmic spectrum $\text{csp}^{\mathfrak{A}}[X]$ of an E -class $X \in \mathcal{E}E$ is the set of cosmic types realized at X :

$$\text{csp}^{\mathfrak{A}}[X] = \left\{ \text{tp}^{\mathfrak{A}}[a, b] \in T_c[\Sigma] \mid a \in X, b \in A \setminus \{a\} \right\} = \left\{ \text{tp}^{\mathfrak{A}}[a, b] \mid a \in X, b \in A \setminus X \right\}.$$

6 Two-variable logics

Note that the cosmic spectrum is empty iff $E = A \times A$. Without loss of generality, we may assume that E is not the full relation, because the full relation is trivially definable in \mathcal{L}^2 . So without loss of generality, we will consider models with nonempty cosmic spectrums of their \mathbf{e} -classes.

Remark 27. If \mathfrak{A} is a full (Π, T) -model, $E = \mathbf{e}^{\mathfrak{A}}$ is not the full relation on A and $X \in \mathcal{E}E$ is a galaxy, then $\text{csp}^{\mathfrak{A}}[X]$ is a cosmic spectrum for (Π, T) .

Proof. We check the conditions for a cosmic spectrum. First, $\sigma = \text{csp}^{\mathfrak{A}}[X]$ is nonempty since E is not the full relation on A .

1. If $\nu \in (\text{tp}_x \upharpoonright \sigma)$ is a noble type, then all elements realizing ν in \mathfrak{A} are in X , hence there is no cosmic $\tau \in \sigma$ having $\text{tp}_y \tau = \nu$.
2. If $\nu \notin (\text{tp}_x \upharpoonright \sigma)$ is a noble type, since \mathfrak{A} is a full model, there is some element $b \in A \setminus X$ realizing ν . Then $\tau = \text{tp}^{\mathfrak{A}}[a, b]$ is a 2-type having $\tau \in \sigma$ and $\text{tp}_y \tau = \nu$.
3. If $\kappa \notin (\text{tp}_x \upharpoonright \sigma)$ is a king type, then there is a unique element $b \in A$ realizing κ . Note that $b \notin X$ and the 2-type $\tau = \text{tp}^{\mathfrak{A}}[a, b]$ is the unique $\tau \in \sigma$ such that $\text{tp}_y \tau = \kappa$.

□

We like to think about a cosmic spectrum of a galaxy as the *reason why* that galaxy is possible. For this we need to define the notion of a *locally-consistent* cosmic spectrum. For this we extract a type instance (Π_σ, T_σ) over the *simpler logic* $\mathcal{L}^2(e-1)\mathbf{E}_{\text{refine}}$ out of the cosmic spectrum σ :

Definition 75. Let (Π, T) be a type instance over $\langle \Sigma, \bar{\mathbf{m}} \rangle$ and $\sigma \subseteq (T \cap T_c[\Sigma])$ be a cosmic spectrum for (Π, T) . Let \mathbf{in} (intended to label the inside of the galaxy) and \mathbf{bh} (intended to label some black hole outside the galaxy) be two new unary predicate symbols. For a 1-type π or a 2-type τ , denote by π^- and τ^- the reducts of π and τ to the language $\Sigma - \{\mathbf{e}\}$. That is, $\pi^- \subset \pi$ and $\tau^- \subset \tau$ consist of those literals that do not feature the predicate symbol \mathbf{e} . We define the spectral type instance (Π_σ, T_σ) corresponding to the cosmic spectrum σ as a type instance over the $\mathcal{L}^2(e-1)\mathbf{E}_{\text{refine}}$ -signature $\langle \Sigma - \{\mathbf{e}\} + \{\mathbf{in}, \mathbf{bh}\}, \bar{\mathbf{m}} \rangle$ as follows:

(I) For every $\pi \in (\text{tp}_x \upharpoonright \sigma)$, the 1-type $\pi^- \cup \{\mathbf{in}(x), \neg \mathbf{bh}(x)\}$ is added to Π_σ .

(O) For every $\pi \in (\text{tp}_y \upharpoonright \sigma)$, the 1-type $\pi^- \cup \{\neg \mathbf{in}(x), \neg \mathbf{bh}(x)\}$ is added to Π_σ .

(B) Let β be an arbitrary 1-type extending $\{\neg \mathbf{in}(x), \mathbf{bh}(x)\}$. Call β the black hole type. The black hole type is added to Π_σ .

(II) For every (galactic) $\tau \in (T \cap T_g[\Sigma])$, the 2-type

$$\tau^- \cup \{\mathbf{in}(x), \neg \mathbf{bh}(x), \mathbf{in}(y), \neg \mathbf{bh}(y)\}$$

is added to T_σ .

6.2 Type realizability of the two-variable first-order logic with equivalences in refinement

(IO),(OI) For every (cosmic) $\tau \in \sigma$, the 2-type

$$\tau^- \cup \{\mathbf{in}(x), \neg\mathbf{bh}(x), \neg\mathbf{in}(y), \neg\mathbf{bh}(y)\}$$

and its inverse are added to T_σ .

(OO) For every unordered pair of (not necessarily distinct) 1-types $\{\pi_\sigma, \pi'_\sigma\} \subseteq \Pi_\sigma$ that are “out”-types: $\{\neg\mathbf{in}(x), \neg\mathbf{bh}(x)\} \subseteq \pi_\sigma$ and $\{\neg\mathbf{in}(x), \neg\mathbf{bh}(x)\} \subseteq \pi'_\sigma$, add an arbitrary 2-type τ connecting them (so $\text{tp}_x \tau = \pi_\sigma$ and $\text{tp}_y \tau = \pi'_\sigma$) and its inverse τ^{-1} to T_σ .

(IB) For every 1-type $\pi_\sigma \in \Pi_\sigma$ that is an “in”-type, so $\{\mathbf{in}(x), \neg\mathbf{bh}(x)\} \subseteq \pi_\sigma$, let τ be an arbitrary 2-type connecting π_σ and the black hole type β that witnesses no messages for π_σ and everything for β : $(\neg\mathbf{m}(x, y)) \in \tau$ and $\mathbf{m}(y, x) \in \tau$ for every $\mathbf{m} \in \bar{\mathbf{m}}$, add τ and τ^{-1} to T_σ .

(OB) For every 1-type $\pi_\sigma \in \Pi_\sigma$ that is an “out”-type, so $\{\neg\mathbf{in}(x), \neg\mathbf{bh}(x)\} \subseteq \pi_\sigma$, let τ be an arbitrary 2-type connecting π_σ and the black hole type β that witnesses every message for π_σ and for β : $\mathbf{m}(x, y) \in \tau$ and $\mathbf{m}(y, x) \in \tau$ for every $\mathbf{m} \in \bar{\mathbf{m}}$, add τ and τ^{-1} to T_σ .

Definition 76. Let (Π, T) be a type instance over the $\mathcal{L}^2 e E_{\text{refine}}$ -classified signature $\langle \Sigma, \bar{\mathbf{m}} \rangle$. The cosmic spectrum $\sigma \subseteq (T \cap T_c[\Sigma])$ is locally consistent if its spectral type instance (Π_σ, T_σ) is fully satisfiable (as a type instance for the $\mathcal{L}^2(e-1)E_{\text{refine}}$ -classified signature $\langle \Sigma - \{\mathbf{e}\} + \{\mathbf{in}, \mathbf{bh}\}, \bar{\mathbf{m}} \rangle$).

Remark 28. Let \mathfrak{A} be a full (Π, T) -model, $E = e^\mathfrak{A}$ is not the full relation of A and $X \in \mathcal{E}E$ be a galaxy. Then $\text{csp}^\mathfrak{A}[X]$ is a locally consistent cosmic spectrum for (Π, T) .

Proof. Let $\Sigma' = \Sigma - \{\mathbf{e}\} + \{\mathbf{in}, \mathbf{bh}\}$. We build a Σ' -structure \mathfrak{A}' based on \mathfrak{A} that fully realizes the spectral type instance (Π_σ, T_σ) . The domain of \mathfrak{A}' is $A' = A \cup \{c\}$, where c is a new black hole element. The 1-type of every element in \mathfrak{A}' is defined as follows:

(I) If $a \in X$ is inside, then $\text{tp}^{\mathfrak{A}'}[a] = \text{tp}^\mathfrak{A}[a]^- \cup \{\mathbf{in}(x), \neg\mathbf{bh}(x)\}$, that is the elements from the galaxy are inside.

(O) If $a \in A \setminus X$ is outside, then $\text{tp}^{\mathfrak{A}'}[a] = \text{tp}^\mathfrak{A}[a]^- \cup \{\neg\mathbf{in}(x), \neg\mathbf{bh}(x)\}$, that is the elements outside the galaxy are outside.

(B) If $a = c$, then $\text{tp}^{\mathfrak{A}'}[a] = \beta$, that is the black hole element has the black hole type.

The 2-type between every pair of distinct elements $a \neq b \in A'$ is defined as follows:

(II) If $a, b \in X$ are inside, then:

$$\text{tp}^{\mathfrak{A}'}[a, b] = \text{tp}^\mathfrak{A}[a, b]^- \cup \{\mathbf{in}(x), \neg\mathbf{bh}(x), \mathbf{in}(y), \neg\mathbf{bh}(y)\}.$$

(IO) If $a \in X$ is inside and $b \in A \setminus X$ is outside, then:

$$\text{tp}^{\mathfrak{A}'}[a, b] = \text{tp}^\mathfrak{A}[a, b]^- \cup \{\mathbf{in}(x), \neg\mathbf{bh}(x), \neg\mathbf{in}(y), \neg\mathbf{bh}(y)\}.$$

(OI) If $a \in A \setminus X$ is outside and $b \in X$ is inside, then:

$$\text{tp}^{\mathfrak{A}'}[a, b] = \text{tp}^{\mathfrak{A}}[a, b]^- \cup \{-in(x), -bh(x), in(y), -bh(y)\}.$$

Note that this produces the 2-type that is exactly the inverse of the previous point.

(OO) If $a, b \in A \setminus X$ are outside, then let $\tau \in T_\sigma$ be the 2-type connecting $\pi_\sigma = \text{tp}^{\mathfrak{A}'}[a]$ and $\pi'_\sigma = \text{tp}^{\mathfrak{A}'}[b]$ and assign $\text{tp}^{\mathfrak{A}'}[a, b] = \tau$. Note that this assignment is symmetric, since the unique 2-type connecting π'_σ and π_σ in T_σ is τ^{-1} .

(IB) If $a \in X$ is inside and $b = c$ is the black hole, then let $\tau \in T_\sigma$ be the unique 2-type connecting $\pi_\sigma = \text{tp}^{\mathfrak{A}'}[a]$ and β and assign $\text{tp}^{\mathfrak{A}'}[a, b] = \tau$.

(OB) If $a \in A \setminus X$ is outside and $b = c$ is the black hole, then let $\tau \in T_\sigma$ be the unique 2-type connecting $\pi_\sigma = \text{tp}^{\mathfrak{A}'}[a]$ and β and assign $\text{tp}^{\mathfrak{A}'}[a, b] = \tau$.

We have to verify that \mathfrak{A}' is indeed a $\langle \Sigma', \bar{m} \rangle$ -structure, that is that the star-type of every element of \mathfrak{A}' contains $m(x, y)$ for every $m \in \bar{m}$.

(I) For the in-elements $a \in X \subset A'$, every $m \in \bar{m}$ that is included in $\text{stp}^{\mathfrak{A}}[a]$ is also included in $\text{stp}^{\mathfrak{A}'}[a]$, since a 2-type τ^- contains $m(x, y)$ iff τ contains $m(x, y)$ ².

(O) For the out-elements $a \in (A \setminus X) \subset A'$, we just need to recall that in the (OB) case, we included a 2-type τ that witnesses every message symbol for both of its ends: $m(x, y) \in \tau$ and $m(y, x) \in \tau$. By construction, this is exactly the 2-type between a and the black hole c .

(B) For the black hole $c \in A'$, we just have to recall that in the (IB) and (OB) cases, all included 2-types connecting to the black hole type contained $m(y, x)$ for every $m \in \bar{m}$.

Since \mathfrak{A} was a full (Π, T) -model, it is clear that \mathfrak{A}' is a full (Π_σ, T_σ) -model. \square

6.3 NOTES

Definition 77. A certificate \mathcal{S} for the type instance (Π, T) is a nonempty set of locally consistent cosmic spectra satisfying the following conditions:

1. If $\pi \in \Pi$, then some (possibly many) $\sigma \in \mathcal{S}$ have $\pi \in (\text{tp}_x \upharpoonright \sigma)$.
2. If $\nu \in \Pi$ is noble, then one $\sigma \in \mathcal{S}$ has $\nu \in (\text{tp}_x \upharpoonright \sigma)$.
3. If $\sigma \in \mathcal{S}$ and $\tau \in \sigma$, then $\tau^{-1} \in \sigma'$ for some $\sigma' \in \mathcal{S}$.

TODO: Certificate extraction: for every cosmic 2-type τ realized in \mathfrak{A} , let a_τ, b_τ be a pair of distinct elements (and in distinct e -classes, since τ is cosmic) realizing it. Take:

$$\mathcal{S} = \left\{ \text{csp}^{\mathfrak{A}}[E[a_\tau]], \text{csp}^{\mathfrak{A}}[E[b_\tau]] \mid \tau \dots \right\}$$

²this is where we use the condition that the message symbols of a classified signature are distinct from the builtin equivalence symbols

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