Compressive Radar Imaging

Richard Baraniuk
Department of Electrical and Computer Engineering
Rice University

Philippe Steeghs
E. P. Wigner Institute
Bucharest, Romania

***Abstract*—We introduce a new approach to radar imaging based on the concept of *compressive sensing* (CS). In CS, a low-dimensional, nonadaptive, linear projection is used to acquire an efficient representation of a compressible signal directly using just a few measurements. The signal is then reconstructed by solving an inverse problem either through a linear program or a greedy pursuit. We demonstrate that CS has the potential to make two significant improvements to radar systems: (*i*) eliminating the need for the pulse compression matched filter at the receiver, and (*ii*) reducing the required receiver analog-to-digital conversion bandwidth so that it need operate only at the radar reflectivity's potentially low "information rate" rather than at its potentially high Nyquist rate. These ideas could enable the design of new, simplified radar systems, shifting the emphasis from expensive receiver hardware to smart signal recovery algorithms.**

I. Introduction

A typical radar system transmits a wideband pulse (linear chirp, coded pulse, pseudonoise (PN) sequence, etc.) and then correlates the received signal with that same pulse in a *matched filter* (effecting *pulse compression*) [1]. A traditional radar receiver consists of either an analog pulse compression system followed by a high-rate analog-to-digital (A/D) converter or a high-rate A/D converter followed by pulse compression in a digital computer (see Figs. 1 and 2); both approaches are complicated and expensive.

Achieving adequate A/D conversion of a wideband PN/chirp radar signal (which is compressed into a short duration pulse by the matched filter) requires both a high sampling frequency and a large dynamic range. Currently available A/D conversion technology is a limiting factor in the design of ultrawideband (high resolution) radar systems, because in many cases the required performance is either beyond what is technologically possible or too expensive.

In this paper, we introduce a new approach to radar imaging based on the concept of *compressive sensing* (CS) [2], [3]. In CS, an *incoherent* linear projection is used to acquire an efficient representation of a compressible signal directly using just a few measurements. Interestingly, random projections play a major role. The signal is then reconstructed by solving an inverse problem either through a linear program or a greedy pursuit.

We demonstrate that when the CS theory applies, significantly fewer samples/measurements of the radar signal need to be acquired in order to obtain an accurate representation for further processing. The potential impacts on radar hardware are promising; we will show that CS can: (*i*) eliminate the need for the matched filter in the radar receiver, and (*ii*) reduce the required receiver A/D conversion bandwidth so that it need operate only at the reflectivity's potentially low "information rate" rather than at its potentially high Nyquist rate (see Fig. 3).

II. Compressive Sensing

Consider a length-N discrete-time signal x of any dimension (without loss of generality, we will focus on one-dimensional (1D) signals for notational simplicity) indexed as $x(n)$, $n = 1, \ldots, N$. We can interpret x as an $N \times 1$ column vector. The signal x is *sparsely representable* if there exists a *sparsity basis* $\{\psi_i\}$ that provides a K-sparse representation of x; that is

$$x = \sum_{i=1}^{N} \theta_i \, \psi_i = \sum_{\ell=1}^{K} \theta(i_\ell) \, \psi_{i_\ell}, \tag{1}$$

where x is a linear combination of K basis vectors chosen from $\{\psi_i\}$, $\{i_\ell\}$ are the indices of those vectors, and $\{\theta_i\}$ are the weighting coefficients. Alternatively, by stacking the basis vectors

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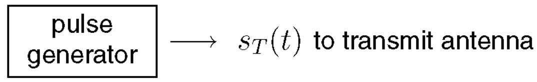


Fig. 1. *Prototypical radar transmitter.*

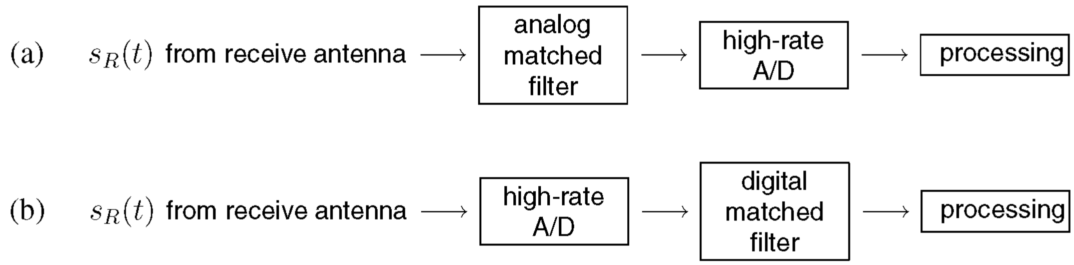


Fig. 2. *Prototypical digital radar receivers for the transmitter in Fig. 1 perform matched filtering either in the (a) analog or (b) digital domain.*

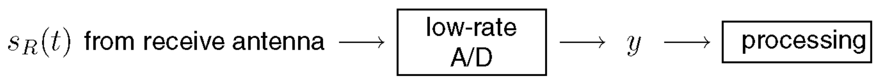


Fig. 3. *Compressive radar receiver for the transmitter in Fig. 1 performs neither matched filtering nor high-rate analog-to-digital conversion.*

as columns into the $N \times N$ sparsity basis matrix $\Psi = [\psi_1 | \ldots | \psi_N]$, we can write in matrix notation

$$x = \Psi\theta, \qquad (2)$$

where θ is an $N \times 1$ column vector with K nonzero elements. Various expansions, including wavelets, the DCT, and Gabor frames [4], are widely used for the representation and compression of natural signals, images, and other data.

Using $\|\cdot\|_p$ to denote the ℓ_p norm, we can write that $\|\theta\|_0 = K$; that is, the ℓ_0 "norm" $\|\theta\|_0$ merely counts the number of nonzero entries in the vector θ. The signal x is *compressible* if the sorted magnitudes of the coefficients $\{|\theta_i|\}$ decay rapidly to zero; this is the case, for example, if $\theta \in \ell_p$ for $p \leq 1$. Compressible signals are well-approximated as sparse.

The standard procedure for compressing sparse signals, known as *transform coding*, is to (*i*) acquire the full N-point signal x via Nyquist-rate sampling; (*ii*) compute the complete set of transform coefficients $\{\theta_i\}$; (*iii*) locate the K largest, significant coefficients and discard the (many) small coefficients; (*iv*) encode the *values and locations* of the largest coefficients.

This procedure has three inherent inefficiencies. First, for a wideband signal, we must start with a large number of Nyquist-rate samples N. Second, the encoder must compute *all* of the N transform coefficients $\{\theta_i\}$, even though it will discard all but K of them. Third, the encoder must encode the locations of the large coefficients since the locations change with each signal.

In compressive sensing (CS), we do not measure or encode the K significant θ_i directly. Rather, we measure and encode $M < N$ linear projections $y(m) = \langle x, \phi_m^T \rangle$ of the signal onto a *second set* of vectors $\{\phi_m\}, m = 1, \ldots, M$, where ϕ_m^T denotes the transpose of ϕ_m and $\langle \cdot, \cdot \rangle$ denotes the inner product. In matrix notation, we measure

$$y = \Phi x, \qquad (3)$$

where y is an $M \times 1$ column vector and the *measurement matrix* Φ is $M \times N$ with each row a measurement vector ϕ_m^T.

Since $M < N$, recovery of the signal x from the measurements y is ill-posed in general. However, the CS theory tells us that when the matrix $\Phi\Psi$ has the *Restricted Isometry Property* (RIP) [2], [3], [5], then it is indeed possible to recover the K largest θ_i's from a similarly sized set of $M = O(K \log(N/K))$ measurements y. The RIP is closely related to an incoherency property between Φ and Ψ, where the rows of Φ do not provide a sparse representation of the columns of Ψ and vice versa. The RIP and incoherency hold for many pairs of bases, including for example, delta spikes and Fourier sinusoids, or sinusoids and wavelets.

An interesting, powerful, and somewhat surprising choice for the measurement matrix Φ is a (pseudo) *random*, noise-like matrix. For example, we may select its MN entries as iid Bernoulli or Gaussian random variables (see Fig. 4). It can be shown that many random measurement matrices are *universal* in the sense that they are incoherent with *any* fixed basis Ψ (spikes, sinusoids, wavelets, Gabor functions, curvelets, and so on) with high probability [2], [3], [5].[1]

When the RIP/incoherency holds, the signal x (via its coefficents θ) can be recovered *exactly* from y by solving an ℓ_1 minimization problem [2], [3]

$$\widehat{\theta} = \arg\min \|\theta\|_1 \quad \text{such that} \quad y = \Phi\Psi\theta. \tag{4}$$

This optimization problem, also known as Basis Pursuit [6], can be solved with traditional linear programming techniques. At the expense of slightly more measurements, iterative greedy algorithms such as Matching Pursuit and Orthogonal Matching Pursuit (OMP) [7] can recover the signal x from the measurements y. The same CS framework of incoherent measurements and optimization-based reconstruction also applies to recovering a close approximation to a compressible signal.

Another choice for the measurement matrix Φ that offers good performance in many cases is a causal, quasi-Toeplitz matrix where each row is an $\lfloor N/M \rfloor$-place right-shift of the row immediately above it; that is, $\phi_{m,n} = p(\lfloor N/M \rfloor m - n)$ for some vector p. In this case, $y = \Phi x$ can be implemented in a streaming fashion as a linear time-invariant filter followed by decimation by $D = \lfloor N/M \rfloor$ [8]

$$y(m) = \sum_{n=1}^{N} p(Dm - n)\, x(n) \tag{5}$$

for $m = 1, \ldots, M$. When p is a PN sequence, we dub this approach *random filtering* (see Fig. 5).

[1]We note that it is critical that the measurement matrix Φ be known to both the encoder and decoder, so in practice it is sufficient to use a pair of pseudo-random number generators at both the encoder and decoder with a common seed known to both.

III. CS-BASED RADAR

In order to illustrate our CS-based radar concept, consider a simplified 1D range imaging model of a target described by $u(r)$ with range variable r. If we let the transmitted radar pulse $s_T(t)$ interact with the target by means of a linear convolution [1], then the received radar signal $s_R(t)$ is given by

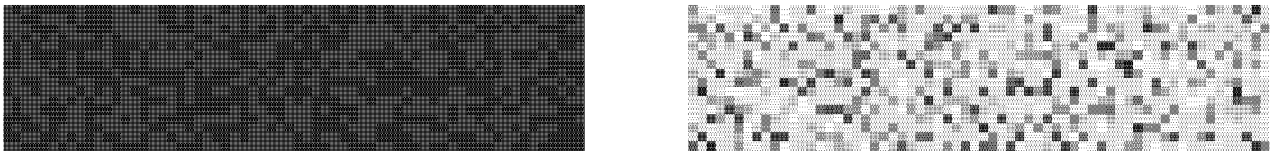
$$s_R(t) = A \int s_T(t - \tau)\, u(\tau)\, d\tau, \tag{6}$$

where we have converted the range variable r to time t using $t = \frac{2r}{c}$, with c the propagation velocity of light, and where A represents attenuation due to propagation and reflection. If the transmitted signal has the property that $s_T(t) * s_T(-t) \approx \delta(t)$ (which is true for PN and chirp signals), then a band-limited measurement of the radar reflectivity $u(t)$ can be obtained by pulse compression, that is, by correlating $s_R(t)$ with $s_T(t)$ in a matched filter (recall Fig. 2) [1]. A/D conversion occurs either before or after the matched filtering, resulting in N Nyquist-rate samples.

Our CS-based radar approach is based on two key observations. First, the target reflectivity functions $u(t)$ that we wish to obtain through the radar process are often *sparse* or *compressible* in some basis. For example, a set of K point targets corresponds to a sparse sum of delta functions as in $u(t) = \sum_{i=1}^{K} a_i\, \delta(t - \kappa_i)$; smooth targets are sparse in the Fourier or wavelet domain; and range-Doppler reflectivities are often sparse in the joint time-frequency (or ambiguity) domain [1]. Such target reflectivity functions $u(t)$ are good candidates for acquisition via CS techniques.

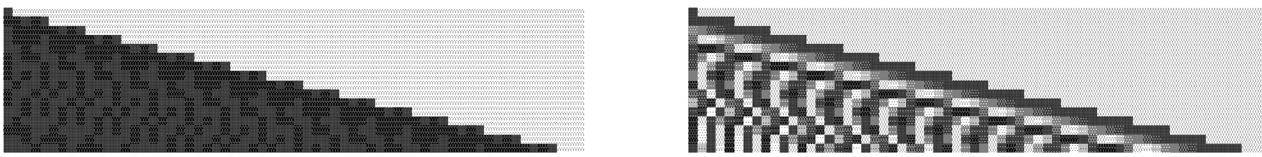
Second, time-translated and frequency-modulated versions of the PN or chirp signals transmitted as radar waveforms $s_T(t)$ form a dictionary (the extension of a basis or frame) that is *incoherent* with the time, frequency, and time-frequency bases that sparsify or compress the above mentioned classes of target reflectivity functions $u(t)$ [8]. This means that PN or chirp signals are good candidates for the rows of a CS acquisition matrix Φ as a "random filter" (recall (5)).

By combining these observations we can both *eliminate the matched filter in the radar receiver* and *lower the receiver A/D converter bandwidth* using CS principles. Consider a new design for a radar



(a) (b)

Fig. 4. *Theoretical compressive sensing (CS) measurement matrices Φ of size $N = 64$ and $M = 16$: (a) iid Bernoulli (± 1) measurements and (b) iid Gaussian measurements. In the color map, blue corresponds to large negative, green to zero, and red to large positive.*



(a) (b)

Fig. 5. *"Random filter" measurement matrices Φ of size $N = 64$ and $M = 16$ based on (a) PN signal and (b) chirp signal. Compare to Fig. 4 and note the causal, quasi-Toeplitz, yet "rich" structure. Same color map as Fig. 4.*

system that consists of the following components. The transmitter is the same as in a classical radar; the transmit antenna emits a PN or chirp signal $s_T(t)$ (recall Fig. 1). However, the receiver does not consist of a matched filter and high-rate A/D converter but rather only a low-rate A/D converter that operates not at the Nyquist rate but at a rate proportional to the target reflectivity's compressibility (see Fig. 3).

We make the connection explicit for a PN-based CS radar with a simple sampling model. Consider a target reflectivity generated from N Nyquist-rate samples $x(n)$ via $u(t) = x(\lceil t/\Delta \rceil)$, $n = 1, \ldots, N$, on the time interval of interest $0 \leq t < N\Delta$. The radar transmits a PN signal generated from a length-N random Bernoulli ± 1 vector $p(n)$ via $s_T(t) = p(\lceil t/\Delta \rceil)$. The received radar signal $s_R(t)$ is given by (6); we sample it not every Δ seconds but rather every $D\Delta$ seconds, where $D = \lfloor N/M \rfloor$ and $M < N$, to obtain the M samples, $m = 1, \ldots, M$,

$$\begin{aligned} y(m) &= s_R(t)|_{t=mD\Delta} \\ &= A \int_0^{N\Delta} s_T(mD\Delta - \tau)\, u(\tau)\, d\tau \\ &= A \sum_{n=1}^{N} p(mD - n) \int_{(n-1)\Delta}^{n\Delta} u(\tau)\, d\tau \\ &= A \sum_{n=1}^{N} p(mD - n)\, x(n), \end{aligned} \tag{7}$$

which are precisely a scaled version of (5). In words, a PN sequence radar implements a random filter in the sense of [8], and hence the low-rate samples y contain sufficient information to reconstruct the signal x corresponding to the Nyquist-rate samples of the reflectivity $u(t)$ via linear programming or a greedy algorithm. Chirp pulses yield similar results.

Figure 6 illustrates the scheme in action. A radar reflectivity profile is probed with a PN pulse sequence, measured at one-half the Nyquist sampling rate, and subsequently recovered exactly using an OMP greedy algorithm and a sparsity frame Ψ combining delta spikes and Haar wavelets.

Additional gains can be expected for 2D CS radar imaging. We illustrate this with a simple simulation of SAR data acquisition and imaging. Figure 7(a) shows the reflectivity function that is to be recovered from the SAR data. We simulated a SAR data acquisition using the method described in [9]. Figure 7(b) shows the result of a 2D CS implementation with four times undersampling, which gives an exact recovery of the reflectivity function. The traditional SAR image (Fig. 7(c)) shows artifacts of the limited aperture of the imaging operator, which are absent in the CS image. The result is similar to what is obtained with the feature-enhanced imaging approach of [10]. However, the CS-based approach has some advantages, such as an almost infinite number of sparse representations to choose from as well as more efficient signal recovery algorithms.

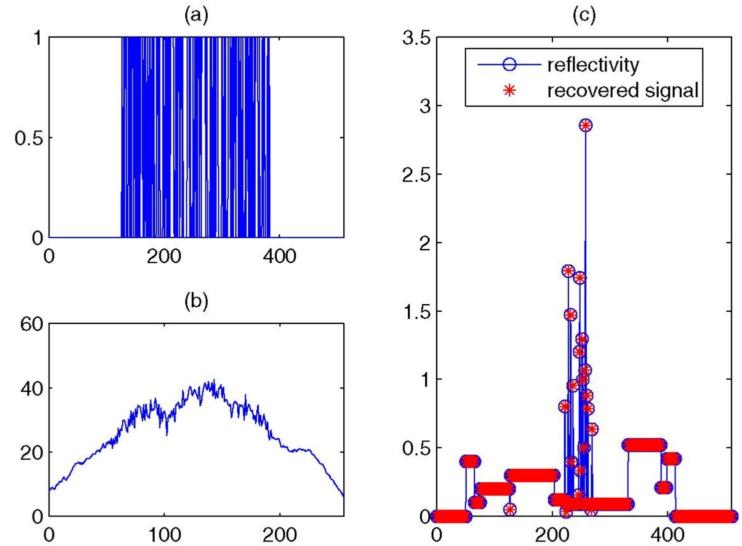


Fig. 6. *CS radar example. (a) Transmitted PN pulse $s_T(t)$, (b) low-rate measurement y, and (c) true and recovered reflectivity profiles $u(t)$.*

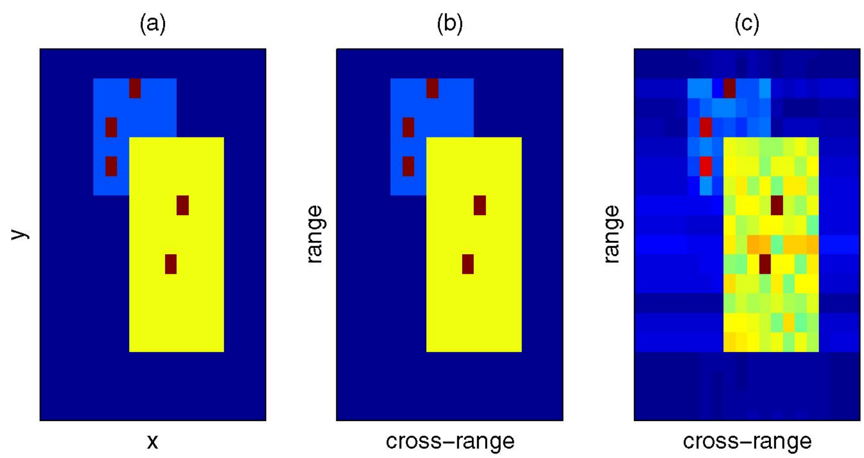


Fig. 7. *CS synthetic aperture radar (SAR) example. (a) 2D reflectivity, (b) CS SAR image, and (c) traditional SAR image.*

IV. Discussion and Conclusions

Using compressive sensing (CS) ideas, we have proposed two potential improvements to a wide class of radar systems: (*i*) we can eliminate the matched filter in the radar receiver, and (*ii*) we can reduce the required sampling rate of the receiver A/D converter so that it need only operate at the target reflectivity's potentially low "information rate" rather than at its potentially high Nyquist rate. For example, for a scene consisting of K point targets, just $M = O(K \log(N/K))$ rather than N measurements will suffice. These ideas could enable the design of new, simplified radar systems, shifting the emphasis from expensive hardware (A/D conversion, matched filtering) to smart signal recovery algorithms. Reconstruction, estimation, detection, and so on are performed using a digital algorithm and can even be performed off-line.

CS techniques are appropriate for monostatic, bistatic, and multiscatic (many receivers and transmitters) scenarios. Since the radar receiver is greatly simplified, CS provides a powerful yet inexpensive framework for multistatic sensor network radars, where the radar signals resulting from one transmitting antenna are received at many receiving antennas that, by virtue of our simplifications, can be made very simply and inexpensively. Our results can also be combined with the theory of distributed compressed sensing (DCS) [11], [12] for array processing and beamforming type applications. The CS-based imaging framework introduced here also applies directly to other modalities such as sonar and synthetic aperture sonar imaging.

While the CS literature has focused almost exclusively on problems in signal reconstruction, approximation, and estimation, CS is *information scalable* to a much wider range of statistical inference tasks. Detection, classification, and recognition do not require a reconstruction of the signal, but only require estimates of the relevant sufficient statistic for the problem at hand [13]. A key point is that it is possible to directly extract these statistics from a small number of random measurements without ever reconstructing the signal. The two upshots are that significantly fewer measurements can be required for signal detection than for signal reconstruction and that the computational complexity of detection can be much reduced compared to reconstruction. Both of these bode well for radar applications, since if we are merely interested in detecting targets rather than reconstructing images of them, then we can use an even lower sampling rate for the CS-based receiver. Moreover, in many radar applications, target detection, classification, and recognition decisions are often made based on the result of some kind of matched filtering or correlation with a set of templates. Information scalability enables us to compute close approximations to these matched filter results directly from incoherent measurements without having to perform expensive reconstruction or approximation computations [14].

There are a number of challenges to be overcome before an actual CS-based radar system will become a reality. First, the target reflectivity being probed must be compressible in some basis, frame, or dictionary. Second, the signal recovery algorithms must be able to handle real-world radar acquisition scenarios with sufficient computational efficiency and robust performance for noisy data. Third, there is a subtle tradeoff to optimize between the reduction in sampling rate $\lfloor N/M \rfloor$ and the dynamic range of the resulting CS system [15]. These are areas of active research for both our team and the broader CS community. In particular, there could be links with recent work on finite rate of innovation sampling for ultrawideband communication systems [16].

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REFERENCES

[1] M. Skolnik, *Radar Handbook*. New York, NY, USA: McGraw Hill, 1970.

[2] E. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. on Information Theory*, vol. 52, no. 2, pp. 489–509, 2006.

[3] D. Donoho, "Compressed sensing," *IEEE Trans. on Information Theory*, vol. 52, no. 4, pp. 5406–5425, 2006.

[4] S. Mallat, *A Wavelet Tour of Signal Processing*. San Diego: Academic Press, 1999.

[5] R. G. Baraniuk, M. Davenport, R. A. DeVore, and M. B. Wakin, "A simplied proof of the restricted isometry property for random matrices (aka the Johnson-Lindenstrauss lemma meets compressed sensing)," *Constructive Approximation*, 2007.

[6] S. Chen, D. Donoho, and M. Saunders, "Atomic decomposition by basis pursuit," *SIAM J. on Sci. Comp.*, vol. 20, no. 1, pp. 33–61, 1998.

[7] J. Tropp and A. C. Gilbert, "Signal recovery from partial information via orthogonal matching pursuit," Apr. 2005. Preprint.

[8] J. Tropp, M. Wakin, M. Duarte, D. Baron, and R. G. Baraniuk, "Random filters for compressive sampling and reconstruction," *Proc. IEEE ICASSP*, 2005.

[9] M. Cetin, *Feature-Enhanced Synthetic Aperture Radar Imaging*. Ph.D. Thesis Boston University, College of Engineering, 2001.

[10] M. Cetin and W. C. Karl, "Feature-enhanced synthetic aperture radar image formation based on non-quadratic regularization," *IEEE Trans. Image Processing*, vol. 10, pp. 623–631, 2001.

[11] D. Baron, M. B. Wakin, M. F. Duarte, S. Sarvotham, and R. G. Baraniuk, "Distributed compressed sensing," 2005. preprint.

[12] M. F. Duarte, M. B. Wakin, D. Baron, and R. G. Baraniuk, "Universal distributed sensing via random projections," in *Proc. Information Processing in Sensor Networks (IPSN)*, 2006.

[13] M. Duarte, M. Davenport, M. Wakin, and R. G. Baraniuk, "Sparse signal detection from incoherent projections," *Proc. IEEE ICASSP*, 2005.

[14] M. Davenport, M. Duarte, M. Wakin, D. Takhar, K. K. Kelly, and R. G. Baraniuk, "The smashed filter for compressive classification and target recognition," in *Proc. IS&T/SPIE Symposium on Electronic Imaging: Computational Imaging*, Jan. 2007.

[15] J. N. Laska, S. Kirolos, M. F. Duarte, T. S. Ragheb, R. G. Baraniuk, and Y. Massoud, "Theory and implementation of an analog-to-information converter using random demodulation," in *IEEE ISCAS*, 2007.

[16] I. Maravic, J. Kusuma, and M. Vetterli, "Low-sampling rate UWB channel characterization and synchronization," *J. Comm. and Networks*, vol. 5, Dec. 2003.