

## Practice with Recurrence Relations (Solutions)

Solve the following recurrence relations using the iteration technique:

1)  $T(n) = T(n - 1) + 2$ ,  $T(1) = 1$

$$T(n) = T(n-1) + 2$$

$$T(1) = 1$$

$$T(n) = T(n-1) + 2 = [T(n-2) + 2] + 2 = T(n-2) + 2 + 2$$

$$T(n) = T(n-2) + 2*2$$

$$T(n) = T(n-2) + 2*2 = [T(n-3) + 2] + 2*2 = T(n-3) + 2 + 2*2$$

$$T(n) = T(n-3) + 2*3$$

$$T(n) = T(n-3) + 2*3 = [T(n-4) + 2] + 2*3 = T(n-4) + 2 + 2*3$$

$$T(n) = T(n-4) + 2*4$$

Substituting Equations

$n \rightarrow n-1$

$$T(n-1) = T(n-2) + 2$$

$$T(n-2) = T(n-3) + 2$$

$$T(n-3) = T(n-4) + 2$$

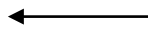
$$T(n-4) = T(n-5) + 2$$

Do it one more time...

$$T(n) = T(n-4) + 2*4$$

So now rewrite these five equations and look for a pattern:

$$T(n) = T(n-1) + 2*1$$



1<sup>st</sup> step of recursion

$$T(n) = T(n-2) + 2*2$$



2<sup>nd</sup> step of recursion

$$T(n) = T(n-3) + 2*3$$



3<sup>rd</sup> step of recursion

$$T(n) = T(n-4) + 2*4$$



4<sup>th</sup> step of recursion

$$T(n) = T(n-5) + 2*5$$



5<sup>th</sup> step of recursion

Generalized recurrence relation at the kth step of the recursion:

$$T(n) = T(n-k) + 2*k$$

We want  $T(1)$ . So we let  $n-k = 1$ . Solving for  $k$ , we get  $k = n - 1$ . Now plug back in.

$$T(n) = T(n-k) + 2*k$$

$$T(n) = T(1) + 2*(n-1), \text{ and we know } T(1) = 1$$

$$T(n) = 2*(n-1) = 2n-2$$

We are done. Right side does not have any  $T(\dots)$ 's. This recurrence relation is now solved in its closed form, and it runs in  $O(n)$  time.

$$2) T(n) = 2T(n/2) + n, T(1) = 1$$

$$T(n) = 2T(n/2) + n$$

$$T(1) = 1$$

Substituting Equations

$$n \rightarrow n/2$$

$$T(n) = 2T(n/2) + n = 2[2T(n/4) + n/2] + n = 4T(n/4) + n + n$$

$$T(n) = 4T(n/4) + 2n$$

$$T(n) = 4T(n/4) + 2n = 4[2T(n/8) + n/4] + 2n = 8T(n/8) + n + 2n$$

$$T(n) = 8T(n/8) + 3n$$

$$T(n) = 8T(n/8) + 3n = 8[2T(n/16) + n/8] + 3n = 16T(n/16) + n + 3n$$

$$T(n) = 16T(n/16) + 4n$$

$$T(n) = 16T(n/16) + 4n = 16[2T(n/32) + n/16] + 4n = 32T(n/32) + n + 4n$$

$$T(n) = 32T(n/32) + 5n$$

$$T(n/2) = 2T(n/4) + n/2$$

$$T(n/4) = 2T(n/8) + n/4$$

$$T(n/8) = 2T(n/16) + n/8$$

$$T(n/16) = 2T(n/32) + n/16$$

So now rewrite these five equations and look for a pattern:

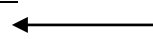
$$T(n) = 2T(n/2) + n = 2^1 T(n/2^1) + 1n$$

$$T(n) = 4T(n/4) + 2n = 2^2 T(n/2^2) + 2n$$

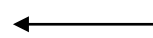
$$T(n) = 8T(n/8) + 3n = 2^3 T(n/2^3) + 3n$$

$$T(n) = 16T(n/16) + 4n = 2^4 T(n/2^4) + 4n$$

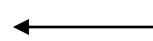
$$T(n) = 32T(n/32) + 5n = 2^5 T(n/2^5) + 5n$$



1<sup>st</sup> step of recursion



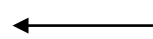
2<sup>nd</sup> step of recursion



3<sup>rd</sup> step of recursion



4<sup>th</sup> step of recursion



5<sup>th</sup> step of recursion

Generalized recurrence relation at the kth step of the recursion:

$$T(n) = 2^k T(n/2^k) + kn$$

We want  $T(1)$ . So we let  $n = 2^k$ . Solving for  $k$ , we get  $k = \log n$ . Now plug back in.

$$T(n) = 2^{\log n} T(2^k/2^k) + (\log n)n = n * T(1) + (\log n)n = n + n \log n$$

$$T(n) = n + n \log n$$

$$3) T(n) = 2T\left(\frac{n}{2}\right) + 1, T(1) = 1$$


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$$T(n) = 2T(n/2) + 1$$

$$T(1) = 1$$

$$T(n) = 2T(n/2) + 1 = 2[2T(n/4) + 1] + 1 = 4T(n/4) + 2 + 1$$

$$T(n) = 4T(n/4) + 3$$

$$T(n) = 4T(n/4) + 3 = 4[2T(n/8) + 1] + 3 = 8T(n/8) + 4 + 3$$

$$T(n) = 8T(n/8) + 7$$

$$T(n) = 8T(n/8) + 7 = 8[2T(n/16) + 1] + 7 = 16T(n/16) + 8 + 7$$

$$T(n) = 16T(n/16) + 15$$

$$T(n) = 16T(n/16) + 15 = 16[2T(n/32) + 1] + 15 = 32T(n/32) + 16 + 15$$

$$T(n) = 32T(n/32) + 31$$

So now rewrite these five equations and look for a pattern:

$T(n) = 2T(n/2) + 1$	$= 2^1T(n/2^1) + 2^1 - 1$	←	1 <sup>st</sup> step of recursion
$T(n) = 4T(n/4) + 3$	$= 2^2T(n/2^2) + 2^2 - 1$	←	2 <sup>nd</sup> step of recursion
$T(n) = 8T(n/8) + 7$	$= 2^3T(n/2^3) + 2^3 - 1$	←	3 <sup>rd</sup> step of recursion
$T(n) = 16T(n/16) + 15$	$= 2^4T(n/2^4) + 2^4 - 1$	←	4 <sup>th</sup> step of recursion
$T(n) = 32T(n/32) + 31$	$= 2^5T(n/2^5) + 2^5 - 1$	←	5 <sup>th</sup> step of recursion

In general, after k iterations, we have:

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2^k - 1$$

We're not done since we still have  $T(\dots)$ 's on the right side of the equation. We need to get down to  $T(1)$ . How?

We have  $T(n/2^k)$ , and we want  $T(1)$ . So let  $n = 2^k$ . We will then have  $T(2^k/2^k)$ , which equals  $T(1)$ . So use that substitution ( $n = 2^k$ ) throughout the entire generalized, kth recurrence relation.

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2^k - 1 = n * T\left(\frac{2^k}{2^k}\right) + n - 1 = n * T(1) + n - 1$$

$$T(n) = n * 1 + n - 1 = 2n - 1$$

So,  $T(n) = 2n - 1$  and runs in  $O(n)$  time.

$$4) T(n) = T(n - 1) + n, T(1) = 1$$

$$T(n) = T(n - 1) + n$$

$$T(1) = 1$$

$$T(n) = T(n-2) + (n-1) + n$$

$$T(n) = T(n-3) + (n-2) + (n-1) + n$$

$$T(n) = T(n-4) + (n-3) + (n-2) + (n-1) + n$$

$$T(n) = T(n-5) + (n-4) + (n-3) + (n-2) + (n-1) + n$$

Substituting Equations

$$n \rightarrow n-1$$

$$T(n-1) = T(n-2) + n-1$$

$$T(n-2) = T(n-3) + n-2$$

$$T(n-3) = T(n-4) + n-3$$

$$T(n-4) = T(n-5) + n-4$$

So now rewrite these five equations and look for a pattern:

$$T(n) = T(n - 1) + n$$

← 1<sup>st</sup> step of recursion

$$T(n) = T(n-2) + (n-1) + n$$

← 2<sup>nd</sup> step of recursion

$$T(n) = T(n-3) + (n-2) + (n-1) + n$$

← 3<sup>rd</sup> step of recursion

$$T(n) = T(n-4) + (n-3) + (n-2) + (n-1) + n$$

← 4<sup>th</sup> step of recursion

$$T(n) = T(n-5) + (n-4) + (n-3) + (n-2) + (n-1) + n$$

← 5<sup>th</sup> step of recursion

Generalized recurrence relation at the kth step of the recursion:

$$T(n) = T(n - k) + (n - k + 1) + (n - k + 2) + \cdots + (n - 1) + n$$

Yes, this looks really ugly, but watch how quickly it cleans up when we try to solve it...

We're not done since we still have  $T(\dots)$ 's on the right side of the equation. We need to get down to  $T(1)$ . How?

We have  $T(n-k)$  and we want  $T(1)$ . So, we let  $n - k = 1$ . Also, solve for  $k$ ,  $k = n - 1$ . Now, plug this in all across the board:

$$T(n) = T(1) + 2 + 3 + \cdots + (n - 1) + n$$

$$T(n) = 1 + 2 + \cdots + (n - 1) + n$$

You should hopefully recognize this sequence, as it was shown in class.

$$T(n) = \frac{n(n + 1)}{2} = O(n^2)$$

5. To see the solution of question number 5, please read the "More Recurrence Relation Examples pdf" file uploaded in webcourse.