Particle Filtering and MCMC Methods for Population Dynamics Modeling

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Overview of the Paper

Main Contributions:

- Proposal of the **Particle Filter Metropolis-Hastings (PFMCMC)** method.
- Application to analyze red kangaroo population dynamics using the general stochastic differential equation (SDE):

$$\frac{dX_t}{X_t} = \left(r + \frac{\sigma^2}{2} - bX_t\right) dt + \sigma dW_t,$$

  with three specific models derived from this equation:

  1. **Density-dependent logistic model (M1):**
     Simulated using **Euler Discretization**, as no closed form exists for $P(X_{t+1}|X_t)$.
  2. **Stochastic exponential growth (M2):**

     $$X_{t+1} = X_t e^{r+\sigma Z_t}, Z_t \sim \mathcal{N}(0, 1)$$

     - Corresponds to $b = 0$ and follows a log-normal distribution.
  3. **Random-walk model (M3):**

     $$X_{t+1} = X_t e^{\sigma Z_t}$$

     - A simplified case with $r = 0$ and $b = 0$.

- Model comparison based on **Bayes factors** and posterior probabilities.

Particle Filter Metropolis-Hastings (PFMCMC)

Motivation:

- Many real-world systems involve **state-space models** with hidden states X_t and observations Y_t.
- Simultaneous estimation of:
  1. **Hidden states** X_t using sequential methods.
  2. **Model parameters** θ using Bayesian inference.
- The challenge: Efficiently approximating the marginal likelihood $p(Y_{1:T}|\theta)$ for complex models.

Solution: PFMCMC

- Combines **Particle Filters** (for X_t estimation) and **MCMC** (for θ estimation).
- Provides a practical framework for state-space models in ecology.

Particle Filter Workflow

Particle Filter Steps:

1. **Prediction:** Simulate particles $X_t^{(i)}$ from the transition model:
$$X_t^{(i)} \sim P_\theta(X_t | X_{t-1}).$$
2. **Weighting:** Compute importance weights:
$$w_t^{(i)} \propto f_\theta(Y_t | X_t^{(i)}),$$
where $f_\theta(Y_t | X_t^{(i)})$ is the observation likelihood.
3. **Resampling:** Resample particles proportional to their weights to approximate the filtering distribution:
$$P(X_t = X_t^{(i)}) = \frac{w_t^{(i)}}{\sum_{j=1}^{N} w_t^{(j)}}.$$

Purpose:

- Provides an unbiased approximation of the likelihood $\hat{p}_\theta(Y_{1:T})$.
- Enables sequential estimation of latent states X_t.

Particle Filter Metropolis-Hastings (PFMCMC) - Definition and Steps

Definition: Combines **Particle Filters (PF)** for state estimation and **Metropolis-Hastings (MCMC)** for parameter inference.

1. **Propose new parameters θ':** Draw candidate parameters using an MCMC proposal distribution.
2. **Estimate the likelihood $p(Y_{1:T}|\theta')$:** Use a Particle Filter to approximate:

$$\hat{p}_\theta(Y_{1:T}) = \prod_{t=1}^{T} \hat{p}_\theta(Y_t|Y_{1:t-1}),$$

   where:

$$\hat{p}_\theta(Y_t|Y_{1:t-1}) \approx \frac{1}{N} \sum_{i=1}^{N} w_t^{(i)}.$$

$w_t^{(i)}$ represents the relative importance of particle $X_t^{(i)}$ given the observation Y_t.

$w_t^{(i)} \propto f_\theta(Y_t|X_t^{(i)})$ is normalized to ensure $\sum_{i=1}^{N} w_t^{(i)} = 1$

Particle Filter Metropolis-Hastings (PFMCMC) - Definition and Steps

3. **Accept/reject θ':** Decide whether to accept the proposed parameter based on the Metropolis-Hastings rule:

$$\alpha = \min\left(1, \frac{\hat{p}_{\theta'}(Y_{1:T})\pi(\theta')g_{\theta'}(\theta^{(k)})}{\hat{p}_{\theta^{(k)}}(Y_{1:T})\pi(\theta^{(k)})g_{\theta^{(k)}}(\theta')}\right),$$

   where:
   - $\hat{p}_{\theta'}(Y_{1:T})$: Approximate likelihood obtained from the PF for parameter θ'.
   - $\pi(\theta')$: Prior distribution of the proposed parameter.
   - $g_{\theta'}(\theta^{(k)})$: Proposal distribution used to propose the current param $\theta^{(k)}$ given θ'.
   - $g_{\theta^{(k)}}(\theta')$: Proposal distribution used to propose θ' from the current param $\theta^{(k)}$.

Advantages of PFMCMC:

- Efficiently handles **non-linear, non-Gaussian** models.
- Combines **sequential state estimation** (via PF) and **Bayesian parameter exploration** (via MCMC).
- Provides a framework for **model comparison** using marginal likelihoods.

Model M1 (Density-Dependent Logistic Growth)

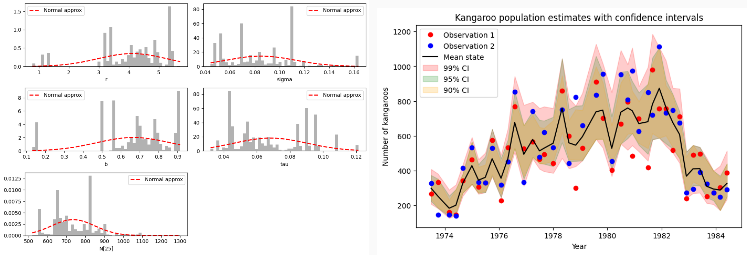

Figure 1: (Left) Trace plots for parameters. (Right) State estimates.

- Parameter estimates show variability, with some distributions appearing multimodal, suggesting potential issues with convergence or identifiability.
- State estimates follow the overall trends in the observations but exhibit oscillations that are not fully aligned with the data.
- Narrow confidence intervals suggest low variability in the estimates, but the model may lack flexibility to capture complex dynamics in the dataset.

Model M2 (Stochastic Exponential Growth)

$$X_{t+1} = X_t e^{r+\sigma Z_t}.$$

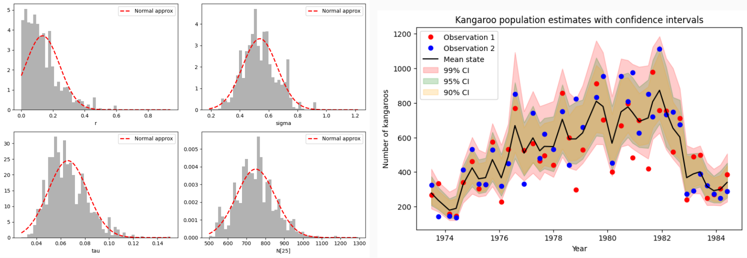

Figure 2: (Left) Trace plots for parameters. (Right) State estimates.

- Parameters converge after burn-in, but the exponential growth model leads to high uncertainty over time.
- Dynamics are overly sensitive to initial conditions and stochastic effects.

Model M3 (Random Walk): Parameter and State Dynamics

$$X_{t+1} = X_t e^{\sigma Z_t}.$$

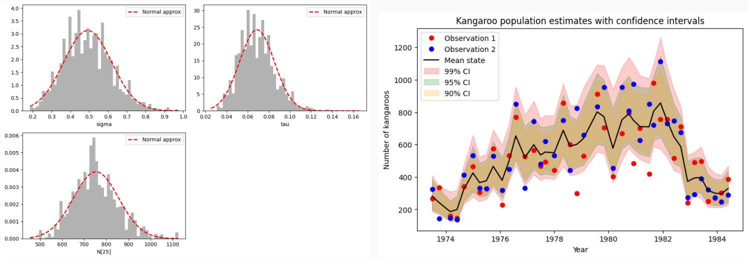

Figure 3: (Left) Trace plots for parameters. (Right) State estimates

- Parameters stabilize after burn-in, showing adequate exploration.
- The random walk model captures observed fluctuations but lacks a mechanism for population growth or decline.

Bayes Factor: Definition and Formula

Definition: The Bayes Factor compares the relative probabilities of the data under two models M_i and M_j:

$$BF_{ij} = \frac{p(Y_{1:T}|M_i)}{p(Y_{1:T}|M_j)}$$

where $p(Y_{1:T}|M)$ is the marginal likelihood of the data under model M.

Interpretation:

- $BF_{ij} > 1$: Model M_i is preferred over M_j.
- Larger values indicate stronger evidence in favor of M_i.

Bayes Factor Matrix:

Model	**M1**	**M2**	**M3**
M1	-	6.90×10^{-2}	6.54×10^{-4}
M2	1.45×10^{1}	-	9.48×10^{-3}
M3	1.53×10^{3}	1.05×10^{2}	-

Table 1: Bayes Factors comparing models M1, M2, and M3.

Observations and Conclusion

Observations:

- **M3 (Random Walk)**: Strongly preferred over both M1 and M2. Demonstrates robustness in capturing short-term fluctuations while being simple and effective.
- **M2 (Exponential Growth)**: Outperforms M1 and is moderately effective, but remains inferior to M3. It shows sensitivity to initial conditions and lacks robustness for long-term predictions.
- **M1 (Logistic Growth)**: The least favored model, consistently outperformed by both M2 and M3. It appears to struggle with alignment to the dataset.

Conclusion:

- Based on the Bayes Factors, the Random Walk is the most suitable model for this dataset, as it offers the best fit and simplicity.
- M2 is moderately effective but less reliable than M3 for this dataset.
- M1 is not recommended, as it consistently underperforms compared to the other models.