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City characteristics, land prices and volatility☆

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ABSTRACT

We develop a model that describes how city characteristics affect the volatility of real estate rents and values. The model includes agglomeration externalities, which amplify the effect of productivity shocks on population growth and rents, as well as city characteristics that constrain population growth. While growth constraints make rents more subject to productivity shocks because of the inelastic supply, they can also suppress the benefits of agglomeration, which has the effect of decreasing the sensitivity of rents to productivity shocks. Our dynamic model exhibits persistent rent growth and rent-to-value ratios that vary across cities and over time. In particular, we show that productivity shocks have a larger initial effect on rents in more constrained cities, but a greater long-term effect in less constrained cities.

1. Introduction

According to a research report by Savills, a UK real estate consultant, at the end of 2022, the value of the world's real estate reached US $380 trillion, which is about 4 times the world's GDP. Real estate is clearly the most important capital asset in the world economy, and these assets either directly or indirectly collateralize a substantial portion of the world's debt. As illustrated by the 2008 financial crisis, understanding the determinants of real estate valuation, and in particular, the volatility of real estate prices, is critical, but still incomplete.

This paper explores how the physical structure of cities interacts with the characteristics of the industries that they host to jointly determine real estate rents, values, and their volatilities. Our model explicitly considers the possibility that agglomeration externalities are larger in cities hosting some industries, e.g., financial and technology industries, as well as the possibility that growth constraints, caused by different geographic constraints or transit technologies, differ across cities. As we show, these exogenous characteristics determine the magnitude of the response of a city's rents and wage to exogenous productivity shocks. Moreover, our dynamic model, which accounts for the possibility that the agglomeration benefits of a growing city take time to materialize, generates persistent changes in rents as well as cross-city differences in rent-to-value ratios.

As a first step, we solve a static model that illustrates how exogenous city characteristics interact with the productivity of the city's firms to determine its population, wages and rents.[1] Within the context of this model, we analytically derive the elasticities of city populations and rents with respect to productivity shocks. Our focus is on how characteristics that constrain the effective supply of residential and commercial land influence these elasticities. We consider alternative transportation technologies that effectively lead to different growth constraints. The idea is that cities with commuter rail infrastructure can economically transport workers from farther away suburbs than can cities that rely exclusively on automobile-based transit.[2] Our analysis of constraints on the supply of commercial land is more straightforward. We consider one case where the supply of CBD land is fixed and another

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[1] Our model is built on a traditional monocentric urban model where the benefits of agglomeration are offset by commuting costs that increase with city populations. These models were initially developed by Alonso (1964), Mills (1967), and Muth (1969).

[2] In the appendix, we compare cities with more versus less undevelopable residential land and show that this source of residential supply constraints generates similar results.

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case where the size of the CBD is determined endogenously to satisfy the condition that commercial land rents equal the rent on residential land adjacent to the CBD.

Our analysis highlights the importance of two opposing channels. On the one hand, there is what we refer to as the scarcity channel. When the supply of land is more constrained, its supply responds less following a positive productivity shock, which implies that land rent will respond more to the shock. On the other hand, there is what we refer to as the agglomeration channel. Because land supply constraints suppress population growth, they also suppress the benefits of agglomeration externalities, which has the effect of decreasing the sensitivity of land rent to productivity shocks.

As we show, commercial rent is more sensitive to productivity shocks when residential land is less constrained and similarly, residential rent is more sensitive to productivity shocks when commercial land is less constrained. However, the sensitivity of residential (commercial) rents to productivity shocks can be either stronger or weaker when residential (commercial) land is more constrained, depending on the strengths of the offsetting channels. For example, in cities with industries that exhibit very weak agglomeration externalities, the scarcity channel dominates the agglomeration channel. As a result, residential supply constraints increase the elasticities of residential rents to productivity shocks. In contrast, in cities with industries that exhibit strong agglomeration externalities, supply constraints decrease the elasticities.[3]

Having established these relationships in our static model, we extend our model to a dynamic setting that explicitly examines the stochastic process that generates changes in both property values and rents. Specifically, we examine the volatilities and serial correlations of changes in rental rates as well as the determinants of rent-to-value ratios. An important feature of the dynamic model is that the benefits of agglomeration take time to materialize, i.e., total factor productivity in the current period depends on the city's population in the prior periods. This assumption, which captures various migration and adjustment costs as well as the time it takes to build infrastructure and develop personal connections, implies that even if the exogenous shocks to a city's productivity are independently distributed, the amplified changes in total factor productivity will be persistent.

A noteworthy feature of our dynamic model is that it allows us to separately explore the implications of productivity shocks on rents and property values. For example, we show that rents in cities that face weak growth constraints, but benefit from strong agglomeration externalities, may initially respond very little to productivity shocks. However, because the growth in rents persist, property values may exhibit a strong immediate response. This in turn implies that following positive productivity shocks, rent-to-value ratios can be quite low in these cities. Moreover, our model suggests that in some situations, real estate prices can be especially volatile in smaller and less constrained cities, even when city fundamentals, like population, wage and rental growth, are relatively stable.

As we mentioned at the outset, our model builds on an urban economics literature that explores the link between agglomeration externalities and the internal structure of cities.[4] In addition to including dynamics, we contribute to this literature by offering a richer framework for exploring how growth constraints and agglomeration externalities affect the volatilities and serial correlations of real estate values and rents as well as rent-to-value ratios.

We also build on the literature that examines how supply constraints affect the sensitivity of housing prices to productivity shocks. In an urban model with no agglomeration externalities, Glaeser et al. (2006) shows that a positive shock to productivity is likely to result in a small increase in population and a large increase in house prices in a supply constrained city. Similarly, Chatterjee and Eyigungor (2017) considers the effect of constraining the physical size of a city and Saiz (2010) studies the effect of geographical and zoning constraints. We believe, however, that we are the first to consider both commercial and residential constraints and study their effects on real estate volatilities and valuation within the context of a dynamic model.

A more recent literature explores how housing supply constraints contribute to the increase in the cross-city dispersion in housing prices. Nieuwerburgh and Weill (2010) develops a dynamic general equilibrium model that illustrates how housing supply constraints can amplify relatively small differences in productivity and create large differences in house prices, and Gyourko et al. (2013) provides a model that describes how supply constraints increase differences in housing prices across cities that attract different tastes for amenities, i.e., certain cities will have amenities that cater to a wealthier clientele who are willing to pay higher housing prices. Using a model featuring the tradeoff between agglomeration benefits and congestion costs, Duranton and Puga (2023) shows a positive correlation between planning regulations and city house prices both theoretically and empirically, and finds substantial gains that arise from a relaxation of growth constraints. We contribute to this strand of the literature by showing that, in addition to causing cross-city differences in housing prices, growth constraints lead to cross-city differences in housing price volatilities and rent-to-value ratios.

Our model also complements another strand of literature, e.g., Davidoff (2013), Gao et al. (2020) and Nathanson and Zwick (2018), that seeks to explain why relatively unconstrained cities, like Las Vegas, experienced large price run ups in the early 2000s. More recently, Chodorow-Reich et al. (2023) shows that the housing boom-bust cycle can be greatly amplified by over-optimism in the boom and fire sale dynamics in the bust, and that high price-to-rent ratios can be generated by economic fundamentals such as income and amenity shocks. In contrast to these models, which rely on behavioral biases, we show that large price run ups can be generated in relatively unconstrained cities in a fully rational model.[5] In our model, productivity shocks are amplified the most in relatively unconstrained cities that host, or hope to host, industries that exhibit strong agglomeration externalities. In such cities, we may observe large increases in prices, even though rents and wages initially respond only modestly. Our numerical model, however, cannot generate the near doubling of prices in cities like Las Vegas, suggesting that our analysis should be viewed as a complement to the above mentioned behavioral models and some rational models such as Favilukis et al. (2017), which considers changes in discount rates.

Finally, we contribute to the literature that incorporates uncertainty and dynamics into spatial models. An early example is Capozza and Helsley (1990), which develops a simple model in which the stochastic labor income of urban residents generates land rent and land price dynamics. Our model also builds on contributions by Rappaport (2004), which provides a non-stochastic growth model where capital adjustment and labor migration costs generate slow responses to productivity

[3] The urban economics literature focuses on the case with weak agglomeration externalities, see Glaeser et al. (2006) and Hilber and Vermeulen (2016) and Glaeser and Gottlieb (2009). The case with strong agglomeration externalities is also considered in Chatterjee and Eyigungor (2017) which derives similar results as ours regarding residential land supply constraints.

[4] See, for example, Lucas and Rossi-Hansberg (2002) and Ahlfeldt et al. (2015). We also draw on the spatial equilibrium literature on system of cities, e.g., Henderson (1974), Rosen (1979), Roback (1982) and Fujita and Ogawa (1982). This literature is part of the broader literature of spatial quantitative economics that provides insights into the spatial distribution of economic activities, as reviewed by Redding and Rossi-Hansberg (2017).

[5] Glaeser and Nathanson (2017) also considers behavioral biases in a dynamic model of the housing market that creates persistent house price changes. Our model generates serially correlated land rent growth rates by assuming that agglomeration externalities take time to materialize, even though we assume that individuals are rational.

shocks, and on Davis et al. (2021a), which similarly generates persistent changes in growth rates, wage rates and rents from independently generated productivity shocks.[6] While it is natural to apply this setting to explore patterns of price changes, we believe that we are the first to consider how city characteristics influence volatilities, serial correlations and rent-to-value ratios.

The rest of the paper is organized as follows: Section 2 presents a number of statistics on rent volatilities, rent serial correlations and rent-to-value ratios to motivate our research. Section 3 introduces the model and examines the elasticities of wage, population and land rents with respect to changes in exogenous shocks to productivity, and how the elasticities depend on the city characteristics. Section 4 introduces the dynamic model and shows the numerical results regarding the volatility and persistence of rent changes, and the rent-to-value ratios. Section 5 concludes the paper and provides a discussion of potential future research.

2. Motivating facts

As we mention at the outset, the goal of this paper is to provide a model that helps us understand the volatility and the persistence of changes in rental rates. To motivate this analysis, we start by reporting relevant summary statistics on data from 54 US office and apartment markets provided by *Real Capital Analytics* for the period between 2001 to 2020. To calculate the statistics, we remove year effects from the data by regressing the rent growth and the rent-to-value ratio (the cap rate) on year dummies with a constant.[7] Results are reported in Table 1.

As the figures in Table 1 indicate, the yearly standard deviation of office rent growth rates ranges from a low of 2.57% (Philadelphia) to a high of 9.69% (Cleveland), with an average of 6.03%. It is noteworthy that while Manhattan exhibits among the highest volatilities, 8.47%, a number of cities with relatively lax land supply constraints, such as Phoenix and Houston, also exhibit large volatilities. The standard deviation of apartment rent growth rates ranges from a low of 3.74% (Minneapolis) to a high of 14.03% (Las Vegas) with an average of 6.66%. In addition to Las Vegas, three cities exhibit apartment rent growth rate volatilities above 10%: Orlando, Palm Beach, and Manhattan. Overall, we observe large cross-city variations in rent growth volatilities, but there are no clear patterns that can be linked to any single city characteristics.

In addition to studying volatilities of rent growth rates, we will be studying their serial correlations. Specifically, we will provide a model where rent growth is persistent. To gauge the persistence of rent growth, Table 1 also reports the serial correlations of the growth rates of office and apartment rents in the US data. The estimated serial correlations are in fact positive for most cities, suggesting that changes in rents do tend to be persistent, and the average serial correlation is 0.15 for offices and 0.19 for apartments.[8] The serial correlations also feature large cross-city variations, with a range between −0.31 (Charlotte) and 0.58 (Pittsburgh) for offices and between −0.61 (East Bay) and 0.89 (Manhattan) for apartments.

If the growth rates of rents are persistent, then we expect rent-to-value ratios to differ across cities as well as across time. Specifically, we expect rent-to-value ratios to be lower in cities that are expected to experience growing rents in the future. Table 1 also reports the average rent-to-value ratios in each market. As the numbers indicate, rent-to-value ratios tend to be larger for office buildings than for apartments.[9] They tend to be lower in large coastal cities which, arguably, attracted the most educated residents during this time period.

We are also interested in the extent to which rent-to-value ratios fluctuate over time, which we capture by the dispersion between the peak and the trough, i.e., $dispersion_i = \max\{x_{i,t}\}_{t=2001}^{2020} - \min_t\{x_{i,t}\}_{t=2001}^{2020}$, where $x_{i,t}$ denotes the rent-to-value ratio of the ith city in year t after the removal of the country-level year effects. The average dispersion is 1.68% for offices, with a minimum value of 0.84% (Minneapolis) and a maximum of 4.4% (Cleveland). The dispersion of rent-to-value ratios for apartments ranges between 0.51% (Philadelphia) and 2.93% (Long Island and Stamford), with an average of 1.26%. The dispersion for apartments in Las Vegas is 1.44% which is above the average in our sample.

Our ultimate goal is to help explain the observed cross-city variations in rent growth volatilities, the serial correlations of rent growth, and rent-to-value ratios. For the most part, we do not observe especially clear patterns in the data. While large volatilities and serial correlations are observed in large and highly constrained markets, like Manhattan, they are also observed in a number of less constrained markets, such as Las Vegas and Phoenix. We do observe one pattern in rent-to-value ratios; they tend to be lower in large coastal cities which tend to have more educated populations. One interpretation of the low rent-to-value ratios in these cities is that they reflect persistently higher expected rental growth rates in cities with industries that benefit more from agglomeration externalities. However, we do not see straightforward patterns in the time-series dispersion in rent-to-value ratios.[10]

3. The static model

We begin with a static model that provides initial insights about how city characteristics affect the elasticities of rents, wages and population with respect to the exogenous productivity. Later, when we extend our model to a dynamic setting we will see how these elasticities provide intuition about volatilities and persistence of rent growth rates.

3.1. Model setup

To derive the elasticities of rents, wages and population, we consider a monocentric city model that illustrates the tradeoff between agglomeration benefits, which increase with city size, and housing and transportation costs, which are greater for larger cities.

3.1.1. City geometry and transportation

The city consists of a commercial CBD of size S surrounded by rings of residential land indexed by i, with the ring nearest to the CBD being $i = 0$. The land area in each ring is normalized to unity.

The distance from a ring to the CBD is measured by the distance between its inner circle to the nearest boundary of the CBD. Thus, for

[6] More recently, some authors explore richer settings that include households' dynamic housing demand decisions and the developer's dynamic housing supply decisions, Han et al. (2018) analyzes how the evolution of economic fundamentals, such as land supply and income, leads to the endogenous changes of house price and rent in a transition economy. Favilukis and Nieuwerburgh (2021) studies how a change in the number of out-of-town house buyers leads to changes in house price, rent and residents' welfare.

[7] We remove the year effects in order to make the data comparable to the simulated data from our model which considers city-level productivity shocks only and ignores shocks that are common across cities.

[8] Without the removal of year effects, the average serial correlations are 0.394 and 0.385 for offices and apartments, respectively.

[9] It should be noted that the higher rent-to-value ratios for office buildings are partly due to the fact that the maintenance on office buildings tends to be somewhat higher.

[10] Bialkowski et al. (2023) studies the cross-sectional and time-series dispersion of rent-to-value ratios in office markets around the world. They find a large part of the cross-sectional and time-series differences can be explained by differences in interest rates, which we ignore in our model. In addition, they find that rent-to-value ratios are lower in financial centers and gateway markets, which is consistent with the lower rent-to-value ratios in the large coastal cities reported in Table 1.

Table 1
Statistics of 54 markets in the US.

	Office				Apartment			
	Volatility (%)	Serial Corr	*Cap Rate* (%)	*Dispersion* (%)	*Volatility* (%)	Serial Corr	*Cap Rate* (%)	*Dispersion* (%)
Atlanta	6.66	−0.04	7.61	1.05	8.25	0.29	6.60	0.99
Austin	6.13	0.28	7.08	1.52	7.22	0.06	6.16	1.03
Baltimore	5.30	0.15	8.04	1.59	5.34	0.44	6.48	1.01
Birmingham (AL)	4.28	0.47	8.14	2.91	7.14	−0.09	6.93	1.04
Boston	4.88	0.15	6.68	1.72	5.29	0.21	5.71	0.90
Broward	5.02	0.20	7.76	1.19	8.66	0.24	6.04	1.65
Charlotte	5.19	−0.31	7.47	1.65	6.70	0.25	6.32	1.33
Chicago	6.09	0.10	7.56	1.06	4.39	−0.07	6.41	1.19
Cincinnati	6.71	0.03	8.30	2.20	7.16	0.13	7.55	1.08
Cleveland	9.69	−0.14	8.19	4.40	4.54	0.09	8.01	0.94
Columbus	6.38	0.41	7.89	1.68	6.04	0.27	7.29	1.10
Dallas	5.62	0.16	7.70	1.17	6.44	−0.22	6.79	1.31
DC	6.23	0.33	6.24	1.39	4.62	0.05	5.14	2.35
DC MD burbs	4.68	0.36	7.41	1.20	5.71	−0.12	6.40	1.47
DC VA burbs	6.47	0.38	7.44	1.67	9.25	0.25	5.77	1.56
Denver	7.79	−0.29	7.70	1.62	5.39	0.45	6.24	0.99
Detroit	7.55	0.35	8.60	1.53	6.26	0.53	7.70	2.48
East Bay	5.14	0.32	6.87	1.13	5.63	−0.61	5.49	0.61
Hartford	5.00	0.24	8.05	3.76	9.70	0.28	7.22	1.06
Houston	7.53	0.22	7.75	0.89	8.45	−0.03	6.73	1.20
Indianapolis	6.01	0.25	8.36	1.11	6.46	0.48	7.13	1.25
Inland Empire	8.74	0.33	7.17	1.59	7.47	0.27	5.93	0.73
Jacksonville	5.20	−0.08	7.68	2.47	7.13	0.10	6.71	1.08
Kansas City	7.35	0.38	8.19	1.54	5.37	−0.27	7.11	0.96
Las Vegas	7.20	0.39	7.64	1.59	14.03	0.40	6.40	1.44
Long Island	4.51	0.04	7.45	1.34	5.45	−0.02	6.01	2.93
Los Angeles	5.30	−0.07	6.64	1.23	3.82	0.38	5.15	0.71
Manhattan	8.47	0.38	5.34	1.46	10.28	0.89	5.05	2.03
Memphis	6.87	0.40	7.88	3.00	8.41	−0.07	7.10	0.96
Miami/Dade Co	6.24	0.02	6.65	1.08	7.34	0.16	6.08	1.05
Minneapolis	6.04	0.20	7.87	0.84	3.74	−0.11	6.42	0.81
Nashville	6.04	0.39	7.77	1.47	4.71	0.08	6.73	1.67
No NJ	8.16	−0.14	7.50	1.15	6.02	−0.10	6.43	0.54
NYC Boroughs	5.83	0.39	6.80	3.45	7.82	0.72	5.96	1.43
Orange Co	6.06	0.16	6.97	0.95	5.12	0.23	5.28	1.04
Orlando	4.50	0.15	8.03	2.04	11.89	−0.11	6.48	1.49
Palm Beach Co	5.32	−0.11	7.42	1.28	11.72	0.06	5.84	1.91
Philadelphia	2.57	−0.08	7.75	1.32	3.91	0.26	6.51	0.51
Phoenix	7.59	0.33	7.59	0.98	9.55	0.43	6.17	1.21
Pittsburgh	7.04	0.58	7.27	3.77	8.29	0.64	6.65	1.54
Portland	4.03	−0.27	7.11	1.51	3.78	0.61	6.09	0.81
Raleigh/Durham	6.51	0.06	7.62	0.95	4.91	−0.03	6.20	0.97
Richmond/Norfolk	6.89	−0.17	8.28	1.85	5.87	0.36	6.87	0.83
Sacramento	4.20	−0.09	7.63	0.86	7.39	0.29	6.09	0.79
Salt Lake City	4.45	0.35	7.74	1.09	5.63	0.44	6.46	0.96
San Antonio	5.65	0.45	7.81	1.23	5.32	0.08	6.66	1.17
San Diego	4.35	−0.16	7.23	1.09	6.75	0.38	5.28	1.07
San Francisco	7.77	0.34	6.32	2.71	7.40	0.11	4.55	1.02
San Jose	8.01	0.38	6.65	1.35	5.76	0.35	5.01	1.76
Seattle	5.36	0.30	6.74	1.30	4.69	0.12	5.62	1.18
St Louis	3.88	−0.23	8.44	1.60	4.61	0.06	7.09	1.06
Stamford	5.13	0.03	7.07	1.70	5.47	0.31	6.22	2.93
Tampa	5.23	−0.10	7.84	1.39	6.96	0.05	6.77	1.27
Westchester	6.62	−0.15	7.37	2.83	4.26	0.25	6.72	1.53
Average	**6.03**	**0.15**	**7.49**	**1.68**	**6.66**	**0.19**	**6.33**	**1.26**
Std. Dev.	**1.40**	**0.23**	**0.62**	**0.81**	**2.18**	**0.27**	**0.71**	**0.52**

Note: Statistics based on 2001–2020 data provided by Real Capital Analytics. Volatility is defined as the standard deviation of the yearly rent growth. Serial correlation is also based on the rent growth. “Dispersion” is the peak-trough dispersion of rent-to-value ratios (cap rates).

the ith ring, the inner circle encompasses a land area of $S+i$, implying a radius of $\sqrt{S+i}/\sqrt{\pi}$, hence its distance is $x = \frac{\sqrt{S+i}-\sqrt{S}}{\sqrt{\pi}}$. Without loss of generality, we use the distance x to denote location, with $x=0$ representing the inner-most ring with a zero distance. It should be noted that x can be infinitesimally small. Thus, with a slight abuse of the notation, we treat x as a continuous variable in the mathematical deviations, but as a discrete variable when we solve the dynamic model in Section 4 numerically.

We use w to denote the wage for all workers, and wage net of transportation costs for workers living at location x is $w \times e^{-f(x,N)}$ where N is the city population.[11] The function $f(x,N)$, which we refer to as the transportation cost function, is assumed to have the following form:

[11] We specify the commuting cost as the melt of wage with distance to capture the idea that commuting time reduces work hours. Similar specification is adopted in Lucas and Rossi-Hansberg (2002) and Chatterjee and Eyigungor (2017). A more common specification in the urban literature is to model commuting cost as a reduction in income in the budget constraint, as in Duranton and Puga (2023). Relative to the more common specification, ours implies a higher commuting cost in more productive cities because we assume commuting cost increases with wage.

$$f(x,N) = \beta_0 + \beta_1 x + \beta_2 xN \quad , \tag{1}$$

where $\beta_1 > 0$ and $\beta_2 > 0$. The congestion effect, captured by β_2, increases with both the distance and the population.

3.1.2. Economic agents and agglomeration

The city is populated by a continuum of firms and a continuum of workers. Both are price takers and together they produce tradable goods. The land and capital are owned by absentee owners who collect rent from either land or capital but do not live in the city.

Each worker is endowed with one unit of labor and earns labor income. Workers living at location x take their wage and the residential rent as given and choose their consumption of land, h, and the tradable good, c to solve the following optimization problem:

$$\max_{c,h} = c^{1-\theta} h^{\theta}$$
$$s.t.\ \ c + p_r(x)h = w \times e^{-f(x,N)} \quad , \tag{2}$$

where $p_r(x)$ is the rental rate of land in location x.

A firm uses the CBD land along with capital and labor to produce the tradable good using a Cobb–Douglas production technology:

$$F(\ell, k, n) = A\ell^{\sigma} k^{\xi} n^{1-\sigma-\xi} \quad , \tag{3}$$

where ℓ, k and n are the land, capital and labor input respectively, and A is total factor productivity (TFP), which varies across cities. Firms take A, land rent p_c, the price of capital r, and wage w as given, and solve the following optimization problem:

$$\max_{\ell,k,n} F(\ell, k, n) - wn - rk - p_c\ell.$$

The agglomeration effect is modeled as production externalities that make TFP an increasing function of the city's population. Specifically, a city's TFP is given by

$$A = \tilde{A}N^{\lambda}, \tag{4}$$

where $\tilde{A}$ is the exogenous productivity of a city, and λ is the agglomeration parameter that determines how the city's TFP increases with the total number of workers in the city.

3.1.3. Bid-rent functions

We derive bid-rent functions for both commercial land and residential land. These functions facilitate the characterization of our equilibrium as well as our analysis of rent elasticities with respect to the productivity.

Commercial Bid-rent function. Because firms are competitive and freely enter and exit the city, they make zero profits, which means that commercial land owners capture all the economic benefits from production. Thus, commercial land rent equals the maximum revenue that can be generated from one unit of land after paying for labor and capital. Production per unit of land is $f(\ell) = Ak^{\xi}n^{1-\sigma-\xi}$, which implies that commercial rent is the solution of the following maximization problem:

$$p_c = \max_{n,k} Ak^{\xi}n^{1-\sigma-\xi} - wn - rk \quad ,$$

from which we obtain the commercial bid-rent function shown below.

$$p_c = \left[\frac{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}{r^{\xi}w^{1-\sigma-\xi}}\right]^{\frac{1}{\sigma}} N^{\frac{\lambda}{\sigma}} \quad , \tag{5}$$

where we have used $A = \tilde{A}N^{\lambda}$ to substitute out A. The city population is present in this bid-rent function because of the agglomeration externality which depends on city population.

Residential Bid-rent functions. Workers are perfectly mobile both within and across cities, which implies that they realize a reservation level of utility wherever they live, denoted $\underline{u}$. By substituting the worker's first-order conditions into the Cobb–Douglas utility function, we can express the worker's reservation utility as a function of rent, wage rate and transportation costs, i.e., $\underline{u} = \frac{(1-\theta)^{1-\theta}\theta^{\theta}}{p_r(x)^{\theta}} we^{-f(x,N)}$ which can be rearranged into the following:

$$\begin{aligned} p_r(x) &= \left[\frac{(1-\theta)^{1-\theta}\theta^{\theta}}{\underline{u}}\right]^{1/\theta} \times \left[we^{-f(x,N)}\right]^{1/\theta} \\ &= B_0 \times \left[we^{-f(x,N)}\right]^{1/\theta}, \end{aligned} \tag{6}$$

where we have defined $B_0 = \left[\frac{(1-\theta)^{1-\theta}\theta^{\theta}}{\underline{u}}\right]^{1/\theta}$ for simplicity of notation.

At the city's edge, the residential rent $p_r(x = X)$ equals the exogenous agricultural rent $\underline{p}$, i.e.

$$\underline{p} = p_{r(x=X)} = B_0\left[we^{-f(X,N)}\right]^{1/\theta}. \tag{7}$$

3.2. The case of exogenous CBD

Our main results depend on how we determine the size of the CBD. We consider two cases: case 1, where the size is exogenous and case 2, where the CBD size is determined by the condition that residential rent near the CBD equals the commercial rent. We will begin with case 1, then proceed to case 2 in Section 3.3.

3.2.1. Aggregate labor supply and demand

Given an exogenous CBD size of S, the model has seven endogenous variables, $\{p_r, p_c, w, N, K, X, A\}$, which are determined by seven equations, as detailed in Appendix A.1.

To facilitate the characterization of equilibrium, we reduce the system of seven equations into the following two equations that describe the relation between the wage rate and population when land markets clear. The derivation is shown in Appendix A.2.

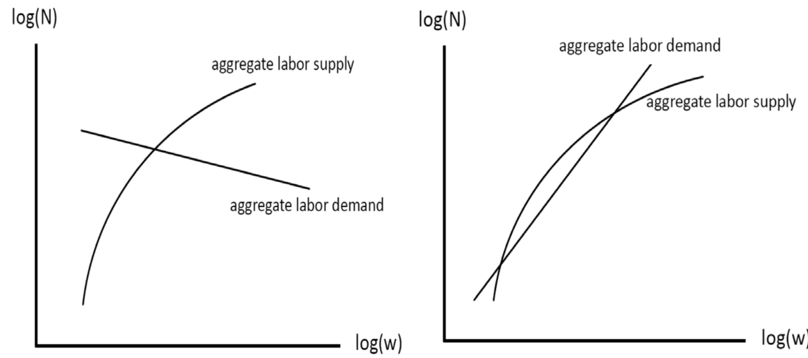
$$log(N) = \frac{1}{\lambda-\sigma} log\left(\frac{r^{\xi}}{\tilde{A}\xi^{\xi}(1-\sigma-\xi)^{1-\xi}S^{\sigma}}\right) + \frac{1-\xi}{\lambda-\sigma} log(w) \tag{8}$$

$$log(N) = log\left(\frac{B_0}{\theta}\right) + \frac{1-\theta}{\theta} log(w) + log\left(\int_0^X e^{-\frac{1-\theta}{\theta}f(x,N)}dx\right). \tag{9}$$

We refer to the above two equations as the aggregate labor demand function and the aggregate labor supply function respectively, which follows the Fujita (1989) characterization of similar "population demand" and "population supply" functions. It should be noted that Eq. (9) contains the city boundary X which itself is determined by w and N, as indicated by Eq. (7).

Eq. (8) presents the number of workers hired by firms, as a function of the wage rate and the productivity of labor. This is not a typical labor demand function because the demand for workers is a function of labor productivity, which is, in turn, determined by population. The feedback from population to productivity arises for two offsetting reasons in this model. The first is agglomeration externalities, which increases total factor productivity as the city grows. The second is that the amount of available commercial land per worker decreases as the city grows, causing the marginal product of labor to decline. Eq. (8), which essentially combines three equations, has a slope of $\frac{1-\xi}{\lambda-\sigma}$, which is positive when the agglomeration effect is stronger than the commercial land effect,(i.e. $\lambda > \sigma$). When this condition holds, firms choose to hire more workers when wages are higher.

Eq. (9) presents the number of workers that can live in the city and receive their reservation utility, as a function of the wage and rental rates. In this sense, it describes the amount of labor supplied



(a) $\sigma \geq \lambda$ (b) $\sigma < \lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$

Fig. 1. Equilibrium. Note: The number of equilibrium (equilibria) is determined by the slopes of aggregate labor supply curve and aggregate labor demand curve.

in a city when the residential land markets clear.[12] As we show in Appendix A.2.2, the slope of the labor supply curve is concave with the following expression:

$$\frac{dlog(N)}{dlog(w)} = \frac{1}{F}, \tag{10}$$

where F captures the degree of residential land constraint, and it has the following expression:

$$F = \frac{\theta}{1-\theta}\left(1 - e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)X}\right)\left(\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N}\right) \tag{11}$$

$$\approx (\beta_1 + 2\beta_2 N)X$$

$$= \frac{\partial f(X,N)}{\partial X}X + \frac{\partial f(X,N)}{\partial N}N. \tag{12}$$

To understand F intuitively, note that for a worker that lives on the city boundary, the change in the wage rate must compensate for the increase in the cost of traveling to the CBD. This increased cost accounts for the increase in travel distance as well as the increase in the cost of congestion, as captured by the two terms in Eq. (12). Thus, we refer to F as the effective residential land supply constraint, which is an endogenous variable that constrains a city's growth.

3.2.2. Characterization of equilibrium

Because the aggregate labor supply curve is concave in our model, multiple equilibria can exist. Fig. 1, which ignores the trivial case where the population and wage are both zeros, illustrates two possibilities. A formal definition of equilibria and a proof of the number of equilibria are given in Appendix A.3.

Panel (a) illustrates the case where $\lambda \leq \sigma$; i.e. the agglomeration effect is relatively weak. In this case there is a unique equilibrium.

Panel (b) shows the case where $\sigma < \lambda < \sigma+(1-\xi)\frac{2\theta}{1-\theta}$. Here the model has two equilibria which we refer to as the small-city equilibrium and the large-city equilibrium, respectively. In this case, the agglomeration parameter should satisfies the following regularity condition which is essentially the "no-black-hole condition" in Fujita et al. (1999):

$$\lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta} \quad , \tag{13}$$

which rules out the unbounded growth of a city arising from the agglomeration externalities that are so high that a sufficiently large city will generate a level of utility for workers that is greater than the reservation utility. Appendix A.4 shows why this non-black-whole condition prevents a city from growing without a bound.

We will ignore the small-city equilibrium in panel (b) because it is not stable — if a small-city finds itself in such an equilibrium, a small increase in population will increase TFP, which causes firms to hire more workers and pay higher wages, which leads to even greater TFP, generating further hiring etc. This feedback loop continues, until the city reaches the large-city equilibrium.

3.2.3. Comparative statics

This section examines elasticities that capture how endogenous variables respond to an exogenous productivity shock. We use $\zeta_w = \frac{dw/w}{d\tilde{A}/\tilde{A}}$, $\zeta_N = \frac{dN/N}{d\tilde{A}/\tilde{A}}$, $\zeta_{p_c} = \frac{dp_c/p_c}{d\tilde{A}/\tilde{A}}$ and $\zeta_{p_r(x)} = \frac{dp_r(x)/p_r(x)}{d\tilde{A}/\tilde{A}}$ to denote the productivity elasticities of the wage rate, population, the commercial land rent, and the residential land rent in location x respectively.

Wage and population elasticities. From Eq. (8), the aggregate labor demand equation, we obtain

$$\zeta_N = -\frac{1}{\lambda-\sigma} + \frac{1-\xi}{\lambda-\sigma}\zeta_w. \tag{14}$$

We obtain the following relationship between ζ_N and ζ_w from Eq. (10).

$$\frac{\zeta_w}{\zeta_N} = F. \tag{15}$$

As the equation indicates, when F is high, which implies that the supply of residential land is more constrained, a productivity shock is translated more into the wage increase than the population increase.

Combining equation (14) with Eq. (15), we obtain the following expression for the elasticity of population:

$$\zeta_N = \frac{1}{-\lambda+\sigma+(1-\xi)F} \tag{16}$$

where $\zeta_N > 0$.[13] The equation indicates that population elasticity increases with the agglomeration parameter λ but decreases with the growth constraint F. Furthermore, population elasticity increases with ξ, the capital use intensity in the production function, but decreases with σ, which captures the importance of land in production.

Land rent elasticities. We differentiate the commercial bid-rent equation with respect to $\tilde{A}$, then use Eqs. (14)–(15) to obtain the following expression for the elasticity of commercial land rent:

$$\zeta_{p_c} = \frac{1+F}{-\lambda+\sigma+(1-\xi)F} \quad . \tag{17}$$

[12] It is worth noting that when the congestion effect is set to zero (i.e. $\beta_2 = 0$) and the city boundary X is exogenous, the aggregate labor supply curve in our model represents a linear relationship between the wage and population. In this case our model is a special case of Lucas and Rossi-Hansberg (2002) with a unique equilibrium.

[13] Since the slope of the aggregate labor supply function is $1/F$ and the slope of the aggregate labor demand function is $\frac{1-\xi}{\lambda-\sigma}$, it is straightforward to show that $-\lambda+\sigma+(1-\xi)F > 0$ holds true in both the unique equilibrium and the large-city equilibrium.

It is straightforward to show that ζ_{p_c} decreases with F. Thus, when residential land supply is less constrained, a shock to productivity causes commercial rents to increase more. This is because the population increases more, thus the productivity is amplified more.

Turning to the elasticity of residential land rent, we differentiate the residential bid-rent equation with respect to $\tilde{A}$ to obtain the expression below.[14]

$$\zeta_{p_r(x)} = \frac{1}{\theta} \times \frac{F - \beta_2 x N}{-\lambda + \sigma + (1-\xi)F} \quad . \tag{18}$$

This equation shows that the elasticity is decreasing in θ and x. The latter result arises from our assumption that congestion creates greater costs for residents living further from the CBD. Without congestion costs, i.e., when $\beta_2 = 0$ in Eq. (18), the elasticity does not depend on the distance to the CBD. Note also that the effect of F on residential land rent elasticity can be negative as well as positive. Taking the partial derivative of $\zeta_{p_r}(x)$ with respect to F, we find that land rent elasticity decreases with F if and only if $\lambda - \sigma > (1-\xi)\beta_2 x N$.

The implications of Eq. (18) is summarized in the following proposition:

Proposition 1. *In the benchmark model, residential land rent elasticity is*

(i) decreasing in θ, the share of land consumption in the worker's utility function;

(ii) decreasing in x, the distance to the CBD;

(iii) decreasing in F if and only if $\lambda - \sigma > (1-\xi)\beta_2 x N$.

The last point of the Proposition describes one of our central results. Residential land rents are not necessarily more sensitive to productivity shocks when residential land supply is more constrained, i.e., when F is larger. For example, land constraints can decrease the sensitivity when the agglomeration parameter, λ, is sufficiently large. The idea is that population will grow more with a positive productivity shock when supply is less constrained, and the growth in population will give productivity a larger follow-on boost when λ is larger. Other parameters, which capture the costs associated with a higher population, offset the agglomeration effect. For example, there is less CBD land per worker when population grows, and this effect is more important when σ is larger. Similarly, the net negative congestion externality that arises from having a higher population, captured by the term $(1-\xi)\beta_2 x N$, is relevant. Finally, it should be noted that these negative externalities associated with a greater population are mitigated if the capital share in production is large, as captured by the term $1-\xi$, since a larger ξ implies fewer workers per unit of output must be transported to the CBD.

Chatterjee and Eyigungor (2017) provides a result that is similar to the last point of Proposition 1. Using a model which ignores congestion externalities and capital input in production, Chatterjee and Eyigungor (2017) compares two cities that are identical except that one of the cities has a fixed boundary. They show that when the agglomeration effect is sufficiently large, rent in the city with the fixed boundary will be less sensitive to exogenous productivity shocks. In our model, congestion and population both act as growth constraints, reflected in the endogenous parameter F. In addition, in the presence of congestion effect captured by β_2, the agglomeration must be even larger for supply constraints to amplify residential rent elasticities, which is captured by condition $\lambda - \sigma > (1-\xi)\beta_2 x N$ in the Proposition.

[14] Differentiating the residential bid-rent function (Eqs. (6)), we get $\zeta_{p_r(x)} = \frac{1}{\theta}\zeta_w - \frac{\beta_2 x N}{\theta}\zeta_N$. Then we use Eqs. (14)–(15) to substitute out ζ_w and ζ_N to get Eq. (18).

3.3. The case of flexible CBD

This subsection considers the case where the CBD can expand or contract endogenously. Specifically, we impose the equilibrium condition that the rent of commercial land equals the rent of residential land adjacent to the CBD. Relative to the case with an exogenous CBD, the model now has one more endogenous variable, which is the CBD size, and it has one more equation, i.e., the commercial rent equal the residential rent near the CBD. Appendix B.1 shows details on the existence of equilibria and the derivation of elasticities of population, wage and rent with respect to productivity. It further shows that the residential land rent elasticity has the same properties as stated in Proposition 1, except that the condition for the elasticity to be decreasing in F becomes $\lambda > \left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right)\beta_2 x N$, rather than $\lambda - \sigma > (1-\xi)\beta_2 x N$.

As we show in the following proposition, under fairly general conditions, residential land rent has a higher elasticity when the CBD is flexible. In addition, when the agglomeration externality is sufficiently strong, allowing the city to increase the supply of commercial space can actually increase the city's commercial land rent elasticity. The proof of the proposition is provided in Appendix B.2.

Proposition 2. *Relative to the exogenous CBD model, for a given population, the following is true when the size of the CBD is flexible,*

(i) residential land rent elasticity is larger if and only if $F < \frac{\theta}{1-\theta}$,

(ii) if $F < \frac{\theta}{1-\theta}$, then commercial land rent elasticity is smaller if and only if $\lambda < (1-\sigma-\xi)F$.

Part (i) of the proposition states that residential land rent elasticity is larger in the flexible CBD model if the residential supply constraint F is small and residential land is important to workers as captured by a large $\frac{\theta}{1-\theta}$. With an flexible CBD, labor output is diminished less by an increase in population because commercial land per unit of labor rises when the CBD expands. This reduced commercial constraint increases residential land rent elasticity provided that the residential land constraint is not too binding, i.e. $F < \frac{\theta}{1-\theta}$.[15]

Part (ii) of the proposition shows that a flexible CBD may reduce the commercial land rent elasticity because it reduces the scarcity of commercial land supply. However, the flexible CBD also amplifies agglomeration effect, which increases the commercial land rent elasticity relative to the case where commercial land is fixed provided that the agglomeration externality parameter λ is sufficiently large, i.e., $\lambda > (1-\sigma-\xi)F$. We note that the condition $\lambda > (1-\sigma-\xi)F$ is generally not satisfied, and the commercial land rent volatility is smaller when the CBD is flexible in our quantitative analysis.

4. Dynamic and quantitative analysis

Up to this point we have explored the link between city characteristics and the elasticities of land rents within the context of a static model. Intuitively, the volatility of rents should be closely linked to these elasticities. To show this, we extend our static model to a dynamic stochastic setting that allows us to explicitly examine these volatilities and study how they are influenced by city characteristics. In addition to volatilities, we will examine the serial correlation of rent growth rates, as well as rent-to-value ratios. As in the static model, we ignore the unstable small city equilibrium.

In addition to providing dynamics, the analysis in this section is a quantitative exercise that generates estimates of the moments of

[15] It is noteworthy that the condition $F < \frac{\theta}{1-\theta}$ generally holds true. The land share in consumption, θ, is 0.3 in our quantitative analysis which is consistent with the literature, thus $\frac{\theta}{1-\theta}$ is about 0.43 which is generally larger than F, (approximately) the travel cost as a fraction of wage for those living in the city edge.

interest. To calibrate the model, we use parameters that roughly match estimates provided in the literature. It is particularly important that the model generates cross-city rent differences that roughly match the rent differences in the US data. When these rent differences are large, a productivity shock of the same size generates vastly different volatilities across cities.

Our ability to match reality is somewhat challenged by our monocentric city assumption, which contrasts with the typical US city where employment is less concentrated. However, as examined in Rappaport (2014), the implications of the monocentric city model should continue to hold in cities that host large numbers of jobs that benefit from agglomeration in their CBDs. Since residents employed in these jobs are required to work in the CBD, they are willing to pay a premium for proximity to the CBD, and thus generate a rent gradient that closely resembles what would be generated in a purely monocentric city. Given this, we might expect to observe rent dynamics that do not substantially differ from what is generated in a more stylized monocentric city model.

4.1. Modifications of the static model

We modify the static model in two important ways. First, we account for the fact that the incremental agglomeration benefits arising from an increase in population take time to materialize. Second, we add physical structures to the model, allowing us to examine building rents, which are observable, in addition to land rents, which generally are not observable.

4.1.1. Agglomeration and productivity

Our dynamic model can be viewed as a series of static models that we link by assuming that the benefits of agglomeration take time to materialize. Specifically, we modify the agglomeration effect in Eq. (4) as follows:

$$A_t = \tilde{A}_t \left(\frac{1}{\tau} \sum_{j=1}^{\tau} N_{t-j} \right)^{\lambda}. \tag{19}$$

where the subscript t denotes years, and the current TFP of the city depends on the average of its population in the last τ years. This assumption, which captures the idea that infrastructure takes time to build and business relationships take time to develop, induces persistence in both population growth and rent increases. This persistence, in turn, generates cross-city differences in rent-to-value ratios. We will show that the degree of persistence and the rent-to-value ratios both depend on city characteristics. In the numerical exercises below, we set $\tau = 3$.

Uncertainty in the model is generated from random draws of the exogenous element of productivity, the logarithm of which is assumed to follow a random walk process:

$$\begin{aligned} \log \tilde{A}_t &= \log \tilde{A}_{t-1} + \epsilon_t, \\ \epsilon_t &\sim \mathcal{N}(0, \sigma_\epsilon^2). \end{aligned} \tag{20}$$

This specification assumes that productivity growth is generated by an i.i.d. process. This assumption is consistent with the Hornbeck and Moretti (2024) empirical evidence showing that the TFP growth rates of the manufacturing sectors of US cities from 1980 to 1990 is largely uncorrelated with their TFP growth rates from 1990 to 2000. As we will show, although TFP growth is assumed to be i.i.d., changes in rents are persistent in our setting because agglomeration externalities depend on lagged population.

4.1.2. Buildings

We provide a very simple way to introduce both commercial and residential structures into the model, and calculate building rents as a combination of land rents and structure rents. In our setting, commercial office buildings can grow vertically, so if the demand for office space increases, the amount of office space per unit of land increases. In contrast, we hold the size of the structure per unit of residential land fixed. Thus, our model is consistent with a city with a residential area that tends to sprawl and a CBD that tends to build vertically as its population grows. The commercial structures in this model are simply a reinterpretation of the capital in the firm's production function, described in our static model, and the land consumption in the static model is replaced by housing consumption.

The rent on commercial buildings, i.e. office rent, is a weighted average of the exogenously fixed structure rent (r) and the endogenously determined land rent (p_c). The weight on the physical structure is also endogenous (the weight is higher for taller buildings), and can be determined from the firm's first-order equation as follows,

$$\frac{k}{\ell + k} = \frac{\xi p_c}{\sigma r + \xi p_c}, \text{ and} \tag{21}$$

$$\frac{\ell}{\ell + k} = \frac{\sigma r}{\sigma r + \xi p_c}. \tag{22}$$

The office rent, denoted q_c, can thus be expressed as

$$q_c = \frac{\ell}{\ell + k} \times p_c + \frac{k}{\ell + k} \times r = \frac{\xi + \sigma}{\sigma r + \xi p_c} \times r \times p_c. \tag{23}$$

For residential buildings, given that we assume that only one housing unit can be put on each unit of residential land, residential rent $q_r(x)$ is simply land rent plus the structure rent which we assume is fixed, i.e.,

$$q_r(x) = \phi + p_r(x), \tag{24}$$

where ϕ is the structure rent per unit of land.

It should be noted that fixing the size of residential structure per unit of land reduces the elasticity of the supply of residential space, since it eliminates the potential for increased density created by building taller structures. We do, however, allow for changes in density that arise because households consume less space when rents increase, and we get additional supply elasticity by allowing the boundaries of the city to expand as rents increase.

Finally, it should be noted that we have simplified the model by assuming that the structures on the land do not depreciate and can be adjusted without costs. This assumption allows us to ignore a real options problem that arises when such adjustments are costly, which would considerably complicate the analysis. These adjustment costs are likely to be especially important in declining cities, as discussed in Duranton and Puga (2014), and can either slow down or speed up the decline in rents depending on the relative importance of the agglomeration channel (the population fall less quickly if housing is durable) and the scarcity channel(because supply adjusts less quickly, rents decline more).[16]

4.2. Calibration

This subsection discusses the calibration of our model. To the extent possible, we select parameters that are consistent with existing estimates reported in the literature or with some key data observations. In addition, we vary the exogenous productivity parameters to generate both large (5 million workers) and small (1 million workers) steady state city sizes. The parameter values are summarized in Table 2.

[16] In general, introducing adjustment costs will affect both the volatility and serial correlation of rents, because it slows down the extent to which the supply of space responds to an increase in the demand.

Table 2
Parameter values.

Symbol	Definition	Value
σ_ϵ	stdev. of productivity shocks	0.003
θ	housing share in preference	0.30
ξ	capital share in production	0.20
σ	land share in production	0.05
$\underline{u}$	reservation utility	0.138
$\underline{p}$	agricultural rent	0.293
ϕ	structure rent per unit of land	0.820
$\tilde{A}$	exogenous productivity (large city)	0.796
r	rental rate of capital	10%

The standard deviation of the productivity shock is set to $\sigma_\epsilon = 0.003$ which is larger than the value of 0.001 used in Davis et al. (2014), but smaller than the value of 0.013 used in Davis et al. (2021a). The share of consumer expenditures on housing is set to $\theta = 0.3$, which is consistent with the expenditure share in the Consumer Expenditure Survey. Based on estimates from Valentinyi and Herrendorf (2008), the capital share in production is set to $\xi = 0.2$, and we note that capital is interpreted as the physical structure on the land. We consider two alternative agglomeration parameters, $\lambda = 0.03$ and $\lambda = 0.08$, which can be regarded as two polar cases of the estimates in literature.[17]

We set the land share in production to $\sigma = 0.05$, which allows us to explore both cases of $\lambda > \sigma$ and $\lambda < \sigma$.[18] The implied labor share in production is $1 - \sigma - \xi = 0.75$, which is higher than the average global labor share documented in Karabarbounis and Neiman (2014). The high labor share is consistent with the CBDs hosting industries in which human capital plays a large role. We take the exogenous rental rate of capital to be $r = 10\%$, which approximates the cost of capital of US firms.

We choose the values for reservation utility $\underline{u}$, agricultural rent $\underline{p}$, and the exogenous productivity $\tilde{A}$ so that the city has a population of 5 million, a CBD of 36 km^2, a distance between the CBD edge and city boundary of 16 km as **the base case.**[19]

The structure rent per unit of residential land is set to $\phi = 0.82$. This leads to an elasticity of housing rent near the city center with respect to population of 0.07, which is the elasticity for US metropolitan areas documented in Duranton and Puga (2023). The implied residential land value is 55% of the house value on average in the base case.[20]

Our numerical model includes commuting costs that are consistent with either car-based or rail-based cities. For the car-based cities, we assume that the fixed component of the commuting cost is zero, but as illustrated in Eq. (25), the cost of commuting increases with distance. In addition, the per mile cost of commuting in the car-based city increases with population, because of road congestion. These parameters for car-based cities are chosen to match two observations in the US cities reported by the US Census Bureau: the average one way commuting time is about 35-40 min in the ten cities with the longest average commutes, and it is about 15–20 min for the ten cities with the shortest average commutes.[21]

In contrast, as shown in Eq. (26), the rail-based transportation cost function has a large fixed component but exhibits smaller increases with both distance and population.[22] These transportation parameters are chosen so that a rail-based city has the same initial population, CBD size and city boundary as a car-based city for the case where $\lambda = 0.08$.

$$f(x, N|car) = 0.000 + 0.0020x + 2.00e^{-9}xN \tag{25}$$

$$f(x, N|rail) = 0.030 + 0.0016x + 1.67e^{-9}xN. \tag{26}$$

4.3. The initial configuration

Our simulations begin with the initial configurations representing cities in their steady states prior to receiving exogenous shocks.

4.3.1. Types of cities

We consider a variety of different city types along the following dimensions: large (5 million workers) versus small (1 million workers), strong versus weak agglomeration, car-based versus rail-based transportation, and fixed versus flexible CBDs. We focus on the comparison between strong agglomeration and weak agglomeration cities, and we introduce their initial configurations respectively, conditional on the sub-types. In the simulation exercises that follow, we report only a subset of the different types of cities to illustrate how these different characteristics influence the evolution of commercial and residential rents, wages and populations.

4.3.2. Strong agglomeration cities

The first row of Table 3 describes the initial configuration of car-based cities for the case of strong agglomeration. The population, CBD size and distance from CBD edge to city boundary are initially specified to be 5 million, 36 km^2 and 16 km, respectively. Wage and rents are endogenously generated from the model. Relative to an annual wage of 4.04, the annual office rent is 7.5 per 100 m^2 (1076.4 square feet), and the annual rent is 1.83 for a house of 100 m^2. Thus, for a large city with an annual wage of \$40,400, the house rent near the CBD is \$18,300 per 100 m^2 per year. Given that the housing share in the utility function is $\theta = 0.3$, housing consumption is 66.23 m^2, which means that the worker will spend about \$12,100 on rent.[23]

[17] As surveyed in Combes and Gobillon (2015), the estimates of the agglomeration parameter in the literature are typically between 0.04–0.07. Ahlfeldt and Pietrostefani (2019) considers 347 estimates of agglomeration parameter in the literature and suggest a citation-weighted average of 0.04.

[18] Based on the US sectoral data, Valentinyi and Herrendorf (2008) estimates a land share in production that is between 0.03–0.06 for non-agricultural sectors. Davis et al. (2014) reaches a point estimate of non-land share in production of 0.974, implying a land share of 0.026.

[19] A CBD of 36 km^2 implies a CBD radius of 3.4 km. Thus, the distance from city center to city boundary is 3.4+16=19.4 km. Given a population of 5 million, the population density is 4.23 thousand per km^2, close to the density of Washington DC and is less than the density of Paris and Hong Kong. Our model better fits cities like New York, Washington D.C. and Boston, rather than newer polycentric cities like Dallas, Houston and Tampa which is the median MSA in the US with a center-periphery distance of 60 kilometers (Duranton and Handbury, 2023). Our simulated cities are quite dense partly because our model does not account for undevelopable land, as well as parks and schools which tend to be land intensive. We have simulated a version of our model where 40% of the residential land is undevelopable, and we find that the population density of such a city with a 5-million population decreases by 30%.

[20] This is consistent with land share for some large cities in the US reported in Davis et al. (2021b).

[21] https://www.census.gov/library/visualizations/interactive/travel-time.html.

[22] The idea is that rail-based commuting has a larger fixed cost in terms of time, is less subject to congestion problems, and can more cheaply transit workers from the periphery of the city to the CBD.

[23] Housing consumption is $h = \frac{\theta w}{q_r(x=0)} = \frac{0.3*4.04}{1.83} = 0.6623$ (100 square meters).

Table 3
Initial configuration of cities with strong agglomeration.

	$\tilde{A}$ (exog)	Pop (mm)	CBD (km^2)	X (km)	$Wage$ (annual)	Annual Rent (per 100 square meters) Office	House	$Land$ (commercial)	$Land$ (residential)
Large City	0.976	5	36	16.0	4.04	7.50	1.83	3.75	1.01
Small City	0.922	1	22	7.65	3.44	3.68	1.41	1.04	0.59
$Small\ City$ (Flexible CBD)	0.896	1	38.99	7.65	3.44	2.38	1.41	0.59	0.59

Note: Initial city configurations. X is the distance from CBD edge to city boundary. House rent and residential land rent are for the location next to the CBD. The unit of rents is numeraire per 100 square meters. Cities are car-based without undevelopable land.

For a small city, the population is one million, and the CBD size 22 km^2 in the case of a fixed CBD. To generate the smaller city, we assume that its exogenous level of productivity is lower, which is shown on the first column of the table.[24] To facilitate the comparison between small and large cities, we assume small cities have the same transportation technology as large cities.

The first two rows of Table 3 compare the physical size and rents of cities with 1 million and 5 million workers. Residential house rent near the CBD is 1.41 for the small city and 1.83 for the large city. The ratio of the percentage difference in rent to the percentage difference in population is 0.07, which roughly corresponds to the elasticity of rent with respect to population in a linear regression.[25] Residential land rent rises from 0.59 to 1.01, implying an elasticity of 0.18 which is below the estimate of 0.3 for French urban areas reported in Combes et al. (2019).[26] Commercial land rent is more than three times bigger in the large city, which is roughly equivalent to an elasticity of 0.65. The distance from the CBD edge to city boundary is 7.65 kilometers for the small city and 16 kilometers for the large city, roughly equivalent to an elasticity of 0.27, which is close to the elasticity of average distance to the center with respect to city population of 0.3 reported in Duranton and Puga (2020).

To understand the role played by the CBD flexibility, we consider a small city where the CBD is endogenous. This is reported in the third row of Table 3. Starting from the initial configuration of a small city with a fixed CBD, we let the CBD expand until commercial rents and residential rents adjacent to the CBD are equal. The initial configuration continues to assume that exogenous productivity is set so that the initial population is 1 million.

For rail-based cities, the initial configuration is the same as the car-based cities except that the distance from CBD edge to city boundary X is larger due to the lower variable cost and smaller congestion effect in the rail-based transportation. Specifically, for rail-based cities the initial distance is $X = 16.24$, $X = 7.70$, and $X = 7.98$ for the cases of large cities with a fixed CBD, small cities with a fixed CBD, and small cities with a flexible CBD, respectively.

4.3.3. Weak agglomeration cities

For weak agglomerations cities ($\lambda = 0.03$), the initial configurations are the same as their strong agglomeration counterparts, except that the exogenous productivity, $\tilde{A}$, is higher. Specifically, we set $\tilde{A}_w = \tilde{A}_s N^{0.08-0.03}$ where $\tilde{A}_w$ and $\tilde{A}_s$ are the exogenous productivity of weak agglomeration and strong agglomeration cities respectively, and $0.08-0.03$ is the difference between agglomeration parameters.[27] As a result, in the case of weak agglomerations, $\tilde{A}$ is set to 2.11, 1.84 and 1.79 respectively for large cities, small cities, and small cities with a flexible CBD.

[24] There are, of course, other reasons why some cities are bigger than others. For example, some cities are larger because they host industries that benefit more from agglomeration externalities or use less land relative to workers.

[25] We have used the ratio as a target in calibrating the structural rent ϕ.

[26] Combes et al. (2019) assumes cities cannot grow at the extensive margin.

[27] The endogenous productivity A needs to be the same across strong agglomeration and weak agglomeration cities in order to keep population, wage and rents the same. Given $A = \tilde{A}_s N^{0.08}$ and $A = \tilde{A}_w N^{0.03}$, we obtain $\tilde{A}_w = \tilde{A}_s N^{0.08-0.03}$.

4.4. Transition dynamics

In this subsection, we numerically generate the dynamic version of the elasticities we studied in our static model. Specifically, we subject each of the different city types to a one time positive shock to $\tilde{A}$, and explore the transition of the endogenous variables in different types of cities.

4.4.1. Impulse responses

Figs. 2 and 3 plot the transition paths for productivity, wages, population, and various rents, with the initial values of the endogenous variables normalized to unity. The shock size is three standard deviation of $\tilde{A}$.

Fig. 2 provides a comparison of the effect of the shock on large car-based and rail-based cities. For the car-based cities, it also provides a comparison of a high and a low agglomeration city. As the figure illustrates, the initial responses of the cities are quite similar, however, populations and rents grow more over the next few years in the city with high agglomeration externalities, and this effect is stronger in the rail-based city, which has lower effective supply constraints.

Fig. 3 provides a comparison of the effect of the shock on a large versus a small city, which each has strong agglomeration externalities. As the figure illustrates, the shock has only a modest effect on the large city. Population, wages and rents change very little. In contrast, the shock has a very large effect on the small city, but the effect takes many years to materialize. The most striking result revealed by this figure is that the population of the small city more than doubles within 20 years, even when the CBD size is fixed, but the population grows by only about 6% in the large city. For the case where the CBD of the small city is flexible, the population more than triples. The flexible CBD city also shows much larger increases in productivity, wages, population and residential house rent.

For small cities, office rents exhibit persistent growth and rise by 65% when the CBD is fixed, but rise by only about 40% when the CBD is flexible. In contrast, office rents increase by only about 3% in the large city. The large sensitivity of commercial rents in smaller cities to the productivity shock is due to the fact that residential land is relatively unconstrained in small cities while the CBD size in one case is fixed and when the CBD is flexible, it still grows less than the residential section.

It is also noteworthy that productivity shocks have very different short-term and long-term effects on residential rents in large and small cities. The immediate response of residential rent to a shock is much stronger in large cities where the effective land supply constraint is larger. Hence, wage and rents initially respond more to a productivity shock in a large city, while in a smaller city, wage and rents initially respond less, but population responds more. However, over time, as the benefits of agglomeration materialize, total factor productivity grows more in the faster growing smaller city, causing their steady-state rents to increase more than in the large city.[28] While our figures illustrate this point numerically, we analytically prove an analogous result within the context of an extended version of our static model in Appendix D.

[28] The above implication, that small cities respond more persistently to productivity shocks and exhibit faster population and rent growth, is consistent with Cuberes (2011) which finds that in highly urbanized economies, the fastest growing cities tend to be small (i.e., low-ranking in the city size hierarchy).

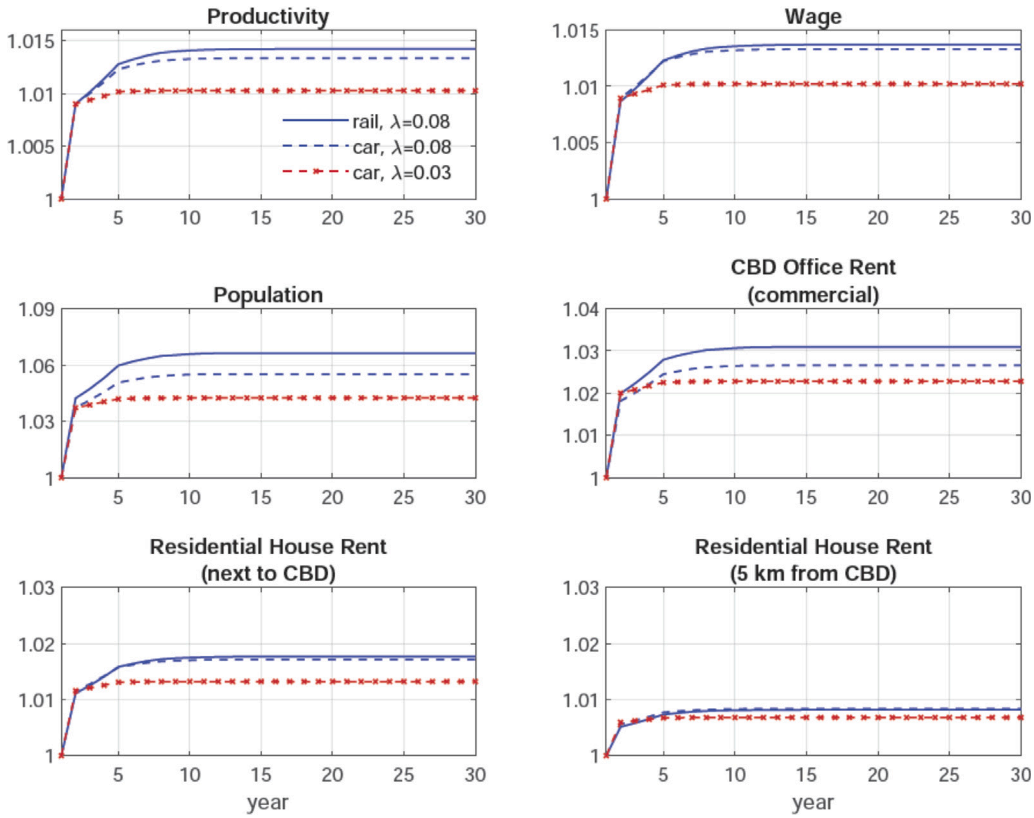


Fig. 2. Transition of large cities after a positive productivity shock
Note: Evolution of the key variables after a positive productivity shock of three standard deviations, with the initial values normalized to unity. The city level TFP depends on city population in the past three years ($\tau = 3$). The cities have an initial configuration as shown in Table 3. The car-based (rail-based) cities with dashed (solid) lines have the transportation cost function given by Eq. (25) (Eq. (26)).

Table 4
Changes (%) in Price After a Productivity Shock.

	$Strong.Agglom$ (λ=0.08)			$Weak.Agglom$ (λ=0.03)		
	ΔP_c $(office)$	$\Delta P_{r,x=0}$ $(house)$	$\Delta P_{r,x=5}$ $(house)$	ΔP_c $(office)$	$\Delta P_{r,x=0}$ $(house)$	$\Delta P_{r,x=5}$ $(house)$
Baseline	2.77	1.85	1.00	2.23	1.48	0.82
Rail	3.18	1.89	0.97	2.47	1.46	0.77
Small City	53.43	3.13	1.51	14.05	0.73	0.42
Small City (Flexible CBD)	38.27	13.96	6.43	8.28	2.87	1.39

Note: Percentage changes in office price and residential house price after a one time positive three standard deviation productivity shock. The city level TFP depends on city population in the past three years ($\tau = 3$). Land price is calculated as the discounted sum of future rents using a discount rate of 7.5%. $x = 0$ and $x = 5$ denote residential locations next to the CBD and 5 kilometers from the CBD, respectively.

4.4.2. Price changes

In contrast to rents, which drift slowly upwards after a positive productivity shock, real estate prices, which equal the present values of all future rents, react immediately to the shock. Table 4 reports the percentage changes in office prices and house prices for the various cities immediately after a one-time three standard deviation increase in productivity. These prices are the discounted sums of rents over the following 120 years using 7.5% as the discount rate.[29] As the table shows, changes in residential house prices, denoted by ΔP_r, are larger in smaller cities with strong agglomeration externalities. This observation is consistent with the experience in cities like Las Vegas in the years prior to the financial crisis, which realized large increases in real estate prices accompanied by relatively modest increases in rents and wage.

It should be emphasized that although our model can generate large increases in real estate prices, it does not come close to matching the doubling of housing prices observed in Las Vegas between 2000 and 2006. To understand why our model is incapable of generating such extreme increases in house values, it is useful to review Table 3 which illustrates that the house rents in a city of 5 million are approximately 30% higher than the house rents in a city of 1 million. What this means is that if a city of 1 million receives a shock that puts it on a path to increase its size by a factor of 5, rents will eventually rise by 30%, resulting in an immediate increase in house prices. However, since these future rents must be discounted, house prices will increase substantially less. Moreover, a fivefold increase in a city's population is extreme. We have picked a shock that eventually leads to a doubling of city size, rather than a fivefold increase, so the increases in price are modest relative to the experience in Las Vegas.

Table 4 also show that residential house prices are much more sensitive to a productivity shock when the CBD is flexible. This is because firms are less space constrained after a positive shock when the CBD is flexible, allowing them to hire more workers, thereby bidding up residential prices. In contrast, since space is less constrained in

[29] In the simulation exercise below, the discount rate of 7.5% generates an average rent-to-value ratio close to the 7% observed in the data.

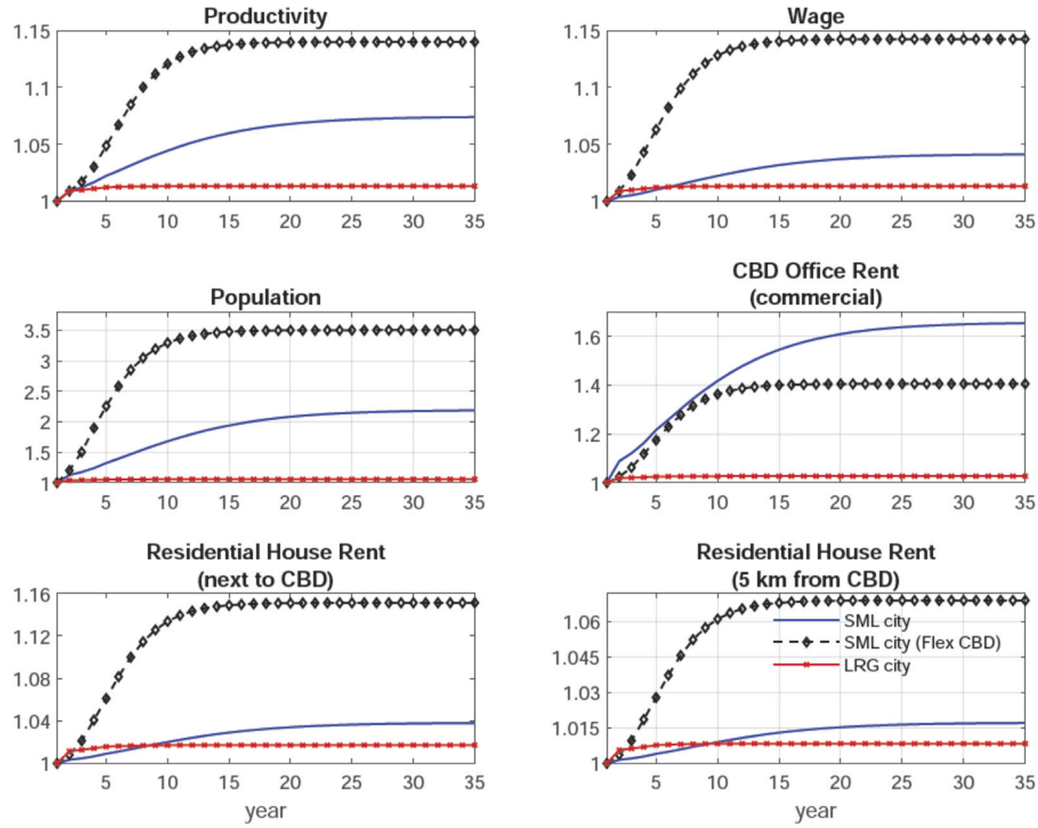


Fig. 3. Transition after a productivity shock: large vs small cities ($\lambda = 0.08$)
Note: Evolution of the key variables after a productivity shock of three standard deviations, with the initial values normalized to unity. The city level TFP depends on city population in the past three years ($\tau = 3$). For all the cities we take $\lambda = 0.08$. The cities are car-based with the initial configurations taken from Table 3.

the flexible CBD cities, office prices increase more a moderately with positive productivity shock.

4.5. Simulations

In this subsection, we use simulations to examine how city characteristics affect the volatilities and serial correlations of rent changes as well as the variation in rent-to-value ratios. For each type of city, we use the stochastic process of exogenous shocks described in Eq. (20) to create three hundred sample paths, each with 120 periods (years). From these sample paths we calculate standard deviations and serial correlations of the rent growth, as well as rent-to-value ratios.[30]

4.5.1. Volatility

We calculate the yearly volatilities of the variables of interest. For each type of city, the volatility of a variable is calculated as the standard deviations of $log\left(\frac{x_{i,t+1}}{x_{i,t}}\right)$ for $i = 1, 2, \ldots, 300$, where $x_{i,t}$ denotes the variable in the ith sample path in year t. Averaging over the three hundred sample paths, we obtain the volatilities in the this city type. Each row of Table 5 reports the volatilities of one type of city.

The simulated sample of the exogenous productivity variable, $\tilde{A}$, has a volatility of 1.24%. As the table reports, the extent to which these shocks are amplified to create a more volatile city-level TFP depends on city characteristics. For example, the volatility of TFP is 1.84% in the large car-based city and 1.98% in the corresponding rail-based city. The amplification effect is larger in small cities where residential land supply is effectively less constrained. The amplification effect is much smaller in cities with weak agglomeration externalities.

The effect of supply constraints can be seen more clearly in the volatilities of the population growth rates in different types of cities. In particular, as we just mentioned, since smaller cities are effectively less constrained, productivity shocks have a greater effect on the growth of their populations. Similarly, the growth rates of cities with rail-based transit are more volatile, and cities with flexible CBDs have more volatile growth rates.

We report volatilities for building rents in column (4)-(6) and land rents in column (7)-(9). In the case of strong agglomeration, it is evident that rent growth volatilities are larger in less constrained cities, i.e., rail-based cities or small cities where congestion is less of a problem. This is true for both commercial rents and residential rents. However, the pattern regarding residential rents is reversed in the case of weak agglomeration — residential rent growth (both house rent and land rent) is less volatile in rail-based cities and small cities. This reversal is consistent with our analysis of elasticities in the static model, as summarized in Proposition 1. The fact that commercial rent growth is always more volatile in less constrained cities is also consistent with the elasticity result in our static model.

Another salient pattern is that the volatility is larger for residential houses near the CBD than for houses close to the periphery of the city. It is also clear that volatilities of land rent are larger than volatilities of building rent, because building rent is a combination of land rent and the rent on the structure, which we assume is a constant. The table also shows the role of the flexibility of CBD. With a flexible CBD, the office rent and commercial land rent are less volatile, but the residential house rent and land rent are significantly more volatile.

[30] It should be noted that cities in this model can be subject to what we would characterize as a run after receiving a series of unfavorable productivity shocks. The negative shocks trigger a decline in population, which further reduces total factor productivity, triggering additional exodus. When this is the case, residential land rent near the CBD collapses to the agricultural rent. In this case we discontinue the simulation path.

Table 5
Volatility (std. of logarithm of rent growth, %)

$Strong.Agglom$ $(\lambda=0.08)$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	A (endog)	$Wage$	Pop	q_c (office)	$q_{r,x=0}$ (house)	$q_{r,x=5}$ (house)	p_c (land)	$p_{r,x=0}$ (land)	$p_{r,x=5}$ (land)
Baseline	1.84	1.83	7.51	3.71	3.43	1.90	9.34	6.10	5.05
Rail	1.98	1.89	9.41	4.67	3.53	1.87	11.29	6.28	5.20
Small City	3.64	2.70	30.15	16.93	4.20	2.15	32.78	8.98	7.01
Small City (Flexible CBD)	3.46	3.71	44.31	9.43	6.33	3.23	12.58	12.58	9.55
$Weak.Agglom$ $(\lambda=0.03)$									
Baseline	1.45	1.44	5.87	2.88	2.70	1.49	7.32	4.81	3.98
Rail	1.47	1.42	6.79	3.24	2.65	1.40	8.20	4.73	3.93
Small City	1.65	0.83	19.77	13.59	1.19	0.58	20.60	2.77	2.09
Small City (Flexible CBD)	1.84	2.01	45.88	5.24	3.05	1.51	6.68	6.68	5.03

Note: Volatility when the agglomeration externality depends on population with a three-year lag (τ=3). Volatility is measured as the standard deviation of the log growth. p_c and p_r denote commercial land rent and residential land rent, respectively. $x = 0$ and $x = 5$ denote residential locations next to the CBD and 5 kilometers from the CBD, respectively.

Note that the above simulations assume that the level of exogenous productivity is higher in large cities (which is why they are large). Our model, however, can generate large cities in a number of ways, e.g., they can host industries that benefit more from agglomeration or require low land use intensity. In unreported simulations, we conducted the experiment where we hold the exogenous productivity of the small city of 1 million population constant and increase the agglomeration parameter to generate a city with a population of 5 million. We find that commercial rents are substantially more volatile in the large city than those reported in Table 5. In another experiment, we take the base case city, increase the land use intensity parameter, σ, and decrease the capital use intensity parameter, until the population is lowered from 5 million to 1 million. In the resulting small city, residential rent volatilities are smaller than in the large city, but commercial rent volatility is only slightly higher.

The simulated office and residential volatilities reported above can be compared to the summary statistics reported previously. As we report in Table 1, the average yearly standard deviation is 6.03% for office rent growth in the US, which is greater than the office rent growth volatilities in the simulated data for large cities, as well as for small cities with weak agglomeration externalities and flexible CBDs. However, it is less than the simulated office volatilities for small cities with high agglomeration and fixed CBDs. The average standard deviation for apartment rent growth in our US sample is 6.66%, which is close to our simulated volatility of 6.33% for residential housing adjacent to the CBD in small cities with flexible CBDs. However, our simulated volatilities are much lower for cities with other characteristics.

There is one important disconnect between the simulated and actual volatilities of office and residential rents. The average standard deviations are roughly equal for apartments and offices in the US data, however, in the simulated data office rent growth are substantially more volatile than residential rent growth. There are a number of potential reasons. First, the model may be generating office rents that are too volatile since we assume that all office buildings are located in the CBD. In reality, we also observe office buildings in outlying areas. Perhaps, the availability of office buildings outside the CBD effectively loosens supply constraints, and thus dampens office rent volatilities. It should also be noted that loosening supply constraints on office space increases the volatility of residential rents, and may thus also explain why our model generates residential volatilities that are too low in larger cities. Second, the model may be generating apartment rents that are not sufficiently volatile because it ignores the fact that locations can have unique amenities; if demand for such amenities fluctuates, residential rents will fluctuate for reasons not captured by our model.

4.5.2. Serial correlations

Table 6 reports the serial correlations of the growth rates of office rents and residential house rents. For a given city type, we calculate the serial correlation of the rent growth, i.e. $\frac{x_{i,t}}{x_{i,t-1}}$, for $i = 1, 300$, then we average over all the 300 sample paths to obtain the serial correlation for the given city type.

The main takeaway from this table is that the serial correlation is much stronger in smaller cities with high agglomeration externalities, especially in those with a flexible CBD. Consistent with the idea that the higher serial correlation arises because of the weaker supply constraints, we also see that serial correlation is larger in rail-based cities. Finally, residential house rent is more serially correlated near the CBD ($x = 0$) than in farther-out locations ($x = 5$).

Recall that the average serial correlation of rental growth rates reported in Table 1is 0.15 for offices and 0.19 for apartments. These magnitudes are roughly consistent with the serial correlations in the simulated data for large cities with strong agglomeration externalities, but they are larger than the serial correlations in weak agglomeration large cities and smaller than the simulated serial correlations in small cities. The simulated serial correlations in small cities with strong agglomeration externalities and fixed CBDs roughly match those observed for Las Vegas.

4.5.3. Rent-to-value ratios

We examine the rent-to-value ratios (cap rates) and their dispersion generated by our model. Cap rates tend to be lower in cities where rents are expected to increase and tend to be higher when rents are expected to decline. We are particularly interested in the extent to which our model generates cap rates that fluctuate differently over time for different city types.

For a given city type, we calculate rent-to-value ratios at year 50 for each of the 300 sample paths. Specifically, for each sample path we divide the realized rent at year 50 by the property value in the same year calculated as the discounted sum of realized rents from year 51 through year 120, using 7.5% as the annual discount rate.[31] Thus for a given city type we generate 300 rent-to-value ratios, which we use to calculate the difference between the 95th percentile and the 5th percentile rent-to-value ratios. This difference captures the extent to which a city's rent-to-value ratios can change over time, i.e., it captures the extent to which cities can have different rent-to-value ratios due to differences in their histories rather than differences in their characteristics.

The first three rows of Table 7 show results for the high agglomeration cities. The dispersion is larger in less constrained cities, including rail-based cities and small cities, and it is smaller for farther-out residential houses.

[31] Thus the land rent-to-value ratio is $0.075/(1 + 0.075) \approx 0.07$ in a steady state, which is about the average rent-to-value ratio in the data provided by *Real Capital Analytics*.

Table 6
Serial correlation of rent growth.

	Strong.Agglom (λ=0.08)			*Weak.Agglom* (λ=0.03)		
	q_c (*office*)	$q_{r,x=0}$ (*house*)	$q_{r,x=5}$ (*house*)	q_c (*office*)	$q_{r,x=0}$ (*house*)	$q_{r,x=5}$ (*house*)
Baseline	0.147	0.139	0.139	0.088	0.084	0.084
Rail	0.174	0.160	0.160	0.095	0.090	0.090
Small City	0.490	0.341	0.335	0.163	0.148	0.141
Small City (Flexible CBD)	0.637	0.612	0.602	0.191	0.187	0.186

Note: Serial correlations of the growth of rents. The agglomeration externality depends on population with a three-year lag (τ=3). $x = 0$ and $x = 5$ denote residential locations next to the CBD and 5 kilometers from the CBD, respectively.

Table 7
Rent-to-value ratio (%)

Strong.Agglom (λ=0.08)	5th percentile			95th percentile			*Dispersion* ($95^{th}-5^{th}$)		
	q_c (*office*)	$q_{r,x=0}$ (*house*)	$q_{r,x=5}$ (*house*)	q_c (*office*)	$q_{r,x=0}$ (*house*)	$q_{r,x=5}$ (*house*)	q_c (*office*)	$q_{r,x=0}$ (*house*)	$q_{r,x=5}$ (*house*)
Baseline	6.78	6.68	6.84	7.54	7.43	7.25	0.76	0.75	0.42
Rail	6.76	6.67	6.84	7.68	7.44	7.25	0.91	0.76	0.41
Small City	5.46	6.45	6.71	11.21	7.25	7.25	5.75	0.80	0.54
Small City (Flexible CBD)	6.15	6.34	6.63	7.53	7.29	7.19	1.38	0.96	0.56
Weak.Agglom (λ=0.03)									
Baseline	6.82	6.75	6.88	7.41	7.34	7.21	0.59	0.59	0.33
Rail	6.80	6.76	6.89	7.46	7.33	7.19	0.66	0.58	0.31
Small City	6.18	6.85	6.94	8.31	7.18	7.11	2.13	0.33	0.17
Small City (Flexible CBD)	6.52	6.72	6.87	8.42	7.66	7.56	1.90	0.94	0.68

Note: Rent-to-value ratios in percentages. For each type of city, given the initial configuration, we simulate the city economy for 120 years. We rank cities based on their land rent-to-value ratios in year 50, and then reports the average rent-to-value ratios for the 5th and 95th percentiles. Cities that collapse are dropped in the calculation.

Similar patterns are found for commercial land in the low agglomeration cities. However, the pattern regarding residential rent is reversed — the dispersion is smaller in less constrained cities, especially in small cities. This reversal of the pattern is similar to our finding with regard to volatility and serial correlation.

Allowing for the CBD flexibility, the dispersion becomes smaller for offices but larger for residential houses, which is consistent with the observation that office rent volatility is dampened but residential house rent volatility is amplified when the CBD is flexible.

The simulated dispersions of rent-to-value ratios can be compared to their counterparts for our sample of US real estate markets. As we report in Table 1, the average dispersion is 1.68 for offices in the US cities, which is larger (smaller) than dispersions for large (small) cities in the simulated data. The average dispersion is 1.26 for apartments in the US data, which is larger than the dispersion for residential houses in simulated data for any type of cities.

5. Conclusion

Although the 2007–2009 global financial crisis had a number of causes, an important contributor was the perception that real estate is a relatively low risk investment. This misperception created an overly levered property sector as well as overly exposed financial institutions, some of which failed.

The model developed in this paper provides a framework that can help us think systematically about the determinants of real estate risk. Specifically, we explore how the design of a city and the type of firms that inhabit it affect both the levels and fluctuations in both rents and values. The key elements in our model include the magnitude of agglomeration externalities and various city characteristics that effectively constrain the supply of both residential and commercial land. As we show, these growth constraints affect the rent growth differently depending on the magnitude of agglomeration externalities.

A key feature of our dynamic model is that agglomeration externalities take time to materialize, and as a result, rents take time to respond fully to a productivity shock. Understanding these rent dynamics provides important insights about why rents and property values react differently to productivity shocks in different cities. We show, for example, that a productivity shock is likely to have a larger initial effect on residential rents in larger cities that are effectively more constrained. However, in small cities that host industries featuring strong agglomeration effects, such a shock may have a smaller initial effect on rents, but the effect will be more persistent, and can lead to long run rent increases that can ultimately exceed the long run rent increase in similar but larger cities.

In addition to providing explanations for why rents can be more volatile and serially correlated in some cities, the dynamic model also generates rent-to-value ratios that differ across cities. These differences arise because forward-looking real estate prices respond immediately to shocks that take time to be fully reflected in rents. For example, if agglomeration externalities are high, house prices in smaller and less constrained cities may respond substantially to a productivity shock even if initial rent and wage responses are modest. This feature of our model provides a partial explanation for the large observed housing price run ups in relatively small cities like Las Vegas, which experienced a housing boom in the early 2000s, but only modest increases in wages and rents.

Our model, however, cannot match the magnitude of the housing price run ups in the early 2000s. There are a number of potentially important factors that we omitted from our analysis that may explain this run up. First, as we mentioned in the introduction, there are a number of behavioral models that describe how these markets may have experienced over reaction during this time period. It is also possible that what appears to be over reaction could reflect anticipated improvements in amenities, like better restaurants and entertainment, that could be associated with a booming city. While it is challenging to include behavioral biases in our dynamic model, it is straightforward to modify our model to account for improvements in amenities. In addition, Favilukis et al. (2017) suggests that a substantial part of the increase in housing prices during this period was due to declines in the risk premium associated with housing, perhaps, due to financial innovations and the flow of foreign capital into the US housing sector. Including fluctuating discount rates to our dynamic model is clearly worthwhile, however, determining the relationship between discount rates and changes in rents will require considerable thought.

Finally, it should be noted that although we motivate our analysis with summary statistics that describe the volatilities and serial correlations of rent growth as well as the rent-to-value ratios in US office and apartment markets, an actual test of our model requires future work. Most notably, as our model illustrates, there are a number of urban characteristics that influence the dynamics of rent growth across cities, within cities, and over time. Given that these characteristics are interdependent and endogenously determined, solving the endogeneity problem is crucial to successfully identifying a causal relationship between urban characteristics, rental movements and property valuations.

CRediT authorship contribution statement

Sheridan Titman: Conceptualization, Data curation, Formal analysis, Writing – original draft, Writing – review & editing. **Guozhong Zhu:** Conceptualization, Data curation, Formal analysis, Writing – original draft, Writing – review & editing.

Appendix A. Equilibrium in the benchmark model

A.1. A system of seven equations

The baseline model with an exogenous CBD has seven endogenous variables, $\{p_r, p_c, w, N, K, X, A\}$, that are determined by seven equations, four of which were introduced in the main text:

(i) Eq. (4), which describes how city-level TFP depends on population;

(ii) Eq. (5), which is the commercial bid-rent function;

(iii) Eq. (6), which is the residential bid-rent function;

(iv) Eq. (7), which determines the city boundary.

Eqs. (27)–(29) represent the remaining three equations.

From the first-order conditions of the firm's optimization problem, we derive the following relative inputs of labor, land, and capital in production **at the city level**:

$$\frac{N}{S} = \frac{1-\sigma-\xi}{\sigma} \times \frac{p_c}{w}, \tag{27}$$

$$\frac{N}{K} = \frac{1-\sigma-\xi}{\xi} \times \frac{r}{w}. \tag{28}$$

Total number of workers that can be housed in a city is the integral of the number of workers in each location in the city which is $1/h(x)$, i.e. the inverse of land demand per worker. Therefore, total number of workers in a city as the function of wage and land rent is:

$$N = \int_{x=0}^{X} \frac{1}{h(x)} dx = \int_{x=0}^{X} \frac{p_r(x)}{\theta w e^{-f(x,N)}} dx. \tag{29}$$

A.2. Aggregate labor supply/demand functions

A.2.1. Deriving two functions

Aggregate labor supply. Aggregate labor supply in a city is defined as the total number of workers that can be housed in a city as a function of wage. Substituting out the land rent in Eq. (29) with the residential bid-rent function, we obtain the following:

$$N = \frac{B_0}{\theta} w^{\frac{1-\theta}{\theta}} \int_0^X e^{-\frac{1-\theta}{\theta} f(x,N)} dx. \tag{30}$$

Taking logarithm of the above equation leads to the aggregate labor supply equation which is Eq. (9).

Aggregate labor demand. We obtain the following expression which describes the total labor input relative to land by substituting out land rent in Eqs. (27) with the commercial bid-rent function:

$$\frac{N}{S} = \left[\frac{\tilde{A}\xi^{\xi}(1-\sigma-\xi)^{1-\xi}}{r^{\xi}}\right]^{\frac{1}{\sigma}} \frac{N^{\frac{\lambda}{\sigma}}}{w^{\frac{1-\xi}{\sigma}}}. \tag{31}$$

Rearranging terms, we re-write Eq. (31) as

$$N = \left[\frac{r^{\xi} w^{1-\xi}}{\tilde{A}\xi^{\xi}(1-\sigma-\xi)^{1-\xi} S^{\sigma}}\right]^{\frac{1}{\lambda-\sigma}}. \tag{32}$$

Taking logarithm of the above equation leads to Eq. (8), the aggregate labor demand function.

Solving for other variables. We solve for the equilibrium wage (w) and population (N) from the aggregate labor demand and supply functions. Using the market clearing N, we then solve for A, the city level TFP, from the agglomeration function $A = \tilde{A}N^{\lambda}$. The remaining endogenous variables, namely p_r, p_c, K, and X, are solved from the commercial bid-rent function (Eq. (5)), the residential bid-rent function (Eq. (6)), and from Eq. (28) and Eq. (39) respectively. ■

A.2.2. Slope of aggregate labor supply curve

Here we show that the slope of the aggregate labor supply function is $\frac{1}{F}$, with the definition of F given in Eq. (11). In addition, we show that the aggregate labor supply curve is upward sloping and concave.

Expression of the slope. Since the transportation cost function is $f(x,N) = \beta_0 + \beta_1 x + \beta_2 xN$, the transportation gradient is $\frac{\partial f(x,N)}{\partial x} = \beta_1 + \beta_2 N$. Thus, the term $\int_0^X e^{-\frac{1-\theta}{\theta} f(x,N)} dx$ in Eq. (9) can be re-written as

$$\begin{aligned}
\int_0^X e^{-\frac{1-\theta}{\theta} f(x,N)} dx &= \int_0^X \frac{1}{-\frac{1-\theta}{\theta}\frac{\partial f(x,N)}{\partial x}} de^{-\frac{1-\theta}{\theta} f(x,N)} \\
&= -\frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)} \int_0^X de^{-\frac{1-\theta}{\theta} f(x,N)} \\
&= -\frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)} \left(e^{-\frac{1-\theta}{\theta} f(X,N)} - e^{-\frac{1-\theta}{\theta} f(0,N)}\right) \\
&= \frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)} \left(e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta} f(X,N)}\right).
\end{aligned} \tag{33}$$

where we have used the condition that $f(0,N) = \beta_0$.

We have shown that $f(X,N) = log(w) - \theta\left(log\frac{p}{B_0}\right)$ based on Eq. (7). Substituting this boundary condition into Eq. (33), we obtain:

$$\int_0^X e^{-\frac{1-\theta}{\theta} f(x,N)} dx = \frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)} \times \left(e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta\left(log\frac{p}{B_0}\right)\right]}\right).$$

With the above equation, the aggregate labor supply function (Eq. (9)) can be rewritten into:

$$\begin{aligned}
log(N) &= log\left(\frac{B_0}{1-\theta}\right) + \frac{1-\theta}{\theta}log(w) - log(\beta_1+\beta_2 N) + log\left(\frac{\theta}{1-\theta}\right) \\
&\quad + log\left[e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}\right].
\end{aligned} \tag{34}$$

From Eq. (34), the derivative of $log(N)$ with respect to $log(w)$ is:

$$\frac{dlog(N)}{dlog(w)} = -\frac{\beta_2 N}{\beta_1+\beta_2 N} \times \frac{dlog(N)}{dlog(w)} + \frac{1-\theta}{\theta}\left(1 + \frac{e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}{e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}\right).$$

After rearranging terms, we have

$$\frac{dlog(N)}{dlog(w)} = \left(\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}\right)\left(\frac{1-\theta}{\theta}\right)\left(\frac{e^{-\frac{1-\theta}{\theta}\beta_0}}{e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}\right)$$

$$= \left(\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}\right)\left(\frac{1-\theta}{\theta}\right)\left(\frac{e^{-\frac{1-\theta}{\theta}f(0,N)}}{e^{-\frac{1-\theta}{\theta}f(0,N)}-e^{-\frac{1-\theta}{\theta}f(X,N)}}\right) \quad (35)$$
$$= \frac{1}{F}.$$

where the last equality comes from Eq. (11) by using $f(0,N)=\beta_0$ and $f(X,N)=\beta_0+\beta_1 X+\beta_2 XN$.

An analysis of Eq. (11) reveals that $\lim_{N\to 0} F = 0$ and

$$\lim_{N\to\infty} F = \frac{2\theta}{1-\theta} \quad (36)$$

Shape of the slope. First, we show that the aggregate labor supply curve is upward sloping. The term $e^{-\frac{1-\theta}{\theta}f(0,N)} - e^{-\frac{1-\theta}{\theta}f(X,N)}$ in Eq. (35) is positive because transportation cost increases in distance, i.e., $f(0,N) < f(X,N)$. Thus, $\frac{1}{F} = \frac{dlog(N)}{dlog(w)} > 0$ and the aggregate labor supply curve is upward sloping.

Second, we show the aggregate labor supply curve is concave, i.e. $d\left[\frac{dlog(N)}{dlog(w)}\right]/d[log(w)] < 0$, which is equivalent to showing $d\left[\frac{dlog(w)}{dlog(N)}\right]/dN = dF/dN > 0$ because $dw/dN > 0$. Using Eq. (35), we have

$$F = \frac{\theta}{1-\theta}\left(1+\frac{\beta_2 N}{\beta_1+\beta_2 N}\right)\left(1-e^{-\frac{1-\theta}{\theta}[f(X,N)-f(0,N)]}\right) \quad (37)$$

Since each term in Eq. (37) is positive, to prove $dF/dN > 0$, it suffices to prove that each term has a positive derivative. It is straightforward to show:

$$\frac{d\left(1+\frac{\beta_2 N}{\beta_1+\beta_2 N}\right)}{dN} = \frac{\beta_1\beta_2}{(\beta_1+\beta_1 N)^2} > 0$$
$$\frac{d\left(1-e^{-\frac{1-\theta}{\theta}[f(X,N)-f(0,N)]}\right)}{dN} = \beta_2 X\left(\frac{1-\theta}{\theta}\right)e^{-\frac{1-\theta}{\theta}[f(X,N)-f(0,N)]} > 0$$

Therefore,

$$dF/dN > 0, \quad (38)$$

which leads to the conclusion that $d\left[\frac{dlog(N)}{dlog(w)}\right]/d[log(w)] < 0$.

A.3. Equilibria

A.3.1. Equilibrium definition

We formally define equilibrium in this economy as follows. An equilibrium is represented by the prices w, p_c and $p_r(x)$, the quantities N, K and X, and the city level TFP, such that:

(i) the wage and population satisfies Eqs. (8)–(9);
(ii) K is determined by equating the marginal product of capital to the rental rate of capital;
(iii) X is determined by equating $p_r(X)$ to the agricultural land rent $\underline{p}$ (Eq. (39));
(iv) the city level TFP satisfies Eq. (4);
(v) commercial land rent satisfies the bid-rent function (5);
(vi) residential land rent satisfies the bid-rent function (6).

The explicit expression of the city boundary X can be derived from the boundary condition of Eq. (7) which is equivalent to $f(X,N) = log(w) + \theta log\left(\frac{B_0}{\underline{p}}\right)$. Using the explicit form of the transportation cost function (Eq. (1)), we obtain

$$X = \frac{log(w)+\theta log\left(\frac{B_0}{\underline{p}}\right)-\beta_0}{\beta_1+\beta_2 N}. \quad (39)$$

A.3.2. The number of equilibria

To understand the number of equilibria in our model, we need to show that the number of crossings of the aggregate labor demand and supply curves depends on their slopes. We have shown in Eq. (35) that the aggregate labor supply curve is upward sloping and concave. In addition:

1. when $log(N)$ and $log(w)$ are small, distance from the CBD to the boundary X is near zero, thus the slope given by Eq. (35) converges to infinity as the term $e^{-\frac{1-\theta}{\theta}f(X,N)}$ converges to $e^{-\frac{1-\theta}{\theta}f(0,N)}$, and the term $\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}$ converges to one.
2. when $log(N)$ and $log(w)$ approach infinity, the slope given by Eq. (35) converges to $\frac{1-\theta}{2\theta}$ because the inverse of the slope, F, converges to $\frac{2\theta}{1-\theta}$ as shown by Eq. (36).

The slope of aggregate labor demand curve, as given by Eq. (8), is $\frac{1-\xi}{\lambda-\sigma}$. When $\lambda < \sigma$, $\frac{1-\xi}{\lambda-\sigma} < 0$ and the curve is downward sloping. In this case the curve has a single crossing with the aggregate labor supply curve, and the equilibrium is unique.

If $\lambda > \sigma$, then the aggregate labor demand curve is upward sloping. It crosses the aggregate labor supply curve at least once because: (i) the latter has a near-infinity slope when wage and population are small; and (ii) the latter goes to negative infinity more quickly than the aggregate labor demand curve when wage tends toward zero.

If λ is larger than σ but not too large so that the slope $\frac{1-\xi}{\lambda-\sigma}$ is larger than $\frac{2\theta}{1-\theta}$ which is the slope of aggregate labor supply curve when wage and population tend toward infinity, then the aggregate labor demand and supply curves will cross twice, leading to two equilibria.

Thus the necessary and sufficient condition for the existence of two equilibria is that the aggregate labor demand curve is steeper than the aggregate labor supply curve when wage and population converge to infinity, i.e. $\frac{1-\xi}{\lambda-\sigma} > \frac{2\theta}{1-\theta}$, which is equivalent to $\sigma < \lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$.

Finally, if $\lambda \geq \sigma + (1-\xi)\frac{2\theta}{1-\theta}$, then the aggregate labor demand curve is flatter than the aggregate labor supply curve, and the city keeps expanding indefinitely. ■

A.4. Regularity condition

We prove that the model has stable large city equilibrium even if $\lambda > \sigma$, given the regularity condition $\lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$ as stated in (13).

Condition (13) is equivalent to $\frac{\lambda-\sigma}{1-\xi} < \frac{2\theta}{1-\theta}$. Given Eq. (36), we have $\frac{\lambda-\sigma}{1-\xi} < \lim_{N\to\infty} F$ which is equivalent to

$$\lim_{N\to\infty}\frac{1}{F} < \frac{1-\xi}{\lambda-\sigma} \quad ,$$

which means the aggregate labor supply curve is flatter than the aggregate labor demand curve (whose slope is represented by the right side of the above equation) when the population converges to infinity, thus the two curves must intersect, resulting in a finite equilibrium population, i.e. a stable equilibrium.

Appendix B. Flexible CBD

This appendix provides some technical details about equilibria and elasticities for cities where the CBD is flexible.

B.1. Equilibria and elasticities

The assumption of a flexible CBD does not affect the aggregate labor supply function (Eq. (9)), because the function is derived from the equilibrium of the residential land market which is not affected by the flexible CBD. As a result, the effective growth constraint F is not affected either. However, the aggregate labor demand function is different because it depend on the size of the CBD land. Land use competition ensures that $p_{r(x=0)} = p_c$, so we can equate the commercial and residential bid-rent functions (Eq. (5) and Eq. (6)) to obtain the following aggregate labor demand function for cities with a flexible CBD.

$$log(N) = \frac{1}{\lambda}log\left(\frac{r^{\xi}(B_0 e^{-\beta_0/\theta})^{\sigma}}{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}\right) + \frac{1}{\lambda}\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)log(w). \quad (40)$$

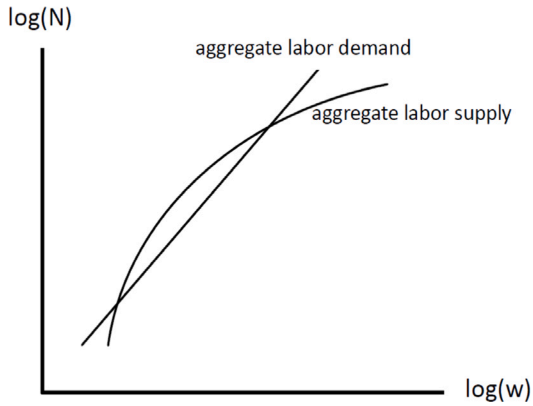


Fig. 4. Equilibria of the model with flexible a CBD.

Equilibria. Because the above aggregate labor demand function is upward sloping given $\lambda > 0$. As a result, the model always has two equilibria as shown in Fig. 4.

We impose the following "no-black-hole" condition to rule out the unbounded growth of a city.

$$\lambda < \frac{2\theta}{1-\theta}\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right), \tag{41}$$

which is obtained by setting the slope of the aggregate labor demand curve to be larger than the slope of the aggregate labor supply curve when N goes to infinity.

Elasticities. We use $\tilde{\zeta}_N$, $\tilde{\zeta}_w$, $\tilde{\zeta}_{p_r}$, $\tilde{\zeta}_{p_c}$ to denote elasticities in the model with an flexible CBD. Differentiating Eq. (40) with respect to $log(\tilde{A})$, we obtain:

$$\tilde{\zeta}_N = \frac{1}{\lambda}\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)\tilde{\zeta}_w - \frac{1}{\lambda}. \tag{42}$$

We still have $\frac{\tilde{\zeta}_w}{\tilde{\zeta}_N} = F$ as in Eq. (15) because it is derived from the aggregate labor supply equation which is independent of the CBD flexibility.

Eq. (42), combined with $\frac{\tilde{\zeta}_w}{\tilde{\zeta}_N} = F$, yields the following population elasticity.

$$\tilde{\zeta}_N = \frac{1}{\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)F-\lambda}, \tag{43}$$

which is positive given the "no-black-hole condition".

Differentiating the residential bid-rent function (Eq. (6)), we get $\tilde{\zeta}_{p_r(x)} = \frac{1}{\theta}\tilde{\zeta}_w - \frac{\beta_2 x N}{\theta}\tilde{\zeta}_N$. Then we substitute out $\tilde{\zeta}_N$ and $\tilde{\zeta}_w$ using $\frac{\tilde{\zeta}_w}{\tilde{\zeta}_N} = F$ and (43) to obtain the residential land rent elasticity:

$$\tilde{\zeta}_{p_r(x)} = \frac{1}{\theta}\times\frac{F-\beta_2 x N}{\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)F-\lambda}, \tag{44}$$

which is also positive given the "no-black-hole condition".

Since the commercial rent always equals the residential rent near the CBD, we simply set $x=0$ in Eq. (44) to obtain $\tilde{\zeta}_{p_c}$. That is

$$\tilde{\zeta}_{p_c} = \frac{1}{\theta}\times\frac{F}{\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)F-\lambda}. \tag{45}$$

B.2. Proof of Proposition 2

Now we prove proposition 2. It should be noted that, for any given population N, the residential supply constraint F is the same in the flexible CBD city as in the exogenous CBD city, because F is the inverse of the slope of the aggregate labor supply function which is independent of the CBD flexibility.

Compared with $\zeta_{p_r(x)}$ in the exogenous CBD model as given by Eq. (18), it is straightforward to see that the elasticity in Eq. (44) is larger if and only if $F < \frac{\theta}{1-\theta}$, i.e.,

$$\tilde{\zeta}_{p_r} > \zeta_{p_r} \quad iff \quad F < \frac{\theta}{1-\theta}.$$

This is point (i) of Proposition 2.

To compare $\tilde{\zeta}_{p_c}$ in Eq. (45) with ζ_{p_c} in Eq. (17), we note that, given the same population, the necessary and sufficient condition for $\tilde{\zeta}_{p_c} < \zeta_{p_c}$ is

$$\begin{aligned}
\frac{1}{\theta}\times\frac{F}{\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)F-\lambda} &< \frac{1+F}{-\lambda+\sigma+(1-\xi)F}\\
\Leftrightarrow \quad -\lambda F+(1-\xi)F^2 &< -\theta\lambda+\theta(1-\xi-\sigma-\lambda)F\\
&\quad +\sigma(1-\theta)F^2+\theta(1-\xi)F^2\\
\Leftrightarrow \quad [(1-\xi)(1-\theta)-\sigma(1-\theta)]F^2 &< -\theta\lambda+[(1-\theta)\lambda+\theta(1-\xi-\sigma)]F\\
\Leftrightarrow \quad \frac{\theta}{1-\theta}\lambda+(1-\sigma-\xi)F^2 &< \left[\lambda+\frac{\theta}{1-\theta}(1-\sigma-\xi)\right]F\\
\Leftrightarrow \quad \frac{\theta}{1-\theta}\lambda+(1-\sigma-\xi)F^2 &< \lambda F+\frac{\theta}{1-\theta}(1-\sigma-\xi)F\\
\Leftrightarrow \quad [(1-\sigma-\xi)F-\lambda]F &< [(1-\sigma-\xi)F-\lambda]\frac{\theta}{1-\theta}.
\end{aligned}$$

Thus if we impose the condition that $F < \frac{\theta}{1-\theta}$, then the above inequality holds if and only if $(1-\sigma-\xi)F-\lambda > 0$, i.e.

$$\tilde{\zeta}_{p_c} < \zeta_{p_c} \quad iff \quad \lambda < (1-\sigma-\xi)F.$$

This proves point (ii) of Proposition 2. ■

Appendix C. Undevelopable land

We use Λ to denote the fraction undevelopable residential land, we show the effects of Λ on the city configuration and land rent elasticities.

C.1. Undevelopable land and city configuration

We have the following proposition regarding how the city configuration depends on Λ.

Proposition 3. *Given the exogenous variables and parameters in the benchmark model, the proportion of undevelopable land Λ has the following effects:*

(i) equilibrium population N decreases with Λ;
(ii) equilibrium wage w decreases with Λ if and only if $\lambda > \sigma$;
(iii) commercial land rent p_c decreases with Λ;
(iv) residential land rent $p_r(x)$ decreases with Λ if and only if $\lambda - \sigma > (1-\xi)\beta_2 x N$.

Proof. In the presence of undevelopable land, the number of workers in each location in the city, which is $(1-\Lambda)/h(x)$ where $h(x)$, is still land demand per worker. Therefore, total number of workers in a city as the function of wage and land rent is:

$$N = \int_{x=0}^{X}\frac{1-\Lambda}{h(x)}dx = \int_{x=0}^{X}\frac{(1-\Lambda)p_r(x)}{\theta w e^{-f(x,N)}}dx,$$

which differs from Eq. (29) only by the term $1-\Lambda$. Consequently, the aggregate labor supply equation becomes

$$log(N) = log\left(\frac{(1-\Lambda)B_0}{\theta}\right)+\frac{1-\theta}{\theta}log(w)+log\left(\int_0^X e^{-\frac{1-\theta}{\theta}f(x,N)}dx\right). \tag{46}$$

Wage and population. Because the presence of undevelopable residential land does not affect firm's production directly, the aggregate labor demand function in the benchmark model is not changed. More undevelopable land is reflected in the downward shift of the aggregate labor supply curve, captured by the $1-\Lambda$ term in Eq. (46), which leads to a smaller equilibrium population. As one can see from panel (b) of Fig. 1, when $\lambda > \sigma$, the shift causes wage to fall if we exclude the small city equilibrium; while the shift causes wage to rise when $\lambda < \sigma$, which can be seen from panel (a) of the figure. This proves point 1 of Proposition 3.

Commercial land rent. To see how Λ affects commercial land rent p_c, we rewrite the commercial bid-rent function as follows:

$$log(p_c) = \frac{1}{\sigma} log\left(\frac{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}{r^{\xi}}\right) + \frac{\lambda}{\sigma} log(N) - \frac{1-\sigma-\xi}{\sigma} log(w).$$

The derivative of $log(p_c)$ with respect to Λ is:

$$\frac{dlog(p_c)}{d\Lambda} = \frac{\lambda}{\sigma} \times \frac{dlog(N)}{dlog(\Lambda)} - \frac{1-\sigma-\xi}{\sigma} \times \frac{dlog(w)}{dlog(\Lambda)}. \tag{47}$$

Based on the aggregate labor demand function, the following relationship exists between $\frac{dlog(N)}{dlog(\Lambda)}$ and $\frac{dlog(w)}{dlog(\Lambda)}$:

$$\frac{dlog(N)}{dlog(\Lambda)} = \frac{1-\xi}{\lambda-\sigma} \times \frac{dlog(w)}{dlog(\Lambda)}. \tag{48}$$

Substituting this relationship into Eq. (47), we obtain

$$\begin{aligned} \frac{dlog(p_c)}{d\Lambda} &= \frac{\lambda}{\sigma} \times \frac{dlog(N)}{dlog(\Lambda)} - \frac{1-\sigma-\xi}{\sigma} \times \frac{\lambda-\sigma}{1-\xi} \times \frac{dlog(N)}{dlog(\Lambda)} \\ &= \frac{\lambda+1-\sigma-\xi}{1-\xi} \times \frac{dlog(N)}{dlog(\Lambda)} \\ &< 0, \end{aligned}$$

where the inequality holds because $\frac{dlog(N)}{dlog(\Lambda)} < 0$ which is true because population falls with Λ. Thus cities with larger shares of undevelopable land always have lower commercial land rents.

Residential land rent. Based on the residential bid-rent function, the logarithm of residential rent is:

$$log(p_{r(x)}) = log(B_0) + \frac{1}{\theta}[log(w) - f(x,N)].$$

The derivative with respect to Λ is:

$$\begin{aligned} \frac{dlog(p_{r(x)})}{d\Lambda} &= \frac{1}{\theta}\left[\frac{dlog(w)}{d\Lambda} - \beta_2 xN \frac{dlog(N)}{d\Lambda}\right] \\ &= \frac{1}{\theta}\left[\frac{\lambda-\sigma}{1-\xi} \times \frac{dlog(N)}{d\Lambda} - \beta_2 xN \frac{dlog(N)}{d\Lambda}\right] \\ &= \frac{\lambda-\sigma-(1-\xi)\beta_2 xN}{\theta(1-\xi)} \times \frac{dlog(N)}{d\Lambda}. \end{aligned}$$

where we have used Eq. (48) to substitute out $\frac{dlog(w)}{d\Lambda}$ in the second equality. Since population falls with Λ so $\frac{dlog(N)}{d\Lambda} < 0$. From the above equation, we conclude that

$$\frac{dlog(p_{r(x)})}{d\Lambda} < 0 \quad iff \quad \lambda-\sigma > (1-\xi)\beta_2 xN. \quad \blacksquare$$

C.2. Effects of undevelopable land on land rent elasticities

If we hold productivity fixed, then cities with more undevelopable land will have a smaller population which itself affects land rent elasticities. To control for this population effect, we assume that the workers are more productive in the city with more undevelopable land. We have the following proposition.

Proposition 4. *Suppose the effect of a larger (smaller) Λ on city population is exactly offset by the higher (lower) productivity $\tilde{A}$, then the following describes how commercial land rent elasticity ζ_{p_c} and residential land rent elasticity ζ_{p_r} depend on the proportion of developable land Λ:*

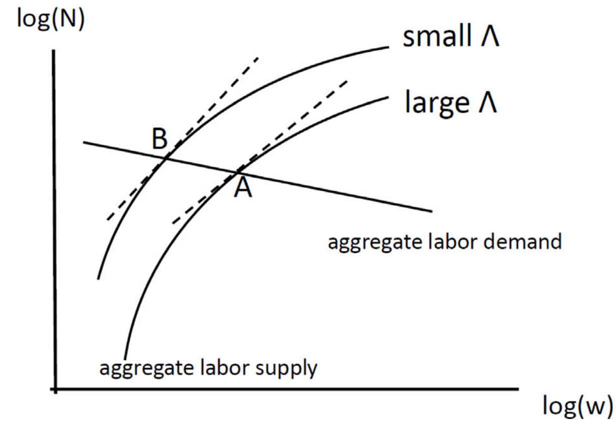


Fig. 5. Effects of more developable land
Note: More developable land (i.e. smaller Λ) shifts the aggregate labor supply curve up, and the equilibrium moves from point A to B, resulting in a flatter slope of aggregate labor supply curve at the point of equilibrium..

(i) ζ_{p_c} is decreasing in Λ.
(ii) for $x > 0$, $\zeta_{p_{r(x)}}$ is decreasing in Λ if and only if $\lambda-\sigma > (1-\xi)\beta_2 xN$.

Proof. Recall that the elasticities are derived from the aggregate labor supply and demand functions. As shown in Eq. (46), the effects of Λ is only captured by a shift of the aggregate labor supply equation. The derivative of Eq. (46) with respect to $\tilde{A}$ is not affected by Λ. Thus the elasticities of land rents are only changed by Λ through F, the effective land supply constraint.

The undevelopable land, when its effect on population is compensated by a higher productivity, leads to a higher F. This is shown by Fig. 5. The aggregate labor supply curve is shifted to the right, leading to a flatter slope at its intersection with the aggregate labor demand curve. To compensate for the declining population we need the exogenous productivity to be higher, which shifts the aggregate labor demand curve rightward, further flattening the slope of the aggregate labor supply curve at the intersection. Recall the slope equals $1/F$. A smaller slope implies that F is larger when Λ is larger.

Therefore, the effect of a larger Λ is equivalent to the effect of a larger F. As we show in the benchmark model, ζ_{p_c} always decreases with F, thus it should decreases with Λ. This proves the first point of Proposition 4. The second point of Proposition 4 is evident from Proposition 1 in the main text. $\blacksquare$

Appendix D. Long-run VS short-run effects

This appendix analyzes how the timing of rent response to a productivity shock depends on city characteristics.

Proposition 5. *Consider residential land rent in cities with different population sizes or effective land supply constraints. Given a productivity increase,*

(i) in the short run before the agglomeration takes effect, land rent increases more in cities that have larger populations or more effective supply constraints;
(ii) in the long run when the economy reach the new steady state equilibrium,
   *(a) land rent increases **less** in larger or more constrained cities if $\lambda > \sigma$,*
   *(b) land rent increases **more** in larger or more constrained cities if $\lambda < \sigma$.*

For exposition purposes, we divide the evolution of the city economy given a productivity shock into three phases. In phase one, the shock affects population, wages and rents, but the agglomeration benefits from the additional population has not yet materialized. In phase two, the agglomeration benefits provides an additional boost to productivity, and feedback between population growth and productivity growth effectively feeds on itself, leading to the persistent rise of the city as illustrated in Fig. 3. In phase three, the process converges and the city is in a new steady state equilibrium.

We compare land rent elasticities in phase one (the short-run) and phase three (the long-run), using $\zeta^0_{p_r}$ and $\zeta^0_{p_c}$ to denote phase-one rent elasticities of residential land and commercial land, respectively. Phase-three rent elasticities are ζ_{p_r} and ζ_{p_c}, which are shown in Eq. (17) and Eq. (18).

To understand land rent elasticities in phase one, we revisit the aggregate labor demand equation. Let N_0 denote city population before the productivity shock occurs. Substituting out land rent in Eqs. (27) with the commercial bid-rent function, we obtain the following aggregate labor demand function after the productivity shock but before the agglomeration benefits materialize.

$$N = \left[\frac{r^{\xi} w^{1-\xi}}{\tilde{A} N_0^{\lambda} \xi^{\xi} (1-\sigma-\xi)^{1-\xi} S^{\sigma}} \right]^{-\frac{1}{\sigma}}, \tag{49}$$

which is comparable to Eq. (8), except that the city level TFP here is $\tilde{A}N_0$ rather than $\tilde{A}N$. We take logarithm of the above equation, then differentiate it with respect to $log(\tilde{A})$ to obtain the following:

$$\zeta_N = \frac{1}{\sigma} - \frac{1-\xi}{\sigma}\zeta_w, \tag{50}$$

which is identical to Eq. (14) if λ is set to zero.

The aggregate labor supply equation captures the number of workers that can be housed in a city, which is not affected by the phase-one assumption that the feedback from increased population is NOT materialized. Thus Eqs. (15) still holds true, which, along with Eq. (50), leading to Eqs. (51)–(52) shown below.

$$\zeta^0_{p_c} = \frac{1+F}{\sigma+(1-\xi)F} \tag{51}$$

$$\zeta^0_{p_r(x)} = \frac{1}{\theta} \times \frac{F-\beta_2 x N}{\sigma+(1-\xi)F} \quad . \tag{52}$$

It is noteworthy that if the λ in phase-three elasticities is set to zero, then it is identical to phase-one elasticities.

Now we are ready to prove Proposition 5. Using Eq. (52), we derive the following derivatives for phase-one elasticities:

$$\frac{d\zeta^0_{p_r}}{dF} = \frac{1}{\theta} \times \frac{\sigma}{[\sigma+(1-\xi)F]^2} \qquad and \tag{53}$$

$$\frac{d\zeta^0_{p_r}}{dN} = \frac{1}{\theta} \times \frac{dF}{dN} \times \frac{\sigma}{[\sigma+(1-\xi)F]^2} \quad , \tag{54}$$

where $\frac{dF}{dN}$ in the right side of Eqs. (54) is positive because $1/F$ is the slope of the aggregate labor supply curve. It follows from Eqs. (53)–(54) that $\frac{d\zeta^0_{p_r}}{dF} > 0$ and $\frac{d\zeta^0_{p_r}}{dN} > 0$, which implies that in the short-run, residential land rents respond more to productivity increases in larger or more effectively constrained cities.

For the long run relationship between land rent elasticity and city size, we take the partial derivative of the phase-three elasticity:

$$\frac{d\zeta_{p_r}}{dF} = \frac{1}{\theta} \times \frac{-\lambda+\sigma}{[-\lambda+\sigma+(1-\xi)F]^2} \qquad and \tag{55}$$

$$\frac{d\zeta_{p_r}}{dN} = \frac{1}{\theta} \times \frac{dF}{dN} \times \frac{-\lambda+\sigma}{[-\lambda+\sigma+(1-\xi)F]^2} \quad . \tag{56}$$

That is, in the long run, $\frac{d\zeta_{p_r}}{dF} < 0$ and $\frac{d\zeta_{p_r}}{dN} < 0$ if $\lambda > \sigma$. If $\lambda < \sigma$, we still have $\frac{d\zeta_{p_r}}{dF} > 0$ and $\frac{d\zeta_{p_r}}{dN} > 0$ as in the short run. ■

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