动态规划

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1. 不确定情况下不能用拉格朗日法
2. McCandless(2008) 的动态规划
3. Wickens(2011) 的动态规划
4. Wash(2017) 的动态规划

最优规划

不确定性下跨期优化问题可以写成，

$$\max \quad E_t[V(x_t)] = E_t\left[\sum_{s=0}^{\infty} \beta^s U(y_{t+s})\right]$$
$$s.t. \quad x_{t+1} = f(x_t, y_t)$$

- x_t 是状态变量，y_t 是控制变量。

拉格朗日函数可以写为，

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} \{\beta^s U(y_{t+s}) + \lambda_{t+s}[f(x_{t+s}, y_{t+s}) - x_{t+s+1}]\}$$

一阶条件为，

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_{t+s}} &= E_t\left\{\lambda_{t+s}\frac{\partial f(x_{t+s}, y_{t+s})}{\partial x_{t+s}} - \lambda_{t+s-1}\right\} = 0, & s > 0 \\
\frac{\partial \mathcal{L}}{\partial y_{t+s}} &= E_t\left\{\beta^s\frac{\partial U(y_{t+s})}{y_{t+s}} + \lambda_{t+s}\frac{\partial f(x_{t+s}, y_{t+s})}{\partial y_{t+s}}\right\} = 0, & s \geq 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_{t+s}} &= E_t[f(x_{t+s}, y_{t+s}) - x_{t+s+1}] = 0, & s \geq 0
\end{aligned}
$$

求解上述方程组时，必然要求解下面的项，

$$E_t\left[\lambda_{t+s}\frac{\partial f(x_{t+s},y_{t+s})}{\partial y_{t+s}}\right]=Cov_t\left[\lambda_{t+s},\frac{\partial f(x_{t+s},y_{t+s})}{\partial y_{t+s}}\right]+E_t(\lambda_{t+s})E_t\left[\frac{\partial f(x_{t+s},y_{t+s})}{\partial y_{t+s}}\right]$$

- 如果 $Cov_t\left[\lambda_{t+s},\frac{\partial f(x_{t+s},y_{t+s})}{\partial y_{t+s}}\right]\neq 0$，该协方差不好计算。从而不能像无不确定性时求解那样消去 λ_{t+s}。
- 此时这样的跨期优化问题就要使用动态规划技术进行。

一般形式

把跨期优化写成递归形式，

$$V(x_t) = \max_{y_t} \; [F(x_t, y_t) + \beta V(x_{t+1})]$$
$$s.t. \;\; x_{t+1} = G(x_t, y_t)$$

- x_t 是状态变量，y_t 是控制变量。

用目标函数对 y_t 求导，得到一阶条件，

$$F_{y_t}(x_t, y_t) + \beta V_{x_{t+1}}(x_{t+1}) \cdot G_{y_t} = 0 \tag{1}$$

- 关键是 $V_{x_{t+1}}(x_{t+1})$ 未知。
- 根据包络定理，

$$V_{x_t}(x_t) = F_{x_t}(x_t, y_t) + \beta V_{x_{t+1}}(x_{t+1}) \cdot G_{x_t}(x_t, y_t) \tag{2}$$

  - 又回到了原处，$V_{x_{t+1}}(x_{t+1})$ 还是未知。
  - 但如果 $G_{x_t}(x_t, y_t) = 0$ 就可以消掉 $V_{x_{t+1}}(x_{t+1})$。

重要技巧

要使 $G_{x_t}(x_t, y_t) = 0$ 关键在于通过把相关变量移入目标函数，从而让预算约束不要包含 t 期的状态变量。

公式

消掉后，通过把式(2)脚标前推一期，一阶条件最终可以写成，

$$F_{y_t}(x_t, y_t) + \beta F_{x_{t+1}}(x_{t+1}, y_{t+1}) = 0 \tag{3}$$

一个例子

$$V = \max \sum_{t=0}^{T} \beta^t \ln c_t$$

$$s.t. \;\; s_{t+1} - s_t = \alpha(s_t - c_t)$$

写成递归形式,

$$\begin{aligned} V(s_0) &= \max \sum_{t=0}^{T} \beta^t \left[\ln\left(s_t - \frac{s_{t+1} - s_t}{\alpha} \right) \right] \\ &= \max \left\{ \ln\left(s_0 - \frac{s_1 - s_0}{\alpha} \right) + \beta \cdot \sum_{t=1}^{T} \beta^{t-1} \left[\ln\left(s_t - \frac{s_{t+1} - s_t}{\alpha} \right) \right] \right\} \\ &= \max \left[\ln\left(s_0 - \frac{s_1 - s_0}{\alpha} \right) + \beta \cdot V(s_1) \right] \end{aligned}$$

第三个等号是因为，

$$V(s_0) = \ln\left(s_0 - \frac{s_1 - s_0}{\alpha}\right) + \beta \ln\left(s_1 - \frac{s_2 - s_1}{\alpha}\right) + \beta^2 \ln\left(s_2 - \frac{s_3 - s_2}{\alpha}\right) + \cdots$$

$$V(s_1) = \ln\left(s_1 - \frac{s_2 - s_1}{\alpha}\right) + \beta \ln\left(s_2 - \frac{s_3 - s_2}{\alpha}\right) + \cdots$$

上述递归可以更一般的写作，

$$\begin{aligned} V(s_t) &= \max \ \left[\ln\left(s_t - \frac{s_{t+1} - s_t}{\alpha}\right) + \beta \cdot V(s_{t+1})\right] \\ &= \max \ \left[\ln c_t + \beta \cdot V(s_{t+1})\right] \end{aligned} \tag{4}$$

控制变量选 s_{t+1}

此时预算约束为，

$$s_{t+1} = x_{t+1} = G(x_t, y_t) = y_t = s_{t+1}$$

- $t+1$ 期的状态变量是 s_{t+1}
- 该状态变量是 t 期的控制变量和状态变量的函数，而 t 期的控制变量又恰好是 k_{t+1}，所以本质上，约束就是，

  $$k_{t+1} = k_{t+1}$$

  并不包含 t 期的状态变量。
- 因此 $G(x_t, y_t) = G(s_t, s_{t+1}) = s_{t+1}$，有，

  $$G_{s_t}(s_t, s_{t+1}) = 0$$

最优规划重写如下，

$$
\begin{aligned}
V(s_t) = \max \ & [\ln c_t + \beta \cdot V(s_{t+1})] \\
s.t. \ & s_{t+1} - s_t = \alpha(s_t - c_t)
\end{aligned}
$$

根据式(3)，上述最优规划，有，

$$
\begin{aligned}
\frac{\partial \ln c_t}{\partial s_{t+1}} + \beta \frac{\partial \ln c_{t+1}}{\partial s_{t+1}} =& 0 \\
\Longrightarrow \frac{1}{c_t} \cdot \left(-\frac{1}{\alpha}\right) + \beta \frac{1}{c_{t+1}} \frac{1+\alpha}{\alpha} =& 0 \quad \text{可从约束条件获得} \\
\Longrightarrow c_{t+1} =& (1+\alpha)\beta c_t
\end{aligned}
$$

控制变量选 c_t

此时，约束条件为，

$$s_{t+1} = G(x_t, y_t) = s_t + \alpha(s_t - c_t)$$

有，

$$G_{x_t} = -\alpha \neq 0$$

- 因此不能选 c_t 做控制变量。

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Wickens(2011) 的动态规划

Wickens(2011) 的思路更绝，直接目标函数对控制变量求导为 0，

$$\frac{\partial V(x_t)}{\partial y_t} = F_{y_t}(x_t, y_t) + \beta V_{y_t}(x_{t+1}) = 0$$

因为

$$V(x_{t+1}) = F(x_{t+1}, y_{t+1}) + \beta V(x_{t+2})$$

那么直接把 $V(x_{t+2})$ 看作给定的，就有，

$$\begin{aligned}\frac{\partial V(x_{t+1})}{\partial y_t} &= \frac{\partial F(x_{t+1}, y_{t+1})}{\partial y_{t+1}} \frac{\partial y_{t+1}}{\partial y_t} \\ &= \frac{\partial F(x_{t+1}, y_{t+1})}{\partial y_{t+1}} \frac{\partial y_{t+1}}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial y_t}\end{aligned}$$

- $\frac{\partial y_{t+1}}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial y_t}$ 可从约束条件直接得到。

为什么把 $V(x_{t+2})$ 看作给定?

注意到,

$$
\begin{aligned}
\frac{\partial V(x_{t+2})}{\partial y_t} &= \frac{\partial F(x_{t+2}, y_{t+2})}{\partial y_{t+2}} \frac{\partial y_{t+2}}{\partial y_t} + V_y(x_{x+3}) \\
&= \frac{\partial F(x_{t+2}, y_{t+2})}{\partial y_{t+2}} \frac{\partial y_{t+2}}{\partial x_{t+2}} \frac{\partial x_{t+2}}{\partial y_t} + V_y(x_{t+3})
\end{aligned}
$$

很显然, $\frac{\partial x_{t+2}}{\partial y_t} = 0$, 不断替换下去, 可知 $V_y(x_{t+2})$ 最终是等于 0 的。

重要公式

因此，Wickens(2011) 的一阶条件公式为，

$$F_{y_t}(x_t, y_t) + \beta \frac{\partial F(x_{t+1}, y_{t+1})}{\partial y_{t+1}} \frac{\partial y_{t+1}}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial y_t} = 0$$

重复前面的例子

$$V(s_t) = \max \quad [\ln c_t + \beta \cdot V(s_{t+1})]$$
$$s.t. \quad s_{t+1} - s_t = \alpha(s_t - c_t)$$

此时，即便把 c_t 看作控制变量也无妨，一阶条件为，

$$\begin{aligned} \frac{V(s_t)}{\partial c_t} &= \frac{\partial \ln c_t}{\partial c_t} + \beta \frac{\partial \ln c_{t+1}}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial s_{t+1}} \frac{\partial s_{t+1}}{\partial c_t} \\ &= \frac{1}{c_t} + \beta \frac{1}{c_{t+1}} \frac{1+\alpha}{\alpha}(-\alpha) = 0 \\ \Longrightarrow c_{t+1} &= (1+\alpha) c_t \beta \end{aligned}$$

Wash(2017) 的动态规划

先把约束写进动态规划，确保目标函数没有下一期的状态变量，

$$V(s_t) = \max \ln c_t + \beta V[\alpha(s_t - c_t) + s_t]$$

然后，对控制变量一阶求导，有，

$$\frac{\partial V(s_t)}{\partial c_t} = \frac{1}{c_t} - \beta V_s(s_{t+1}) \cdot \alpha = 0 \tag{5}$$

最后利用包络定理，有，

$$V_s(s_t) = \beta V_s(s_{t+1}) \cdot (\alpha + 1) \tag{6}$$

那么联立式(5)和式(6)，消掉 $V_s(\cdot)$ 即可。

注意事项

注意

因为控制变量是由状态变量决定，因此在消除 $V_s(\cdot)$ 时，要切记，只能把 c_t 时间脚标往前移，不能往后移。

式(5)意味着，

$$V_s(s_{t+1}) = \frac{1}{c_t \beta \alpha}$$
$$V_s(s_{t+2}) = \frac{1}{c_{t+1} \beta \alpha}$$

把式(6)的脚标往前推一期，有，

$$V_s(s_{t+1}) = \beta V_s(s_{t+2}) \cdot (\alpha + 1)$$

于是，有，

$$\frac{1}{c_t\beta\alpha} = \frac{\alpha+1}{\alpha c_{t+1}}$$
$$\Longrightarrow c_{t+1} = (1+\alpha)c_t\beta$$

参考文献

1. Walsh C E. Monetary Theory and Policy[M]. 4th ed. The MIT Press, 2017.
2. Wickens M. Macroeconomic Theory: A Dynamic General Equilibrium Approach[M]. 2nd ed. Princeton: Princeton University Press, 2011.
3. McCandless G. The ABCs of RBCs[M]. Harvard University Press, 2008.

感谢您的聆听！
Questions?

The Frog Prince