

PARABOLIC BIFURCATION LOCI IN THE SPACES OF RATIONAL FUNCTIONS

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ABSTRACT. We give a geometric description of the parabolic bifurcation locus in the space Rat_d of all rational functions on \mathbb{P}^1 of degree $d > 1$, generalizing the study by Morton and Vivaldi in the case of monic polynomials. The results are new even for quadratic rational functions.

1. INTRODUCTION

In studying a (topologically parametrized) family of (discrete) dynamical systems on a (common) phase space, once a stability notion for this family is fixed, one of main topics is classifying the (in)stability phenomena under a perturbation/variation of the dynamical system. We also call an instability phenomenon a bifurcation. For example, we say a family of dynamical systems is structurally stable at a dynamical system f if any small perturbation of f is conjugate to the initial f by some automorphism of the phase space. The structural stability is an open condition, but the structurally stable locus in the parameter space is not necessarily dense even if it is non-empty. An obvious obstacle to the structural stability (or a possible unstable limit of structurally stable dynamics) is (informally) a collision between two distinct periodic points (possibly belonging to a same cycle) of a dynamical system. In other (but still naive) words, by perturbing such an unstable dynamical system appropriately, some multiple cycle of it splits into more than one cycles (having the same period) or turns to a single but simple cycle (having a greater period), and we call such a (classical) bifurcation phenomenon a parabolic bifurcation.

Even for the (iterations of the monic centered complex) quadratic polynomial family

$$z \mapsto z^2 + t \quad \text{on } \mathbb{C}$$

(holomorphically) parametrized by $t \in \mathbb{C}$ (which is a toy model of the dynamical moduli space M_d of degree $d > 1$ rational functions on \mathbb{P}^1 below), the theory of stability/bifurcation is much richer than what would be expected. For each individual $P_t(z) = z^2 + t$, which is regarded as an endomorphism of \mathbb{C} (or the projective line $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ equipped with the chordal metric), the phase space \mathbb{C} is partitioned into a non-equicontinuity part (the chaotic locus in the sense of Devaney) and an equicontinuity part (the region of normality in the sense of Montel) for the sequence of the iterations $P_t^n = P_t^{\circ n} = P_t \circ \cdots \circ P_t$ (n times), $n \in \mathbb{N}$, which are respectively called the Julia set J_t and the Fatou set F_t of P_t and are both totally invariant under P_t (for complex dynamics, we refer to Milnor [20]). We say the family $(P_t)_{t \in \mathbb{C}}$ is J -stable at the parameter $t = t_0 \in \mathbb{C}$ if the (compact set valued) function $t \mapsto J_t$ is continuous at $t = t_0$ in the Hausdorff topology. By a general theory due to Mañé–Sad–Sullivan, Lyubich, and McMullen ([17, 16], and see also [18, §4]) on a holomorphic family of rational functions on $\mathbb{P}^1(\mathbb{C})$ of (common) degree $d > 1$, the J -unstable (or bifurcation) locus \mathcal{B}_2 in the parameter space \mathbb{C} coincides with the

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(fractal-shaped) boundary $\partial\mathcal{C}_2$ of the so called Mandelbrot set

$$\mathcal{C}_2 = \left\{ t \in \mathbb{C} : \limsup_{n \rightarrow \infty} |P_t^n(0)| < +\infty \right\}$$

(where we note that the point $z = 0$ is the unique critical point of P_t in the phase space \mathbb{C}), the J -stable locus $\mathcal{S}_2 = \mathbb{C} \setminus \partial\mathcal{C}_2$ in the parameter space \mathbb{C} is (open and) dense, and the difference between \mathcal{S}_2 and the structurally stable locus in the parameter space \mathbb{C} of $(P_t)_{t \in \mathbb{C}}$ is the set of all the (so called) superattracting parameters t , that is, the roots t in \mathbb{C} of the polynomial

$$P_t^n(0) = (P_t^n(z) - z)|_{z=0} \in \mathbb{Z}[t]$$

for some $n \in \mathbb{N}$. We say a parameter $t \in \mathbb{C}$ is hyperbolic if the (restricted dynamics) $P_t : J_t \rightarrow J_t$ is uniformly expanding. The superattracting parameters are hyperbolic ones. The hyperbolicity locus \mathcal{H}_2 is clopen in \mathcal{S}_2 . Conjecturally, $\mathcal{H}_2 = \mathcal{S}_2$.

The bifurcation locus $\mathcal{B}_2 = \partial\mathcal{C}_2$ also densely contains various kinds of distinguished parameters t (at least nearly) algebraically defined. Most importantly, \mathcal{B}_2 densely contains all the parabolic bifurcation parameters mentioned at the beginning; for example, at the parameter $t = 1/4$, $P_{1/4}(z) - z = z^2 - z + 1/4$ has the multiple root $z = 1/2$. Formally, the parabolic bifurcation (collision) parameters t for the quadratic polynomial family $(P_t)_{t \in \mathbb{C}}$ should have been defined by the roots t in \mathbb{C} of the discriminants

$$\text{Disc}(P_t^n(z) - z) \in \mathbb{Z}[t]$$

of $P_t^n(z) - z \in (\mathbb{Z}[t])[z]$, $n \in \mathbb{N}$, which contains all the roots t in \mathbb{C} of the resultant

$$\text{Res}(P_t^n(z) - z, P_t^\ell(z) - z) \in \mathbb{Z}[t]$$

between $P_t^n(z) - z$ and $P_t^\ell(z) - z \in (\mathbb{Z}[t])[z]$, $n \in \mathbb{N}$ and $\ell \in \mathbb{N}$ dividing n and less than n . Another example of an (almost) algebraic parameter $t \in \mathcal{B}_2$ is a Misiurewicz one, for which $z = 0$ is in J_t as well as is (strictly) preperiodic, i.e. t is a root of $P_t^n(0) - P_t^m(0) \in \mathbb{Z}[t]$ for some integers $n > m > 0$, and it is observed by a computer drawing (and shown by renormalization) that \mathcal{B}_2 near a Misiurewicz parameter t is asymptotically similar to J_t near the (so called postcritical) orbit $z = P_t^n(0)$ for $n \gg 1$ (since [27]; for a recent potential theoretic development, see [2]).

Although both parabolic bifurcation parameters and Misiurewicz parameters are dense in \mathcal{B}_2 , it has not been completely understood how they are equidistributed towards the harmonic measure on \mathcal{B}_2 quantitatively (in terms of electrostatic or potential theory, since [15]; for recent developments, see e.g. [5, 6, 11, 12]). Other interesting species also live in \mathcal{B}_2 (see e.g. [28, 19, 9]).

The study of arithmetic dynamics (dynamics of rational functions on \mathbb{P}^1 defined over a number field e.g. \mathbb{Q} which has the p -adic (non-archimedean) norm $|\cdot|_p$ for each prime p as well as the Euclidean (archimedean) norm $|\cdot|_\infty$) plays a more and more important role in the study of complex dynamics (i.e. dynamics of rational functions on \mathbb{P}^1 defined over \mathbb{C}); it might be surprising that the theory of heights in arithmetic is closely related to electrostatic or potential theory (since Baker–Rumely, Chambert-Loir, and Favre–Rivera-Letelier [4, 7, 10]). The (J -)stability and bifurcation in non-archimedean setting are more subtle (e.g. see [26] for a non-archimedean counterpart to the Mañé–Sad–Sullivan and Lyubich theory). Morton and Vivaldi [21] studied, towards a general bifurcation theory in (merely) algebraic dynamics, the parabolic bifurcation locus (or hypersurfaces) in the (d -dimensional) parameter space R^d of monic polynomials

$$z^d + b_{d-1}z^{d-1} + \cdots + b_1z + b_0, \quad (b_0, \dots, b_{d-1}) \in R^d$$

of degree d , $d > 1$, defined over an integral domain R (e.g. the ring \mathbb{Z}). Our aim is to contribute to the study of stability and bifurcation in algebraic(, complex, and non-archimedean) dynamics by developing the Morton–Vivaldi-type theory in the space Rat_d of degree d rational functions on \mathbb{P}^1 (and also in the dynamical moduli space M_d of them), so in particular by giving a geometric description of the parabolic bifurcation hypersurfaces in Rat_d and M_d .

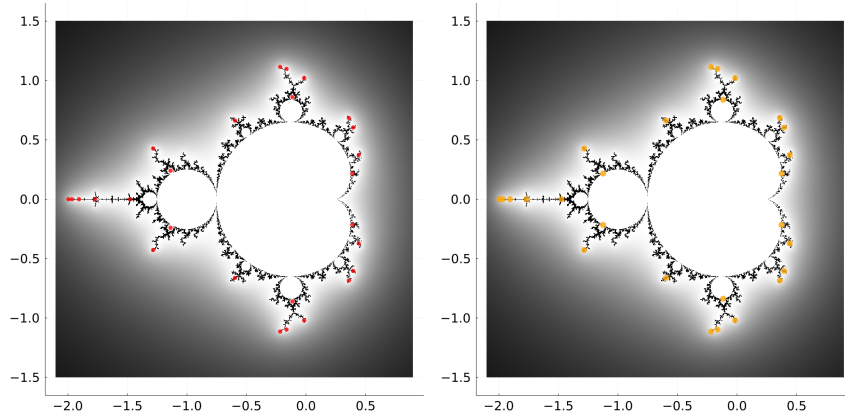


FIGURE 1. The Mandelbrot set \mathcal{C}_2 (white), the superattracting parameters t (red dots, living in $\text{int } \mathcal{C}_2 \subset \mathcal{S}_2$), and the parabolic bifurcation parameters t (orange dots, living in $\mathcal{B}_2 = \partial \mathcal{C}_2$) for the (formally exact) periods $n = 6$; those two computer pictures exhibit that the distribution of the superattracting parameters and that of the parabolic bifurcation parameters are similar, and indeed the averaged mass distributions/counting measures of both converge to the (same) harmonic measure/equilibrium mass distribution $\mu_{\mathcal{B}_2, \infty}$ on \mathcal{B}_2 with pole ∞ as the (formally exact) period $n \rightarrow \infty$. However, no (non-trivial) speed estimates of convergence of the parabolic bifurcation parameters towards $\mu_{\mathcal{B}_2, \infty}$ have been known, while that of the superattracting ones has been known quite precisely (see e.g. [12]); in other words, no non-trivial quantitative asymptotic estimates of the difference between the (averaged distributions of the) superattracting and parabolic bifurcation parameters as the period $n \rightarrow \infty$ have been known.

Organization of the paper. In Section 2, we introduce the necessary notions and state the main results (Theorem 1 and Corollary 1). In Section 3, we recall some complex dynamical and potential theoretic calculus on $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$, and also some details on the Möbius function/inversion and on homogeneous resultants/discriminants. In Section 4, we show Theorem 1. In Section 5, we conclude with a few computations in the case where $d = 2$.

2. MAIN RESULTS

2.1. The space Rat_d of degree d rational functions on \mathbb{P}^1 . The space of rational functions on \mathbb{P}^1 is by no means (a little) more complicate than that of polynomials. For each integer $d \geq 0$, over the complex number field \mathbb{C} , the set $\text{Rat}_d(\mathbb{C})$ of all rational functions

$$f(z) = P(z)/Q(z), \quad P, Q \in \mathbb{C}[z] \text{ are coprime, and } \max\{\deg P, \deg Q\} = d$$

on $\mathbb{P}^1(\mathbb{C})$ of degree d is regarded as a complex manifold $\mathbb{P}^{2d+1}(\mathbb{C}) \setminus V_d(\mathbb{C})$, by identifying f with the ratio

$$[a : b] = [a_0 : \dots : a_d : b_0 : \dots : b_d] \in \mathbb{P}^{2d+1}(\mathbb{C}), \quad a = (a_0, \dots, a_d), b = (b_0, \dots, b_d) \text{ for short,}$$

of the $2d + 2$ coefficients $a_0, \dots, a_d, b_0, \dots, b_d \in \mathbb{C}$ of the denominator and the numerator

$$Q(z) = \sum_{j=0}^d a_j z^j \quad \text{and} \quad P(z) = \sum_{k=0}^d b_k z^k$$

of f , but by removing a (complex analytic) hypersurface $V_d(\mathbb{C})$ from the ambient $2d + 1$ dimensional projective space $\mathbb{P}^{2d+1}(\mathbb{C})$ so that P, Q are coprime. We note that $\text{Rat}_0(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$ (and $V_0(\mathbb{C}) = \emptyset$), and that the $\text{Rat}_1(\mathbb{C})$, the Möbius transformation/projective coordinate change group on $\mathbb{P}^1(\mathbb{C})$, is $\cong \text{PSL}(2, \mathbb{C})$.

From now on, we assume $d > 1$.

Notation 2.1. For a (commutative) ring R , R^* denotes the unit (or invertible element) group.

Notation 2.2. For each ring R and each $d_0 \in \mathbb{N} \cup \{0\}$, the set of all homogeneous polynomials in $R[X_0, \dots, X_n]$ of degree d_0 is denoted by $R[X_0, \dots, X_n]_{d_0}$. Here, $H \in R[X_0, \dots, X_n]$ is said to be homogeneous if H is the sum of monomials in X_0, \dots, X_n the sum of the powers of X_0, \dots, X_n of any of which (is identical and) equals $\deg H$.

Then we (abstractly) introduce the space of degree d rational functions on \mathbb{P}^1

$$\text{Rat}_d := \mathbb{P}_{\mathbb{Z}}^{2d+1} \setminus V_d$$

as a scheme over the scheme $\text{Spec } \mathbb{Z}$, where $V_d := V(\rho_d)$ is the hypersurface in $\mathbb{P}_{\mathbb{Z}}^{2d+1}$ defined by (the zeros in $\mathbb{P}_{\mathbb{Z}}^{2d+1}$ of) the d -th homogeneous “resultant” form

$$\rho_d(a, b) := \det \begin{pmatrix} a_0 & \cdots & a_{d-1} & a_d & & \\ 0 & \ddots & \vdots & \vdots & \ddots & 0 \\ & & a_0 & a_1 & \cdots & a_d \\ b_0 & \cdots & b_{d-1} & b_d & & \\ 0 & \ddots & \vdots & \vdots & \ddots & 0 \\ & & b_0 & b_1 & \cdots & b_d \end{pmatrix} \in \mathbb{Z}[a_0, \dots, a_d, b_0, \dots, b_d]_{2d}$$

on $\mathbb{P}_{\mathbb{Z}}^{2d+1}$ (or on $\mathbb{A}_{\mathbb{Z}}^{2d+2}$), still writing $a = (a_0, \dots, a_d)$ and $b = (b_0, \dots, b_d)$ for short.

We introduced Rat_d as a (not only quasi-projective but also affine) scheme (defined over the ring \mathbb{Z}), which could be thought of as some topological space similar to an algebraic variety (defined over an algebraically closed field, e.g., \mathbb{C}) but still equipped with the ring

$$(2.1) \quad \mathbb{Z}(\text{Rat}_d) = \left\{ \rho_d^{-\frac{\deg P}{2d}} \cdot P : P \in \mathbb{Z}[a, b] \setminus \{0\} \text{ is homogeneous and } (2d) | (\deg P) \right\} \cup \{0\}$$

of regular functions on Rat_d ; see [24, §1] or [25, §4.3] (we refer to Hartshorne [14, §1-§2] for schemes/algebraic varieties). Since the above $\rho_d(a, b)$ is irreducible in $\mathbb{Z}[a, b]$ (see [13, Chapter 13]) and the ring $\mathbb{Z}[a, b]$ is a uniquely factorization domain (UFD), we have

$$(2.2) \quad (\mathbb{Z}(\text{Rat}_d))^* = \mathbb{Z}^* (= \{\pm 1\})$$

(see Proposition 3.2 below). On the other hand, we also denote by $\mathcal{O}_{\text{Rat}_d(\mathbb{C}), \text{an}}(D)$ the ring of complex analytic functions on each non-empty open subset D in the complex manifold $\text{Rat}_d(\mathbb{C})$.

Remark 2.3. We would do calculus only on the complex manifold $\text{Rat}_d(\mathbb{C})$ to avoid too many technicalities while, for the formulation and the proof of our main result, the above scheme-theoretic introduction of Rat_d (and M_d below) is not only natural but also necessary.

2.2. Variation of periodic points/cycles. Computation of the iterations of rational functions on \mathbb{P}^1 is also by no means more complicate than that of polynomials. As in the previous section, we identify each $[a : b] = [a_0 : \dots : a_d : b_0 : \dots : b_d] \in \mathbb{P}_{\mathbb{Z}}^{2d+1} \setminus V_d$ with a rational function

$$f(z) = f_{[a:b]}(z) = f_{[a_0:\dots:a_d:b_0:\dots:b_d]}(z) = \frac{\sum_{j=0}^d b_j z^j}{\sum_{i=0}^d a_i z^i} \quad \text{on } \mathbb{P}_{\mathbb{Z}}^1 (= \text{Proj}(\mathbb{Z}[X, Y])),$$

where $z = Y/X$ is an affine coordinate of $\mathbb{P}_{\mathbb{Z}}^1$, and then the pair of the homogenizations

$$(2.3) \quad \begin{aligned} F(X, Y) &= F(X, Y; a, b) = (F_0(X, Y; a, b), F_1(X, Y; a, b)) \\ &= (F_0(X, Y), F_1(X, Y)) = \left(\sum_{i=0}^d a_i X^{d-i} Y^i, \sum_{j=0}^d b_j X^{d-j} Y^j \right) \in ((\mathbb{Z}[a, b]_1)[X, Y]_d)^2 \end{aligned}$$

of the denominator and the numerator of $f = f_{[a:b]}$ (which is identified with the point $(a, b) \in \mathbb{A}_{\mathbb{Z}}^{2d+2}$, and is unique up to the “multiplication by a non-zero constant” action on $\mathbb{A}_{\mathbb{Z}}^{2d+2}$ of the multiplication group scheme $\mathbb{G}_{m, \mathbb{Z}}$) is an endomorphism of the affine plane $\mathbb{A}_{\mathbb{Z}}^2 (= \text{Spec}(\mathbb{Z}[X, Y]))$ of degree d and is called a lift (to $\mathbb{A}_{\mathbb{Z}}^2$) of $f = f_{[a:b]}$. Such a lifting f (to $\mathbb{A}_{\mathbb{Z}}^2$) is useful in

computation, noting that for each $n \in \mathbb{N}$, the n -th iteration $F^n = F \circ \cdots \circ F$ (n times) of F , which we write as

$$(2.4) \quad F^n = (F_0^{(n)}(X, Y; a, b), F_1^{(n)}(X, Y; a, b)) \in ((\mathbb{Z}[a, b]_{(d^n-1)/(d-1)})[X, Y]_{d^n})^2,$$

is a lift (to $\mathbb{A}_{\mathbb{Z}}^2$) of the n -th iteration $f^n = f \circ \cdots \circ f$ (n times) in Rat_{d^n} of $f = f_{[a:b]}$.

Letting Ω be an algebraically closed field of characteristic 0, e.g. \mathbb{C} , for each individual rational function $f = f_{[a:b]} \in \text{Rat}_d(\Omega)$ on $\mathbb{P}^1(\Omega)$ of degree d defined over Ω , an endomorphism $F = F(X, Y; a, b) \in (\Omega[X, Y]_d)^2$ of $\mathbb{A}^2(\Omega)$, where $(a, b) \in \Omega^{2d+2}$, is also called a lift (to Ω^2) of this $f = f_{[a:b]}$ and is unique up to a multiplication in $\Omega^* = \Omega \setminus \{0\}$.

Recall that we are interested in when f has a multiple cycle, that is, when or where a collision of distinct periodic points in \mathbb{P}^1 of a rational function $f_{[a:b]}$ happens as the point $[a : b]$ varies in Rat_d . The following notion from arithmetic is useful for our purpose. For the Möbius function $\mu : \mathbb{N} \rightarrow \{0, \pm 1\}$ and the (additive/multiplicative) Möbius inversion of a sequence in an abelian/commutative group, see Subsection 3.2 below.

Notation 2.4. For each $i \in \mathbb{N}$, the i -th cyclotomic polynomial

$$C_i(T) := \prod_{k|i} (T^k - 1)^{\mu(\frac{i}{k})} = \prod_{\omega: \text{an } i\text{-th primitive root of unity (in } \overline{\mathbb{Q}})} (T - \omega) \in \mathbb{Z}[T]$$

in T is the Möbius inversion of the sequence $(T^i - 1)_i$ (in $(\mathbb{Q}(T))^* = \mathbb{Q}(T) \setminus \{0\}$).

First, let Ω be an algebraically closed field of characteristic 0. For every individual $f \in \text{Rat}_d(\Omega)$ and every $n \in \mathbb{N}$, the set $\text{Fix}(f^n)$ of all fixed points in $\mathbb{P}^1(\Omega)$ of f^n , i.e. all solutions of the equation $f^n(z) = z$ in $\mathbb{P}^1(\Omega)$, is the zero locus in $\mathbb{P}^1(\Omega)$ of the homogeneous polynomial

$$Y \cdot F_0^{(n)}(X, Y) - X \cdot F_1^{(n)}(X, Y) \in \Omega[X, Y]_{d^n+1},$$

where F is a lift (to Ω^2) of f (recalling the notation in (2.4)); then we call

$$\text{Fix}^*(f^n) := \text{Fix}(f^n) \setminus \bigcup_{m: |n \text{ and } < n} \text{Fix}(f^m)$$

the set of all periodic points of f in $\mathbb{P}^1(\Omega)$ having the exact period n , and denoting, for each $m \in \mathbb{N}$ and each $z \in \text{Fix}^*(f^m)$, by $(f^m)'(z) = (f^m)'(f(z)) = \cdots = (f^m)'(f^{m-1}(z)) \in \Omega$ the multiplier of the cycle $z, f(z), \dots, f^{m-1}(z)$ of f in $\mathbb{P}^1(\Omega)$ (even if $z = \infty$ for simplicity), we call

$$(2.5) \quad \text{Fix}^{**}(f^n) := \text{Fix}^*(f^n) \sqcup \bigsqcup_{m: |n \text{ and } < n} \left\{ z \in \text{Fix}^*(f^m) : C_{\frac{n}{m}}((f^m)'(z)) = 0 \right\}$$

the set of all periodic points of f in $\mathbb{P}^1(\Omega)$ having the *formally exact* period n (which might be less studied but is more adequate for our purpose. The terminology “formally exact period n ” is a slight modification of the original “formal period n ” in [25, §4.1]). We note that

$$\text{Fix}^*(f^n) \subset \text{Fix}^{**}(f^n) \subset \text{Fix}(f^n),$$

the first inclusion in which is the equality for generic $f \in \text{Rat}_d(\Omega)$.

Remark 2.5. Our uses of the superscripts “*” and “**” are reverse to those in [25, §4.1] but in this article, the less used “**” is attached to the less studied “ $\text{Fix}^{**}(f^n)$ ”. The superscript “***” would be added to the notations coming from $\text{Fix}^{**}(f^n)$.

Next, for each $n \in \mathbb{N}$, let Fix_n be the hypersurface $V(\Phi_n)$ in the (fibered) product (scheme)

$$\mathbb{P}_{\text{Rat}_d}^1 := \mathbb{P}_{\mathbb{Z}}^1 \times \text{Rat}_d$$

defined by (the zeros in $\mathbb{P}_{\text{Rat}_d}^1$ of) the polynomial

$$\begin{aligned} \Phi_n(X, Y; a, b) &= \Phi_{F,n}(X, Y) \\ &:= Y \cdot F_0^{(n)}(X, Y; a, b) - X \cdot F_1^{(n)}(X, Y; a, b) \in (\mathbb{Z}[a, b]_{(d^n-1)/(d-1)})[X, Y]_{d^n+1}, \end{aligned}$$

in terms of the notations in (2.4). This hypersurface $\text{Fix}_n = V(\Phi_n)$ in $\mathbb{P}_{\text{Rat}_d}^1$ is regarded as a $d^n + 1$ -(multi)valued section of the projection $\mathbb{P}_{\text{Rat}_d}^1 \rightarrow \text{Rat}_d$ and is encoding the variation of the $d^n + 1$ periodic points in \mathbb{P}^1 of $f = f_{[a:b]}$ having the exact periods dividing n when the $[a : b]$ varies in Rat_d , in working over an algebraically closed field Ω of characteristic 0.

Finally, for each $n \in \mathbb{N}$, we also set the (additive) Möbius inversion

$$d_n := \sum_{m|n} \mu\left(\frac{n}{m}\right)(d^m + 1) \quad (\text{so, e.g., } d_p = d^p - d \text{ for a prime number } p)$$

of the sequence $(d^n + 1)_n$ (in the abelian group \mathbb{Z}), and set the Möbius inversion

$$\Phi_n^{**}(X, Y; a, b) = \Phi_{F,n}^{**}(X, Y) := \prod_{m|n} (\Phi_{F,m}(X, Y))^{\mu(\frac{n}{m})} \in \begin{cases} (\mathbb{Z}[a, b]_1)[X, Y]_{d+1} & \text{if } n = 1, \\ (\mathbb{Z}[a, b]_{d_n/(d-1)}[X, Y]_{d_n} & \text{if } n > 1, \end{cases}$$

which is a priori a rational function but indeed a (homogeneous) polynomial, of the sequence $(\Phi_n)_n$ (in $((\mathbb{Z}[a, b])(X, Y))^* = ((\mathbb{Z}[a, b])(X, Y)) \setminus \{0\}$); letting Ω be an algebraically closed field of characteristic 0, for each individual $f \in \text{Rat}_d(\Omega)$, the homogeneous polynomial $\Phi_{F,n}^{**}(X, Y) \in \Omega[X, Y]_{d_n}$ is called the n -th dynatomic polynomial for a lift F (to Ω^2) of f , and recalling (2.5), we indeed have

$$(2.6) \quad \text{Fix}^{**}(f^n) = (\text{the zero locus in } \mathbb{P}^1(\Omega) \text{ of } \Phi_{F,n}^{**} \in \Omega[X, Y]_{d_n}).$$

For each $n \in \mathbb{N}$, we have $\Phi_n = \prod_{m|n} \Phi_m^{**}$, and letting Fix_n^{**} be the hypersurface $V(\Phi_n^{**})$ in $\mathbb{P}_{\text{Rat}_d}^1$ defined by (the zeros in $\mathbb{P}_{\text{Rat}_d}^1$ of) the polynomial $\Phi_n^{**}(X, Y; a, b)$, we also have

$$\text{Fix}_n = \bigcup_{m|n} \text{Fix}_m^{**} \quad \text{in } \mathbb{P}_{\text{Rat}_d}^1.$$

The hypersurface $\text{Fix}_n^{**} = V(\Phi_n^{**})$ in $\mathbb{P}_{\text{Rat}_d}^1$ is regarded as a d_n -(multi)valued section of the projection $\mathbb{P}_{\text{Rat}_d}^1 \rightarrow \text{Rat}_d$ and is indeed encoding the variation of the d_n periodic points in \mathbb{P}^1 of $f = f_{[a:b]}$ having the formally exact periods n when the $[a : b]$ varies in Rat_d , in working over an algebraically closed field Ω of characteristic 0.

See [24, Theorem 4.4 and its Remark] for a full account over \mathbb{Z} , and also [25, §4.1] over \mathbb{Q} .

2.3. Collision of periodic points: parabolic bifurcation locus in Rat_d . For the homogeneous resultant/discriminant of homogeneous polynomials in X, Y , see Subsection 3.3.

First, for each $n \in \mathbb{N}$, the homogeneous discriminant $\text{Disc}(\Phi_n^{**})$ of the homogeneous polynomial $\Phi_n^{**} \in (\mathbb{Z}[a, b])[X, Y]_{d_n}$ in X, Y is a priori in the quotient field $\mathbb{Z}(a, b)$ of the polynomial ring $\mathbb{Z}[a, b]$, and indeed

$$\text{Disc}(\Phi_n^{**}) \in \begin{cases} \mathbb{Z}[a, b]_{2d} & \text{if } n = 1 \\ \mathbb{Z}[a, b]_{2d_n(d_n-1)/(d-1)} & \text{if } n > 1 \end{cases}$$

(see Remark 4.1 below). For each $n \in \mathbb{N}$ and each $\ell \in \mathbb{N}$ (not necessarily dividing n and) less than n , we have the homogeneous resultant

$$\text{Res}(\Phi_n^{**}, \Phi_\ell^{**}) \in \begin{cases} \mathbb{Z}[a, b]_{2d_n d/(d-1)} & \text{if } \ell = 1 \\ \mathbb{Z}[a, b]_{2d_n d_\ell/(d-1)} & \text{if } \ell > 1 \end{cases}$$

between the homogeneous polynomials Φ_n^{**} and Φ_ℓ^{**} in X, Y .

Recall the descriptions (2.1) of the ring $\mathbb{Z}[\text{Rat}_d]$ of regular functions on Rat_d and (2.2) of the unit group $(\mathbb{Z}[\text{Rat}_d])^*$ of it, and note that $(d(d-1)) \mid d_n$ if $n > 1$ (by (3.10) below). For every $n \in \mathbb{N}$ (resp. for every $n \in \mathbb{N}$ and every $\ell \in \mathbb{N}$ less than n), setting

$$N_d(n) := (2d)^{-1} \deg(\text{Disc}(\Phi_n^{**})) \in \mathbb{N} \quad (\text{resp. } N_d(n, \ell) := (2d)^{-1} \deg(\text{Res}(\Phi_n^{**}, \Phi_\ell^{**})) \in \mathbb{N}),$$

we have the regular function on Rat_d

$$(2.7) \quad D_n^{**} := \rho_d^{-N_d(n)} \text{Disc}(\Phi_n^{**}) \in \mathbb{Z}[\text{Rat}_d] \quad \left(\text{resp. } R_{n,\ell}^{**} := \rho_d^{-N_d(n,\ell)} \text{Res}(\Phi_n^{**}, \Phi_\ell^{**}) \in \mathbb{Z}[\text{Rat}_d] \right).$$

From (2.6), we have $R_{n,\ell}^{**} \in (\mathbb{Z}[\text{Rat}_d])^* (= \mathbb{Z}^* = \{\pm 1\})$ if $\ell \nmid n$ and $\ell < n$.

For every $n \in \mathbb{N}$, we call the hypersurface $V(D_n^{**})$ in Rat_d defined by (the zeros of) $D_n^{**} \in \mathbb{Z}[\text{Rat}_d]$ the n -th formally exact parabolic bifurcation hypersurface in Rat_d , which contains all the hypersurfaces $V(R_{n,\ell}^{**})$ in Rat_d defined by (the zeros of) $R_{n,\ell}^{**} \in \mathbb{Z}[\text{Rat}_d]$ for every $\ell \in \mathbb{N}$ dividing n and less than n (by (2.6) and the chain rule), and call the union

$$\bigcup_{n \in \mathbb{N}} V(D_n^{**})$$

the parabolic bifurcation locus in Rat_d .

Next, for every $n \in \mathbb{N}$, there is a so called n -th formally exact multiplier polynomial

$$p_n^{**}(T; [a : b]) \in (\mathbb{Z}[\text{Rat}_d])[T]$$

in T of degree d_n/n so that, letting Ω be an algebraically closed field of characteristic 0 (e.g. \mathbb{C}), for every individual $f = f_{[a:b]} \in \text{Rat}_d(\Omega)$, we have

$$(2.8) \quad (p_n^{**}(T; f))^n = \prod_{j=1}^{d_n} ((f^n)'(z_{f,j}^{(n)}) - T) \in \Omega[T]$$

(the left hand side in which is not the n -th iteration but the n -th power of $p_n^{**}(T; f)$), where we write as $[\Phi_{F,n}^{**} = 0] = \sum_{j=1}^{d_n} [z_{f,j}^{(n)}]$ the effective divisor on $\mathbb{P}^1(\Omega)$ defined by the d_n roots in $\mathbb{P}^1(\Omega)$ of $\Phi_{F,n}^{**} \in \Omega[X, Y]_{d_n}$ for a lift F (to Ω^2) of f , taking into account their multiplicities (so the sequence $(z_{f,j}^{(n)})_{j=1}^{d_n}$ in $\mathbb{P}^1(\Omega)$ is independent of the choice of F). See [24, Theorem 4.5 and the paragraph before that] for a full account over \mathbb{Z} , and also [25, §4.5] over \mathbb{Q} .

When $T = 1$, we have $V(p_n^{**}(1; \cdot)) = V(D_n^{**})$ (up to multiplicities), where $V(p_n^{**}(1; \cdot))$ is the hypersurface in Rat_d defined by (the zeros of) $p_n^{**}(1; \cdot) \in \mathbb{Z}[\text{Rat}_d]$. On the other hand, for every $n \in \mathbb{N}$ and every $\ell \in \mathbb{N}$ dividing n and less than n , the (so called) $\frac{n}{\ell}$ -th cyclic resultant for the polynomial $p_\ell^{**}(T; [a : b])$ in T gives the regular function

$$[a : b] \mapsto \Delta_{n,\ell}(f_{[a:b]}) := \text{Res}(C_{\frac{n}{\ell}}(T), p_\ell^{**}(T; [a : b])) \in \mathbb{Z}[\text{Rat}_d]$$

on Rat_d ; letting Ω be an algebraically closed field of characteristic 0, for every individual $f \in \text{Rat}_d(\Omega)$, in terms of the above $(z_{f,j}^{(\ell)})_{j=1}^{d_\ell}$ in $\mathbb{P}^1(\Omega)$, we have

$$(2.9) \quad (\Delta_{n,\ell}(f))^\ell = \prod_{j=1}^{d_\ell} C_{\frac{n}{\ell}}((f^\ell)'(z_{f,j}^{(\ell)})) = \prod_{\omega: \text{an } \frac{n}{\ell}\text{-th primitive root of unity (in } \overline{\mathbb{Q}})} (p_\ell^{**}(\omega; f))^\ell.$$

In particular, $V(\Delta_{n,\ell}) = V(R_{n,\ell}^{**})$ (up to multiplicities), where $V(\Delta_{n,\ell})$ is the hypersurface in Rat_d defined by (the zeros of) $\Delta_{n,\ell} \in \mathbb{Z}[\text{Rat}_d]$. For every $n \in \mathbb{N}$, we also set a rational function

$$\Delta_{n,n} := \frac{p_n^{**}(1; \cdot)}{\prod_{\ell: |n \text{ and } < n} \Delta_{n,\ell}} \quad \text{on } \text{Rat}_d.$$

Remark 2.6. Under our notational convention, it would be more consistent to write $\Delta_{n,n}, \Delta_{n,\ell}$ as $\Delta_{n,n}^{**}, \Delta_{n,\ell}^{**}$, respectively, but for simplicity, we omit “**” for them here and below.

2.4. Main results. Our principal result is the following computation of both the defining regular function D_n^{**} on Rat_d of the n -th formally exact parabolic bifurcation hypersurface $V(D_n^{**})$, $n \in \mathbb{N}$, and the regular function $R_{n,\ell}^{**}$ on Rat_d , $\ell \mid n$ and $< n$, in terms of $\Delta_{n,\ell}$ and $\Delta_{n,n}$.

Theorem 1. *Fix an integer $d > 1$. Then for every $n \in \mathbb{N}$, we have*

$$D_n^{**} = \pm (\Delta_{n,n})^n \cdot \prod_{\ell: |n \text{ and } < n} (\Delta_{n,\ell})^{n-\ell} \quad \text{on } \text{Rat}_d,$$

and for every $n \in \mathbb{N}$ and every $\ell \in \mathbb{N}$ dividing n and less than n , we have

$$R_{n,\ell}^{**} = \pm (\Delta_{n,\ell})^\ell \quad \text{on } \text{Rat}_d,$$

where by “ \pm ” we mean “up to sign”. Moreover, for every $n \in \mathbb{N}$, we have $\Delta_{n,n} \in \mathbb{Z}[\text{Rat}_d]$.

The proof of Theorem 1 is based on (complex) potential theoretic computations while the resulting formulas in Theorem 1 are algebraic. For a (monic) polynomial, Theorem 1 recovers [21, the first equality in Theorem A and Theorem 2.2]. As a consequence of Theorem 1, we have the following geometric descriptions of both the n -th formally exact parabolic bifurcation hypersurface $V(D_n^{**})$ in Rat_d , $n \in \mathbb{N}$, and its subhypersurfaces $V(R_{n,\ell}^{**})$ in Rat_d , $\ell \mid n$ and $\ell < n$, in terms of effective divisors in Rat_d .

Corollary 1. *Fixing $d > 1$, for every $n \in \mathbb{N}$, we have*

$$\begin{aligned} [D_n^{**} = 0] &= n[\Delta_{n,n} = 0] + \sum_{\ell: \ell \mid n \text{ and } \ell < n} (n - \ell)[\Delta_{n,\ell} = 0] \quad \text{and} \\ [p_n^{**}(1; \cdot) = 0] &= [\Delta_{n,n} = 0] + \sum_{\ell: \ell \mid n \text{ and } \ell < n} [\Delta_{n,\ell} = 0] \quad \text{on } \text{Rat}_d, \end{aligned}$$

and for every $\ell \in \mathbb{N}$ dividing n and less than n , $[R_{n,\ell}^{**} = 0] = \ell[\Delta_{n,\ell} = 0]$ on Rat_d .

From a viewpoint of dynamics, it would be more natural to study each f up to a projective coordinate changes of the source and target copies of \mathbb{P}^1 rather than f itself. Over \mathbb{C} , the set $M_d(\mathbb{C}) = \text{Rat}_d(\mathbb{C})/\text{PSL}(2, \mathbb{C})$ of all Möbius transformation conjugacy classes $[f]$ of rational functions $f \in \text{Rat}_d(\mathbb{C})$ is called the dynamical moduli space of degree d rational functions on $\mathbb{P}^1(\mathbb{C})$. To equip this set $M_d(\mathbb{C})$ with a natural complex affine algebraic variety structure, we (abstractly) introduce the dynamical moduli space

$$M_d := \text{Rat}_d / \text{SL}_2$$

of degree d rational functions on \mathbb{P}^1 as the (GIT-)quotient of Rat_d by the (GIT-stable) conjugation action on Rat_d of the linear algebraic group scheme SL_2 , and then the variety (rather than just a set) $M_d(\mathbb{C})$ is obtained by the base extension (from $\text{Spec } \mathbb{Z}$ to $\text{Spec } \mathbb{C}$) of the scheme M_d ; see [24, §2] over \mathbb{Z} , and also [25, §4.4] over \mathbb{Q} .

The regular functions $p_n^{**}(1; \cdot)$, $\Delta_{n,n}$, $\Delta_{n,\ell}$, D_n^{**} , and $R_{n,\ell}^{**}$ on Rat_d are all invariant under the action of SL_2 on Rat_d (see Remark 3.1 below for the SL_2 -invariance of D_n^{**} , $n \in \mathbb{N}$, and that of $R_{n,\ell}^{**}$, $n > \ell$), and descend to M_d as regular functions there. Hence the assertions similar to those in Theorem 1 and Corollary 1 also hold when replacing Rat_d with M_d .

3. BACKGROUND

3.1. Complex dynamics and potential theory. Let

$$\pi : \mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{P}^1 (= \mathbb{P}^1(\mathbb{C}))$$

be the canonical projection so that $\pi(0, 1) = \infty$ (and that $\pi(Z_0, Z_1) = Z_1/Z_0$ if $Z_0 \neq 0$), and ω be the Fubini-Study area element on \mathbb{P}^1 normalized as $\omega(\mathbb{P}^1) = 1$. Set

$$(Z_0, Z_1) \wedge (W_0, W_1) = Z_0 W_1 - Z_1 W_0 \in \mathbb{Z}[Z_0, Z_1, W_0, W_1]_2,$$

so that for $Z, W \in \mathbb{C}^2$ and $A \in \text{GL}(2, \mathbb{C})$,

$$(3.1) \quad (AZ) \wedge (AW) = (\det A)(Z \wedge W),$$

and let $\|\cdot\|$ be the Euclidean norm on \mathbb{C}^2 . The normalized chordal metric on \mathbb{P}^1 is defined as

$$[z, w]_{\mathbb{P}^1} = \frac{|Z \wedge W|}{\|Z\| \cdot \|W\|} \quad \text{on } \mathbb{P}^1 \times \mathbb{P}^1,$$

where $Z \in \pi^{-1}(z)$, $W \in \pi^{-1}(w)$, so that $[0, \infty]_{\mathbb{P}^1} = 1$. The (generalized) Laplacian Δ on \mathbb{P}^1 is normalized so that $\Delta(\log[\cdot, \infty]_{\mathbb{P}^1}) = \delta_\infty - \omega$ on \mathbb{P}^1 , where δ_z is the Dirac measure on \mathbb{P}^1 at $z \in \mathbb{P}^1$.

Pick $f \in \text{Rat}_d(\mathbb{C})$, $d > 1$, and a lift $F \in (\mathbb{C}[X, Y]_d)^2$ (to \mathbb{C}^2) of f , which is unique up to multiplication in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then for every $n \in \mathbb{N}$ and every $c \in \mathbb{C}^*$, $(cF)^n = c^{(d^n-1)/(d-1)} F^n \in (\mathbb{C}[X, Y]_{d^n})^2$ is a lift (to \mathbb{C}^2) of f^n . From the homogeneity of F , the uniform limit

$$G^F := \lim_{n \rightarrow \infty} \frac{\log \|F^n\|}{d^n} \quad \text{on } \mathbb{C}^2 \setminus \{(0, 0)\} \quad (F^n = F \circ \dots \circ F, \text{ } n \text{ times})$$

exists and is called the escaping rate function of F on $\mathbb{C}^2 \setminus \{(0, 0)\}$. We note that for every $c \in \mathbb{C}^*$, $G^F(cZ) = G^F(Z) + \log|c|$ and $G^{cF} = G^F + (\log|c|)/(d-1)$ on $\mathbb{C}^2 \setminus \{(0, 0)\}$, that

$$(3.2) \quad G^F \circ F = d \cdot G^F \quad \text{on } \mathbb{C}^2 \setminus \{(0, 0)\},$$

and that for every $n \in \mathbb{N}$, $G^{F^n} = G^F$ on $\mathbb{C}^2 \setminus \{(0, 0)\}$.

The continuous function

$$g_F := G^F - \log \|\cdot\| \quad \text{on } \mathbb{P}^1$$

is called the dynamical Green function of F on \mathbb{P}^1 , and the probability Radon measure $\mu_f := \Delta g_F + \omega$ on \mathbb{P}^1 is independent of the choice of F and is called the equilibrium (or canonical) measure of f on \mathbb{P}^1 . The homogeneous resultant of a lift (to \mathbb{C}^2) $F = F(X, Y; a, b) = (F_0, F_1)$ of $f_{[a:b]} \in \text{Rat}_d(\mathbb{C})$ ($a = (a_0, \dots, a_d), b = (b_0, \dots, b_d)$ for short) is

$$\text{Res } F := \text{Res}(F_0, F_1) = \rho_d(a, b) \in \mathbb{C}^*,$$

where $\text{Res}(F_0, F_1)$ is the homogeneous resultant between $F_0, F_1 \in \mathbb{C}[X, Y]_d$ (see Subsection 3.3 below). For every $c \in \mathbb{C}^*$, we also have $\text{Res}(cF) = c^{2d} \text{Res } F$, so that the function

$$g_f := g_F - \frac{\log |\text{Res } F|}{2d(d-1)} \quad \text{on } \mathbb{P}^1$$

is independent of the choice of F and is called the dynamical Green function of f on \mathbb{P}^1 . We also note that for every $n \in \mathbb{N}$, $g_{f^n} = g_f$ on \mathbb{P}^1 or equivalently $|\text{Res}(F^n)| = |\text{Res } F|^{(d^n(d^n-1))/(d(d-1))}$.

For each lift F (to \mathbb{C}^2) of f , the g_F -kernel function on \mathbb{P}^1 is defined by the function $\Phi_{g_F}(z, w) := \log[z, w]_{\mathbb{P}^1} - g_F(z) - g_F(w)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ (there would be no confusions between this and dynatomic polynomials, in notations), so that $\Delta(\Phi_{g_F}(\cdot, w)) = \delta_w - \mu_f$ on \mathbb{P}^1 for every $w \in \mathbb{P}^1$. We have $\int_{\mathbb{P}^1} \Phi_{g_f}(\cdot, w) \mu_f(w) \equiv -(\log |\text{Res } F|)/(d(d-1))$ on \mathbb{P}^1 since $\Delta \int_{\mathbb{P}^1} \Phi_{g_f}(\cdot, w) \mu_f(w) = \mu_f - \mu_f = 0$ on \mathbb{P}^1 (by the Fubini theorem) and moreover the energy/capacity formula

$$\int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_{g_F}(\mu_f \times \mu_f) = -\frac{\log |\text{Res } F|}{d(d-1)}$$

also holds (due to DeMarco [8]; for a simple proof which also works in non-archimedean setting, see [3, Appendix A] or [22, Appendix]). In particular, the g_f -kernel function

$$\Phi_{g_f}(z, w) := \log[z, w]_{\mathbb{P}^1} - g_f(z) - g_f(w) \quad \text{on } \mathbb{P}^1 \times \mathbb{P}^1$$

satisfies $\int_{\mathbb{P}^1} \Phi_{g_f}(\cdot, w) \mu_f(w) \equiv 0$ on \mathbb{P}^1 .

The following Riesz decomposition-type formula

$$(3.3) \quad \Phi_{g_f}(f, \text{Id}_{\mathbb{P}^1}) = \int_{\mathbb{P}^1} \Phi_{g_f}(\cdot, w) [f = \text{Id}_{\mathbb{P}^1}](w) \quad \text{on } \mathbb{P}^1$$

([23, Lemma 4.2], the most involved point is that the harmonic part $\int_{\mathbb{P}^1} \Phi_{g_f}(f, \text{Id}_{\mathbb{P}^1}) \mu_f \in \mathbb{R}$ of $\Phi_{g_f}(f, \text{Id}_{\mathbb{P}^1})$ also vanishes on \mathbb{P}^1) plays a key role. Here, writing

$$(3.4) \quad F(Z) \wedge Z = \prod_{j=1}^{d+1} (Z \wedge q_{F,j}) \quad \text{on } \mathbb{C}^2$$

for some $q_{F,1}, \dots, q_{F,d+1} \in \mathbb{C}^2 \setminus \{(0, 0)\}$, the sequence $(\pi(q_{F,j}))_{j=1}^{d+1}$ is independent of the choice of F , up to permutation, and the effective divisor $[f = \text{Id}_{\mathbb{P}^1}] = \sum_{j=1}^{d+1} [\pi(q_{F,j})]$ on \mathbb{P}^1 defined by

the solutions in \mathbb{P}^1 of the equation $f = \text{Id}_{\mathbb{P}^1}$ taking into account their multiplicities is regarded as a positive Radon measure

$$[f = \text{Id}_{\mathbb{P}^1}] = \sum_{j=1}^{d+1} \delta_{\pi(q_{F,j})} \quad \text{on } \mathbb{P}^1.$$

For every $z \in \text{Fix}(f)$, (denoting the multiplier of the cycle z of f (having the period 1) by $f'(z)$ even when $z = \infty$, for simplicity,) if this z is simple in that $f'(z) \neq 1$, then we have

$$(3.5) \quad \lim_{\mathbb{P}^1 \ni w \rightarrow z} (\Phi_{g_f}(f(w), w) - \Phi_{g_f}(z, w)) = \lim_{\mathbb{P}^1 \ni w \rightarrow z} \log \frac{[f(w), w]_{\mathbb{P}^1}}{[z, w]_{\mathbb{P}^1}} = \log |f'(z) - 1|,$$

which together with (3.3) yields

$$(3.6) \quad \log |f'(z) - 1| = \int_{\mathbb{P}^1 \setminus \{z\}} \Phi_{g_f}(z, w) [f = \text{Id}_{\mathbb{P}^1}](w).$$

In the rest of this subsection, let us assume that F satisfies $|\text{Res } F| = 1$, so that

$$(3.7) \quad \Phi_{g_f}(\pi(Z), \pi(W)) = \log |Z \wedge W| - G^F(Z) - G^F(W) \quad \text{on } (\mathbb{C}^2 \setminus \{(0, 0)\})^2.$$

Another consequence of (3.3), (3.7), the factorization (3.4) of $F \wedge \text{Id}_{\mathbb{C}^2}$, and (3.2)) is the vanishing

$$(3.8) \quad \sum_{j=1}^{d+1} G^F(q_{F,j}) = 0.$$

Hence for every $n \in \mathbb{N}$, noting that $F^n \wedge \text{Id}_{\mathbb{C}^2} = \prod_{m|n} \Phi_{F,m}^{**}$ (by Möbius inversion) and writing each $\Phi_{F,m}^{**}$ as $\Phi_{F,m}^{**}(Z) = \prod_{j=1}^{d_m} (Z \wedge Z_{F,j}^{(m)})$ for some $Z_{F,1}^{(m)}, \dots, Z_{F,d_m}^{(m)} \in \mathbb{C}^2 \setminus \{(0, 0)\}$, by the vanishing (3.8) applied to F^n and by Möbius inversion, we also have

$$(3.9) \quad \sum_{j=1}^{d_n} G^F(Z_{F,j}^{(n)}) = 0,$$

which will be frequently used in this paper.

3.2. Möbius function and inversions. For the details, see e.g. [1, §2.7]. The Möbius function $\mu : \mathbb{N} \rightarrow \{0, \pm 1\}$ is defined so that $\mu(n) = 1 \cdot (-1)^N$ if n is a product of some N distinct positive prime numbers, $N \in \mathbb{N} \cup \{0\}$, (so $\mu(1) = 1$) and $\mu(n) = 0$ in other cases. We repeatedly use

$$(3.10) \quad \sum_{m|n} \mu(m) = 0 \quad \text{if } n > 1$$

in this paper. For any sequences $(A_k)_{k \in \mathbb{N}}, (B_\ell)_{\ell \in \mathbb{N}}$ in a commutative group,

$$A_k = \prod_{\ell|k} B_\ell \text{ for every } k \in \mathbb{N} \quad \text{if and only if} \quad B_\ell = \prod_{k|\ell} (A_k)^{\mu(\frac{\ell}{k})} \text{ for every } \ell \in \mathbb{N},$$

and then we say the sequences $(A_k), (B_\ell)$ are obtained by (multiplicative) Möbius inversions of each other. The same thing could also be stated additively (in an abelian group).

3.3. Homogeneous resultant and discriminant. For any homogeneous polynomials $P = \sum_{i=0}^d a_i X^{d-i} Y^i \in R[X, Y]_d$ and $Q = \sum_{j=0}^e b_j X^{e-j} Y^j \in R[X, Y]_e$ over an integral domain R , the homogeneous resultant between P and Q is defined as

$$\text{Res}(P, Q) := \det \begin{pmatrix} a_0 & \cdots & a_{e-1} & \cdots & a_{d-1} & a_d & & & \\ 0 & \ddots & \vdots & & & & \ddots & & 0 \\ & & a_0 & a_1 & \cdots & \cdots & a_{d-1} & a_d & \\ b_0 & \cdots & b_{e-1} & b_e & & & & & \\ 0 & \ddots & & & \vdots & & \ddots & & 0 \\ & & & & b_0 & \cdots & b_{e-1} & b_e & \end{pmatrix} \in R,$$

so that $\text{Res}(P, Q) = (-1)^{de} \text{Res}(Q, P)$ and that $\text{Res}(cP, Q) = c^e \text{Res}(P, Q)$ for every $c \in R$.

Letting F be the quotient field of R , any $P \in R[X, Y]_d$ factors as

$$P(X, Y) = \prod_{i=1}^d ((X, Y) \wedge A_i)$$

over an algebraic closure \bar{F} of F , where $A_i \in \bar{F}^2 \setminus \{(0, 0)\}$ for each $i \in \{1, \dots, d\}$. Then for every $Q \in R[X, Y]_d$,

$$(3.11) \quad \text{Res}(P, Q) = \prod_{i=1}^d Q(A_i),$$

and the homogeneous discriminant of P is defined as

$$(3.12) \quad \text{Disc}(P) := \prod_{i=1}^d \prod_{\ell \in \{1, \dots, d\} \setminus \{i\}} (A_i \wedge A_\ell) \in F,$$

so that $\text{Disc}(cP) = c^{2(d-1)} \text{Disc}(P)$ for every $c \in R$. For more details, see e.g. [25, §2.4].

Remark 3.1. In terms of the notations in Subsections 3.1 and 2.2, for every $A = (a_{st}) \in \text{GL}(2, \mathbb{C})$, regarding A as a linear automorphism $(X, Y) \mapsto (a_{11}X + a_{12}Y, a_{21}X + a_{22}Y)$ of \mathbb{C}^2 and setting a Möbius transformation $a(z) := (a_{21} + a_{22}z)/(a_{11} + a_{12}z)$ of $\mathbb{P}^1(\mathbb{C})$, the conjugation $A \circ F \circ A^{-1} \in (\mathbb{C}[X, Y]_d)^2$ of F is a lift (to \mathbb{C}^2) of the conjugation $a \circ f \circ a^{-1} \in \text{Rat}_d(\mathbb{C})$ of f , and using (3.1) repeatedly, we have

$$((A \circ F \circ A^{-1})(Z)) \wedge Z = (\det A)^{-d} \prod_{j=1}^{d+1} (Z \wedge (Aq_{F,j}))$$

on \mathbb{C}^2 . This together with (3.1), (3.11), (3.12), and

$$\text{Res}(A \circ F \circ A^{-1}) = (\det A)^{d-d^2} \text{Res } F$$

for every $A \in \text{GL}(2, \mathbb{C})$ (see, e.g., [25, Exercise 2.12]) concludes the SL_2 -invariance of D_n^{**} , $n \in \mathbb{N}$, and that of $R_{n,\ell}^{**}$, $n > \ell$, on Rat_d .

3.4. A proposition from algebra. For completeness, we include a proof of the following.

Proposition 3.2. *Let R be a uniquely factorization domain (UFD), $N \in \mathbb{N}$, and F be an irreducible homogeneous polynomial in $R[X_0, \dots, X_N]$ of degree > 0 , and set $U := \mathbb{P}_R^N \setminus V(F)$, where $V(F)$ be the hypersurface in \mathbb{P}_R^N defined by (the zeros in \mathbb{P}_R^N of) F . Then the unit group $(R[U])^*$ of the ring $R[U]$ of regular functions on U equals R^* .*

Proof. Pick $\phi \in (R[U])^*$. Then $\phi = P/F^{(\deg P)/(\deg F)}$ and $\phi^{-1} = Q/F^{(\deg Q)/(\deg F)}$ for some homogeneous $P, Q \in R[X_0, \dots, X_N]$ such that $(\deg P)/(\deg F), (\deg Q)/(\deg F) \in \mathbb{N} \cup \{0\}$. Since $R[X_0, \dots, X_N]$ is also UFD, $PQ = F^{(\deg P)/(\deg F) + (\deg Q)/(\deg F)}$ implies $P = rF^m$ for some $r \in (R[X_0, \dots, X_N])^* = R^*$ and some $m \in \mathbb{N} \cup \{0\}$, so $\phi = (rF^m)/F^{(m \deg F)/\deg F} = r \in R^*$. \square

4. PROOF OF THEOREM 1

It is enough to show Theorem 1 working over \mathbb{C} . Fix $d > 1$, and pick $n \in \mathbb{N}$. For every individual (generic) $f \in \text{Rat}_d(\mathbb{C})$ and every lift F (to \mathbb{C}^2) of f , as in Section 3.1, we have a factorization $F^n(Z) \wedge Z = \prod_{j=1}^{d^n+1} (Z \wedge q_{F,j}^{(n)})$ on \mathbb{C}^2 of $F^n \wedge \text{Id}_{\mathbb{C}^2}$, where $q_{F,j}^{(n)} \in \mathbb{C}^2 \setminus \{(0,0)\}$ for every $j \in \{1, \dots, d^n + 1\}$. Moreover, for every $\ell \in \mathbb{N}$, we also have a factorization

$$\Phi_{F,\ell}^{**}(Z) = \prod_{j=1}^{d_\ell} (Z \wedge Z_{F,j}^{(\ell)})$$

on \mathbb{C}^2 of $\Phi_{F,\ell}^{**}$, where $Z_{F,j}^{(\ell)} \in \mathbb{C}^2 \setminus \{(0,0)\}$ for every $j \in \{1, \dots, d_\ell\}$, and setting

$$(4.1) \quad z_{f,j}^{(\ell)} := \pi(Z_{F,j}^{(\ell)}) \in \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C}) \quad \text{for each } j \in \{1, \dots, d_\ell\},$$

the sequence $(z_{f,j}^{(\ell)})_{j=1}^{d_\ell}$ is independent of the choice of F , up to permutation, so that $[\Phi_{F,\ell}^{**} = 0] = \sum_{j=1}^{d_\ell} [z_{f,j}^{(\ell)}]$ (as an effective divisor) on \mathbb{P}^1 .

Without loss of generality, we normalize the labeling of the sequence $(z_{f,j}^{(n)})_{j=1}^{d_n}$ so that for every $r \in \{0, 1, \dots, \frac{d_n}{n} - 1\}$ and every $k \in \{1, \dots, n-1\}$,

$$(4.2) \quad f^k(z_{f,1+nr}^{(n)}) = z_{f,1+nr+k}^{(n)} \quad \left(\text{and} \quad f^n(z_{f,1+nr}^{(n)}) = z_{f,1+nr}^{(n)} \right),$$

and then for every $r \in \{0, 1, \dots, \frac{d_n}{n} - 1\}$, set the homogeneous polynomial

$$(4.3) \quad \Lambda_{F,r}^{(n)}(X, Y) := \prod_{k=0}^{n-1} ((X, Y) \wedge Z_{F,1+nr+k}^{(n)}) \in \mathbb{C}[X, Y]_n \quad \left(\text{so that } \Phi_{F,n}^{**} = \prod_{r=0}^{\frac{d_n}{n}-1} \Lambda_{F,r}^{(n)} \right).$$

The equalities (4.4) and (4.5) below are substantial.

4.1. We first claim that for every $\ell \in \mathbb{N}$ dividing n and less than n ,

$$(4.4) \quad \left| \frac{(\text{Res } F)^{-N_d(n,\ell)} \text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})}{(\Delta_{n,\ell}(f))^\ell} \right| = 1,$$

which yields $R_{n,\ell}^{**}/(\Delta_{n,\ell})^\ell \in (\mathbb{Z}[\text{Rat}_d])^* (= \mathbb{Z}^* = \{\pm 1\})$ recalling (2.2) and the definition of $R_{n,\ell}^{**}$ in (2.7); for, we note that

$$\Phi_{F,n}^{**}(Z) \left(= \prod_{m|n} (F^m(Z) \wedge Z)^{\mu(\frac{n}{m})} \right) = \prod_{m: \ell \nmid m|n} (F^m(Z) \wedge Z)^{\mu(\frac{n}{m})} \times \prod_{m: \ell | m|n} (F^m(Z) \wedge Z)^{\mu(\frac{n}{m})}$$

on \mathbb{C}^2 , and compute

$$\begin{aligned} & \prod_{m: \ell \nmid m|n} (F^m(Z) \wedge Z)^{\mu(\frac{n}{m})} = \prod_{m: \ell \nmid m|n} \left(\prod_{m': m'|m} \Phi_{F,m'}^{**}(Z) \right)^{\mu(\frac{n}{m})} \\ &= \prod_{m': \ell \nmid m'|n} \left(\prod_{m: m'|m|n \text{ and } \ell \nmid m} \Phi_{F,m'}^{**}(Z)^{\mu(\frac{n}{m})} \right) \\ &= \prod_{m': m' \nmid \ell \nmid m'|n} \left(\prod_{m: m'|m|n \text{ and } \ell \nmid m} \Phi_{F,m'}^{**}(Z)^{\mu(\frac{n}{m})} \right) \times \prod_{m': \ell | m' \text{ and } < \ell} \left(\prod_{m: m'|m|n \text{ and } \ell \nmid m} \Phi_{F,m'}^{**}(Z)^{\mu(\frac{n}{m})} \right), \end{aligned}$$

where the first equality is from $F^m(Z) \wedge Z = \prod_{m' \mid m} \Phi_{F, m'}^{**}(Z)$ by Möbius inversion. Moreover, we have

$$\begin{aligned} \prod_{m': \ell \text{ and } < \ell} \left(\prod_{m: m' \mid m \mid n \text{ and } \ell \nmid m} \Phi_{F, m'}^{**}(Z)^{\mu(\frac{n}{m})} \right) &= \prod_{m': \ell \text{ and } < \ell} \left(\prod_{m'': \frac{\ell}{m'} \mid m'' \mid \frac{n}{m'}} \Phi_{F, m'}^{**}(Z)^{\mu(\frac{n/m'}{m''})} \right) \\ &= \prod_{m': \ell \text{ and } < \ell} \Phi_{F, m'}^{**}(Z)^{\sum_{m'' \mid \frac{n}{m'}} \mu(\frac{n/m'}{m''}) - \sum_{m'': \frac{\ell}{m'} \mid m'' \mid \frac{n}{m'}} \mu(\frac{n/m'}{m''})} \equiv 1, \end{aligned}$$

noting that by (3.10),

$$\begin{cases} \sum_{m'' \mid \frac{n}{m'}} \mu(\frac{n/m'}{m''}) = 0 & (\text{also by } \frac{n}{m'} > 1) \quad \text{and} \\ \sum_{m'': \frac{\ell}{m'} \mid m'' \mid \frac{n}{m'}} \mu(\frac{n/m'}{m''}) = \sum_{m'' \mid \frac{n}{\ell}} \mu(\frac{n/\ell}{m''}) = 0 & (\text{also by } \frac{n}{\ell} > 1) \end{cases}$$

when $m' \in \mathbb{N}$ divides ℓ and is $< \ell$ (this purely algebraic computation is similar to that in [21, Proof of Theorem 2.2]).

On the other hand, for every $m \in \mathbb{N}$, assuming $|\text{Res } F| = 1$, we compute

$$\begin{aligned} \log \left| \prod_{m: \ell \mid m \mid n} (F^m(Z) \wedge Z)^{\mu(\frac{n}{m})} \right| &= \sum_{k \mid \frac{n}{\ell}} \mu\left(\frac{n/\ell}{k}\right) \log |F^{k\ell}(Z) \wedge Z| \\ &= \sum_{k \mid \frac{n}{\ell}} \mu\left(\frac{n/\ell}{k}\right) \left(\Phi_{g_f}(f^{k\ell}(\pi(Z)), \pi(Z)) + G^F(F^{k\ell}(Z)) + G^F(Z) \right) \quad (\text{by (3.7)}) \\ &= \sum_{k \mid \frac{n}{\ell}} \mu\left(\frac{n/\ell}{k}\right) \left(\Phi_{g_f}(f^{k\ell}(\pi(Z)), \pi(Z)) + ((d^\ell)^k + 1) \cdot G^F(Z) \right) \quad (\text{by (3.2)}) \\ &= \sum_{k \mid \frac{n}{\ell}} \mu\left(\frac{n/\ell}{k}\right) \Phi_{g_f}(f^{k\ell}(\pi(Z)), \pi(Z)) + (d^\ell)_{\frac{n}{\ell}} \cdot G^F(Z) \end{aligned}$$

generically on \mathbb{C}^2 , where $(d^\ell)_{\frac{n}{\ell}} = \sum_{k \mid \frac{n}{\ell}} \mu(\frac{n/\ell}{k})((d^\ell)^k + 1)$; recalling the definition (4.1) of the sequence $(z_{f,j}^{(m)})_{j=1}^{d_m}$, for every $j \in \{1, \dots, d_\ell\}$, we also have

$$\begin{aligned} &\lim_{\mathbb{C}^2 \ni Z \rightarrow Z_j^{(\ell)}} \sum_{k \mid \frac{n}{\ell}} \mu\left(\frac{n/\ell}{k}\right) \Phi_{g_f}(f^{k\ell}(\pi(Z)), \pi(Z)) \\ &= \lim_{\mathbb{C}^2 \ni Z \rightarrow Z_j^{(\ell)}} \sum_{k \mid \frac{n}{\ell}} \mu\left(\frac{n/\ell}{k}\right) \left(\Phi_{g_f}(f^{k\ell}(\pi(Z)), \pi(Z)) - \Phi_{g_f}(\pi(Z), \pi(Z_j^{(\ell)})) \right) \\ &= \sum_{k \mid \frac{n}{\ell}} \mu\left(\frac{n/\ell}{k}\right) \log |(f^{k\ell})'(z_{f,j}^{(\ell)}) - 1| \quad (\text{by (3.5)}) \\ &= \sum_{k \mid \frac{n}{\ell}} \mu\left(\frac{n/\ell}{k}\right) \log |((f^\ell)'(z_{f,j}^{(\ell)}))^k - 1| = \log |C_{\frac{n}{\ell}}((f^\ell)'(z_{f,j}^{(\ell)}))| \quad (\text{using the chain rule}), \end{aligned}$$

where the second equality follows from $\sum_{m \mid \frac{n}{\ell}} \mu(\frac{n/\ell}{m}) = 0$ also by (3.10) and $n/\ell > 1$.

From the above computations, when $|\text{Res } F| = 1$, using (3.11) repeatedly, we have

$$\begin{aligned}
& \log |\text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})| \\
&= \sum_{j=1}^{d_\ell} \log |\Phi_{F,n}^{**}(Z_{F,j}^{(\ell)})| = \log \left| \prod_{m': m' \nmid \ell \nmid m' | n} \left(\prod_{m: m' | m | n \text{ and } \ell \nmid m} \text{Res}(\Phi_{F,m'}^{**}, \Phi_{F,\ell}^{**})^{\mu(\frac{n}{m})} \right) \right| + 0 + \\
& \quad + \log \left| \prod_{j=1}^{d_\ell} C_{\frac{n}{\ell}}((f^\ell)'(z_{f,j}^{(\ell)})) \right| + (d^\ell)_{\frac{n}{\ell}} \sum_{j=1}^{d_\ell} G^F(Z_{F,j}^{(\ell)}) = \\
&= \log \prod_{m': m' \nmid \ell \nmid m' | n} \left(\prod_{m: m' | m | n \text{ and } \ell \nmid m} |\text{Res}(\Phi_{F,m'}^{**}, \Phi_{F,\ell}^{**})|^{\mu(\frac{n}{m})} \right) + \log |(\Delta_{n,\ell}(f))^\ell| + 0 \quad (\text{by (2.9), (3.9)}),
\end{aligned}$$

so that, no matter whether $|\text{Res } F| = 1$, we have

$$\begin{aligned}
& \left| \frac{(\text{Res } F)^{-N_d(n,\ell)} \text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})}{(\Delta_{n,\ell}(f))^\ell} \right| \\
&= \prod_{m': m' \nmid \ell \nmid m' | n} \left(\prod_{m: m' | m | n \text{ and } \ell \nmid m} |(\text{Res } F)^{-N_d(m',\ell)} \text{Res}(\Phi_{F,m'}^{**}, \Phi_{F,\ell}^{**})|^{\mu(\frac{n}{m})} \right) = 1
\end{aligned}$$

since if $m' \nmid \ell \nmid m'$, then $(\text{Res } F)^{-N_d(m',\ell)} \text{Res}(\Phi_{F,m'}^{**}, \Phi_{F,\ell}^{**}) \in (\mathbb{Z}[\text{Rat}_d])^* (= \mathbb{Z}^* = \{\pm 1\})$. Hence the desired (4.4) holds.

4.2. Next, we claim that

$$(4.5) \quad |\Delta_{n,n}(f)|^n = \left| \frac{(\text{Res } F)^{-N_d(n)} \text{Disc}(\Phi_{F,n}^{**})}{\prod_{\ell: |n \text{ and } < n} (\Delta_{n,\ell}(f))^{n-\ell}} \right| = \prod_{r=0}^{\frac{d_n}{n}-1} \prod_{s \neq r} \left| (\text{Res } F)^{-M_d(n)} \text{Res}(\Lambda_{F,r}^{(n)}, \Lambda_{F,s}^{(n)}) \right|,$$

the first equality in which yields $D_n^{**}/((\Delta_{n,n})^n \cdot \prod_{\ell: |n \text{ and } < n} (\Delta_{n,\ell})^{n-\ell}) \in (\mathbb{Z}[\text{Rat}_d])^* (= \mathbb{Z}^* = \{\pm 1\})$ recalling (2.2) and the definition of D_n^{**} in (2.7), and in the second one in which, setting

$$M_d(n) := \begin{cases} 1 & \text{if } n = 1 \\ d_n(d_n - n)/(d(d-1)) & \text{if } n > 1 \end{cases} \in \mathbb{N},$$

the product $(\text{Res } F)^{-M_d(n)} \prod_{r=0}^{\frac{d_n}{n}-1} \prod_{s \neq r} \text{Res}(\Lambda_{F,r}^{(n)}, \Lambda_{F,s}^{(n)})$ is independent of the choice of F from $\text{Res}(cF) = c^{2d} \text{Res } F$ and $(cF)^n = c^{(d^n-1)/(d-1)} \text{Res } F$ for every $c \in \mathbb{C}^*$;

(i) for, by $F^n(Z) \wedge Z = \prod_{\ell|n} \Phi_{F,\ell}^{**}(Z)$ (by Möbius inversion) and the definition (3.12) of the homogeneous discriminants, we have

$$\begin{aligned}
\text{Disc}(F^n \wedge \text{Id}_{\mathbb{C}^2}) &= \prod_{\ell|n} \prod_{j=1}^{d_\ell} \left(\prod_{\ell': |n \text{ and } \neq \ell} \prod_{k=1}^{d_{\ell'}} (Z_{F,j}^{(\ell)} \wedge Z_{F,k}^{(\ell')}) \times \prod_{k \in \{1, \dots, d_\ell\} \setminus \{j\}} (Z_{F,j}^{(\ell)} \wedge Z_{F,k}^{(\ell)}) \right) \\
&= \prod_{\ell|n} \prod_{\ell': |n \text{ and } \neq \ell} \prod_{j=1}^{d_\ell} \prod_{k=1}^{d_{\ell'}} (Z_{F,j}^{(\ell)} \wedge Z_{F,k}^{(\ell')}) \times \prod_{\ell|n} \text{Disc}(\Phi_{F,\ell}^{**}) \\
&= \prod_{(\ell, \ell') \in \mathbb{N}^2: \ell|n, \ell'|n, \text{ and } \ell \neq \ell'} \text{Res}(\Phi_{F,\ell}^{**}, \Phi_{F,\ell'}^{**}) \times \prod_{\ell|n} \text{Disc}(\Phi_{F,\ell}^{**}),
\end{aligned}$$

using (3.11) repeatedly. Assume now that $|\text{Res } F| = 1$. Then we also compute not only

$$\begin{aligned}
& \log |\text{Disc}(F^n \wedge \text{Id}_{\mathbb{C}^2})| \\
&= \sum_{j=1}^{d^n+1} \sum_{k \in \{1, \dots, d^n+1\} \setminus \{j\}} \Phi_{g_f}(\pi(q_{F,j}^{(n)}), \pi(q_{F,k}^{(n)})) + (d^n+1) \sum_{j=1}^{d^n+1} G^F(q_{F,j}^{(n)}) \quad (\text{by (3.7)}) \\
&= \sum_{j=1}^{d^n+1} \log |(f^n)'(\pi(q_{F,j}^{(n)})) - 1| + 0 \quad (\text{by (3.6) and (3.8)}) \\
&= \sum_{\ell|n} \sum_{j=1}^{d_\ell} \log |((f^\ell)'(z_{f,j}^{(\ell)}))^{\frac{n}{\ell}} - 1| \quad (\text{by } F^n(Z) \wedge Z = \prod_{\ell|n} \Phi_{F,\ell}^{**}(Z) \text{ and the chain rule}) \\
&= \sum_{\ell|n} \sum_{j=1}^{d_\ell} \log \left| \prod_{\ell'|\frac{n}{\ell}} C_{\ell'}((f^\ell)'(z_{f,j}^{(\ell)})) \right| \quad (\text{by Möbius inversion of the sequence } (C_i)_i) \\
&= \sum_{\ell|n} \sum_{\ell'|\frac{n}{\ell}} \log \left| \prod_{j=1}^{d_\ell} C_{\ell'}((f^\ell)'(z_{f,j}^{(\ell)})) \right| = \sum_{\ell|n} \left(\log |p_{\ell}^{**}(1; f)|^\ell + \sum_{\ell': 1 < \ell'|\frac{n}{\ell}} \log |\Delta_{(\ell', \ell), \ell}(f)|^\ell \right) \\
&= \sum_{\ell|n} \log |p_{\ell}^{**}(1; f)|^\ell + \sum_{(\ell, \ell'') \in \mathbb{N}^2: \ell|\ell''|n \text{ and } \ell < \ell''} \log |\Delta_{\ell'', \ell}(f)|^\ell
\end{aligned}$$

(where the 6th equality is by the definition (2.8) of p_{ℓ}^{**} and the first equality in (2.9)) but also

$$\begin{aligned}
& \prod_{(\ell, \ell') \in \mathbb{N}^2: \ell|n, \ell'|n \text{ and } \ell \neq \ell'} |\text{Res}(\Phi_{F,\ell}^{**}, \Phi_{F,\ell'}^{**})| \\
&= \prod_{(\ell, \ell') \in \mathbb{N}^2: \ell|\ell'|n \text{ and } \ell < \ell'} |\text{Res}(\Phi_{F,\ell}^{**}, \Phi_{F,\ell'}^{**})|^2 \times \prod_{(\ell, \ell') \in \mathbb{N}^2: \ell|n, \ell'|n \text{ and } \ell \nmid \ell' \nmid \ell} |\text{Res}(\Phi_{F,\ell}^{**}, \Phi_{F,\ell'}^{**})|.
\end{aligned}$$

The above three computations yield

$$\begin{aligned}
& \prod_{\ell|n} \left(|p_{\ell}^{**}(1; f)|^\ell \cdot \prod_{\ell': \ell \text{ and } < \ell'} |\Delta_{\ell, \ell'}(f)|^{\ell'} \right) \\
&= \prod_{\ell|n} \left(|\text{Disc}(\Phi_{F,\ell}^{**})| \cdot \prod_{\ell': \ell \text{ and } < \ell} |\text{Res}(\Phi_{F,\ell}^{**}, \Phi_{F,\ell'}^{**})|^2 \times \prod_{\ell': \ell'|n \text{ and } \ell \nmid \ell' \nmid \ell} |\text{Res}(\Phi_{F,\ell}^{**}, \Phi_{F,\ell'}^{**})| \right)
\end{aligned}$$

(for every $n \in \mathbb{N}$), and in turn, by Möbius inversions of both sides, we have

$$|p_n^{**}(1; f)|^n \cdot \prod_{\ell: \ell|n \text{ and } < n} |\Delta_{n, \ell}(f)|^\ell = |\text{Disc}(\Phi_{F,n}^{**})| \cdot \prod_{\ell: \ell|n \text{ and } < n} |\text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})|^2,$$

which is, by the definition of $\Delta_{n,n}$, no matter whether $|\text{Res } F| = 1$, equivalent to

$$(4.6) \quad |\Delta_{n,n}(f)|^n = \left| \frac{(\text{Res } F)^{-N_d(n)} \text{Disc}(\Phi_{F,n}^{**})}{\prod_{\ell: \ell|n \text{ and } < n} (\Delta_{n, \ell}(f))^{n-\ell}} \right| \cdot \prod_{\ell: \ell|n \text{ and } < n} \left| \frac{(\text{Res } F)^{-N_d(n, \ell)} \text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})}{(\Delta_{n, \ell}(f))^\ell} \right|^2.$$

(ii) By the definition (3.12) of the homogeneous discriminants, the labeling (4.2) of the sequence $(z_{F,j}^{(n)})_{j=1}^{d_n}$, and the definition (4.3) of $\Lambda_{F,r}^{(n)}$, we have

$$\begin{aligned} \text{Disc}(\Phi_{F,n}^{**}) &= \prod_{r=0}^{\frac{d_n}{n}-1} \prod_{k=0}^{n-1} \left(\prod_{\ell \in \{0, \dots, n-1\} \setminus \{k\}} (Z_{F,1+nr+k}^{(n)} \wedge Z_{F,1+nr+\ell}^{(n)}) \times \prod_{s \neq r} \prod_{\ell=0}^{n-1} (Z_{F,1+nr+k}^{(n)} \wedge Z_{F,1+ns+\ell}^{(n)}) \right) \\ &= \prod_{r=0}^{\frac{d_n}{n}-1} \text{Disc}(\Lambda_{F,r}^{(n)}) \times \prod_{r=0}^{\frac{d_n}{n}-1} \prod_{s \neq r} \prod_{k=0}^{n-1} \prod_{\ell=0}^{n-1} (Z_{F,1+nr+k}^{(n)} \wedge Z_{F,1+ns+\ell}^{(n)}) \\ &= \prod_{r=0}^{\frac{d_n}{n}-1} \text{Disc}(\Lambda_{F,r}^{(n)}) \times \prod_{r=0}^{\frac{d_n}{n}-1} \prod_{s \neq r} \text{Res}(\Lambda_{F,r}^{(n)}, \Lambda_{F,s}^{(n)}), \end{aligned}$$

using (3.11) repeatedly. If $|\text{Res } F| = 1$, then we also have

$$\begin{aligned} &\log \left| \prod_{r=0}^{\frac{d_n}{n}-1} \text{Disc}(\Lambda_{F,r}^{(n)}) \right| \\ &= \sum_{r=0}^{\frac{d_n}{n}-1} \sum_{k=0}^{n-1} \left(\sum_{m \in \{0, 1, \dots, n-1\} \setminus \{k\}} \Phi_{g_f}(z_{f,1+nr+k}^{(n)}, z_{f,1+nr+m}^{(n)}) + n \cdot G^F(Z_{F,1+nr+k}^{(n)}) \right) \quad (\text{by (3.7)}) \\ &= \sum_{r=0}^{\frac{d_n}{n}-1} \sum_{k=0}^{n-1} \sum_{m=1}^{n-1} \Phi_{g_f}(z_{f,1+nr+k}^{(n)}, f^m(z_{f,1+nr+k}^{(n)})) + 0 \quad (\text{by (3.9) and the labeling (4.2) of } (z_{F,j}^{(n)})) \\ &= \sum_{r=0}^{\frac{d_n}{n}-1} \sum_{k=0}^{n-1} \sum_{m=1}^{n-1} \left(\log |Z_{F,1+nr+k}^{(n)} \wedge F^m(Z_{F,1+nr+k}^{(n)})| - G^F(Z_{F,1+nr+k}^{(n)}) - G^F(F^m(Z_{F,1+nr+k}^{(n)})) \right) \\ &= \sum_{r=0}^{\frac{d_n}{n}-1} \sum_{k=0}^{n-1} \sum_{m=1}^{n-1} \sum_{\ell | m} \log |\Phi_{F,\ell}^{**}(Z_{F,1+nr+k}^{(n)})| - \left(\sum_{m=1}^{n-1} (d^m + 1) \right) \sum_{j=1}^{d_n} G^F(Z_{F,j}^{(n)}) \\ &= \log \left| \prod_{m=1}^{n-1} \prod_{\ell | m} \text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**}) \right| - 0 \quad (\text{by (3.11) repeatedly and (3.9)}) \\ &= \log \prod_{\ell: |n \text{ and } < n} |\text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})|^{\frac{n}{\ell}-1} + \log \prod_{\ell: \nmid n \text{ and } < n} |\text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})|^{\lfloor \frac{n}{\ell} \rfloor}, \end{aligned}$$

where the third equality is by (3.7), and the fourth one is by $F^m(Z) \wedge Z = \prod_{\ell | m} \Phi_{F,\ell}^{**}(Z)$ and (3.2). Hence no matter whether $|\text{Res } F| = 1$, we have

$$\begin{aligned} (4.7) \quad &\left| \frac{(\text{Res } F)^{-N_d(n)} \text{Disc}(\Phi_{F,n}^{**})}{\prod_{\ell: |n \text{ and } < n} (\Delta_{n,\ell}(f))^{n-\ell}} \right| \\ &= \prod_{\ell: |n \text{ and } < n} \left| \frac{(\text{Res } F)^{-N_d(n,\ell)} \text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})}{(\Delta_{n,\ell}(f))^\ell} \right|^{\frac{n}{\ell}-1} \times \prod_{\ell: \nmid n \text{ and } < n} |(\text{Res } F)^{-N_d(n,\ell)} \text{Res}(\Phi_{F,n}^{**}, \Phi_{F,\ell}^{**})|^{\lfloor \frac{n}{\ell} \rfloor} \\ &\quad \times \prod_{r=0}^{\frac{d_n}{n}-1} \prod_{s \neq r} |(\text{Res } F)^{-M_d(n)} \text{Res}(\Lambda_{F,r}^{(n)}, \Lambda_{F,s}^{(n)})|. \end{aligned}$$

(iii) Now the desired (4.5) holds by (4.4), (4.6), (4.7), and $R_{n,\ell}^{**} \in (\mathbb{Z}[\text{Rat}_d])^* (= \mathbb{Z}^* = \{\pm 1\})$ if $\ell \nmid n$ and $\ell < n$.

4.3. The two equalities in Theorem 1 have already been shown in Subsections 4.1 and 4.2.

Finally, pick $\tilde{F} = \tilde{F}(X, Y; f) =: \tilde{F}_f(X, Y) \in ((\mathcal{O}_{\text{Rat}_d(\mathbb{C}), \text{an}}(D))[X, Y]_d)^2$ on any simply connected domain D in the complex manifold $\text{Rat}_d(\mathbb{C})$ such that for any individual $f \in D$, \tilde{F}_f is a lift (to \mathbb{C}^2) of f . Then there is a (finitely sheeted) possibly branched analytic covering $\eta : M \rightarrow D$ of D by a complex manifold M such that for every $j \in \{1, \dots, d_n\}$, the mapping $Z_{\tilde{F}_{\eta, j}}^{(n)} : M \rightarrow \mathbb{C}^2 \setminus \{(0, 0)\}$ is complex analytic (, or informally, $Z_{\tilde{F}_{\eta, j}}^{(n)}$ is a marked point in $\mathbb{C}^2 \setminus \{(0, 0)\}$ complex analytically parametrized by M). Hence recalling the definition (4.3) of $\Lambda_{\tilde{F}_{\eta, r}}^{(n)}$, for any distinct $r, s \in \{1, \dots, \frac{d_n}{n} - 1\}$, $\text{Res}(\Lambda_{\tilde{F}_{\eta, r}}^{(n)}, \Lambda_{\tilde{F}_{\eta, s}}^{(n)})$ is a (complex analytic so) locally bounded function on M , and then the second equality in (4.5) yields $\Delta_{n, n} \in \mathbb{Z}[\text{Rat}_d]$. Now the proof of Theorem 1 is complete. \square

Remark 4.1. It also follows from an argument similar to that in the final paragraph in Subsection 4.3 that $\text{Disc}(\Phi_n^{**})$ is in $\mathbb{Z}[a, b]$ (rather than merely in the field $\mathbb{Z}(a, b)$).

Remark 4.2. All the potential theoretic computations from Subsection 3.1 and used in the proof of Theorem 1 still work over an algebraically closed field K that is complete with respect to a non-trivial and possibly non-archimedean absolute value $|\cdot|$ (e.g., over the field \mathbb{C}_p equipped with the (extended) p -adic norm $|\cdot|_p$, or the field \mathbb{L} of “formal” Puiseux series ϕ centered at the origin $z = 0$ in \mathbb{C} equipped with the norm induced by the orders of ϕ at $z = 0$).

5. A FEW COMPUTATIONS IN Rat_2

Focusing on the case where $d = 2$, we conclude with a few computation in Rat_2 . Here we write each quadratic rational function f on \mathbb{P}^1 as $[aX^2 + bXY + cY^2 : pX^2 + qXY + rY^2] = [a : b : c : p : q : r] \in \text{Rat}_2$.

5.1. the $n = 1$ case. When $n = 1$, the hypersurface $V(\Delta_{1,1})$ coincides with the hypersurface $V(\text{Disc}(\Phi_1^{**}))$ in $\text{Rat}_2 = \mathbb{P}^5 \setminus V(\rho_2)$; they are the loci in Rat_2 for every $[a : b : c : p : q : r]$ in which, at least one of the $2_1 = 2 + 1 = 3$ fixed points in \mathbb{P}^1 of $f = f_{[a:b:c:p:q:r]}$ is multiple (when working in an algebraically closed field). We indeed compute

$$\rho_2 = a^2r^2 - abqr - 2acpr + acq^2 + b^2pr - bcpq + c^2p^2$$

and

$$\begin{aligned} \text{Disc}(\Phi_1^{**}) = & 4a^3c - a^2b^2 + 2a^2br - 12a^2cq - a^2r^2 + 2ab^2q + 18abcp - 4abqr - 18acpr + 12acq^2 + 2aqr^2 \\ & - 4b^3p + 12b^2pr - b^2q^2 - 18bcpq - 12bpr^2 + 2bq^2r + 27c^2p^2 + 18cpqr - 4cq^3 + 4pr^3 - q^2r^2, \end{aligned}$$

so that, by the first equality in Theorem 1,

$$\begin{aligned} \Delta_{1,1} &= \pm D_1^{**} \\ &= \pm (\rho_2)^{-1} \text{Disc}(\Phi_1^{**}) \\ &= \pm (4a^3c - a^2b^2 + 2a^2br - 12a^2cq - a^2r^2 + 2ab^2q + 18abcp - 4abqr - 18acpr + 12acq^2 + 2aqr^2 \\ &\quad - 4b^3p + 12b^2pr - b^2q^2 - 18bcpq - 12bpr^2 + 2bq^2r + 27c^2p^2 + 18cpqr - 4cq^3 + 4pr^3 - q^2r^2) / \rho_2 \end{aligned}$$

on Rat_2 .

5.2. the $n = 2$ case. When $n = 2$, since $2_2 = (2^2 + 1) - (2^1 + 1) = 2$, for each $[a : b : c : p : q : r] \in \text{Rat}_2$, there are exactly one cycle C_2 in \mathbb{P}^1 of $f = f_{[a:b:c:p:q:r]}$ having the formally exact periods $n = 2$, and the hypersurface $V(\text{Disc}(\Phi_2^{**}))$ coincides with the hypersurface $V(\Delta_{2,1})$ in Rat_2 ; they are the loci in Rat_2 for every $[a : b : c : p : q : r]$ in which, this cycle C_2 in \mathbb{P}^1 of $f = f_{[a:b:c:p:q:r]}$ reduces to a fixed point of f (when working in an algebraically closed field). We indeed compute

$$\begin{aligned} \text{Disc}(\Phi_2^{**}) = & 4a^3c - a^2b^2 + 2a^2br + 4a^2cq + 3a^2r^2 - 2ab^2q + 2abcp + 6acpr \\ & + 2aqr^2 - b^2q^2 + 2bcpq + 4bpr^2 - 2bq^2r - c^2p^2 + 2cpqr + 4pr^3 - q^2r^2 \end{aligned}$$

and

$$\begin{aligned}
\text{Res}(\Phi_2^{**}, \Phi_1^{**}) &= 4a^5cr^2 - a^4b^2r^2 - 4a^4bcqr + 2a^4br^3 - 8a^4c^2pr + 4a^4c^2q^2 + 4a^4cqr^2 + 3a^4r^4 \\
&\quad + a^3b^3qr + 6a^3b^2cpr - a^3b^2cq^2 - 4a^3b^2qr^2 - 4a^3bc^2pq - 2a^3bcpr^2 - 2a^3bcq^2r \\
&\quad - 3a^3bqr^3 + 4a^3c^3p^2 - 8a^3c^2pqr + 4a^3c^2q^3 + 3a^3cq^2r^2 + 2a^3qr^4 - a^2b^4pr \\
&\quad + a^2b^3cpq + 2a^2b^3pr^2 + 2a^2b^3q^2r - a^2b^2c^2p^2 + 4a^2b^2cpqr - 2a^2b^2cq^3 + 3a^2b^2pr^3 \\
&\quad - a^2b^2q^2r^2 - 2a^2bc^2p^2r - 2a^2bc^2pq^2 - 7a^2bcpr^2 + 4a^2bpr^4 - 4a^2bq^2r^3 + 4a^2c^3p^2q \\
&\quad - 10a^2c^2p^2r^2 + 6a^2c^2pq^2r - 2a^2cpqr^3 + 2a^2cq^3r^2 + 4a^2pr^5 - a^2q^2r^4 - 2ab^4pqr \\
&\quad + 2ab^3cp^2r + 2ab^3cpq^2 + ab^3q^3r - 4ab^2c^2p^2q + 6ab^2cp^2r^2 - ab^2cq^4 - 2ab^2pqr^3 \\
&\quad + 2ab^2q^3r^2 + 2abc^3p^3 - 9abc^2p^2qr + 2abc^2pq^3 - 8abc^2p^2r^3 + 4abcpq^2r^2 - 2abcq^4r \\
&\quad - 4abpqr^4 + abq^3r^3 + 8ac^3p^3r - ac^3p^2q^2 - 2ac^2p^2qr^2 + 2ac^2pq^3r - 8acp^2r^4 \\
&\quad + 6acpq^2r^3 - acq^4r^2 - b^4pq^2r + 2b^3cp^2qr + b^3cpq^3 + 4b^3p^2r^3 - 2b^3pq^2r^2 - b^2c^2p^3r \\
&\quad - 3b^2c^2p^2q^2 - 2b^2cp^2qr^2 + 2b^2cpq^3r + 4b^2p^2r^4 - b^2pq^2r^3 + 3bc^3p^3q + 4bc^2p^3r^2 \\
&\quad - 4bc^2p^2q^2r - 4bc^2pqr^3 + bcpq^3r^2 - c^4p^4 + 2c^3p^3qr + 4c^2p^3r^3 - c^2p^2q^2r^2 \\
&= \rho_2 \cdot \text{Disc}(\Phi_2^{**}),
\end{aligned}$$

so that, also by the equalities in Theorem 1,

$$\begin{aligned}
\Delta_{2,1} &= \pm R_{2,1}^{**} \\
&= \pm(\rho_2)^{-2} \text{Res}(\Phi_2^{**}, \Phi_1^{**}) \\
&= \pm(\rho_2)^{-1} \text{Disc}(\Phi_2^{**}) \\
&= \pm D_2^{**} \\
&= \pm(4a^3c - a^2b^2 + 2a^2br + 4a^2cq + 3a^2r^2 - 2ab^2q + 2abcp + 6acpr + 2aqr^2 - b^2q^2 + 2bcprq \\
&\quad + 4bpr^2 - 2bq^2r - c^2p^2 + 2cpqr + 4pr^3 - q^2r^2)/\rho_2
\end{aligned}$$

on Rat_2 and

$$(\Delta_{2,2})^2 = \pm \frac{D_2^{**}}{\Delta_{2,1}} = 1 \quad \text{or equivalently} \quad \Delta_{2,2} = \pm 1$$

on Rat_2 .

5.3. the $n = 3$ case. When $n = 3$, we similarly compute

$$\begin{aligned}
\Delta_{3,1} &= \pm R_{3,1}^{**} \\
&= \pm(\rho_2)^{-6} \text{Res}(\Phi_3^{**}, \Phi_1^{**}) \quad (\text{by the second equality in Theorem 1}) \\
&= \pm(16a^6c^2 - 8a^5b^2c + 16a^5bcr + 4a^5cr^2 + a^4b^4 - 4a^4b^3r - 8a^4b^2cq + 3a^4b^2r^2 + 48a^4bc^2p \\
&\quad + 4a^4bcqr + 2a^4br^3 + 24a^4c^2pr + 12a^4c^2q^2 + 16a^4cqr^2 + 7a^4r^4 + 2a^3b^4q - 20a^3b^3cp \\
&\quad - 5a^3b^3qr + 30a^3b^2cpr - 11a^3b^2cq^2 - 3a^3b^2qr^2 - 12a^3bc^2pq + 18a^3bcpr^2 - 2a^3bcq^2r \\
&\quad - 5a^3bqr^3 + 36a^3c^3p^2 + 48a^3c^2pqr - 8a^3c^2q^3 + 8a^3cpr^3 + 7a^3cq^2r^2 + 2a^3qr^4 + 2a^2b^5p \\
&\quad - 7a^2b^4pr + 3a^2b^4q^2 - 9a^2b^3cpq + 5a^2b^3pr^2 - 3a^2b^3q^2r + 27a^2b^2c^2p^2 - 18a^2b^2cpqr - a^2b^2cq^3 \\
&\quad + 7a^2b^2pr^3 + 54a^2bc^2p^2r + 18a^2bc^2pq^2 + 63a^2bcpr^2 - 16a^2bcq^3r + 16a^2bpr^4 - 3a^2bq^2r^3 \\
&\quad + 54a^2c^2p^2r^2 - 18a^2c^2pq^2r + 9a^2c^2q^4 + 18a^2cpqr^3 + 5a^2cq^3r^2 + 4a^2pr^5 + 3a^2q^2r^4 + 2ab^5pq \\
&\quad - 12ab^4cp^2 - 4ab^4pqr + 2ab^4q^3 + 12ab^3cp^2r - 12ab^3cpq^2 - 16ab^3pqr^2 + ab^3q^3r - 27ab^2c^2p^2q \\
&\quad - 18ab^2cp^2r^2 - ab^2cq^4 - 2ab^2pqr^3 - 3ab^2q^3r^2 + 54abc^3p^3 + 81abc^2p^2qr - 30abc^2pq^3 \\
&\quad + 48abcp^2r^3 - 18abcpq^2r^2 - 4abcq^4r + 4abpqr^4 - 5abq^3r^3 + 27ac^3p^2q^2 + 54ac^2p^2qr^2 \\
&\quad + 12ac^2pq^3r - 3ac^2q^5 + 24acp^2r^4 + 30acpq^2r^3 - 7acq^4r^2 + 16apqr^5 - 4aq^3r^4 + b^6p^2 \\
&\quad - 3b^5p^2r + 2b^5pq^2 + 3b^4cp^2q + 9b^4p^2r^2 - b^4pq^2r + b^4q^4 - 9b^3c^2p^3 - 30b^3cp^2qr + 5b^3cpq^3 \\
&\quad - 8b^3p^2r^3 - b^3pq^2r^2 + 2b^3q^4r + 27b^2c^2p^3r + 18b^2cp^2qr^2 - 12b^2cpq^3r + 2b^2cq^5 + 12b^2p^2r^4 \\
&\quad - 11b^2pq^2r^3 + 3b^2q^4r^2 - 27bc^3p^3q - 27bc^2p^2q^2r + 3bc^2pq^4 - 12bc^2pqr^3 - 9bcpr^3r^2 \\
&\quad + 2bcq^5r - 8bpq^2r^4 + 2bq^4r^3 + 27c^4p^4 + 54c^3p^3qr - 9c^3p^2q^3 + 36c^2p^3r^3 + 27c^2p^2q^2r^2 \\
&\quad - 12c^2pq^4r + c^2q^6 + 48cp^2qr^4 - 20cpq^3r^3 + 2cq^5r^2 + 16p^2r^6 - 8pq^2r^5 + q^4r^4)/(\rho_2)^2
\end{aligned}$$

on Rat_2 ; the computations of $D_3^{**} = (\rho_2)^{-15} \text{Disc}(\Phi_3^{**})$ and $\Delta_{3,3}$ on Rat_2 are more lengthy.

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