Math 2A03 Midterm Exam

Wednesday 18 May, 2010

Name: ______________________________

Student number: ____________________

Instructor: Dr. Trevor Arnold

Duration of examination: 2 Hours.
McMaster University midterm examination.

This examination paper includes 5 pages and 12 questions. You are responsible for ensuring your copy of the paper is complete. Bring any discrepancy to the attention of the invigilator.

Special instructions: Use of the McMaster standard Casio FX991 calculator *only* is allowed.

Question	Marks	Max. Marks
1		5
2		6
3		6
4		8
5		10
6		9
7		8
8		8
9		8
10		10
11		10
12		12
Total		100
Mark		12

1. Suppose $z = e^{xy}$. If $x = t^2$ and $y = t^3$, use the chain rule to find $\frac{dz}{dt}$ as a function of t.

2. Find the length of the curve

$$\mathbf{r}(t) = \langle t, \sqrt{2}e^t, \frac{1}{2}e^{2t} \rangle, \quad 0 \leq t \leq \ln 2.$$

Note: you can probably expect to be taking the square root of a perfect square.

3. In what direction does the function

$$f(x, y, z) = x^2 e^{yz}$$

increase most rapidly starting from the point $(2, 1, 0)$? What is the maximum rate of increase?

4. Explain why the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 2y^2}$$

does not exist.

5. Let

$$f(x, y) = x^3 - 6xy + 8y^3.$$

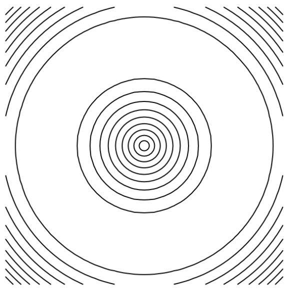
(a) Find all the critical points of f and classify as local maxima/minima or saddle points.

(b) Does f attain a global maximum or minimum at any point? Provide a brief but convincing argument for why your answer is correct. (It might help to consider how f behaves when you look at (x, y) lying on the coordinate axes.)

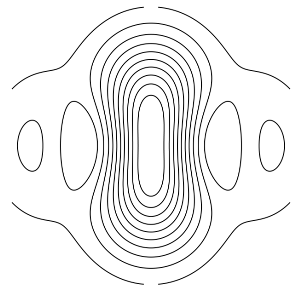
6. Find the maximum and minimum values and the point(s) where they occur for the function $f(x, y, z) = x + 2y$ subject to the constraints $x^2 + y^2 = 16$ and $x + y + z = 6$.

7. Matching! For each contour plot, identify the corresponding 3D graph.

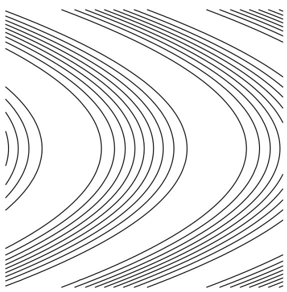
(a)



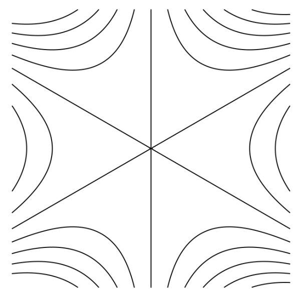
(b)



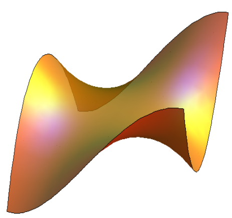
(c)



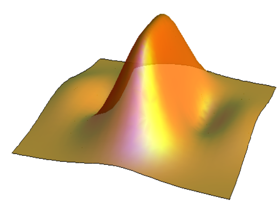
(d)



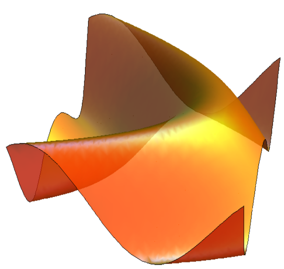
(i)



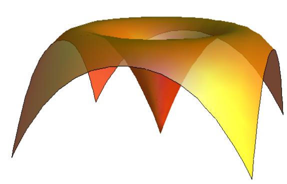
(ii)



(iii)



(iv)



8. Find parametric equations for the tangent line to the curve

$$x = 2\sin t, \quad y = 2\sin(2t), \quad z = 2\sin(3t)$$

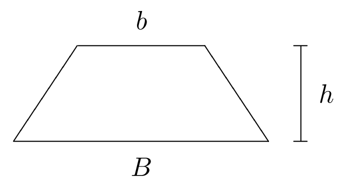
at the point $(1, \sqrt{3}, 2)$.

9. Reparametrize the space curve

$$\mathbf{r}(t) = e^t\,\mathbf{i} + e^t \sin t\,\mathbf{j} + e^t \cos t\,\mathbf{k}.$$

with respect to arclength s starting from the point $(1, 0, 1)$ (corresponding to $t = 0$).

10. Recall that the area of a trapezoid is $A = \frac{1}{2}(B + b)h$, where B and b are the parallel edges and h is the distance between them, as illustrated in the picture.



(a) Suppose the dimensions of the trapezoid pictured are $B = 2$ m, $b = 1$ m, and $h = 1.5$ m. If the lower base (B) is growing at a rate of 3 m/s, the upper base (b) is growing at a rate of 4 m/s, and the height (h) is growing at a rate of 6 m/s, then at what rate is the area (A) growing? Give units for your answer.

(b) What is the largest possible area of a trapezoid with dimensions satisfying the constraint $\frac{1}{2}B^2 + b^2 + 3h^2 = 6$? What are the dimensions of the trapezoid of this largest possible area? Note: you don't need to consider negative values for B, b, and h.

11. Let $f(x, y, z) = \frac{1}{4}x^2 + (y - 1)^2 + \frac{1}{2}z^2$.

(a) Compute $\mathrm{D}_{\mathbf{u}}\, f(2, 2, 2)$ for $\mathbf{u} = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$.

(b) Show that the tangent plane to the level surface $f(\mathbf{x}) = 4$ at the point $(2, 2, 2)$ is parallel to the plane through $(0, 0, 0)$ spanned by the vectors $(0, 1, -1)$ and $(2, -1, 0)$.

(c) Are there any other points on the level surface $f(\mathbf{x}) = 4$ where the tangent plane is parallel to the plane described in (b)? If there are such points, find them.

12. Consider the curves

$$\mathbf{q}(t) = \langle t, t^2 - t, t^2 - t \rangle \quad \text{and} \quad \mathbf{r}(t) = \langle t^2 + t, -t, t^2 + t \rangle.$$

(a) Show that the point $(2, 2, 2)$ lies on both curves, i.e., they intersect at that point. Do these two curves intersect at any other points? If so, find these other points of intersection.

(b) Find the angle between the curves at $(2, 2, 2)$, i.e., the angle between their tangent vectors at that point.

(c) Show that the curves $\mathbf{q}$ and $\mathbf{r}$ lie completely on the surface

$$x + y = (z - x - y)^2.$$

(d) Find the equation of the tangent plane to

$$x + y = (z - x - y)^2$$

at the point $(2, 2, 2)$. You can use your computations from (b) to save yourself some work, but you don't have to.