LAST (family) NAME: ______________________ **Test # 1**

FIRST (given) NAME: ______________________ **Math 2P04**

ID # : __________________

SAMPLE TEST 1 (b) SOLUTIONS

Instructions: You **must** use permanent ink. Tests submitted in pencil will not be considered later for remarking. This exam consists of 8 problems on 10 pages (make sure you have all 10 pages). The last page is for scratch or overflow work. The total number of points is 50. Do not add or remove pages from your test. No books, notes, or "cheat sheets" allowed. The only calculator permitted is the McMaster Standard Calculator, the Casio fx 991.

GOOD LUCK!

Points:

1. ________ (4)

2. ________ (4)

3. ________ (4)

4. ________ (4) *SOLUTIONS*

5. ________ (10)

6. ________ (8)

7. ________ (6)

8. ________ (10)

TOTAL: ________ (50)

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PART I: Multiple choice. Indicate your choice very clearly. There is only one correct answer in each multiple-choice problem. Circle the letter (a,b,c,d or e) corresponding to your choice. Ambiguous answers will be marked as wrong.

1. *(4 pts.)* Let $y(x)$ be the unique solution to the initial value problem

$$\begin{cases} y' = y^2 \cos(y), \\ y(0) = a. \end{cases}$$

For which one of the values of a listed below, does the solution $y(x)$ satisfy

$$\lim_{x \to +\infty} y(x) = 0?$$

(**Hint**: Do not try to solve the ODE)

(a) $a = -3.$

(b) $a = -2.$

$\rightarrow$**(c)** $a = -1.$

(d) $a = 1.$

(e) $a = 2.$

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2. *(4 pts.)* Find the general solution of the differential equation

$$y^{(6)} + 4\,y^{(4)} - 16\,y^{(2)} - 64\,y = 0.$$

(**Hint**: $m^6 + 4\,m^4 - 16\,m^2 - 64 = (m^2 - 4)\,(m^4 + 8\,m^2 + 16)$)
The answer (where $C_1, \ldots, C_6$ denote arbitrary constants) is:

$\rightarrow$ **(a)** $y(x) = C_1\,e^{2x} + C_2\,e^{-2x} + C_3\,\cos(2x) + C_4\,\sin(2x) + C_5\,x\,\cos(2x) + C_6\,x\,\sin(2x)$.

(b) $y(x) = C_1\,e^{2x} + C_2\,e^{-2x} + C_3\,x\,e^{2x} + C_4\,x\,e^{-2x} + C_5\,\cos(2x) + C_6\,\sin(2x)$.

(c) $y(x) = C_1\,e^{2x} + C_2\,e^{-2x} + C_3\,\cos(2x) + C_4\,\sin(2x) + C_5\,\cos(4x) + C_6\,\sin(4x)$.

(d) $y(x) = C_1\,e^{2x} + C_2\,e^{-2x} + C_3\,\cos(3x) + C_4\,\sin(3x) + C_5\,x\,\cos(3x) + C_6\,x\,\sin(3x)$.

(e) $y(x) = C_1\,e^{4x} + C_2\,e^{-4x} + C_3\,\cos(2x) + C_4\,\sin(2x) + C_5\,\cos(4x) + C_6\,\sin(4x)$.

We have

$$\begin{aligned} m^6 + 4\,m^4 - 16\,m^2 - 64 &= (m^2 - 4)\,(m^4 + 8\,m^2 + 16)) = (m^2 - 4)\,(m^2 + 4)^2 \\ &= (m + 2)\,(m - 2)\,(m + 2i)^2\,(m - 2i)^2. \end{aligned}$$

Thus $m = \pm 2$ are both simple roots and $m = \pm 2\,i$ are both complex roots of multiplicity 2.

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3. *(4 pts.)* Let $y(x)$ be the unique solution of the initial value problem:

$$\begin{cases} \frac{dy}{dx} = -2\, e^x\, y^3, \\ y(0) = 1. \end{cases}$$

Then, the largest interval centered at $x = 0$ where the solution is defined and continuous is:

(a) $(-\infty, \infty)$.
(b) $(-e^{-1}, e^{-1})m$.
(c) $(-\frac{1}{4}, \frac{1}{4})$.
→**(d)** $(-\ln(\frac{4}{3}), \ln(\frac{4}{3}))$.
(e) $(-\ln(\frac{2}{3}) - 2, \ln(\frac{2}{3})) + 2)$.

The DE is separable:m

$$-\frac{1}{2\,y^3}\,\frac{dy}{dx} = e^x$$

Integrating, we get

$$\frac{1}{4\,y^2} = e^x + C$$

Since $y(0) = 1$, $C = -\frac{3}{4}$, so $y = \pm\frac{1}{\sqrt{4\,e^x - 3}}$. Since $y(0) = 1 > 0$, we have

$$y = \frac{1}{\sqrt{4\,e^x - 3}}$$

The solution is defined as long as $4\,e^x - 3 > 0$ or $x > \ln(3/4) = -\ln(4/3)$. The largest interval centered at $x = 0$ where the solution is defined is thus the interval $(-\ln(\frac{4}{3}), \ln(\frac{4}{3}))$.

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4. *(4 pts.)* Let $g(x)$ be the unique solution of the initial value problem:

$$\begin{cases} \frac{dy}{dx} = y\,(1-y)^2, \\ y(0) = \frac{1}{5}. \end{cases}$$

Then, the graph of the solution $g(x)$ has an inflection point at the point (x_0, y_0) where y_0 is the following:
(**Hint**: Do not try to solve the ODE)

(a) $y_0 = -1.$
(b) $y_0 = \frac{1}{2}.$
(c) $y_0 = \frac{1}{8}$.m
(d) $y_0 = \frac{3}{4}.$
→ **(e)** $y_0 = \frac{1}{3}.$

Letting $F(y) = y\,(1-y)^2$, we have $\frac{dy}{dx} = F(y)$ and

$$\begin{aligned} \frac{d^2y}{dx^2} &= F'(y)\,\frac{dy}{dx} = F'(y)\,F(y) \\ &= [(1-y)^2 - 2\,y\,(1-y)]\,y\,(1-y)^2 = (1-3\,y)\,y\,(1-y)^3. \end{aligned}$$

In particular, if $y_0 = y(x_0)$, $\frac{d^2y}{dx^2}|_{x=x_0} = 0$ when $y_0 = \frac{1}{3}$. Note that since $y(0) = \frac{1}{5}$, the solution is strictly increasing and approaches the value 1 as $x \to +\infty$. When y reaches the value $\frac{1}{3}$, the concavity changes from positive to negative. There is thus am inflection point at the corresponding point $(x_0, \frac{1}{3})$.

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Part II: Provide all details and fully justify your answer in order to receive credit.

5. *(10 pts.)* Compute in explicit form the solution of the initial value problem

$$\begin{cases} y' - 2\,y = g(x), \\ \quad y(0) = 1. \end{cases}$$

where

$$g(x) = \begin{cases} -e^x, & x < 1, \\ (1-x)\,e^x, & x > 1. \end{cases}$$

(**Hint**: Make sure that your solution is continuous)

The ODE is of 1st order and linear. The integrating factor is

$$u(x) = e^{\int -2\,dx} = e^{-2x}.$$

For $x < 1$, we have $y' - 2\,y = -e^x$ and, multiplying both sides by the integrating factor the DE become

$$(e^{-2x}\,y)' = -e^{-x}.$$

Integrating, we obtain,

$$e^{-2x}\,y = e^{-x} + C.$$

Since $y(0) = 1$, $C = 0$, so $y(x) = e^x$, for $x \leq 1$. Note in particular that $y(1) = e$. For $x > 1$, we have $y' - 2\,y = (1-x)\,e^x$ and, multiplying both sides by the integrating factor, the DE become

$$(e^{-2x}\,y)' = (1-x)\,e^{-x}.$$

Since $\int (1-x)\,e^{-x}\,dx = x\,e^{-x} + C$, we obtain after integrating both sides that

$$e^{-2x}\,y = x\,e^{-x} + C.$$

Thus

$$y = x\,e^x + C\,e^{2x}$$

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Since $y(1) = e$, we have $e + C\,e^2 = e$ so $C = 0$. Hence, $y = x\,e^x$, for $x > 1$. To summarize, the solution is

$$y(x) = \begin{cases} e^x, & x \le 1, \\ x\,e^x, & x > 1. \end{cases}$$

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6. *(8 pts.)* Compute the general solution of the linear DE

$$y^{(3)} - y^{(2)} - y' + y = 9\,x\,e^{2\,x}.$$

(**Hint**: $m = -1$ is root of the auxiliary equation.)

The auxilliary equation is $m^3 - m^2 - m + 1 = 0$ which can be written in factorized form as $(m+1)(m^2 - 2\,m + 1) = 0$ or $(m+1)\,(m-1)^2 = 0$. Thus $m = -1$ is a simple root and $m = 1$ is a double root. The complementary solution is given by

$$y_c(x) = C_1\,e^{-x} + C_2\,e^{x} + C_3\,x\,e^{x}, \quad C_1, C_2, C_3 \,\text{arbitrary constants.}$$

A particular solution has the form

$$y_p(x) = (A\,x + B)\,e^{2\,x}.$$

We have

$$\begin{aligned} y_p' &= [2\,A\,x + (A + 2\,B)]\,e^{2\,x}, \\ y_p'' &= [4\,A\,x + (4\,A + 4\,B)]\,e^{2\,x}, \\ y_p''' &= [8\,A\,x + (12\,A + 8\,B)]\,e^{2\,x}. \end{aligned}$$

Thus,

$$y_p^{(3)} - y_p^{(2)} - y_p' + y_p = [3\,A\,x + (7\,A + 3\,B)]\,e^{2\,x} = 9\,x\,e^{2\,x}.$$

Solving for the constants, we obtain $A = 3$, $B = -7$. Hence, $y_p(x) = (3\,x - 7)\,e^{2\,x}$ and the general solution is

$$y(x) = y_p(x) + y_c(x) = (3\,x - 7)\,e^{2\,x} + C_1\,e^{-x} + C_2\,e^{x} + C_3\,x\,e^{x},$$

where C_1, C_2, C_3 are arbitrary constants.

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7. *(6 pts.)* Find the form of a particular solution of the DE

$$y^{(4)} - 2\,y^{(3)} + 2\,y^{(2)} - 2\,y' + y = x\,e^{2x} + x^2 - \cos(x).$$

obtained from the method of undetermined coefficients.
(**Hint**: the polynomial $m^4 - 2\,m^3 + 2\,m^2 - 2\,m + 1$ factors as $(m-1)^2\,(m^2+1)$.)
Do **not** solve for the coefficients!

The auxilliary equation is $m^4 - 2\,m^3 + 2\,m^2 - 2\,m + 1 = 0$ or $(m-1)^2\,(m^2+1) = 0$. Thus $m = 1$ is a double root and $m = \pm i$ are both complex roots of multiplicity one. A particular solution y_p has the form

$$y_p(x) = (A\,x + B)\,e^{2x} + (C\,x^2 + D\,x + E) + F\,x\,\cos(x) + G\,x\,\sin(x).$$

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8. *(10 pts.)* Use the **variation of parameters** method to find the general solution of the differential equation

$$4\,y'' - 4\,y' + y = \frac{4\,e^{x/2}}{1+x}, \qquad x > -1.$$

The ODE is linear with constant coefficients. The auxilliary equation is written as $4\,m^2 - 4\,m + 1 = 0$ or $4\,(m - \frac{1}{2})^2 = 0$. Thus $m = \frac{1}{2}$ is double root of the auxilliary equation and the complementary solution has the form $y_c(x) = C_1\,e^{x/2} + x\,C_2\,e^{x/2}$, where C_1 and C_2 are arbitrary constants. In particular,we can choose the functions $y_1(x) = e^{x/2}$ and $y_2(x) = x\,e^{x/2}$ to form a fundamental system of solutions for the associated homogeneous equation. The corresponding Wronskian is

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x/2} & x\,e^{x/2} \\ \frac{1}{2}\,e^{x/2} & (1+\frac{x}{2})\,e^{x/2} \end{vmatrix} = e^x.$$

In standard form, the RHS of the DE becomes $f(x) = \frac{e^{x/2}}{1+x}$. Hence,

$$\begin{aligned} y_p(x) &= -\left[\int \frac{f(x)\,y_2(x)}{W(y_1, y_2)(x)}\,dx\right]\,y_1(x) + \left[\int \frac{f(x)\,y_1(x)}{W(y_1, y_2)(x)}\,dx\right]\,y_2(x) \\ &= -\left[\int \frac{e^{x/2}\,x\,e^{x/2}}{(1+x)\,e^x}\,dx\right]\,e^{x/2} + \left[\int \frac{e^{x/2}\,e^{x/2}}{(1+x)\,e^x}\,dx\right]\,x\,e^{x/2} \\ &= -\left[\int \frac{x}{1+x}\,dx\right]\,e^{x/2} + \left[\int \frac{1}{1+x}\,dx\right]\,x\,e^{x/2} \end{aligned}$$

Since

$$\int \frac{1}{1+x}\,dx = \ln(1+x) \quad \text{and} \quad \int \frac{x}{1+x}\,dx = \int 1 - \frac{1}{1+x}\,dx = x - \ln(1+x),$$

we have

$$y_p(x) = -[x - \ln(1+x)]\, e^{x/2} + \ln(1+x)\, x\, e^{x/2} = \ln(1+x)\,(x+1)\, e^{x/2} - x\, e^{x/2},$$

and the general solution is

$$y(x) = y_p(x) + y_c(x) = \ln(1+x)\,(x+1)\, e^{x/2} - x\, e^{x/2} + C_1\, e^{x/2} + C_2\, x\, e^{x/2},$$

where C_1 and C_2 are arbitrary constants, or, more simply,

$$y(x) = \ln(1+x)\,(x+1)\, e^{x/2} + C_1\, e^{x/2} + C_2\, x\, e^{x/2}.$$

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SCRATCH

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