# l'Hospital's Rule: A Classic Example

Let's compute the limit: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Notice, if I plug in the "$\infty$ ", we, formally, get:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \left(1+\frac{1}{\infty}\right)^\infty = \left(1+0\right)^\infty = 1^\infty$$

But recognize, the base isn't quite 1. And the exponent, of course isn't even a real number, so what's going on?
Well, any number slightly larger than 1, to a large power, would expand to $\infty$ .
Conversely, any number slightly smaller than 1 would shrink away to zero.
And of course, an *exact* value, taken to any power, even a very large one, is still 1.

So as it stands, it appears we have an *indeterminate form*.

Now, is there any way to change this indeterminate form into one we know how to deal with?
Well, since the problem is with the exponent, let's bring it down with a ln-operation.

If we let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ then

$$\ln(y) = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\ln\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$$

Again, plugging in our "$\infty$ ", then

$$\ln(y) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \infty\cdot\ln(1) = \infty\cdot 0$$

Now, this form looks familiar. We know how to deal with this: we can convert this, with a little algebra, to one of the standard l'Hopital's rule forms, $0/0$ or $\infty/\infty$.

$$\ln(y) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x} = \text{"}\frac{0}{0}\text{"}$$

Now we can use l'Hopital's rule,

$$\ln(y) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)}{\frac{d}{dx}1/x} = \lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}\cdot\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)} = \frac{1}{\left(1+\frac{1}{\infty}\right)} = \frac{1}{\left(1+\frac{1}{\infty}\right)} = \frac{1}{(1+0)} = 1$$

But, now we have that $\ln(y) = 1$, so $y = e^1 = e$

And remember, $y$ was the name we gave the original limit, so we've discovered:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

This is a classic result, which can be used to generate approximations of this famous number.