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Math objects made using MathType; graphs made using Winplot.

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System of Frequency

$x(t) = e^{i\omega t}$

$\Rightarrow y(t) = H(\omega)e^{i\omega t}$

$x(t) = \cos(\omega t)$

$$y(t) = \underbrace{|H(\omega)|}_{\text{gain}} \cos\left(\omega t + \underbrace{\sphericalangle H(\omega)}_{\text{phase shift}} + \psi\right)$$

All gain is stored from 0-π.

e.g. 1) Radar

An antenna array (many small antennas) sends out multiple sines. The phase shift (gain) of the original signal says how far the object is away.

Doppler radar: changes the frequency so you can see the speed that things are moving

Discrete Fourier Series (DFS)

If x(n) is p-periodic, $x(n) = \sum_{k=0}^{p-1} X_k e^{i\omega_0 kn}, \omega_0 = \frac{2\pi}{p}$

Associated Analysis Equation

$$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n) e^{-i\omega_0 kn}$$

e.g. 2)

x(n):

0	1	0	−1

p = 4

$$X_0 = \tfrac{1}{4}\begin{pmatrix}0 & 1 & 0 & -1\end{pmatrix} = 0$$

$$X_1 = \tfrac{1}{4}\begin{pmatrix}0 & \underbrace{e^{-i\frac{\pi}{2}}}_{-i} & 0 & -e^{i-\frac{3\pi}{2}}\end{pmatrix} = \frac{-i}{2}$$

$$X_2 = \tfrac{1}{4}\begin{pmatrix}0 & e^{-i\frac{2\pi}{2}} & 0 & -e^{-i2\cdot\frac{3\pi}{2}}\end{pmatrix} = 0$$

$$X_3 = \tfrac{1}{4}\begin{pmatrix}0 & \underbrace{e^{i\frac{2\pi}{2}}}_{i} & 0 & -e^{-i3\cdot\frac{3\pi}{2}}\end{pmatrix} = \frac{i}{2} = -\frac{1}{2i}$$

- Coefficient in front of the brackets is 1/p
- Since the first and 3rd x values are 0, so are the first and third parts of each X

$$\begin{aligned} x(n) &= \tfrac{1}{2i} e^{i\frac{\pi}{2}n} + \tfrac{-1}{2i}\overbrace{e^{i\frac{3\pi}{2}n}}^{=e^{-i\frac{\pi}{2}n}} \\ &= \sin\left(\tfrac{\pi}{2} n\right) \end{aligned}$$

i is like a phase shift of π/2

$$X(0) = \tfrac{1}{2i} + \tfrac{-1}{2i} = 0$$

Fourier Series

$$X(t) \approx \sum_{k=-\infty}^{\infty} X_k e^{i\omega_0 kt}$$

This is not guaranteed to converge, which is why we use a ≈, instead of an equal sign, like we did for the Discrete Fourier Series.

$$X_k = \frac{1}{p}\int_0^p x(t) e^{-i\omega_0 kt}$$

It’s very difficult to build a stabilizer. You need a material that has resistance that increases with heat, like platinum, unlike silicon.

e.g. 3) DTFT

Using an LTI system:

$$x(n)=e^{i\omega n}$$

$$H(\omega)\Rightarrow y(n)=H(\omega)e^{i\omega n}$$

$$h(n)\Rightarrow y(n)=\sum_{k=-\infty}^{\infty}h(n)x(n-k)\ \text{(Convolution)}$$

Although the system was designed in the frequency domain, you will need to compute the output to any input by going back to the time domain.

$$\begin{aligned}y(n)&=\sum_{k=-\infty}^{\infty}h(k)e^{i\omega(n-k)}\\&=e^{i\omega n}\sum_{k=-\infty}^{\infty}h(k)e^{-ik\omega}\end{aligned}$$

Discrete-Time Fourier Transform (DTFT): $\boxed{H(\omega)=\sum_{k=-\infty}^{\infty}h(k)e^{-i\omega k}}$

What can we do with this?

e.g. 4)

$$y(n)=\frac{1}{3}x(n)+\frac{1}{3}x(n-1)+\frac{1}{3}x(n-2)$$

$$h(n)=\begin{cases}\frac{1}{3}, & n=0,1,2\\ 0, & \text{else}\end{cases}$$

$$x(n)=e^{i\omega n}x(n-1)=e^{-i\omega}e^{i\omega n}$$

$$y(n)=H(\omega)e^{i\omega n}$$

$$H(\omega)=\frac{1}{3}+\frac{1}{3}e^{-i\omega}+\frac{1}{3}e^{-i2\omega}$$

More DTFT

Continuation of the example, deriving the Discrete-Time Fourier Transform:

$$\begin{aligned}H(\omega)&=\sum_{k=-\infty}^{\infty}h(k)e^{-i\omega k}\\&=h(0)e^{0}+h(1)e^{-i\omega}+h(2)e^{-i2\omega}\\&=\tfrac{1}{3}+\tfrac{1}{3}e^{-i\omega}+\tfrac{1}{3}e^{-i2\omega}\end{aligned}$$

Converting to Continuous

$x(t) = e^{i\omega t}$

$y(t) = H(\omega)e^{i\omega t}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau$$
$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{i\omega(t-\tau)}\mathrm{d}\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)e^{-i\omega\tau}\mathrm{d}\tau e^{i\omega t}$$

Continuous Time Fourier Transform (**CTFT**): $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\mathrm{d}t$