Lecture 2013-10-09

Least Squares

A m×n, m>n, b∊R^m

r = b – Ax

$$\min \varphi(x) = \frac{1}{2}\|b - AX\|_2$$

$$\sum_{i=1}^{m} a_k \sum_{j=1}^{n} a_{ij} x_j = \sum_{i=1}^{k} a_{ik} b_i, i = 1..n$$

$$A^T Ax = A^T b$$

normal equations

Linear Regression

(t_i,b_i) i = 1,..,m

p(x) = x_1 + x_2t

$$p(t_1) = x_1 + x_2 t_1 \approx b_1$$

$$p(t_2) = x_1 + x_2 t_2 \approx b_2$$

$$\vdots$$

$$p(t_m) = x_1 + x_2 t_m \approx b_m$$

You can re-write this in matrix form:

$$A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_m \end{bmatrix} \cdot \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{m} 1 & \sum_{i=1}^{m} t_i \\ \sum_{i=1}^{m} t_i & \sum_{i=1}^{m} t_i^2 \end{bmatrix}$$

$$= B$$

$$A^{\mathrm{T}}B = \begin{bmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{m} b_i \\ \sum_{i=1}^{m} t_i b_i \end{bmatrix}$$

$$\underbrace{mx_1}_{\sum_{i=1}^{m} 1} + \left(\sum_{i=1}^{m} t_i \right) x_2 = \sum_{i=1}^{m} b_i$$

$$\left(\sum_{i=1}^{m} t_i \right) x_1 + \left(\sum_{i=1}^{m} t_i^2 \right) x_2 = \sum_{i=1}^{m} t_i b_i$$

t_i	0	1	2
b_i	0.1	0.9	2.0

$$3x_1 + 3x_2 = 3$$

$$3x_1 + \underbrace{5}_{0^2+1^2+2^2} x_2 = 4.9$$

$$\left(A^{\mathrm{T}}A \right) x = \left(A^{\mathrm{T}}b \right)$$

$$x_1 = 0.05$$

$$x_2 = 0.95$$

$$p(t) = 0.05 + 0.95t$$

$$p_{n-1}(t) = x_1 + x_2 t + \dots + x_n t^{n-1}$$

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \dots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \dots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \dots & t_m^{n-1} \end{bmatrix} \leftarrow \text{Verdermonde}$$

$$p(t) = x_1 + x_2 t + x_3 t^2$$

$$\begin{aligned} b_1 &\approx 1 \cdot x_1 + t_1 \cdot t_2 + t_1^2 \cdot x_3 \\ b_2 &\approx 1 \cdot x_1 + t_2 t_2 + t_2^2 \cdot x_3 \\ &\vdots \\ b_m &\approx x_1 + t_m x_2 + t_m^2 x_3 \end{aligned}$$

$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & & \end{bmatrix}$$