MATHEMATICS 2C03

DAY SECTIONS 01, 02
DURATION of FINAL EXAM: 3 HOURS
McMASTER UNIVERSITY

THIS EXAMINATION INCLUDES 20 PAGES AND 20 QUESTIONS. IT IS POSSIBLE TO OBTAIN A TOTAL OF 102 MARKS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR. A **Table of Laplace Transforms** is provided on the top of page 3.

SPECIAL INSTRUCTIONS:
Page 20 is to be used for continuation of a problem if you run out of space. **Ask the invigilators** for **SCRAP paper** if required. **NO NOTES OR AIDS OR PIECES OF PAPER OF ANY KIND (other than that distributed by the invigilator) ARE PERMITTED.**

You are **NOT permitted** to have any **ELECTRONIC DEVICES** of any kind, including calculators and cell phones.

PART I is made up of **14 Multiple choice questions**. **MARK YOUR ANSWERS ON THE OMR EXAMINATION SHEET with HB pencil ONLY.** For this part, only the OMR Examination sheet will be marked. Each multiple choice question is worth **3 marks**. There is **no penalty** for an incorrect answer.

PART II is made up of **6 Complete Answer Questions**. Each question is of equal value. Only the solutions to the **5 questions** that you indicate will be marked and count towards your final score. You must **indicate clearly which 5 questions you want to count by circling the question number on the page with the solution** or we will count the first 5 questions attempted.

You must print your name and ID number at the **top of each complete answer page** in the space provided **as well as** on this page below. **You MUST hand in both the OMR EXAMINATION SHEET** and **this examination paper.**

NAME:______________________________

ID #:____________________ Tutorial #:____________________

Questions	Mark	Out of
15		12
16		12
17		12
18		12
19		12
20		12
TOTAL		60

GOOD LUCK!

continued ...

Multiple Choice:

1) E
2) C
3) A
4) B
5) A
6) A
7) B
8) B
9) D
10) C
11) A
12) C
13) C
14) D

continued ...

Table of Laplace Transforms

$$\mathcal{L}\{f(t)\} = F(s)$$

$\mathcal{L}\{1\} = \frac{1}{s}$ $\quad$ $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

$\mathcal{L}\{\sin(\beta t)\} = \frac{\beta}{s^2+\beta^2}$ $\quad$ $\mathcal{L}\{\cos(\beta t)\} = \frac{s}{s^2+\beta^2}$ $\quad$ $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $n \geq 1$ integer

$\mathcal{L}\{e^{kt}f(t)\} = F(s-k)$ $\quad$ $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$ $\quad$ $\mathcal{L}\{\delta(t-a)\} = e^{-as}$, $a \geq 0$

$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$

$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s)$, $c \geq 0$, where $u(t)$ is the unit step function or Heaviside function.

If $f(t+T) = f(t)$ for all t, $\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-sT}}$

$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{\int_o^t f(t-v)g(v)\, dv\} = F(s)G(s)$

$\mathcal{L}\{t^r\} = \frac{\Gamma(r+1)}{s^{r+1}}$, $r > -1$ real, where $\Gamma(t) = \int_0^\infty e^{-u}u^{t-1}\, du$, $t > 0$

PART I: MULTIPLE CHOICE

There are 14 multiple choice questions. All questions have the same value. A correct answer scores 3 and an incorrect answer scores zero. Record your answer on the OMR Examination Answer Sheet provided. Follow the instructions on page 2 carefully. **Use HB pencil only.** Ask the invigilators for scrap paper if required.

1. The differential equation $\quad (2xy + x^2 + x^4)dx + (1 + x^2)dy = 0, \quad$ is:

   (A) homogeneous and exact.

   (B) separable and homogeneous.

   (C) separable and exact.

   (D) linear and homogeneous.

   (E) linear and exact.

continued ...

2. An integrating factor of the form $\mu(x)$ for $(2x^2 + y)dx + x(xy - 1)dy = 0$ is:

(A) x^2

(B) x^{-1}

(C) x^{-2}

(D) e^x

(E) xe^x

3. If $y(x)$ is the solution of the initial value problem,

$x^2y'' - 3xy' + 4y = 0,\ y(1) = 0,\ y'(1) = 2,$ then $y(e)$ equals:

(A) $2e^2$

(B) $e^2 + 1$

(C) $2(e^2 + 1)$

(D) $e^2 + 2e$

(E) $e^2 + 2$

continued ...

4. If $y(x)$ is the solution of the initial value problem,
   $y' + \frac{1}{x}y = x^2y^2,\ y(1) = 2$, then $y(2)$ equals:

   (A) $-\frac{3}{2}$

   (B) $-\frac{1}{2}$

   (C) $-\frac{1}{4}$

   (D) 0

   (E) $\frac{2}{3}$

5. Find the general solution of the fourth-order homogeneous linear differential equation

$$y^{(4)} + 18y'' + 81y = 0.$$

   (A) $y(x) = (c_1 + c_2x)\cos(3x) + (c_3 + c_4x)\sin(3x)$.

   (B) $y(x) = c_1\cos(3x) + c_2\sin(3x)$.

   (C) $y(x) = c_1 + c_2x + c_3\cos(3x) + c_4\sin(3x)$.

   (D) $y(x) = c_1e^{3x} + c_2xe^{3x} + c_3\cos(3x) + c_4\sin(3x)$.

   (E) $y(x) = (c_1x + c_2x^2)\cos(3x) + (c_3x + c_4x^2)\sin(3x)$.

continued ...

6. Find the Laplace transform of $(t^2+3)u(t-2)$, where $u(t)$ is the unit step function.

(A) $e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{7}{s}\right)$
(B) $e^{-2s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{5}{s}\right)$
(C) $e^{-2s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{3}{s}\right)$
(D) $e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{9}{s}\right)$
(E) $e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s}\right)$

7. Let $F(s)$ denote the Laplace transform of $f(t)=\int_0^t e^{-\tau}\cos(\tau)\,d\tau.$

Then, $F(2)$ is equal to:

(A) $\frac{1}{10}$
(B) $\frac{3}{20}$
(C) $\frac{1}{2}$
(D) $-\frac{1}{e}$
(E) 0

8. The inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+11}\right\}$ is:

(A) $\cos(\sqrt{2}(t-3))-\frac{3}{2}\sin(\sqrt{2}(t-3))$
(B) $e^{-3t}(\cos(\sqrt{2}\,t)-\frac{3}{\sqrt{2}}\sin(\sqrt{2}\,t))$
(C) $t\cos(\sqrt{2}\,t)\,u(t-\sqrt{2})$
(D) $e^{-3t}(\cos(\sqrt{2}\,t)-\sin(\sqrt{2}\,t))$
(E) $e^{-3t}\cos(\sqrt{2}\,t)$

continued ...

9. Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$, if $F(s) = \frac{s^2}{(s+1)(s^2+4)}$.

(A) $e^{-t} + \sin(2t)$

(B) $\frac{1}{2}(e^{-t} + 3\cos(2t) - 3\sin(2t))$

(C) $\frac{1}{4}(e^{-t} + 2\cos(2t) + 3\sin(2t))$

(D) $\frac{1}{5}(e^{-t} + 4\cos(2t) - 2\sin(2t))$

(E) $e^{-t} + 2\cos(2t) - 2\sin(2t)$

10. The recursive formula for the coefficients a_n of the power series solution about $x_0 = 0$ of the differential equation $y'' - xy = 0$, is given by:

(A) $a_{n+2} = \frac{2na_{n-1}}{(n+2)(n+1)},\ n \geq 1.$

(B) $a_{n+2} = \frac{2(n-1)a_{n-1}}{n(n+1)},\ n \geq 1.$

(C) $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)},\ n \geq 1.$

(D) $a_{n+2} = \frac{na_{n-1}}{(n+1)},\ n \geq 1.$

(E) $a_{n+2} = \frac{a_{n-1}}{(n)(n+1)},\ n \geq 1.$

continued ...

11. Which one of the functions listed below is a power series solution about the ordinary point $x_0 = 0$, of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, with radius of convergence $R = \infty$, for the differential equation $(1 + 2x^2)y'' + A(x)y' + B(x)y = 0$, given only that $A(x)$ and $B(x)$ are analytic functions, the coefficients a_0 and a_1 are arbitrary, and the recurrence formula is:

$$a_n = \frac{-(2n-5)(n-5)}{n(n-1)}\, a_{n-2}, \quad n \geq 2.$$

(A) $3x + x^3$

(B) $2 + \frac{1}{3}x^3$

(C) $3x + \frac{1}{3}x^3 + \frac{2}{20}x^5$

(D) $1 + \frac{1}{2}x^2$

(E) $\sum_{k=1}^{\infty} (-1)^k \frac{[(-1)(3)(7)\cdots(4k-5)]x^{2k}}{k!\,(2k-3)(2k-1)}$

12. The *best* one can say, without actually solving the differential equation,

$(x^2 + 6x + 8)y'' + 6(x-3)y' + 4y = 0$, is that the radius of convergence of the power series solution about, $x_0 = 3$, is *at least*:

(A) ∞

(B) 6

(C) 5

(D) 4

(E) 3

continued ...

13. A complete list of the *regular singular points* of the differential equation

$$x(x+2)^2(x-5)^2(x-1)^4y'' + (x-5)^2(x+3)(x-1)^3y' + (x-1)^2(x+2)y = 0$$

is given by:

(A) 1, 5

(B) 0, 1

(C) 0, 1, 5

(D) 0, 1, −2, 5

(E) 0, 1, −2, −3, 5

14. The roots of the indicial equation for the point $x_0 = 0$ of the differential equation $x^2y'' + xy' + (x^2 - 1)y = 0$ are:

(A) $-1, 0$

(B) $-1, \frac{1}{2}$

(C) $\frac{1}{2}, \frac{3}{2}$

(D) $-1, 1$

(E) $0, \frac{1}{2}$

PART II: COMPLETE ANSWER QUESTIONS

Questions 15-20 Answer any 5 of the 6 questions in this part. Each question is of equal value. Only the solutions to the **5 questions** that you indicate will be marked and count towards your final score. You must **indicate clearly which 5 questions you want to count by circling the question number on the page with the solution** or we will count the first 5 questions attempted.

You will be graded on the clarity and presentation of the solution, not just upon whether or not you obtain the correct answer.

15. [12 marks] Show that the following differential equation is exact, and then find an explicit solution.

$$\underbrace{(1 + e^x y + xe^x y)}_{M(x,y)}dx + (\underbrace{xe^x + 2}_{N(x,y)})dy = 0.$$

Exact since $\frac{\partial M}{\partial y} = e^x + xe^x$ $\quad \frac{\partial N}{\partial x} = e^x + xe^x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the DE is exact

Let,

$N(x,y) = \frac{\partial F}{\partial y}$. $\therefore$ $F(x,y) = \int xe^x + 2 \, dy$

$= xe^x y + 2y + g(x)$

$\frac{\partial F}{\partial x} = e^x y + xe^x y + g'(x) = M(x,y) = 1 + e^x y + xe^x$

$\therefore$ $g'(x) = 1 \Rightarrow g(x) = x$.

$F(x,y) = xe^x y + 2y + x$, Implicit Sol'n: $xe^x y + 2y + x = C$.

An explicit solution is: $y = (C - x)/(2 + xe^x)$

$M(x,y) = \frac{\partial F}{\partial x}$ $\therefore$ $F(x,y) = \int 1 + e^x y + xe^x y \, dx$

$= x + e^x y + e^x(x-1)y + g(y)$

$= x + xe^x y + g(y)$

$\frac{\partial F}{\partial y} = xe^x + g'(y) = xe^x + 2 \Rightarrow g'(y) = 2 \Rightarrow g(y) = 2y$

$\therefore$ $F(x,y) = x + xe^x y + 2y = C \Rightarrow y = \frac{C - x}{2 + xe^x}$.

continued ...

16. [12 marks] Note that there is a **Table of Laplace Transforms** on page 3.

The motion of a spring-mass system that is given a sharp blow at time $t = \pi$ is described by the initial-value problem:

$$y''(t) + 2y'(t) + 10y(t) = K\delta(t - \pi),\ y(0) = 0, y'(0) = 3.$$

For what value of K will the system become motionless, i.e., $y(t) = 0$ for all $t > \pi$.

$$\left(s^2 Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) + 10Y(s) = Ke^{-\pi s}$$

$$Y(s)(s^2 + 2s + 10) = Ke^{-\pi s} + 3$$

$$Y(s) = \frac{Ke^{-\pi s}}{(s+1)^2 + 9} + \frac{3}{(s+1)^2 + 9}$$

$$Y(s) = \frac{e^{\pi} K e^{-\pi(s+1)}}{(s+1)^2 + 9} + \frac{3}{(s+1)^2 + 9}$$

$$y(t) = \frac{e^{\pi}}{3} K e^{-t} u(t-\pi) \sin(3(t-\pi)) + 3e^{-t} \sin(3t)$$

If $t > \pi$, then

$$y(t) = -\frac{e^{\pi}}{3} K e^{-t} \sin(3t) + e^{-t} \sin(3t).$$

$\therefore$ $y(t) = 0$ for all $t > \pi$ if $K = 3e^{-\pi}$.

continued ...

17. [12 marks]

(a) [6 marks] If the *Method of Undetermined Coefficients* is used to solve the differential equation:

$$y''' - y'' + 4\,y' - 4\,y = 5e^{x} - 2xe^{2x}\sin(3x).$$

what is the correct form to use to find a particular solution? (DO NOT SOLVE.)

(b) [6 marks] Find the general solution of the following differential equation using the *Method of Undetermined Coefficients*:

$$y'' + y' - 2\,y = 8 - 40\cos(2x).$$

(a) $r^3 - r^2 + 4r - 4 = 0$

i.e. $(r-1)(r^2+4) = 0$, $r = 1, \pm 2i$

$y_p(x) = Axe^{x} + (Bx+C)e^{2x}\sin(3x)$
$\quad + (Dx+E)e^{2x}\cos(3x)$

(b) $r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$, $r = -2, 1.$

$y_c(x) = c_1 e^{x} + c_2 e^{-2x}$

$y_p(x) = A + B\cos(2x) + C\sin(2x)$

$y_p'(x) = -2B\sin(2x) + 2C\cos(2x)$

$y_p''(x) = -4B\cos(2x) - 4C\sin(2x)$

$$\left(-4B\cos(2x) - 4C\sin(2x)\right) + \left(-2B\sin(2x) + 2C\cos(2x)\right)$$
$$- 2\left(A + B\cos(2x) + C\sin(2x)\right)$$
$$= 8 - 40\cos(2x).$$

continued ...

Continuation of solution to problem 17.

Coeff of constant term:

$$-2A = 8 \Rightarrow A = -4.$$

Coeff of cos(2x):

$$-6B + 2C = -40$$

Coeff of sin(2x):

$$-2B - 6C = 0$$

$$-6B - 18C = 0$$

$$20C = -40$$

$$C = -2$$

$$B = -3C = 6$$

$$\therefore \quad y_p(x) = -4 + 6\cos(2x) - 2\sin(2x)$$

General solution is:

$$y(x) = y_c(x) + y_p(x).$$

$$= c_1 e^{x} + c_2 e^{-2x} - 4 + 6\cos(2x) - 2\sin(2x).$$

18. [12 marks] Find the general solution of the differential equation,

$$4x^2y'' - 4xy' + 3y = 4x^{\frac{5}{2}}e^x, \; x > 0,$$

given that $y_1(x) = x^{\frac{1}{2}}$ and $y_2(x) = x^{\frac{3}{2}}$ solve the associated homogeneous equation:

$$4x^2y'' - 4xy' + 3y = 0, \; x > 0.$$

Variation of Parameters:

Standard form: $y'' - \frac{1}{x}y' + \frac{3}{4x^2}y = x^{1/2}e^x$

$$W(x^{1/2}, x^{3/2}) = \begin{vmatrix} x^{1/2} & x^{3/2} \\ \frac{1}{2}x^{-1/2} & \frac{3}{2}x^{1/2} \end{vmatrix} = \frac{3}{2}x - \frac{1}{2}x = x.$$

$$y(x) = v_1(x)x^{1/2} + v_2(x)x^{3/2}$$

$$v_1'(x) = \frac{\begin{vmatrix} 0 & x^{3/2} \\ x^{1/2}e^x & \frac{3}{2}x^{1/2} \end{vmatrix}}{x} = -\frac{x^2e^x}{x} = -xe^x$$

$$v_2'(x) = \frac{\begin{vmatrix} x^{1/2} & 0 \\ \frac{1}{2}x^{-1/2} & x^{1/2}e^x \end{vmatrix}}{x} = \frac{xe^x}{x} = e^x$$

$$v_1(x) = \int -xe^x\,dx = -e^x(x-1)$$

$$v_2(x) = \int e^x\,dx = e^x$$

continued ...

Continuation of solution to problem 18.

$\therefore$ the general solution is:

$$y(x) = c_1 x^{1/2} + c_2 x^{3/2} + (x^{1/2})(-e^x(x-1)) + x^{3/2}(e^x)$$

$$= c_1 x^{1/2} + c_2 x^{3/2} + e^x x^{1/2}.$$

OR Variation of Parameters

Let $y_1(x) = x^{1/2}$, $y_2(x) = x^{3/2}$,

$g(x) = x^{1/2} e^x$

$y(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$

$W(y_1, y_2) = x$

$$v_1(x) = \int -\frac{g(x) y_2(x)}{W(y_1,y_2)} dx, \quad v_2(x) = \int \frac{g(x) y_1(x)}{W(y_1,y_2)} dx.$$

$$v_1(x) = \int -x e^x \, dx \quad dx, \quad v_2(x) = \int e^x \, dx = e^x$$

$$= -e^x(x-1).$$

$$y(x) = c_1 x^{1/2} + c_2 x^{3/2} + x^{1/2}(-e^x(x-1)) + x^{3/2} e^x$$

$$= c_1 x^{1/2} + c_2 x^{3/2} + e^x x^{1/2}.$$

OR Reduction of Order:

$y_1(x) = x^{1/2}$, $y(x) = x^{1/2} v$

$y'(x) = \frac{1}{2} x^{-1/2} v + x^{1/2} v'$

$y''(x) = -\frac{1}{4} x^{-3/2} v + x^{-1/2} v' + x^{1/2} v''$

continued ...

$$4x^2\left(-\frac{1}{4}x^{-3/2}v + x^{-1/2}v' + x^{1/2}v''\right)$$
$$-4x\left(\frac{1}{2}x^{-1/2}v + x^{1/2}v'\right)$$
$$+3x^{1/2}v \quad = 4x^{5/2}e^x$$

$$4x^{5/2}v'' + v'\left(4x^{3/2} - 4x^{3/2}\right) + v\left(-x^{1/2} - 2x^{1/2} + 3x^{1/2}\right)$$
$$= 4x^{5/2}e^x$$

$$v'' = e^x$$
$$v' = e^x + c_1$$
$$v = e^x + xc_1 + c_2$$

$$\therefore y(x) = y_1(x)v(x)$$
$$= x^{1/2}e^x + c_1x^{3/2} + c_2x^{1/2}$$

OR Reduction of Order

$$y_1(x) = x^{3/2}, \quad y(x) = v(x)x^{3/2}.$$

$$y'(x) = \frac{3}{2}x^{1/2}v(x) + x^{3/2}v'(x)$$

$$y''(x) = \frac{3}{4}x^{-1/2}v(x) + 3x^{1/2}v'(x) + x^{3/2}v''(x)$$

$$4x^2\left(\frac{3}{4}x^{-1/2}v(x) + 3x^{1/2}v'(x) + x^{3/2}v''(x)\right)$$
$$-4x\left(\frac{3}{2}x^{1/2}v(x) + x^{3/2}v'(x)\right)$$
$$+3x^{3/2}v \quad = 4x^{5/2}e^x.$$

$$4x^{7/2}v'' + v'\left(12x^{5/2} - 4x^{5/2}\right) + v\left(3x^{3/2} - 6x^{3/2} + 3x^{3/2}\right)$$

$$= 4x^{5/2}e^x$$

$$xv'' + 2v' = e^x$$

$$v'' + \frac{2}{x}v' = x^{-1}e^x$$

$$\mu(x) = e^{\int \frac{2}{x}dx} = x^2$$

$$(x^2 v')' = x^2 e^x / x^{-1} = xe^x$$

$$x^2 v' = \int x e^x \, dx = e^x(x-1) + C$$

$$v' = x^{-1}e^x - x^{-2}e^x + C_1 x^{-2}$$

$$v = \int e^x(x^{-1} - x^{-2})\,dx + C_1 x^{-1} + C_2$$

$$= \frac{e^x}{x} + C_1 x^{-1} + C_2$$

$$y(x) = x^{1/2}e^x + C_1 x^{1/2} + C_2 x^{3/2}$$

19. [12 marks] Consider the following differential equation:

$$y'' - y' + 4xy = 0.$$

Find the first 3 nonzero terms for each of two linearly independent solutions in the form of power series about $x_0 = 0$.

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n-1} + 4x \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} (n-1)a_{n-1} x^{n-2} + \sum_{n=0}^{\infty} 4a_n x^{n+1} = 0$$

$$\sum_{\cancel{n=1}\; n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{\cancel{n=1}\; n=2}^{\infty} (n-1)a_{n-1} x^{n-2} + \sum_{n=3}^{\infty} 4a_{n-3} x^{n-2} = 0$$

$n=2$ $\quad 2(2-1)a_2 - (2-1)a_1 = 0$

$$a_2 = \frac{a_1}{2}$$

$n \geq 3$ $\quad n(n-1)a_n - (n-1)a_{n-1} + 4a_{n-3} = 0$

$$a_n = \frac{(n-1)a_{n-1} - 4a_{n-3}}{n(n-1)}, \quad n \geq 3$$

a_0 arb., a_1 arb., $a_2 = a_1/2$

$$a_3 = \frac{2a_2 - 4a_0}{(3)(2)} = \frac{a_1 - 4a_0}{6}$$

continued ...

Continuation of the solution to problem 19.

$$a_4 = \frac{3a_3 - 4a_1}{4(3)} = \frac{\frac{(a_1 - 4a_0)}{2} - 4a_1}{12}$$

$$= \frac{-7a_1 - 4a_0}{24}$$

$$y(x) = a_0 + a_1 x + x^2 \left(a_1/2\right) + x^3 \left(\frac{a_1 - 4a_0}{6}\right) + x^4 \left(\frac{-7a_1 - 4a_0}{24}\right) + \cdots$$

a_0, a_1 arbitrary.

$$\begin{cases} y_1(x) = a_0 - \frac{2}{3} a_0 x^3 - \frac{1}{6} a_0 x^4 + \cdots \\ y_2(x) = a_1 x + \frac{a_1}{2} x^2 + \frac{a_1}{6} x^3 + \cdots \end{cases}$$

2 linearly independent solutions.

20. [12 marks] Note that there is a **Table of Laplace Transforms** on page 3.

(a) [4 marks] Given that $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a \geq 0$, using the definition of the Laplace transform, prove that if $b > 0$, then $\mathcal{L}\{f(bt)\} = \frac{1}{b}F\left(\frac{s}{b}\right)$, $s > ab$.

(b) [4 marks] Derive an expression for $\mathcal{L}\{\, t \int_0^t f(t-v)g(v)\, dv \,\}$.

(c) [4 marks] Find $\mathcal{L}\{\, t \int_0^t (t-v)^2 \cos(v)\, dv \,\}$. (*Hint:* Use (b)).

(a) $\mathcal{L}\{f(bt)\} = \int_0^\infty e^{-st} f(bt)\, dt$

let $u = bt$
$du = b\,dt$

$$= \frac{1}{b}\int_0^\infty e^{-\frac{su}{b}} f(u)\, du$$

$$= \frac{1}{b} F\left(\frac{s}{b}\right), \quad \frac{s}{b} > a$$

$$= \frac{1}{b} F\left(\frac{s}{b}\right), \quad s > ab.$$

(b) $\mathcal{L}\left\{t \int_0^t f(t-v)g(v)\,dv\right\}$

$$= -\frac{d}{dt}\left(F(s)G(s)\right)$$

$$= -\left(F'(s)G(s) + F(s)G'(s)\right)$$

(c) $\mathcal{L}\left\{t\int_0^t (t-v)^2 \cos(v)\, dv\right\} =$

$$= -\left(\frac{d}{ds}\left(\frac{2}{s^3}\right)\frac{s}{s^2+1} + \frac{2}{s^3}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right)$$

continued ...

Continuation of solution to problem 20.

$$= \left(\frac{6}{s^4}\right)\left(\frac{s}{s^2+1}\right) - \frac{2}{s^3}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)$$

$$= \frac{6}{s^3(s^2+1)} + \frac{2s^2-2}{s^3(s^2+1)^2}$$

$$= \frac{6(s^2+1)+2s^2-2}{s^3(s^2+1)^2}$$

$$= \frac{8s^2+4}{s^3(s^2+1)^2}$$

$$\mathcal{L}\left\{t\int_0^t (t-v)^2\cos(v)\,dv\right\}$$

$$= -\frac{d}{ds}\left(\left(\frac{2}{s^3}\right)\left(\frac{s}{s^2+1}\right)\right) = -\frac{d}{ds}\left(\frac{2}{s^2(s^2+1)}\right)$$

$$= +2\left(\frac{2s(s^2+1)+2s^3}{s^4(s^2+1)^2}\right)$$

$$= \frac{2(4s^2+2)}{s^3(s^2+1)^2} = \frac{8s^2+4}{s^3(s^2+1)^2}.$$