The Relation Between Sums and Series

The Basics:

The concept of series is deeply intertwined with the concept of sequences, so to understand their connection, we have to look very carefully what a series really is.

Given a *sequence*, say $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, a_4, \ldots\}$ then we can define:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4, \ldots + a_n$$

The sum of the first n terms in our sequence of "a"-values. These S_n are called the "*parial sums*" of our series.

Notice that these too form a new sequence!

$$\{S_n\}_{n=1}^{\infty} = \{S_1, S_2, S_3, S_4, \ldots\}$$

Where:

$$S_1 = \sum_{i=1}^{1} a_i = a_1$$

$$S_2 = \sum_{i=1}^{2} a_i = a_1 + a_2$$

$$S_3 = \sum_{i=1}^{3} a_i = a_1 + a_2 + a_3$$

$$S_4 = \sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4$$

$$\vdots$$

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4, \ldots + a_n$$

The *Series*, informally is described as the sum of our sequence, ie all the infinite a_n's.

Rigourously, and more usefully it is described as the limit of the S_n's.

$$\sum_{i=1}^{\infty} a_i = S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

So, if the limit of the *sequence of partial sums* exists, S exists, so the series is said to converge. If the limit of the partial sum does not exist, then by definition S does not exist, and the series is said to diverge.

Again, it is important to note that the covergence or divergence of series **IS** is the convergence or divergence of the *sequence of partial sums*. NOT the convergence of the original sequnce of a_n's.

The Interconnections:

Because of our definition, the sequences and series are even more intertwined.

Consider the form of S_n from the definition:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4, \ldots + a_{n-1} + a_n$$

So of course, for any given $n > 1$, the previous partial sum would have been:

$$S_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + a_3 + a_4, \ldots + a_{n-1}$$

And we notice that, since each time we increment our n-value, we're adding more a-terms.
So if we take the difference,

$$S_n - S_{n=1} = \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i = \right) = \left(a_1 + a_2 + a_3 + a_4, \ldots + a_{n-1} + a_n\right) - \left(a_1 + a_2 + a_3 + a_4, \ldots + a_{n-1}\right)$$
$$= a_n$$

What does this mean?

Well beyond the obvious that given equations for our partial sum terms, we could deduce our original sequence, a_n we can also see that the limits of S_n and a_n, that is, the convergence of our sequence and our series are related.

The Divergence Theorem:

The limits of the series (ie the sequence of partial sums) and the originating sequence are related, but certainly not identical.

Notice that if the series does converge, this means:

$$\lim_{n \to \infty} S_n = S$$

So using our relation betwen subsequent S_n we notice the following limit:

$$\lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \lim_{n\to\infty} a_n$$

but since our S exists, we also have:

$$\lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \left(\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}\right) = S - S = 0 = \lim_{n\to\infty} a_n$$

So what does that mean?

It means that if the series converges, then $\lim_{n\to\infty} a_n = 0.$

But the converse certainly isn't true. The property, $\lim_{n\to\infty} a_n = 0,$ does not guarentee a covergent series.

For the classic example of this, consider a series called the "harminic series".

$$\sum_{n=1}^{\infty}\frac{1}{n} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

This series has as its originating sequence: $a_n = \dfrac{1}{n}$

Clearly, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{1}{n} = 0,$ but we can show it diverges (Warning: Boring!)

Consider:

$$\sum_{n=1}^{\infty}\frac{1}{n} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\cdots$$
$$=\frac{1}{1}+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\cdots$$

Looking at each bracketed set, each is bigger than $1/2$ and one can keep going, grouping the terms in the sum between $\dfrac{1}{2^k}$ and $\dfrac{1}{2^{k+1}}$ to add up to values $> 1/2$.

So if our series were to converge, we would have:

$$\sum_{n=1}^{\infty}\frac{1}{n} > \frac{1}{1}+\frac{1}{2}+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\cdots$$

The value of which clearly blows up very quickly, so our series diverges.

So in all, we have ourselves a theorem:

Divergence Theorem

$$\lim_{n \to 0} a_n \neq 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

That is, if the sequence does not go to to zero, the series converges. But if the sequence does go to zero, this theorem tells us noting whatsoever. Things can still go wrong.

This is the classic example of the scenario known in logic of a necessary but not sufficient condition.

We need to have a zero limit for the series to converge, but if we do get zero, this is not enough. Things may still go wrong (c.f. the Harmonic series above.)

Summary:

Given $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, a_4, \ldots\}$

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4, \ldots + a_n$$

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$

$$S_n - S_{n-1} = \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i \right) = a_n$$

Divergence Theorem

$$\lim_{n \to 0} a_n \neq 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

$$(\textit{Rembember}: \lim_{n \to 0} a_n = 0 \ \textit{tells us } \textbf{\textit{NOTHING}})$$