Fourier Series - Joseph Fourier (1768-1830)

A periodic signal $x : \mathit{Reals} \to \mathit{Reals}$ with period $p \in \mathit{Reals}$ can usually be described as a constant term plus a sum of sinusoids:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$$

- x - Fourier series
- A_k, ϕ_k - amplitude and phase of cosine function (depend on x)
- ω_0 - fundamental frequency ($\omega_0 = 2\pi/p$)
- A_0 - DC term
- cosine terms for $k \geq 2$ - harmonics

Uniqueness of Fourier series

They are unique... what does that mean?

$$
\begin{aligned}
x(t) &= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k) \\
x(t) &= B_0 + \sum_{k=1}^{\infty} B_k \cos(k\omega_0 t + \theta_k)
\end{aligned}
$$

Then $A_0 = B_0$, $A_k = B_k$, $\phi_k = \theta_k \mod 2\pi$.

Discrete-time signals

NOT ALL FXNs ARE PERIODIC

EASIER TO WORK W/ CONTINUOUS

$$x(n+p) = x(n),$$

but n is now an integer. Subtlety: not all sinusoidal discrete-time signals are periodic. Consider:

$$x(n) = \cos(2\pi f n).$$

Need

$$x(n+p) = \cos(2\pi f n + 2\pi f p) = \cos(2\pi f n) = x(n).$$

In other words, $2\pi f p$ is an integer multiple of 2π, i.e. there exist nonzero integers p and m such that $f = m/p$.

Linear time-invariant (LTI) systems

- Key property: given a sinusoidal input, output is a sinusoidal signal with the same frequency but possibly a different amplitude and a different phase
- filtering (adjusting signal) is more naturally a frequency-based operation
- for example, removing abrupt changes corresponds to removing high frequency components
- will look at "frequency response" for systems

GET RID OF SHARP SOUNDS BY

REDUCING HIGH FREQS

Time Invariance - Delay Systems

differential equations,

$D_2(x) = x(t-2)$

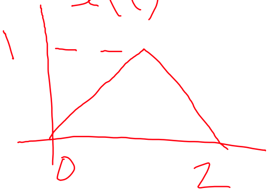
"−" implies in past, "+" ' delay

We start with continuous-time systems. Consider a *delay system* (D_τ). This system takes an input x and produces an output y which is x delayed by τ time units.

$D_{-1}(x+1)$

$x(t)$

$$y(t) = x(t - \tau)$$



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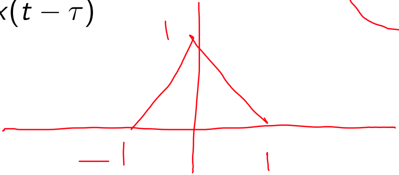

Time invariant systems - definition

A time-invariant system S satisfies

$$\forall t \in Reals, \quad S \circ D_\tau = D_\tau \circ S.$$

SoD means add system
then add on delay,
whereas DoS is delay
then add system

- $S \circ D_\tau$ - delay input by τ, then apply S
- $D_\tau \circ S$ - apply S to the input, then delay by τ

Interchanging the two systems must yield the same output, i.e. a delayed input $D_\tau(x)$ produces a delayed output $D_\tau(y)$.

Example 1

constant

$$y(t) = Cx(t)$$

Let $\hat{x}(t) = D_\tau(x(t))$. Then

$$\begin{aligned} \hat{y}(t) &= C\hat{x}(t) \\ &= Cx(t-\tau) \\ &= y(t-\tau) \end{aligned}$$

delay on system

Example 2

$$y(t) = Cx^3(t)$$

$$\begin{aligned}
\hat{y}(t) &= C(\hat{x}(t))^3 \\
&= C(x(t-\tau))^3 \\
&= y(t-\tau)
\end{aligned}$$

Example 3

reverses signal in time.

$$y(t) = x(-t)$$

$$\begin{aligned} \hat{y}(t) &= \hat{x}(-t) \\ &= x(-t-\tau) \end{aligned}$$

But $y(t-\tau) = x(-t+\tau)$. not time invariant

Exercise 1

Is the following time-invariant?

$$y(t) = 10x(t) + 5x(t-2)$$

$\hat{x}(t) = x(t-\tau)$

$\hat{y}(t) = 10x(t-T) + 5x(t-2-\tau)$

*time invariant

Exercise 2

time invariant :

Is the following time-invariant?

$$y(t) = tx(t)$$

if coefficient depends on the time, it is not time invariant

Discrete-time time-invariance

TI = TIME INVARIANT

$D_M(x(n)) = x(n-M)$ — INT, # of samples

$$S(D_M(x)) = D_M(S(x))$$

Examples:

doesn't depend on 'n'

$$\begin{aligned} y(n) &= x(n) + 0.1(x(n))^2 \\ y(n) &= \cos(2\pi n)x(n) = x(n) \leftarrow \text{TI} \\ y(n) &= \cos(2\pi n/9)x(n) \end{aligned}$$

INT

doesn't affect n

NOT TI

Review - Complex Numbers

- Define i to be the square root of -1, in other words, $(i)(i) = -1$.
- A *complex number* is the sum of a real number and an imaginary number, written $z = x + iy$, where the *real part* of z ($Re\{z\}$) is x and the *imaginary part* of z ($Im\{z\}$) is y.

Arithmetic

- Addition/subtraction. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \\ z_1 - z_2 &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

- Multiplication.

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Complex conjugate

$$
\begin{aligned}
z &= x + iy \\
z^* &= x - iy \\
z + z^* &= 2Re\{z\} \\
z - z^* &= 2iIm\{z\} \\
zz^* &= x^2 + y^2 \\
|z| &= \sqrt{zz^*}
\end{aligned}
$$

Complex exponentials

The Taylor series expansion for an exponential function is

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Let z be an imaginary number, $z = i\theta$.

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= \left[1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] \\ &= \cos(\theta) + i\sin(\theta) \end{aligned}$$

Polar representation

$$z = x + iy = re^{i\theta}$$

where

$$r = |z| = \sqrt{x^2 + y^2}$$

and

$$\theta = \arg(z) = \tan^{-1}(y/x).$$

Complex systems

Let $x(t)$ have range equal to the complex numbers, and a be a complex constant. Define

$$\begin{aligned} (ax)(t) &= a(x(t)) \\ (x_1 + x_2)(t) &= x_1(t) + x_2(t) \end{aligned}$$

Let S map complex-valued functions to complex-valued functions. These are called *complex systems*.

Linearity

- Homogeneity: $S(ax) = aS(x)$
- Additivity: $S(x_1 + x_2) = S(x_1) + S(x_2)$
- Linearity: Homogeneity and Additivity.

Example 1.

$$y(t) = 10x(t) + 5x(t-2).$$