47. Consider the following multi-objective problem:

$$\begin{cases} \min\ (3\,x_1 + 2\,x_2\ ,\ \ -x_1 - 2\,x_2) \\ x_1 + 2\,x_2 \leq 0 \\ -x_1 \leq -1 \\ x_1 - 2\,x_2 \leq 4 \end{cases}$$

a) Is it a convex problem? Why?

b) Do minima exist? Why?

c) Is the point $(2, -1)$ a minimum? Why?

d) Is the point $(1, -1)$ a weak minimum? Why?

e) Find the set of all weak minima by using the scalarization method.

f) Find the set of all minima by using the scalarization method.

g) Find the ideal point.

h) Apply the goal method with $\|\cdot\|_2$.

THE PROBLEM IS LINEAR SINCE EACH OBJECTIVE FUNCTION IS LINEAR AND ALL THE CONSTRAINTS ARE LINEAR

$$\begin{cases} \min\ \alpha_1(3x_1 + 2x_2) + \alpha_2(-x_1 - 2x_2) \\ x_1 + 2x_2 \leq 0 \\ -x_1 \leq -1 \\ x_1 - 2x_2 \leq 4 \\ \alpha \geq 0 \\ \alpha_1 + \alpha_2 = 1 \end{cases}$$

KKT SYSTEM

$$\begin{cases} 3\alpha_1 - (1-\alpha_1) + \lambda_1 - \lambda_2 + \lambda_3 = 0 \\ 2\alpha_1 - 2(1-\alpha_1) + 2\lambda_1 - 2\lambda_3 = 0 \\ \lambda_1(x_1 + 2x_2) = 0 \\ \lambda_2(-x_1 + 1) = 0 \\ \lambda_3(x_1 - 2x_2 - 4) = 0 \\ x_1 + 2x_2 \leq 0 \\ -x_1 + 1 \leq 0 \\ x_1 - 2x_2 - 4 \leq 0 \end{cases}$$

$\alpha_1 = 0$, $\lambda \geq 0$ I) $\lambda_2 = 0$

$-1 - \lambda_2 + \lambda_3 + \lambda_1 = 0 \rightarrow \cancel{-1} - \lambda_2 + \lambda_3 + \cancel{1} + \lambda_3 = 0 \rightarrow 2\lambda_3 = \lambda_2$

$-2 - 2\lambda_3 + 2\lambda_1 = 0 \rightarrow \lambda_1 - \lambda_3 = 1 \rightarrow \lambda_1 = 1 + \lambda_3$

I) {WEAK MINIMA OF $P\alpha_1$} $= \begin{cases} x_1 = -2x_2 \\ -x_1 + 1 \leq 0 \\ x_1 - 2x_2 - 4 \leq 0 \end{cases}$ $-1 \leq x_2 \leq -1/2$

$0 < \alpha_1 \leq 1$

$3\alpha_1 - (1-\alpha_1) + \lambda_1 - \lambda_2 + \lambda_3 = 0$

$2\alpha_1 - 2(1-\alpha_1) + 2\lambda_1 - 2\lambda_3 = 0$

$1 - \lambda_1 - \lambda_2 + 3\lambda_3 = 0$ $4\alpha_1 - 2\lambda_2 + 4\lambda_3 = 0$

$\lambda_1 = 1 - \lambda_2 + 3\lambda_3$

$4\alpha_1 - \cancel{2} + \cancel{2} - 2\lambda_2 + 4\lambda_3 = 0$

$\lambda_1 = 0$ $\begin{cases} 1 - \lambda_2 + 3\lambda_3 = 0 \\ 4\alpha_1 - 2 - 2\lambda_3 = 0 \end{cases}$

$\alpha_1 = \frac{1 + \lambda_3}{2}$

$\begin{cases} x_1 - 2x_2 \leq 0 \\ x_1 = 1 \\ x_1 - 2x_2 - 4 \leq 0 \end{cases}$ $-\frac{3}{2} \leq x_2 \leq -1/2$

$\lambda_2 = 0 \rightarrow \lambda_3 = -1/3$ IMP.

$\lambda_2 \neq 0 \rightarrow \lambda_3 = \frac{\lambda_2 - 1}{3} \rightarrow \begin{cases} \lambda_2 = 1 , \lambda_3 = 0 \\ \lambda_2 > 1 , \lambda_3 \neq 0 \end{cases}$

$\lambda_2 = 0$

$\lambda_1 = 1 - 3\lambda_3$; $4\alpha_1 + 4\lambda_3 = 0 \rightarrow \alpha_1 = -\lambda_3 \iff \alpha_1 = 0 , \lambda_3 = 0 , \lambda_1 = 1$

$\lambda_3 = 0$

$\lambda_1 = 1 - \lambda_2$; $4\alpha_1 - 2\lambda_2 = 0 \rightarrow \alpha_1 = \frac{1}{2}\lambda_2$

$\lambda_2 \neq 0$
$\lambda_1 \neq 0$

$\begin{cases} x_1 = -2x_2 \\ x_1 = 1 \\ x_1 - 2x_2 - 4 \leq 0 \end{cases}$ $\bar{x} = (1, -1/2)$

$\lambda_2 \neq 0$
$\lambda_1 = 0$
$\lambda_2 = 1$

$\begin{cases} x_1 + 2x_2 \leq 0 \rightarrow 1 + 2x_2 \leq 0 \rightarrow x_2 \leq -1/2 \\ x_1 = 1 \\ x_1 - 2x_2 - 4 \leq 0 \rightarrow 1 - 2x_2 - 4 \leq 0 \rightarrow x_2 \geq -\frac{3}{2} \end{cases}$

$$\{\text{WEAK MINIMA OF } P_\alpha\} = \begin{cases} x_1 = -2x_2 \\ -1 \le x_2 \le -1/2 \end{cases} \cup \begin{cases} x_1 = 1 \\ -\frac{3}{2} \le x_2 \le -1/2 \end{cases}$$

$$\{\text{MINIMA OF } P_\alpha\} = \begin{cases} x_1 = 1 \\ -\frac{3}{2} \le x_2 \le -1/2 \end{cases}$$