GAME THEORY

$X_i :=$ SET OF STRATEGIES $\qquad f_i : X_1, X_2, \dots, X_n \to \mathbb{R}$

$$\begin{cases} \min f_i(\dots x_i \dots) \\ x_i \in X_i \end{cases}$$

$\longrightarrow$ MINIMIZE THE COST FUNCTION GIVEN THE OTHER PLAYERS' STRATEGY

NON-COOPERIVE GAMES WITH TWO PLAYERS

1) $\begin{cases} \min\limits_x f_1(x,y) \\ x \in X \end{cases}$ $\qquad$ 2) $\min\limits_y \begin{cases} f_2(x,y) \\ y \in Y \end{cases}$

NASH EQUILIBRIUM

$$(\bar{x}, \bar{y}) = \min_x f_1(x, \bar{y}) = \min_y f_2(\bar{x}, y)$$

MATRIX GAME

- ZERO SUM GAME $:= f_2 = -f_1$
- X, Y ARE FINITE SETS

$$f_1(\bar{x}, \bar{y}) := \min_{x \in X} f_1(x, \bar{y}) = \max_{y \in Y} f_1(\bar{x}, y)$$

$$f_1(i, j) = c_{ij}$$

MIXED STRATEGIES

$$X := \left\{ x \in \mathbb{R}^m : x \geq 0, \sum_{i=1}^{m} x_i = 1 \right\} \qquad Y := \left\{ y \in \mathbb{R}^n : y \geq 0, \sum_{i=1}^{n} y_i = 1 \right\}$$

$$f_1(x,y) = x^T C y \qquad f_2(x,y) = -x^T C y$$

$(\bar{x}, \bar{y})$ IS A SADDLE POINT OF THE FUNCTION $f_1(x,y) = x^T C y$

ANY MATRIX GAME HAS AT LEAST A MIXED STRATEGY NASH EQUILIBRIUM

$$\bar{x} = \min_{x \in X} \max_{y \in Y} x^T C y \longrightarrow \begin{cases} \min v \\ v \geq \sum_{i=1}^{m} c_{ij} x_i \quad \forall j = 1, \dots, n \\ x \geq 0, \ \sum_{i=1}^{m} x_i = 1 \end{cases}$$

$(\max w)$

$\left(w \leq \sum_{j=1}^{n} c_{ij} w_i \ \forall i = 1, \dots, m \right)$

BIMATRIX GAME

NON-ZERO SUM GAME $:= f_2 \neq -f_1$, $\quad f_1(x,y) = x^T C_1 y \qquad f_2(x,y) = x^T C_2 y$

$\longrightarrow :=$ DIRECTION OF MINIMIZATION

ES: $\quad C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$

- IF C_1 HAS A STRICTLY DOMINATED STRATEGIES THEN BOTH ROWS IN C_1 AND C_2 MUST BE DELETED
- IF C_2 HAS A STRICTLY DOMINATED STRATEGIES THEN BOTH COLUMNS IN C_2 AND C_1 MUST BE DELETED

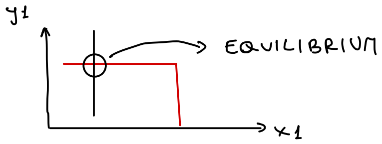
BEST RESPONSE MAPPING

$$B_1(y) = \left\{ \text{optimal solutions of } \min_{x \in X} \; x^T C_1 y \right\},$$

$$B_2(x) = \left\{ \text{optimal solutions of } \min_{y \in Y} \; x^T C_2 y \right\},$$

$(\bar{x}, \bar{y})$ IS A NASH EQUILIBRIUM $\iff$ $\bar{x} \in B_1(\bar{y})$

$\bar{y} \in B_2(\bar{x})$



CONVEX GAME

Player 1: $\begin{cases} \min_x \; f_1(x,y) \\ g_i^1(x) \le 0 \;\; \forall\, i = 1,..,p \end{cases}$ Player 2: $\begin{cases} \min_y \; f_2(x,y) \\ g_j^2(y) \le 0 \;\; \forall\, j = 1,..,q \end{cases}$

IF f_1 AND f_2 ARE QUASICONVEX AND X, Y ARE CONVEX, CLOSED AND BOUNDED THEN THERE EXISTS AT LEAST A NASH EQUILIBRIUM