Light and Matter

$c = f\lambda$ for waves

$E = hf$ for photons

$p = mv$ for particles

$p = \frac{h}{\lambda}$ for waves

Wave Function

$P(x,t) = |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$

Valid wave functions are "square integrable":

$$\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)\,dx = N = 1$$

Normalization: $\Psi_{norm}(x,t) = \frac{\Psi(x,t)}{\sqrt{N}}$

where $N \neq 1$.

Probability

$$\langle j \rangle = \frac{\sum_j j\,n(j)}{\text{trials}} = \sum_j j\,P(j)$$

$$\binom{x}{y} := \frac{x!}{y!(x-y)!}$$

Schrodinger's eq'n

$$-\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \Psi(x,t)}{\partial x^2} + U(x,t)\Psi(x,t) = \frac{i\hbar\,\partial \Psi(x,t)}{\partial t}$$

$K_E\Psi + P_E\Psi = E\Psi \quad K_E = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{\hbar k^2}{2m}$

Momentum

$$p = -i\hbar\frac{\partial}{\partial x}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} p(x)\,P(x)\,dx = \int_{-\infty}^{\infty} \Psi^*(x,t)\,\frac{-i\hbar\,\partial \Psi(x,t)}{\partial x}\,dx$$

Momentum Wave Function

$$\xi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-i/\hbar\,(xp)}\,\Psi(x,t)\,dx$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{i/\hbar\,(xp)}\,\xi(p,t)\,dp$$

$$\langle x \rangle = \int_{-\infty}^{\infty} \xi^*(p,t)\,\frac{i\hbar\,\partial \xi(p,t)}{\partial p}\,dp$$

Stationary States

$U(x,t) = U(x)$

$P(x,t) = P(x,0)$

$$\Psi(x,t) = \phi(x)\,\chi(t=0)\,e^{-i/\hbar\,Et}$$

$$\langle E \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,E\,\Psi(x,t)\,dx$$

Hamiltonian

$$H = \left[-\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} + U(x)\right]$$

$$H\phi(x) = E\phi(x)$$

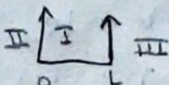
Superposition

If $\phi(x)\chi(0)e^{-i/\hbar\,Et}$ is a sol'n to the TDSE, the follow superpositions are also:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\phi_n(x)\,e^{-i/\hbar\,E_n t}$$

Infinite Square Well Potential

$$U(x)\begin{cases} 0 & 0 \le x \le L \\ 1 & \text{otherwise} \end{cases}$$



In region I: $\phi_I = A\cos(kx) + B\sin(kx)$

$k^2 = \frac{2mE}{\hbar^2} \quad \phi_I(0) = \phi_{II}(0)$

$k = \frac{n\pi}{L} \quad \phi_I(L) = \phi_{III}(L)$

$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad \phi_n(x) = B\sin(k_n x);\ n \in 1,2,3\ldots$

Normalization: $\frac{B^2 L}{2} = 1$

Commutators

$$[x,p] = xp - px$$

Orthogonality

$$\delta_{mn} = \int_{-\infty}^{\infty} \phi^*_n(x)\,\phi_m(x)\,dx = 0 \quad \forall m \neq n$$

$$\therefore \int_{-\infty}^{\infty} \phi^*_m(x)\,\Psi(x)\,dx = c_m$$

Finite Square Well Potential

$$U(x)\begin{cases} 0 & 0 \le x \le L \\ U_0 & \text{otherwise} \end{cases}$$

In region I: $\phi_I = Ce^{ikx} + De^{-ikx}$

In regions II, III: $Fe^{iKx} + Ge^{-iKx}$, $He^{iKx} + Je^{-iKx}$

$K^2 = \frac{2m}{\hbar^2}(E - U_0) \quad \nu = K/i$

$\phi_I(0) = \phi_{II}(0) \rightarrow C + D = F + G,\ A = G$

$\phi_I(L) = \phi_{III}(L) \rightarrow Ce^{ikL} + De^{-ikL} = He^{-\nu L} + Je^{\nu L}$

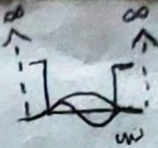
$\phi_{II}(-\infty) = 0 \rightarrow F = 0 \quad \phi_{III}(\infty) = 0 \rightarrow J = 0$

$\phi'_I(0) = \phi'_{II}(0) \rightarrow ikC - ikD = \nu G,\ kB = \nu G$

$\phi'_I(L) = \phi'_{III}(L) \rightarrow ikCe^{ikL} - ikDe^{-ikL} = -\nu He^{-\nu L},$

$-kA\sin(kL) + kB\cos(kL) = -\nu He^{-\nu L}$

This has no analytical sol'n.



Fe^{ikx} ← incoming wave

Ge^{-ikx} ← reflected wave

He^{ikx} ← transmitted wave

$$\delta = \frac{\hbar}{2m(U_0 - E)}$$

$$P(\text{trans}) = \frac{|\phi_{III}(x)|^2}{|Fe^{ikx}|^2} = \frac{|H|^2}{|F|^2} = T$$

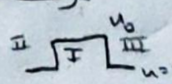
$$P(\text{reflect}) = \frac{|G|^2}{|F|^2} = R$$

$$R + T = 1$$

Resonance:

$$\frac{1}{P(\text{trans})} = 1 + \frac{U_0^2}{4E(E-U_0)^2}\sin^2\!\left(\frac{L}{\hbar}\sqrt{2mE}\right)$$

Tunneling



$R = \frac{|G|^2}{|F|^2}$

$T = \frac{|H|^2}{|F|^2}$

$$\frac{1}{T} = 1 + \frac{1}{4}\left[\frac{U_0^2}{E(U_0 - E)}\right]\left[\frac{e^{-\nu L} + e^{\nu L}}{2}\right]^2$$

For a high barrier:

$$T \approx e^{-\alpha L}, \quad \alpha = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)}$$

For a wide barrier:

$$T \rightarrow 0$$

Harmonic Oscillators

$E = (n + \frac{1}{2})\hbar\omega$

$x = N(a + a^\dagger)$

$p = N(a - a^\dagger)$

$$U(x) = \frac{1}{2}m\omega^2 x^2$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi(x) + \frac{1}{2}m\omega^2 x^2\phi(x) = E\phi(x)$$

raising operator: $a = \sqrt{\frac{m\omega}{2\hbar}}\,x + i\frac{p}{\sqrt{2m\omega\hbar}}$

lowering operator: $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,x - i\frac{p}{\sqrt{2m\omega\hbar}}$

$H|\phi_+(x)\rangle = (E - \hbar\omega)|\phi_+(x)\rangle$

where $|\phi_+(x)\rangle = a|\phi(x)\rangle$

$a|\phi_0(x)\rangle = 0 \rightarrow E_0 = \frac{\hbar\omega}{2}$

$\phi_0(x) = N_0 e^{-x^2/x_0^2}$

where $N = a^\dagger a$

Free Particles

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi(x) = E\phi(x)$$

$\phi(x) = Ae^{ikx} \quad k = \frac{\sqrt{2mE}}{\hbar}$

$E = \frac{\hbar^2 k^2}{2m}$

$\Psi(x,t) = Ae^{i(kx - \omega t)}$

$\Psi(x,0) = \phi(x) = \int A_k e^{ikx}\,dx$

Free Particles Cont'd

$\Psi(x,0) = \int_{-\infty}^{\infty} \phi_p e^{i/\hbar\, px}\, dp$

$\phi(p) = \int_{-\infty}^{\infty} \Phi(x) e^{-i/\hbar\, px}\, dx$

Each particle is actually a wave packet that spreads.

phase velocity: $v_p = \frac{\omega}{k}$

group velocity: $v_g = \frac{d\omega}{dk} = v_p + k\frac{\partial v_p}{\partial k}$

Hilbert Space

All square integrable f'ns.

If $A|\psi_1\rangle = |\psi_2\rangle$ then $\langle \psi_1|\psi_2\rangle = A$

Hermitian Operators

$A = A^\dagger$

$\langle A\psi_1|\psi_2\rangle = \langle\psi_1|A\psi_2\rangle$

$\det(A-\lambda I)=0$

Unitary Operators

$A^\dagger = A^{-1}$

$A^\dagger A = I$

If H is Hermitian then: $e^{-i/\hbar\, Ht}$ is unitary.

Commutators

$[x_i, P_i] = i\hbar\delta_{ij}$

$[A,B] = -[B,A]$

$[A, B+C] = [A,B]+[A,C]$

$[A,BC] = [A,B]C + B[A,C]$

$\sigma_A = \langle A^2\rangle - \langle A\rangle^2$

$\Delta A = \sigma_A^2$

Schwartz Inequality

$\sigma_A\sigma_B \geq |\langle \bar A\psi|\bar B\psi\rangle|^2$

$;\ \bar A = A - \langle A\rangle,\ \bar B = B - \langle B\rangle$

Generalized Uncertainty

$\Delta A \Delta B \geq \frac{\langle [A,B]\rangle}{2i}$

$\Delta E\Delta t \geq \frac{\hbar}{2}$

$[x,P_x] = i\hbar \Rightarrow \Delta x\Delta p \geq \frac{i\hbar}{2i} = \frac{\hbar}{2}$

Angular Momentum

$\hat L = \hat r \times \hat p$

$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$

$\Delta L_i \Delta L_j = \frac{\hbar}{2}|\langle L_k\rangle|$

$L^2 = L_x^2+L_y^2+L_z^2$

$[L^2,L_x]=[L^2,L_y]=[L^2,L_z]=0$

Hydrogen Atom

$\psi(r,\theta,\phi) = R_{nl}(r) Y_{lm}(\theta,\phi)$

$P(r)\,dr = r^2|R_{nl}(r)|^2\, dr$

$P(r,\theta,\phi)\,dr\,d\theta\,d\phi = |\psi(r,\theta,\phi)|^2 r^2\sin\theta\, dr\,d\theta\,d\phi$

$R_{nl}(r) = \frac{u_{nl}}{r} = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho)$

$v(\rho) = \sum_{k=0} c_k\rho^k \quad V = \frac{-e^2}{4\pi\epsilon_0 r}$

$c_{k+1} = \frac{2(k+l+1)-\rho_0}{(k+1)(k+2l+2)}c_k$

$\rho = Kr$

$K = \frac{\sqrt{-2mE}}{\hbar} = \frac{1}{a_0}\frac{1}{n}$

$a_0 = 0.0529\,nm$

$$\psi_{nlm}(r,\theta,\phi) = \sqrt{\left(\frac{2}{na_0}\right)^3\frac{(n-l-1)!}{2n[(n-l)!]^3}}\, e^{-r/na_0} \left(\frac{2r}{na_0}\right)^l\left[L^{2l+1}_{n-l-1}(2r/na_0)\right]Y_{lm}(\theta,\phi)$$

$L^{(p)}_{q-p}(x) = (-1)^p\frac{d^p}{dx^p}L_q(x)$

$L_q(x) = e^x\frac{d^q}{dx^q}(e^{-x}x^q)$

$Y_{lm}(\theta,\phi) = N_{lm}P_{lm}(\cos\theta)e^{im\phi}$

$P_{lm}(w) = (1-w^2)^{|m|/2}\left(\frac{d}{dw}\right)^{|m|}P_{l0}(w)$

$P_{l0}(w) = \frac{1}{2^l l!}\left(\frac{d}{dw}\right)^l(w^2-1)^l$

$\lambda_l = l(l+1)\hbar\ ;\ l=0,1,2,\ldots$

$$N_{lm} = \begin{cases} (-1)^m\sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+m)!}} & ; m\geq 0 \\ \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+m)!}} & ; m<0 \end{cases}$$

$\psi(r,\theta,\phi,t) = \sum_{nlm} c_{nlm} e^{-it/\hbar\, E_n}\psi_{nlm}(r,\theta,\phi)$

$c_{nlm} = \langle\psi_{nlm}|\psi_0\rangle$

$\psi_0 = \psi(r,\theta,\phi)$

$E_n = \frac{-me^4}{32\pi^2\epsilon_0^2\hbar^2n^2} = \frac{-13.6eV}{n^2}$

$H = -\frac{\hbar^2}{2m}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\right\} + U(r)$

$U(r) = -\frac{ke^2}{r}\ ;\ k = \frac{1}{4\pi\epsilon_0}$

$\mu_m = m\hbar\ ;\ m = \pm l$

Generalized Angular Momentum

$J_+ = J_x + iJ_y \quad J_- = J_x - iJ_y$

$[J_i,J_j] = i\hbar\epsilon_{ijk}J_k$

$J^2|j,m\rangle = \lambda_j|j,m\rangle$

$J_z|j,m\rangle = \mu_m|j,m\rangle$

$\Delta J_i\Delta J_j = \frac{\hbar}{2}|\langle J_k\rangle|$

$\lambda_j = j(j+1)\hbar^2\ ;\ j\in[0,\frac12,1,1\frac12,\ldots]$

$\mu_m \in [-j\hbar, -j\hbar+\hbar,\ldots,j\hbar]$

$J_\pm|j,m\rangle = C_\pm|j,m\pm1\rangle$

$C_\pm = \frac{1}{\sqrt{\langle j,m|J_\pm^\dagger J_\pm|j,m\rangle}}$

For any angular momentum operator A:

$A_{mm'} = \langle j,m'|A|j,m\rangle$

Electrons

$j = \frac12 \equiv S$

$\det(S_z-\lambda I)=0 \Rightarrow \lambda = \pm\frac12\hbar$

$S_z = \frac34\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$S_x = \begin{bmatrix}0 & c_+/2\\ c_-/2 & 0\end{bmatrix} \quad c_+ = \sqrt{s(s+1)-m_s(m_s+1)}\,\hbar,\ c_- = \sqrt{s(s+1)-m_s(m_s-1)}\,\hbar$

S_x, S_y, S_z are the Pauli matrices!

$\chi_+^{(x)} = \frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix} \quad \chi_-^{(x)} = \frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}$

$\chi_+^{(y)} = \frac{1}{\sqrt2}\begin{pmatrix}1\\i\end{pmatrix} \quad \chi_-^{(y)} = \frac{1}{\sqrt2}\begin{pmatrix}1\\-i\end{pmatrix}$

$\chi_+^{(z)} = \begin{pmatrix}1\\0\end{pmatrix} \quad \chi_-^{(z)} = \begin{pmatrix}0\\1\end{pmatrix}$

In general:

$\chi_+ = \begin{bmatrix}\cos(\theta/2)\\ \sin(\theta/2)e^{i\phi}\end{bmatrix}$

$\chi_- = \begin{bmatrix}-\sin(\theta/2)e^{-i\phi}\\ \cos(\theta/2)\end{bmatrix}$

$\chi(t) = \begin{bmatrix}\cos(\alpha/2)e^{i\gamma B_0 t/2}\\ \sin(\alpha/2)e^{-i\gamma B_0 t/2}\end{bmatrix}$

ex. $P_+^{(x)}(t) = |c_+^{(x)}|^2 = |\chi_+^{(x)\dagger}\chi|^2$

Probabilistic Model

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

bit $|v\rangle = \begin{pmatrix}a\\b\end{pmatrix} = a|0\rangle + b|1\rangle$

; $a = P(v=0)$, $b = P(v=1)$

Measurement

$M_0 = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

$M_1 = \begin{pmatrix}0&0\\0&1\end{pmatrix}$

Quantum Model

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\alpha|^2 = P(\psi = 0)$

$|\beta|^2 = P(\psi = 1)$

$|\psi\rangle$ is normalized if $|\alpha|^2 + |\beta|^2 = 1$

Normalization

$|\psi_N\rangle = \frac{1}{\sqrt{N}}|\psi\rangle$; $N = |\alpha|^2 + |\beta|^2$

Complex Numbers

$\alpha = \alpha_R + i\alpha_I = |\alpha|e^{i\theta} = |\alpha|\cos\theta + i|\alpha|\sin\theta$

$i^2 = -1$ $\quad (-i)(i) = 1$ $\quad \alpha^2 = \alpha^*\alpha$

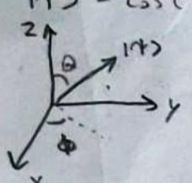
$\alpha^* = |\alpha|e^{-i\theta} = \alpha_R - i\alpha_I$

Polar Qubits

$|\alpha|^2 = \cos^2(\theta/2)$

$|\beta|^2 = \sin^2(\theta/2)$

$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$



Unitary Operators

$U^\dagger U = I$

$U^\dagger = (U^*)^T$

$U = e^{i\delta}[\cos(\gamma/2) I + i\sin(\gamma/2)(n_x X + n_y Y + n_z Z)]$

U rotates a vector by γ about (n_x, n_y, n_z).

Pauli Matrices

$X = \begin{pmatrix}0&1\\1&0\end{pmatrix} \to$ not

$Y = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \to i\,XZ$

$Z = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \to$ phase flip

$I = \begin{pmatrix}1&0\\0&1\end{pmatrix} \to$ identity

$X|0\rangle = |1\rangle$

$X|1\rangle = |0\rangle$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

$Y = -iXZ$

Hadamard Gate

$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$ $\quad H|x_{n-1} \ldots x_0\rangle$

$= \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{x\cdot y}|y\rangle$

$HH|\psi\rangle = |\psi\rangle$

Two Qubit Systems

$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

$P(00) = |\alpha_1\alpha_2|^2 = |\alpha_{00}|^2$

A valid state has the form:

$|\psi_{12}\rangle = \sum_{i,j=0}^{1}(\alpha_{ij}|ij\rangle)$

$\sum_{i,j}|\alpha_{i,j}|^2 = 1$

Entangled states

$|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_{12}\rangle$

brings a contradiction.

Bell State:

$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$|\psi_{12}\rangle = CNOT\,H_1 I_2|00\rangle$

Two Qubit Measurement

$|\psi\rangle^\dagger = \langle\psi|$

$P(ij) = \langle\psi|ij\rangle\langle ij|\psi\rangle$

$M_{ij} = |ij\rangle\langle ij|$

Superdense Coding

0. start w/ $|\psi_{12}\rangle = CNOT\,H_1 I_2|00\rangle$
1. Give $|\psi_2\rangle$ to Bob
2. $a = 1 \to Z|\psi_1\rangle$
3. $b = 1 \to X|\psi_1\rangle$
4. Send $|\psi_1\rangle$ to Bob
5. $CNOT_{12}|\psi_1\psi_2\rangle$
6. $H|\psi_1\rangle$
7. Measure $|\psi_1\rangle, |\psi_3\rangle$

Teleportation

$(CZ)_{32}C_{12}M_2M_1H_3C_{31}C_{12}H_1|0_1 0_2\psi_3\rangle$

is the single bit teleportation operation.

Quantum Computing

$U|x\rangle|y\rangle = |x\rangle|f(x) \text{ XOR } y\rangle$

$U: x \to f(x)$

Cloning Disproof

let $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$

Then $U|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle$

$U(a|\psi\rangle + b|\phi\rangle)|0\rangle$

$= a|\psi\rangle|\psi\rangle + b|\phi\rangle|\phi\rangle$

$= Ua|\psi\rangle|0\rangle + Ub|\phi\rangle|0\rangle$

$\neq (a|\psi\rangle + b|\phi\rangle)(a|\psi\rangle + b|\phi\rangle)$

Bernstein-Vazirani Algorithm

$|0_{n-1} \ldots 0_0\rangle = |z\rangle$

$U: a \cdot x$

$H_{n-1} \ldots H_0 H_y U H_{n-1} \ldots H_0 H_y X_y |z\rangle|0_y\rangle$

$= \sum_{z=0}^{2^n-1}\sum_{w=0}^{2^n-1}(-1)^{(a+w)\cdot z}|w\rangle|1_y\rangle$

$(-1)^{(a+w)\cdot z} = (1+(-1)^{(a_{n-1}+w_{n-1})}) \cdots (1+(-1)^{(a_0+w_0)})$

$\therefore$ when $a_i = w_i$, the corresponding coefficient is zero.

$\therefore w = a$.

$\therefore \sum_{z=0}^{2^n-1}\sum_{w=0}^{2^n-1}(-1)^{(a+w)\cdot z}|w\rangle|1_y\rangle = |a\rangle|1_y\rangle$

Phase Kickback

ex. $UH_yX_y|x\rangle|0_y\rangle = U|x\rangle\left(\frac{|0_y\rangle - |1_y\rangle}{\sqrt{2}}\right)$

$= (-1)^{f(x)}|x\rangle\left(\frac{|0_y\rangle - |1_y\rangle}{\sqrt{2}}\right)$

The output of $f(x)$ isn't in the output register but in an overall phase.

Our Sun's Position

Distance to galactic center: $d_0 = 27.2 \pm 1.1$ kly
Galactic rotation period: 240 Myr

Galactic Components of the Milky Way

Disk: all matter confined to the plane of the galaxy
Spiral Arms: part of the disk, spiral pattern of stars, gas, dust.
Spherical: all matter in a spherical distro from the galaxy's center.
Halo: part of the spherical component, the spherical region of a spiral galaxy with a scattering of stars
Central Bulge: dense cloud of stars at galaxy's center. Diameter of 65001y.

Galactic Mass

$M \gtrsim 100$ billion solar masses.
Dark matter: non-luminous matter detected only through its gravitational influence.
Dark halo: non-luminous extension of galactic halo composed of dark matter.

Star Formation in Spiral Arms

Density wave theory: spiral arms are compressions of the interstellar medium in the galactic disk. Star formation occurs in regions of gas cloud compression.

The Age of the Milky Way

Oldest open clusters about 9–10 billion yrs old.
Oldest globular clusters about 13 billion yrs old.

Stellar Populations

	pop 1 extreme	pop 1 intermediate	pop 2 extreme	pop 2 intermediate
Location	spiral	disk	bulge	halo
Metals (%)	3	1.6	0.8	<0.8
Shape of orbit	circular	slight ellipse	moderate ellipse	highly elliptical
Age (yrs)	0.2 billion	0.2–10 billion	2–10 billion	10–13 billion

Milky Way History

Monolithic Collapse Model: says galaxy formed from collapse of a single large cloud of turbulent gas.

Black Holes

If a star with >3M solar masses collapses, no force can stop the collapse and it becomes a singularity. Around a singularity (= infinite mass and zero radius) is a "black hole", from which nothing can escape.
Schwartzschild Radius: the radius of the event horizon, proportional to the black hole's solar mass.

R_S (km) $\approx 3 \times M$ (solar masses)

1st Order Perturbation Theory

For an e^- in a potential box with potential teE_x added, the gnd state wave fn with 1st correction is:

$$\psi_n^1 = \psi_1^{(0)} + \sum_{k \neq 1} \frac{\langle \psi_k^0 | H' | \psi_1^0 \rangle}{(E_1^0 - E_k^0)} \psi_n^0$$

$$\therefore \psi_n^1 = \sum_{m \neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{(E_n^0 - E_m^0)} \psi_n^0$$

The first term: $\dfrac{\langle \psi_2^{(0)} | H' | \psi_1^{(0)} \rangle}{(E_1^0 - E_2^0)}$

$$= \frac{\frac{2eE}{a} \int_0^a \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) x \, dx}{\left(\frac{k^2\pi^2}{2ma^2} - \frac{4\pi^2 k^2}{2ma^2}\right)}$$

let $y = \frac{\pi x}{a}$, $x = \frac{a}{\pi} y$

$$= \left(\frac{2eE}{a}\right)\left(\frac{a}{\pi}\right)^2 \int_0^\pi \sin(y) \sin(2y) \, y \, dy \cdot -\frac{2ma^2}{3\pi^2 \hbar^2}$$

$$= 2\left(\frac{2eE}{a}\right)\left(\frac{a}{\pi}\right)^2 \left[-2 - \tfrac{14}{9}\right] \cdot \left(\frac{-2ma^2}{3\pi^2 \hbar^2}\right)$$

$$= \left(\frac{-2ma^2}{3\pi^2 \hbar^2}\right)\left(-\frac{8}{9}\right)\left(\frac{a}{\pi}\right)^2 \left(\frac{2eE}{a}\right)$$

$$\therefore \psi_1 \approx \psi_1^{(0)} + 0.\overline{8} \left(\frac{2eEa}{3\pi^2}\right)\left(\frac{ma^2}{\hbar^2}\right) \psi_2^{(0)}$$

For a particle in a 2-D symmetric well:

$H' = cxy$ unperturbed: $E_{n_1 n_2}^{(0)} = \frac{\pi^2 \hbar^2}{2mL^2}(n_1^2 + n_2^2)$

$$\psi_{n_1 n_2}^{(0)}(x, y) = \frac{2}{L} \sin\left(\frac{n_1 \pi x}{L}\right) \sin\left(\frac{n_2 \pi x}{L}\right)$$

$$\Delta E_{n_1 n_2}^{(1)} = \langle \psi_{n_1 n_2}^{(0)} | cxy | \psi_{n_1 n_2}^{(0)} \rangle$$

$$= \frac{4c}{L^2} \int_0^L x \sin^2\left(\frac{n_1 \pi x}{L}\right) dx \int_0^L y \sin^2\left(\frac{n_2 \pi x}{L}\right) dy$$

$$= \frac{cL^2}{4}$$

$$\therefore E_{n_1 n_2}^{(1)} = E_{n_1 n_2}^{(0)} + \frac{cL^2}{4} = \frac{\pi^2 \hbar^2}{2mL^2}(n_1^2 + n_2^2)$$

In general: $H = H_0 + \lambda \overbrace{H'}^{\text{perturbation}}$; $\lambda \ll 1$

$$\psi_n = \psi_n^0 + \lambda \psi_n^1 + \lambda^2 \psi_n^2 + \ldots$$

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \ldots$$

Quantum 2

Multiple Particles

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2, t)$$

$$P(d^3\vec{r}_2) = |\Psi(\vec{r}_1, \vec{r}_2, t)|^2 \, d^3\vec{r}_1 d^3\vec{r}_2$$

If time independent:

$$\Psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2) e^{-iEt/\hbar}$$

Non-interacting particles in a square well:

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}\right) \quad E_{n_1 n_2} = (n_1^2 + n_2^2)\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$$

$$\psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$$

Gnd state: $\psi_{11} = \left(\frac{2}{a}\right)\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$

1st excited state:

$$\psi_{12} = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)$$

$$E_{21} = \frac{5\pi\hbar^2}{2ma^2}$$

N-particle systems:

$$\hat{H} = \sum_{i=1}^{N} \frac{-\hbar^2}{2m_i}\nabla_i^2 + \hat{V}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t)$$

$$[x_j, \hat{p}_{x_k}] = i\hbar\delta_{jk}$$

$$\langle\psi|\hat{A}|\psi\rangle = \int d^3\vec{r}_1 \int d^3\vec{r}_2 \int \psi^*(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) A\psi(\vec{r}_1, \dots, \vec{r}_N)\, d^3\vec{r}_N$$

Multielectron Systems

A z-electron, relativistic atom:

$$\left[-\frac{\hbar}{2m_e}\sum_{i=1}^{z}\vec{\nabla}^2_{r_i} - \frac{\hbar^2}{2M}\vec{\nabla}_R^2 - \sum_{i=1}^{z}\frac{ze^2}{|\vec{r}_i - \vec{R}|} + \sum_{i>j}\frac{e^2}{|\vec{r}_i - \vec{r}_j|}\right]\psi(\vec{r}_1, \dots, \vec{r}_z)$$

$$= E\psi(\vec{r}_1, \dots, \vec{r}_z, \vec{R})$$

; where M = nucleus mass,

$\frac{\hbar^2}{2M}\vec{\nabla}_R^2 = KE$,

$-\sum_{i=1}^{z}\frac{ze^2}{|\vec{r}_i - \vec{R}|}$ = attract of e^- w nucleus

$\sum_{i>j}\frac{e^2}{|\vec{r}_i - \vec{r}_j|}$ = interaction between e^-

Interchange Symmetry

$\hat{P}_{ij}$ is a swap operator of particles i and j.

$$[\hat{P}_{ij}, \hat{P}_{kl}] \neq 0 \text{ iff } ij \neq kl$$

$$\hat{P}_{ij}^2 \equiv 1 \quad \hat{P}_{ij}^2\psi = \pm\psi$$

Distinguishable and Non-interacting particles

$$\hat{H} = \sum_{i=1}^{N} H_i = \sum_{i=1}^{N}\left\{-\frac{\hbar^2}{2m_i}\nabla_i^2 + \hat{V}_i(\xi_i)\right\}$$

$$[\hat{H}_i, \hat{H}_j] = 0 \quad [x_i, x_j] = [p_i, p_j] = 0$$

$$\psi_{n_1 n_2 \dots n_N}(\xi_1, \xi_2 \dots \xi_N) = \prod_{i=1}^{N}\psi_{n_i}(\xi_i)$$

In an infinite well (spinless particles):

$$E_{n_1 n_2 n_3 n_4} = \frac{\hbar^2\pi^2}{2a^2}\left\{\frac{n_1^2}{m_1} + \frac{n_2^2}{m_2} + \frac{n_3^2}{m_3} + \frac{n_4^2}{m_4}\right\}$$

$$\psi_{n_1 n_2 n_3 n_4}(x_1, x_2, x_3, x_4) = \frac{4}{a^2}\sin\left(\frac{n_1\pi}{a}x_1\right)\sin\left(\frac{n_2\pi}{a}x_2\right)\sin\left(\frac{n_3\pi}{a}x_3\right)\sin\left(\frac{n_4\pi}{a}x_4\right)$$

Systems of Identical Particles

$$\psi(\xi_i, \xi_j, \dots, \xi_N) = \pm\psi(\xi_j, \xi_i, \dots, \xi_N)$$

Symmetric, Asymmetric WFs

$$\psi_+(\xi_1, \xi_2) = \frac{1}{\sqrt{2}}\left\{\psi(\xi_1, \xi_2) + \psi(\xi_2, \xi_1)\right\}$$

$$\psi_-(\xi_1, \xi_2) = \frac{1}{\sqrt{2}}\left\{\psi(\xi_1, \xi_2) - \psi(\xi_2, \xi_1)\right\}$$

In general, 2-particle systems:

$$\psi_+(\xi_1, \xi_2) = \frac{1}{\sqrt{2!}}\underbrace{\sum_{P}}_{\text{all permutations}}\hat{P}\,\psi_{n_1}(\xi_1)\psi_{n_2}(\xi_2)$$

$$\psi_-(\xi_1, \xi_2) = \frac{1}{\sqrt{2!}}\begin{vmatrix}\psi_{n_1}(\xi_1) & \psi_{n_1}(\xi_2) \\ \psi_{n_2}(\xi_1) & \psi_{n_2}(\xi_2)\end{vmatrix}$$

Systems of 3-particles

$$\psi_+(\xi_1\xi_2\xi_3) = \frac{1}{\sqrt{3!}} \sum_p \hat{P}\, \psi_{n_1}(\xi_1)\psi_{n_2}(\xi_2)\psi_{n_3}(\xi_3)$$

If $n_1 = n_2 = n_3$:

$$\psi_-(\xi_1\xi_2\xi_3) = \frac{1}{\sqrt{3!}} \sum_p (-1)^p \hat{P}\, \psi_{n_1}(\xi_1)\psi_{n_2}(\xi_2)\psi_{n_3}(\xi_3)$$

; $(-1)^p = \begin{cases} 1 & \text{if } p \text{ is an even permutation} \\ -1 & \text{if } p \text{ is odd} \end{cases}$

Exchange Forces for Distinguishable Particles

$$\langle x_1 x_2\rangle = \langle x_1\rangle_a \langle x_2\rangle_b = \langle x\rangle_a \langle x\rangle_b$$

$$\langle (x_1 - x_2)^2\rangle = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b$$

Exchange Forces for Indistinguishable Particles

Symmetric and asymmetric:

$$\langle x_1 x_2\rangle = \langle x\rangle_a\langle x\rangle_b \pm |\langle x_{ab}\rangle|^2$$

$$\langle (\Delta x)^2\rangle_\pm = \langle (\Delta x)^2\rangle_d \mp 2|\langle x\rangle_{ab}|^2$$

Atoms

A neutral atom w/ atomic # z, ze^-, mass m and a heavy nucleus:

$$H = \left\{ \sum_{j=1}^{z} -\frac{\hbar^2}{2m}\nabla_j^2 - \frac{1}{4\pi\varepsilon_0}\frac{ze^2}{r_j} \right\} + \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)\sum_{j\neq k}^{z} \frac{e^2}{|\vec{r}_j - \vec{r}_k|}$$

3-D wells

A neutron in the upper third of a box with a proton that could be anywhere:

$$\int_{x_n=0}^{a}\int_{y_n=0}^{b}\int_{z_n=\frac{2c}{3}}^{c}\int_0^a\int_0^b\int_0^c |\psi(\vec{r}_n,\vec{r}_p,t)|^2\, d\tau_n\, d\tau_p$$

; where $d\tau = dx\,dy\,dz$

If separation of variables is possible:

$$\psi(\vec{r}_n,\vec{r}_p,t) = \psi_n(\vec{r}_n,t)\,\psi_p(\vec{r}_p,t)$$

Systems of 4 particles

If they are identical: $\psi_a(\xi_1\xi_2\xi_3\xi_4) = \frac{1}{\sqrt{n!}}\sum_p (-1)^p \hat{P}\, \psi_{n_1}(\xi_1)\psi_{n_2}(\xi_2)\psi_{n_3}(\xi_3)\psi_{n_4}(\xi_4)$,

$$\psi_s(\xi_1\xi_2\xi_3\xi_4) = \frac{1}{\sqrt{n!}}\sum_p \hat{P}\, \psi_{n_1}(\xi_1)\psi_{n_2}(\xi_2)\psi_{n_3}(\xi_3)\psi_{n_4}(\xi_4) \quad ; n=4,\ n_1\neq n_2\neq n_3\neq n_4$$

1-D Wells with N particles

$$E_0 = \frac{N\hbar^2\pi^2}{2ma^2}$$

Bosons:

$$E_0 = NE_1 = \frac{N\hbar^2\pi^2}{2ma^2}$$

Fermions:

$$E_0 = \sum_{n=1}^{N/2} n^2 \frac{\hbar^2\pi^2}{2ma^2}$$

Bosons:

$$\psi_0 = \prod_{i=1}^{N}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi n}{a}x_i\right)$$

Indistinguishable Particles

$$\psi_\pm(\vec{r}_1,\vec{r}_2) = A\{\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\}$$

Pauli exclusion principle:

$$A\{\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_1)\psi_a(\vec{r}_2)\} = 0$$

1st Order Perturbation Theory

Perturbed energy is given by: $E = E^{(0)} + E^{(1)}$

In a 1-D well: $E^{(1)} = \langle \psi_1^{(0)}|H'|\psi_1^{(0)}\rangle$

$$= \int V_0 |\psi_1^{(0)}|^2\, dx \qquad E^{(0)} = \frac{\pi^2\hbar^2}{2ma^2}$$

Note: $\int_0^a \cos^2\left(\frac{\pi x}{6a}\right)dx = \left[\frac{x}{2} + \frac{3a}{2\pi}\sin\left(\frac{\pi x}{3a}\right)\right]_0^a$

For an e^- in a potential box with potential added $+eEx$, the lowest allowed energy to the 1st order is given by:

$$E_n = E_n^{(0)} + E_n^{(1)} = \int_0^a \psi_n^0\, eEx\, \psi_n^0\, dx$$

also $\hat{H} = \hat{H}_0 + \hat{H}'$ $\quad \hat{H}^0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

$\hat{H}' = eEx \quad E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$

$\psi_n^{(0)} = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

Note: $E_n = eE\int_0^a \frac{2}{a}x\sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{2eE}{a}\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right)dx$

$$= \frac{2eE}{a}\left[\frac{x^2}{4} - \frac{a^2}{8n^2\pi^2}\cos\left(\frac{2n\pi x}{a}\right) - \frac{ax}{4n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{1}{2}eEa$$

$$E = E_1^{(0)} + E_1^{(1)} = \frac{\pi^2\hbar^2}{2ma^2} + \frac{1}{2}eEa$$

Perfect secrecy

$p(M|C) = p(M)$

Baye's Theorem

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Shannon Entropy

$$H = -\sum_i p_i \log_b p_i$$

Von Neumann Entropy

$$S = -\mathrm{Tr}[\rho \ln \rho]$$
$$= -\sum_j c_j \ln c_j$$
$$; \rho = \sum_j c_j |j\rangle\langle j|.$$

Quantum generalization of Shannon entropy. When ρ is diagonal the Shannon and Von Neumann entropy are the same.

Codes

A code is the set of codewords that are the non-trivial linear combs of G.

$$G = \begin{pmatrix} \mathbb{I} \\ H' \end{pmatrix}, \quad H = (H' \ \mathbb{I})$$

Source replacement

i.e. replacing 3D system A with 2D system A':

$$|\psi\rangle_{AA'} = \frac{1}{\sqrt{3}}|0\rangle|0\rangle + \frac{1}{\sqrt{3}}|1\rangle|1\rangle + \frac{1}{\sqrt{3}}|2\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right)$$

Generally:

$$|\psi\rangle_{AA'} = \sum_i \sqrt{p_i}\,|i\rangle_A |\psi_i\rangle_{A'}$$

Tomographic Completeness

POVM $\{F_x\}$ is complete if after tomography:

$$p(x) = \mathrm{Tr}[\rho F_x] \ \forall x.$$
$$= \langle \tilde{\rho} | \tilde{F}_x \rangle.$$

Shannon Entropy

$$H(x) = -\sum_{x \in X} p(x) \log_2 p(x).$$

Cryptograms

$$C = \underset{\text{message}}{M} \underset{\text{xor}}{\oplus} \underset{\text{key}}{K}$$

Hamming Weight

The number of 1's in a binary vector, ω_H.

Hamming Distance

$$d_H(c_i, c_j) = \omega_H(c_i \oplus c_j)$$

Code Minimum Distance

$$d_{min} = \min_{c_i \neq c_j} [d_H(c_i, c_j)]$$

Syndromes

$$s = Hc' = H(\underset{\text{codeword}}{c} \oplus \overset{\text{error}}{e})$$

parity check

$s = 0$ indicates perfect transmission.

$s \in H_{cols}$ indicates flip error in corresponding position.

$s \neq 0 \notin H$ incorrectible

Schmidt Decomposition

Any bi-partide state can be written as:

$$\sum_{j=1}^{m} \sqrt{p_j}\,|j\rangle_S |j\rangle_E$$

; $|j\rangle_S \perp$ in S,

$|j\rangle_E \perp$ in E,

$m \leq \min(\dim E, \dim S)$

Imperfect Secrecy

$$\rho_E^x = \mathrm{Tr}_S[\rho_{SE}\, F_x \otimes \mathbb{I}]$$

If $\rho_E^x \perp \rho_E^{x'}$ then Eve knows x from x'.

If $\rho_E^x = \rho_E^{x'}$ Eve knows nothing after the POVM is applied.

Conditional Shannon Entropy

$$H(X|Y) = \sum_{y \in Y} p(y)\left(-\sum_x p(x|y) \log_2 p(x|y)\right)$$

Shannon Information

$$I(x,y) = H(x) + H(y) - H(x,y).$$
$$= H(y) - H(y|x).$$

Leaked Information

$$\frac{1}{n}\delta_{leak} \geq H(X|Y) + \sqrt{\frac{3\log(2/\epsilon_{EC})}{n}}\log(S).$$

Two-universal Hashes

A two-universal hash f of family F with probability dist. P_F that takes $X \to Z$ satisfies:

$$\Pr_f[f(x) = f(x')] \leq \frac{1}{|Z|}.$$

Fock Basis

$|n\rangle \ ; \ n \ni \mathbb{Z}$

$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{a}|0\rangle = 0$

$\hat{n} = \hat{a}^\dagger \hat{a}$

$\hat{n}|n\rangle = n|n\rangle$

Coherent States

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \quad ; \alpha \in \mathbb{C}.$$

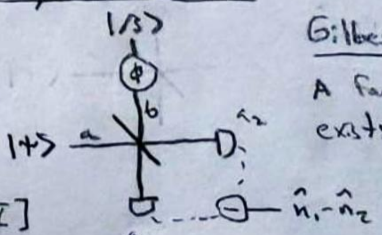
$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$

Energy is: $\hbar\omega|\alpha|^2$.

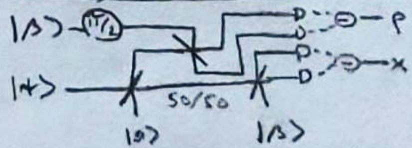
$\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$.

Homodyne Measurements



$$\langle\psi|\langle\beta|(\hat{n}_1 - \hat{n}_2)|\beta\rangle|\psi\rangle$$
$$= \sqrt{2}\,\beta\langle\psi|\hat{x}_\phi|\psi\rangle$$
$$x_\phi = \frac{1}{\sqrt{2}}(a e^{-i\phi} + a^\dagger e^{i\phi})$$

Heterodyne Measurement



POVM elements are projections onto coherent states:

$$F_{xp} = \frac{1}{\sqrt{\pi}}|\alpha\rangle\langle\alpha| \quad ; \alpha = \sqrt{2}(x + ip).$$

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = \mathbb{1}$$

$$\mathrm{Tr}[\rho F_x \rho] = \underset{\text{quasi-probability.}}{Q(\alpha)}$$

Private states

$$\tilde{\rho}_{ABA'B'} = U(|\phi^+\rangle\langle\phi^+|_{AB} \otimes \rho_{A'B'})U^\dagger$$

$$U = \sum_{ij}(|i\rangle\langle i|_A \otimes |j\rangle\langle j|_B \otimes U^{ij}_{A'B'})$$

Wyner's Theorem

If $e_{AB} < e_{AE}$ then Alice and Bob can generate a secret key.

Entanglement Distillation

$$E_D(\rho_{AB}) = \lim_{n\to\infty} \frac{D(\rho_{AB}^{\otimes n})}{n}$$

For pure states:

$$E_D(|\psi\rangle_{AB}) = S(\rho_A)$$

Gilbert-Varshamov Bound

A family of codes C_k exists such that:

$$\lim_{k\to\infty} R_k = 1 - h(2e)$$

Holevo Quantity

$$\chi = S(\rho) - \sum_i p_i S(\rho_i)$$

Key Rate

$R = \frac{\text{keys}}{\text{signals}}$ infinite keys limit

$R_\infty = \min_n \{I(A:B)\} - \chi(A:E)\}$

$R_\infty = \frac{1}{2}(1 - 2h(e)) \leftarrow$ shannon limit.

$R_\infty = \frac{1}{2}(1 - \delta_{leak} - h(e)) \leftarrow$ BB84

QKD

If entanglement-based:

1) distribute n copies of ρ_{AB}
2) measure w $\{F_x\}$ and $\{G_y\}$.

Otherwise:

1) send n states chosen from $|\psi_i\rangle$ w prob. $p(i)$.
2) Bob measures using $\{G_y\}$.

3) A, B jointly select random test sample, learning $p(x,y)$.
4) A, B announce measurement results.
5) A or B performs key map of results into raw key var. The other participant recovers the same result.

Minimum Uncertainty States

$$\Delta A \Delta B \geq \frac{1}{2} |\langle\psi| [A,B] |\psi\rangle|$$

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

Interference

Two-beam:

$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$

$\delta = k(s_2 - s_1)$

constructive:

$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$

$\delta = 2m\pi$

destructive:

$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$

$\delta = (2m+1)\pi$

if $I_1 = I_2 = I_0$,

then $I = 4I_0\cos^2(\frac{\delta}{2})$

visibility $= \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$

Young's:

$\Delta = s_2 - s_1 = a\sin\theta$

$\sin\theta \approx \tan\theta = y/L$

$\delta = k(s_2 - s_1)\frac{2\pi}{\lambda}\Delta$

Irradiance at screen:

$I = 4I_0\cos^2(\frac{\delta}{2})$

$\delta = \frac{2\pi a\sin\theta}{\lambda}$

bright fringes:

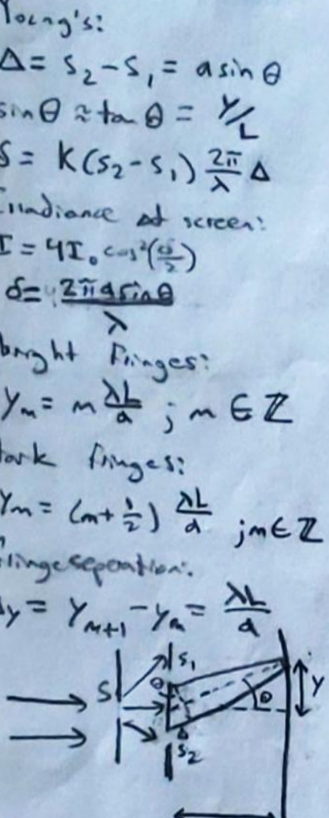
$y_m = m\frac{\lambda L}{a}$; $m \in \mathbb{Z}$

dark fringes:

$y_m = (m + \frac{1}{2})\frac{\lambda L}{a}$; $m \in \mathbb{Z}$

fringe separation:

$\Delta y = y_{m+1} - y_m = \frac{\lambda L}{a}$



a is slit separation.

Two-beam, through film:

$\Delta_p = 2n_f t\cos\theta$; $\Delta_p = OPD$

constructive interference:

$\Delta_p + \Delta_r = m\lambda$; $\Delta_r = OPD$ due to phase shifts from reflection

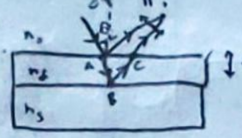
destructive interference:

$\Delta_p + \Delta_r = (m + \frac{1}{2})\lambda$

phase shift due to reflections:

2 external $(n_0 < n_f < n_s)$ or 2 internal $(n_0 > n_f > n_s)$ yield no shift

one each $(n_f > n_0, n_f > n_s)$ or $(n_f < n_0, n_f < n_s)$ yields a shift of π.



if $\theta_i = 0$, then $\Delta = n_f(AB + BC)$
$= 2n_f t$

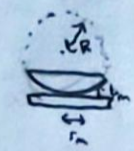
Fringes of equal thickness:

called "Fizeau fringes".

$\Delta_p = 2n_f t = 2n_f x\alpha$; x = fringe position, α = wedge angle

$\Delta x = \lambda_f / 2\alpha$

Newton's rings:



$R^2 = r_m^2 + (R - t_m)^2$

Maxima: $2n_f t_m = (m - \frac{1}{2})\lambda_0$

$r_m = \sqrt{(m - \frac{1}{2})\lambda_f R}$

Minima: $r_m = \sqrt{m\lambda_f R}$

if $(n_f < n_2, n_f < n_1)$ or $(n_f > n_2, n_f > n_1)$ then the center is dark.

Stoke's relations:

$r = -r'$; r is reflection coefficient from n_1 to n_2, r' is from n_2 to n_1

$tt' = 1 - r^2$; t is transmission coefficient

Multi-beam in parallel plate:

$\delta = k\Delta \quad \Delta = 2n_f t\cos\theta_t$

$E_N = E_0 t (r')^{2N-3} t' e^{i[\omega t - (N-1)\delta]}$

$E_R = \sum_{N=1}^{\infty} E_N = E_0 e^{i\omega t}\left[\frac{r(1 - e^{-i\delta})}{1 - r^2 e^{-i\delta}}\right]$

$I_R = E_R \cdot E_R^* = I_i \cdot \left[\frac{2r^2(1 - \cos\delta)}{(1 + r^4) - 2r^2\cos\delta}\right]$

$I_T = I_i - I_R$

Inferometry

Michelson inferometers:

$\Delta_p = 2d\cos\theta$

$\Delta_r = \lambda/2$

dark fringes: $2d\cos\theta = m\lambda$

Fabry-Perot inferometers:

$T = \frac{1}{1 + F\sin^2(\frac{\delta}{2})}$

$F = \frac{4r^2}{(1 - r^2)^2}$; r is reflectivity of mirrors

$F = \frac{T_{max} - T_{min}}{T_{min}}$

$\mathcal{F} = 2\pi / (2\delta_{1/2})$; $\delta_{1/2}$ = half-width at half-max

$\mathcal{F} = \frac{\pi}{2\sin^{-1}(\frac{1}{\sqrt{F}})}$

if F is very large:

$\mathcal{F} = \pi\sqrt{F}/2 = \frac{\pi r}{1 - r^2}$

Coherence

enables stationary interference.

temporal: tells us how monochromatic source is.

spatial: tells us the size of the source.

For a wave w/ lifetime $\tilde{\tau}_0$, bandwidth $\Delta\omega$ is required to represent the finite harmonic wave train:

$\Delta\omega = \frac{2\pi}{\tilde{\tau}_0}$, $\Delta\nu = \frac{1}{\tilde{\tau}_0}$

coherent length $l_t = c\tilde{\tau}_0$

line width $\Delta\lambda = \frac{\lambda^2}{l_t}$; λ is center wavelength of the source

frequency range $\Delta\nu = \frac{c}{\lambda^2}\Delta\lambda$

Interference fringe visibility:

$V = 1 - \frac{\tilde{\tau}}{\tilde{\tau}_0} = 1 - \frac{\Delta}{l_t} = 1 - \frac{\Delta \cdot \Delta\lambda}{\lambda^2}$

Complete incoherence: $\tilde{\tau} \geq \tilde{\tau}_0$ or $\Delta \leq l_t$

complete coherence: $\tilde{\tau} = 0$ or $\Delta = 0$

Fraunhofer Diffraction

Diffraction from a single slit:

$I = \left(\frac{\varepsilon_0 c}{2}\right)E_0^2 = I_0\left(\frac{\sin^2\beta}{\beta^2}\right)$

$\beta = \frac{1}{2}kb\sin\theta$

if $\theta = 0$, then $I = I_0$ ← at principle maximum

minima: $b\sin\theta_m = m\lambda$

zero irradiance at: $y_m = \frac{m\lambda f}{b}$
$= f\sin\theta_m$

secondary maxima at: $\tan(\beta) = \beta$; 1.4, 2.45, 3.47

Diffraction from rectangular slit:

a = slit width b = slit length

$I(\theta) = I_0\left(\frac{\sin^2\alpha}{\alpha^2}\right)\left(\frac{\sin^2\beta}{\beta^2}\right)$

$= I_0\,\mathrm{sinc}^2\alpha\,\mathrm{sinc}^2\beta$

$\alpha = \frac{1}{2}ka\sin\theta \quad \beta = \frac{1}{2}kb\sin\theta$

diffraction from circular aperture:

D = diameter.

$$I = I_0\left(\frac{2J_1(\gamma)}{\gamma}\right)^2$$

$$\gamma = \frac{1}{2}kD\sin\theta$$

Max I at center.

zero irradiance at: $D\sin\theta = 1.22\lambda$

Resolution:

Rayleigh's criterion: the max of one pattern falls directly over the 1st min of the other.

Limit of resolution: $(\Delta\theta)_{min} = \dfrac{1.22\lambda}{D}$

Double slit diffraction:

$$I = 4I_0\left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha$$

$\left(\frac{\sin\beta}{\beta}\right)^2$ ← diffraction; $\cos^2\alpha$ ← interference; $4I_0$ ← contributions from either slit

$\alpha = \left(\frac{ka}{2}\right)\sin\theta \quad \beta = \left(\frac{kb}{2}\right)\sin\theta$

interference max: $\alpha = p\pi$ or $a\sin\theta = p\lambda$

diffraction minima: $\beta = m\pi$ or $b\sin\theta = m\lambda$

missing order at: $\frac{a}{b} = \frac{p}{m}$

N-slit diffraction:

$$I(\theta) = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

$\left(\frac{\sin\beta}{\beta}\right)^2$ ← diffraction; $\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$ ← interference

principle maxima: $\alpha = \frac{p\pi}{N} \quad (p = 0, \pm N, \pm 2N)$

secondary minima: $p \in \mathbb{Z},\ p \notin [0, \pm N, \pm 2N]$

<u>Diffraction Gratings</u>

$a(\sin\theta_i + \sin\theta_m) = m\lambda$

Resolving power $R = \dfrac{\lambda}{(\Delta\lambda)_{min}} = mN$

<u>Polarized Light</u>

Jones Vector: $\tilde{E}_0 = \begin{bmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y}\end{bmatrix}$

Linear polarization ($\Delta\phi = m\pi$):

vertical: $\tilde{E}_0 = \begin{bmatrix}0\\1\end{bmatrix}$

horizontal: $\tilde{E}_0 = \begin{bmatrix}1\\0\end{bmatrix}$

$+45°$: $\tilde{E}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$-45°$: $\tilde{E}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

Circular polarization ($\Delta\phi = \frac{\pi}{2}$):

left: $\tilde{E}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$

right: $\tilde{E}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$

Elliptical polarizations

no slant: ($\Delta\phi = (m+\frac{1}{2})\pi$)

left: $\hat{E}_0 = \frac{1}{\sqrt{A^2+B^2}}\begin{bmatrix}A\\iB\end{bmatrix}$

right: $\hat{E}_0 = \frac{1}{\sqrt{A^2+B^2}}\begin{bmatrix}A\\-iB\end{bmatrix}$

slanted: ($\Delta\phi \neq \begin{cases} m\pi \\ (m+\frac{1}{2})\pi\end{cases}$)

left: $\hat{E}_0 = \frac{1}{\sqrt{A^2+B^2+C^2}}\begin{bmatrix}A\\B+iC\end{bmatrix}$

right: $\hat{E}_0 = \frac{1}{\sqrt{A^2+B^2+C^2}}\begin{bmatrix}A\\B-iC\end{bmatrix}$

$A, B, C > 0$

if elliptically polarized at angle α above x-axis:

$$\tan(2\alpha) = \frac{2E_{0x}E_{0y}\cos(\varepsilon)}{E_{0x}^2 - E_{0y}^2}$$

$E_{0x} = A \quad E_{0y} = \sqrt{B^2+C^2}$

$\varepsilon = \tan^{-1}\left(\frac{C}{B}\right)$

elliptical rotation directions:

A is always positive.

$C^+ \to$ CCW

$C^- \to$ CW

Polarizers:

$$M = \begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix},$$

if $\theta = 90°$, then:

$$M = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

θ is angle of polarizer relative to x-axis.

Phase Retarder:

quarter-wave plate:

$\Delta = (4m+1)\lambda/4$

$\Delta\phi = \pm\pi/2$

half-wave plate:

$\Delta = (4m+1)\lambda/2$

$\Delta\phi = \pm\pi$

full-wave plate:

$\Delta = (4m+1)\lambda$

$\Delta\phi = \pm 2\pi$

$M = \begin{bmatrix}e^{i\varepsilon_x} & 0\\ 0 & e^{i\varepsilon_y}\end{bmatrix}$

Rotator:

$$M = \begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$$

multiple elements' effects:

$M_T = M_1 M_2 M_3 \cdots M_n$

Constants

Plank length: $l_{pl} = 1.616 \times 10^{-35}$ m

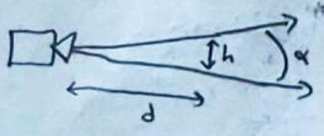
Milky Way length: $l_{MW} = 10^{21}$ m

Universe's boundary: $10^{-35}\text{ m} < R_{uni} < 10^{26}\text{ m}$

$R_{uni} = 10^{61}\, l_{pl}$

Field of View

$\alpha = \tan^{-1}\left(\frac{h}{2d}\right) \cdot 2$



Earth's Motion

diameter of Earth: $d_E \doteq 13{,}000$ km

West ⟶ East

distance to moon: $x_m = 380{,}000$ km $\doteq \frac{1}{4} d_E$

Astronomical Unit

$1\text{AU} = 150{,}000{,}000\text{ km} = \frac{1}{63240}$ ly

Orbits: Mercury 0.387 AU, Venus 0.723AU, Earth 1AU, Mars 1.524AU, Jupiter 5.203AU, Saturn 9.523AU, Uranus 19.203AU, Neptune 30.087AU, Pluto 39.746AU.

AU is not constant in time:

$$\frac{d(AU)}{dt} = 15 \pm 4 \text{ cm/cy}$$

This due to a) universe's expansion

b) loss of solar mass
c) loss of electromagnetic radiation } 0.3 cm/cy

Alpha Centauri

Closest sun to ours, 4.366 ly away.

Milky Way

diameter: 80,000 ly = 5.06×10^9 AU

stars: 100×10^9.

In a cluster of 4×10^6 ly, the "Local Group".

Filaments and Walls

"Filaments" of galaxies exist between galactic superclusters. Between filaments are voids of 11 to 150 megaparsec diameters.

Bootes Void

700 million ly away.

300 million ly diameter.

The Universe

Observable up to 13.8 billion ly away.

Universe's age: $t_{UN} = 13.8$ billion yrs.

Brightness of Stars

Apparent visual magnitude m_V is that seen on the Earth by human eyes.

Flux is the energy crossing a unit area in time as photons travel.

$$F = \frac{L}{4\pi r^2}$$

Observable Motion Terms

Zenith - above observer

Nadir - below

Precession - change in Earth's axis of rotation. Cycles every 26,000 yrs.

Sun

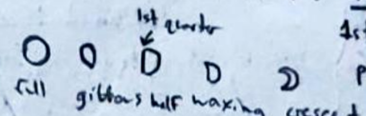
Equinox - Earth not tilted w/ respect to sun.

W ⟶ E on celestial sphere,

E ⟶ W in the sky.

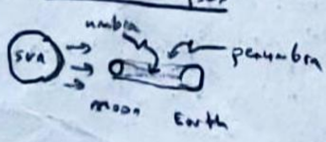
Moon

One side always faces Earth.

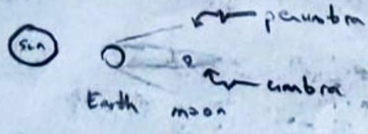


A four week cycle.

Solar Eclipses



Lunar Eclipse



Saros Cycle

The cycle of eclipses repeats every 18 years, 11 days and 8 hours.

Stellar Coordinates

Declination δ - angular N→S degrees from celestial equator.

Right Ascension α - hh:mm:ss E→W from Vernal equinox.

Timekeeping

Solar Day: Sun crosses local meridian.

Sidereal Day: Any star crosses the local meridian. (23h, 56m, 4.09s)

Synodic month: Complete lunar cycle. (29.5mo)

Sidereal month: Moon orbits Earth relative to any star. (27.3 days)

Tropical (Solar) Year: Spring equinox.

Sidereal Year: Earth orbits sun relative to any star.

History of Astronomy

900BC

Thales of Miletus - predicted solar eclipse.

Pythagoras - Earth is sphere.

Eratosthenes - calculated Earth's circumference.

0CE

Alexander the Great - founded Alexandria in Egypt.

Hypatia - female director of Alexandrian observatory.

Aryabhata - Indian astronomer.

Aristotle

1500CE

1st principle: the heavens are perfect.

Believed in Geocentric model where all objects had constant speed.

Claudius Ptolemy

Gave mathematical form to Aristotle's ideas. Attempted to explain retrograde motion.

Retrograde Motion

The apparent backwards (westward) motion of planets against the background of stars.

Epicycles - small circular paths thought to be followed by planets.

Deferent - large circular paths thought to be followed by epicycles orbiting the Earth.

Copernicus

Proposed helio-centric model.
Retrograde motion easily explained.

Tycho Brahe

Built observatory, measured star and planet positions for 20 years and hired Kepler.

Kepler's Laws

1. planets' orbits are ellipses w/ sun at one focus point.
2. A line between a planet to the sun sweeps over equal areas in equal intervals of time.
3. Orbital period is proportional to distance from the sun.

$$P_y^2 = a_{AU}^3$$

Galileo

Discovered the moon isn't perfect, Jupiter has four moons.

Discovered Venus had phases like the moon. Supported Copernicus' model.

Newton's Laws of Motion

1. An object stays at rest or in uniform motion unless acted on by an outside force.
2. An object's change in motion is proportional to the force acting on it
3. Every action has an equal opposite reaction.

Universal Theory of Gravitation

$$F = -G\frac{m_1 m_2}{r^2}$$

Tides

Spring Tide – large, occurs at full moon

Neap Tide – small, 1/4 and 3/4 moon.

Telescopes

$$P_{gathering} = \left(\frac{D_{objective}}{7}\right)^2$$

Primary lens: largest lens for refracting scope or the mirror for a reflecting scope.

Focal length: the distance from the primary lens to where the image is focused.

Resolving power: ability to resolve fine detail.

Diffraction fringe: blurred fringe surrounding any image.

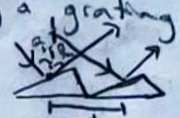
Astronomical instruments

photometer: photographic plates.

CCD: charge-coupled device of light-sensitive elements. Can be digitized.

Spectographs

Separates light into spectrum using a grating.



path difference = $d\sin(\theta_i) \pm \sin\theta_r$

Space Telescopes

Hubble telescope
James Webb
Kepler
Herschell Space Observatory
Chandra X-ray Observatory

Gamma-Ray Telescopes

Compton Gamma-Ray Observatory
European INTEGRAL Satellite
The GLAST

The Sun

Distance: 150×10^6 km

Diameter: $109 \times d_E$

Mass: $333 \times 10^3 \times m_E$

Density: a bit denser than water

Wien's Law

The hotter an object, the shorter the λ of it's maximum intensity.

Stefan-Boltzmann Law

Hotter objects emit more energy than cooler objects of the same size.

Photosphere

Visible surface of sun.
Avg. temp: 5800K

Sunspots: darker, produced by intense magnetic fields.

Granulation: caused by convection currents of gas rising from convective zone

Doppler Effect

Blueshift: shift towards shorter λ by source approaching observer.
Redshift: shift toward longer λ.

Kirchhoff's Laws

1. A solid, liquid or dense gas will emit a continuous spectrum.
2. A low-density gas will emit at specific λ.
3. A continuous light spectrum passing through a cool, low-density gas results in an absorption spectrum.

Charging

By Friction: - charge glass (+) = silk (-)
- charge plastic (-) = wool (+)

Net Charge = $q = (N_p - N_e)e$

$e = 1.6\times10^{-19}\,C$

Coulomb's Law

$$F = k\cdot\frac{|q_1||q_2|}{r^2} = \frac{|q_1||q_2|}{4\pi\varepsilon_0 r^2}$$

$1C = 1A\cdot 1s$

$\varepsilon_0 = 8.85\times10^{-12}\ C^2/N\cdot m^2$

$k = \frac{1}{4\pi\varepsilon_0} = 8.99\times10^{9}\ N\cdot m^2/C^2$

Shell Thm's

- A shell of uniform charge can be represented by a point charge at its center.
- A shell of uniform charge exerts no electrostatic force on a particle enclosed by it.

Electric Fields

$$\vec{q}_1 \to \vec{E}_1 = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{|q_1|}{r^2}\right)$$

$\vec{F}_{21} = q_2\vec{E}_1$

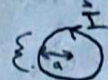
$\vec{E} = \frac{\vec{F}}{q_0}$ ← test charge

$\vec{E}$ due to electric dipole

$$|\vec{E}| = \left(\frac{1}{2\pi\varepsilon_0}\right)\left(\frac{|\vec{p}|}{z^3}\right)$$ ← distance between poles

$|\vec{p}| = qd$

Magnetic Dipoles

coil 

$|\vec{M}| = |\vec{I}|\pi a^2$

direction of M is according to right-hand rule

Ring of Uniform Positive Charge

$$E \approx \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q}{z^2}\right)$$ ← distance to particle from ring center

$$E = \int_0^{2\pi R} \cos\theta\, dE$$

E direction according to right-hand rule

Partial Ring

$$E = \int_{-\lim}^{+\lim} \cos\theta\, dE = \frac{\lambda\sin\theta}{4\pi\varepsilon_0 r}\Big|_{-\lim}^{+\lim}$$

in degrees

Rings by Layers

$$dE = k\cdot\left(\frac{z\sigma\cdot 2\pi r\cdot dr}{(z^2+r^2)^{3/2}}\right)$$

$\therefore E = \frac{\sigma}{2\varepsilon_0}$

$\vec{E}$ due to linear charged object

$$\vec{E}\int_0^L \frac{\lambda\cdot dx}{4\pi\varepsilon_0 d^2}$$

Dipole in an Electric Field

$\vec{\tau} = \vec{p}\times\vec{E} = pE\sin\theta$

$U = -\vec{p}\cdot\vec{E} = -pE\cos\theta$

$\Delta U = U_f - U_i$

$W = -\Delta U$

$W_{app} = \Delta U$

Electric Flux

$\Phi = \vec{E}\cdot\vec{A} = EA\cos\theta\quad N\cdot m^2/C$

area of surface, ⊥ to it, point outward

For a Gaussian surface:

$$\Phi = \oint \vec{E}\cdot dA$$

Gauss' Law

$\varepsilon_0\Phi = q_{enc}$

Charged Isolated Conductor

$|\vec{E}_\perp| = \frac{\sigma}{\varepsilon_0}$

$|\vec{E}_\parallel| = 0$

All unbalanced charge is on the conductor's surface.

Infinite Charged Rods

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Infinite Non-conducting Sheet

$$E = \frac{\sigma}{2\varepsilon_0}$$

Parallel Conducting Plates

= same charge:

$$E = \frac{\sigma_1}{\varepsilon_0}$$

= opposite charge:

$$E = \frac{2\sigma}{\varepsilon_0}$$

Spherical Shell

$E = \frac{q}{4\pi\varepsilon_0 r^2}$; outside the shell

$E = 0$; inside the shell

Solid Spheres

$E = \frac{q}{4\pi\varepsilon_0 R^2}$; outside ← sphere radius

$E = \frac{q\cdot r}{4\pi\varepsilon_0 R^3}$; inside ← inner test charge position

$q_{enc} = \left(\frac{r}{R}\right)^3 q$

Electric Potential Energy

$U = qV$ $\qquad 1eV = e(1V) = 1.6\times10^{-19}\,J$

$\Delta U = q(V_f - V_i)$

$W = -\Delta U$

$W_{app} = \Delta U$

$W_{electric} = \vec{F}_{ext}\cdot\vec{d}$

Electric Potential

$V = \frac{U}{q}$

$\Delta V = \frac{\Delta U}{q} = \frac{-W}{q}$

V is the potential energy per unit charge at a point in an electric field.

Variable Electric Fields

$$\Delta V = -\int_i^f \vec{E}\cdot d\vec{s}$$

- V due to a charged particle:

$$V = \frac{q}{4\pi\varepsilon_0 r}$$

- V due to a group of charges:

$$V = \sum_{i=1}^{n} V_i = k\cdot\sum_{i=1}^{n}\frac{q_i}{r_i}$$

Capacitance

$C = \frac{q}{V}$

Parallel Plate Caps

$V = E\cdot d$

$E = \frac{\sigma}{\varepsilon_0}$

$C = \frac{\varepsilon_0 A}{d}$

Cylindrical Caps

$E = \frac{q}{\varepsilon_0 2\pi r L}$ ← capacitor's length; ← radius

$$C = 2\pi\varepsilon_0\left(\frac{L}{\ln(b/a)}\right)$$

Spherical Caps

$C = 4\pi\varepsilon_0 R$

Potential Energy of a system of 2 charges

$$U = W = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

Caps in Parallel

$V_{eq} = V_1 = V_2 = \ldots = V_n$

$q_{eq} = q_1 + q_2 + \ldots + q_n$

$C_{eq} = \frac{q_{eq}}{V_{eq}} = C_1 + C_2 + \ldots + C_n$

Caps in Series

$V_{eq} = V_1 + V_2 + \ldots + V_n$

$q_{eq} = q_1 = q_2 = \ldots = q_n$

$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$

Ampere's Law

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i_{enc}$$

Caps in Dielectrics

$E = \frac{1}{4\pi K \varepsilon_0} \cdot \frac{q}{V}$

$q' = q\left(1 - \frac{1}{K}\right)$

induced charge on dielectric

$K\varepsilon_0 \Phi = q_{enc}$

Charging Caps

V is fixed

$U = \frac{1}{2} C V^2$

Discharging Caps

q is fixed

$U = \frac{1}{2} \cdot \frac{Q^2}{C}$

Current and Current Density

$i = \frac{dq}{dt}$

$i = \int \vec{J} \cdot dA$

$J = \frac{i}{A}$; if i is uniform and parallel to dA.

Resistance and Resistivity

resistivity $R = \frac{\rho L}{A}$ ← length; A ← cross-sectional area

$\rho_{iron} = 9.68 \times 10^{-8}\ \Omega m$

Power and Resistivity

$P = i^2 R = \frac{V^2}{R}$, relates resistance to generated E_{th}

Work, Energy, Emf

$\mathcal{E} = \frac{dW}{dq}$

RC Circuits

Charging:

$q = C\mathcal{E}(1 - e^{-t/\tau})$

$i = \frac{dq}{dt} = \left(\frac{\mathcal{E}}{R}\right) e^{-t/\tau}$

$\tau = RC$

Discharging:

$q = q_0 e^{-t/\tau}$

$i = \frac{dq}{dt} = -\left(\frac{q_0}{\tau}\right) e^{-t/\tau}$

$q_0 = CV_0$

Magnetic Fields

$B = \frac{F_B}{|q| v}$

$\vec{F}_B = q\vec{v} \times \vec{B}$

$|F_B| = |q| v B \sin\theta$

$1\ \text{Tesla} = 1T = 1 \frac{N}{A \cdot m}$

Circulating Charged Particles

$|q| v B = m \frac{v^2}{r}$

$r = \frac{mv}{|q|B}$ $\quad \omega = 2\pi f$

$T = \frac{2\pi m}{|q|B}$ $\quad f = \frac{1}{T}$

pitch $= v_{\parallel} T = v\cos(\theta)(2\pi m / qB)$

Biot-Savart Law

current in a straight wire:

$B = \frac{\mu_0 i}{2\pi R}$ ← distance from wire ⊥

$\therefore dB = \frac{\mu_0}{4\pi} \cdot \frac{i \cdot ds \cdot r \sin\theta}{r^3}$, s = length of wire

$B = \int_{-\infty}^{\infty} (dB)\, ds$

$\mu_0 = 4\pi \times 10^{-7}\ T \cdot \frac{m}{A}$

$\theta = R/r$

$r = \sqrt{s^2 + R^2}$

current in a curved wire:

$B = \left(\frac{\mu_0 i}{4\pi R}\right)\theta$

$\therefore dB = \frac{\mu_0}{4\pi} \cdot \frac{i \cdot ds \cdot r \sin\theta}{r^3}$

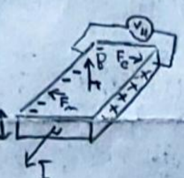
Thompson's Apparatus

$\frac{m}{|q|} = \frac{B^2 L^2}{2yE}$; for a particle of mass m, and charge q, exposed to B and E. Has displacement y after travelling distance L.

Hall Effect

$V_H = \frac{IB}{dne}$

$V_H = \vec{v} B d$



n = charge-carrier density

v_d = drift speed $\quad V_H$ = Hall voltage

$\vec{F}_B$ = magnetic force on −ve charge carriers

$\vec{F}_E$ = electric force from charge buildup

Magnetic Force on Current-Carrying Wire

$\vec{F} = i\vec{L} \times \vec{B} = iLB\sin\theta$

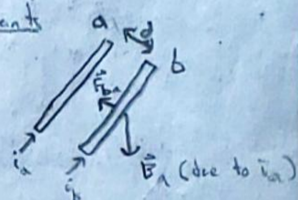
$d\vec{F}_B = i\, d\vec{L} \times B$

Force between Parallel Currents

$B_a = \frac{\mu_0 i_a}{2\pi d}$

$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a$

$\vec{F}_{ba} = \frac{\mu_0 L i_a i_b}{2\pi d}$



Inside a Current Carrying Wire

$B = \left(\frac{\mu_0 i}{2\pi R^2}\right) r$ ← location of Amperian loop; R ← radius of wire

Outside a Current Carrying Wire

$B = \left(\frac{\mu_0 i}{2\pi r}\right)$

Solenoids

$B = \mu_0 n i$; n = turns/length

B is constant throughout inside of solenoid

Toroids

$B = \frac{\mu_0 i N}{2\pi r}$; N = turns

B is related to r.

Faraday's Law of Induction

$\mathcal{E} = -\frac{d\Phi_B}{dt}$; for one loop

$\mathcal{E} = -N\left(\frac{d\Phi_B}{dt}\right)$; for a coil

$\Phi_B = BA\cos\theta$

The direction of $\mathcal{E}$ opposes the change in B.

Inductors

$L = \frac{N\Phi_B}{i}$

solenoids:

$L = \mu_0 n^2 l A$

$1\ \text{Henry} = 1H = 1T \cdot \frac{m^2}{A}$

AC Generators

$\Phi_B = BA\cos\theta$

$\theta = \omega t$

$\mathcal{E} = -N\left(\frac{d\Phi_B}{dt}\right) = -NBA\omega\sin(\omega t)$

ω is the angular velocity of the coil about its axis of rotation

Self-Induction

$\mathcal{E}_L = -L\frac{di}{dt}$

RL Circuits

Introducing Emf:

$L\frac{di}{dt} + iR = \mathcal{E}$

$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L})$

$V_R = \mathcal{E}(1 - e^{-t/\tau_L})$

$V_L = \mathcal{E} e^{-t/\tau_L}$

$\mathcal{E}$ is constant

Removing Emf:

$L\frac{di}{dt} + iR = 0$

$i = i_0 e^{-t/\tau_L} = \frac{\mathcal{E}}{R} e^{-t/\tau_L}$

$V_R = \mathcal{E} e^{-t/\tau_L} = -V_L$

$\tau_L = \frac{L}{R}$

Induction

$\mathcal{E} = BLv$; v = speed of conductor; L = length of rectangular loop

Induction Furnaces

$\mathcal{E} = E 2\pi r$

Energy and Energy Density in a Magnetic Field

total energy stored:

$U_B = \frac{1}{2} L i^2 \longleftrightarrow U_E = \frac{q^2}{2C}$

energy density at a point:

$u_B = \frac{B^2}{2\mu_0} \longleftrightarrow u_E = \frac{1}{2}\varepsilon_0 E^2$

Inductors vs. Capacitors

Caps:

$C = \frac{Q}{V}$

$U_C = \frac{1}{2} C V^2$

Inductors:

$L = \frac{N\Phi_B}{i}$

$U_B = \frac{1}{2} L i^2$

Nature of Light

$E = h\nu$ ← photon energy

$h = 6.62 \times 10^{-34}$ J·s

$1 eV = 1.6 \times 10^{-19}$ J

in air: $c = \lambda \nu$

in media: $v = c/n$

$\lambda = \frac{v}{\nu} = \frac{c}{n\nu} = \frac{\lambda_0}{n}$

Apertures

Controlling Image Brightness:

AS – the real element that limits the cone of rays accepted by the system

EnP – the image of the AS in optical elements that precede it.

ExP – the image of the AS in optical elements that follow it.

"Chief Ray" goes through the center of EnP, AS, and ExP.

Limiting Field of View:

FS – The real element that limits the angular field of view formed by a system. Subtends the smallest ∠ from the center of the EnP.

EnW – The image of the FS in the optical elements that precede it.

ExW – The image of the FS in the optical elements that follow it.

Cameras

$s_1 = \frac{s_0 f(f + A d)}{f^2 + A d s_0}$ f-number → $A = \frac{f}{D}$ ← diameter of lens

$s_2 = \frac{s_0 f(f - A d)}{f^2 - A d s_0}$ $E_e \propto \frac{D^2}{f^2}$

$E_e \cdot t$ = exposure

Field depth $= \frac{2 D d s_0 (s_0 - f) f^2}{f^4 - D^2 d^2 s_0^2}$ ← blurring diameter

Geometrical Optics:

$\theta_i = \theta_r$ ← law of reflection

$n_i \cdot \sin\theta_i = n_t \cdot \sin\theta_t$ ← Snell's

Huygen's Principle:

Every point on a wavefront serves as the source of secondary wavelets. The wavefront at some future time is the envelope of these wavelets.

Principle of Least Time:

Light always takes the path that takes the least time.

Fermat's Principle: $t = OPL/c$

Total Internal Reflection:

$n_1 > n_2$ ← condition

$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

Refraction through Plane Surfaces:

$s' = \left(\frac{n_2}{n_1}\right) s$

Reflection at a Spherical Surface:

$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$ ← mirror eq'n

$f = -\frac{R}{2}$ ← radius

convex mirror: $f < 0$

lateral magnification: $m = \frac{h_i}{h_o} = -\frac{s'}{s}$

$s^+ \rightarrow$ left $s'^+ \rightarrow$ left

$f^+ \rightarrow$ left $R^+ \rightarrow$ right center

Refraction at a Spherical Surface:

$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$ ← refraction eq'n

if $R \rightarrow \infty$: $s' = -\left(\frac{n_2}{n_1}\right)s$

lateral magnification: $m = \frac{h_i}{h_o} = \frac{n_1 s'}{n_2 s}$

$s^+ \rightarrow$ left, real $s'^+ \rightarrow$ right, real

$R^+ \rightarrow$ right center $m^+ \rightarrow$ erect

Thin Lenses:

$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$ $m = -\frac{f}{x} = \frac{h_i}{h_o} = -\frac{s'}{s} = -\frac{x'}{f}$

$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

$x x' = f^2$ ← Newtonian eq'n

Prisms

Deviation:

$\delta = \theta_1 + \theta_2 - (\theta_1' + \theta_2') = \theta_1 + \theta_2 - A$

$\delta = -A + \theta_1 + \sin^{-1}\left(\sqrt{n^2 - \sin^2\theta_1}\ \sin A - \sin\theta_1 \cos A\right)$

$n = \frac{\sin[(\delta_{min} + A)/2]}{\sin(A/2)}$ ← application for finding n

Cauchy Relation:

$n_\lambda = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots$

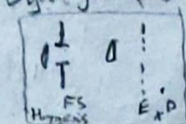
Dispersion: $\frac{dn}{d\lambda} \doteq -\frac{2B}{\lambda^3}$

Resolving Power: $R = \frac{\lambda}{(\Delta\lambda)_{min}} = b\frac{dn}{d\lambda}$

Simple Magnifiers

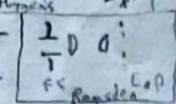
Image viewed at ∞ → Object at f point, $\therefore M = \frac{25}{f}$

I viewed at near point → Object just past f, $\therefore M = \frac{25}{f} + 1$



Eyepieces / Oculars

$\frac{1}{f_{eff}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_1 f_2}$



$\frac{1}{f_1} = (n-1)\left(\frac{1}{R_{11}} - \frac{1}{R_{12}}\right) = (n-1)K_1$

$\frac{1}{f_2} = (n-1)\left(\frac{1}{R_{21}} - \frac{1}{R_{22}}\right) = (n-1)K_2$

to correct aberration: $\frac{d(\frac{1}{f})}{dn} = 0$

Optimum separation: $L = \frac{1}{2}(f_1 + f_2)$

Microscopes

$d = f_o + L + f_e$ ← lens separation

$M = M_o M_e = -\left(\frac{L}{f_o}\right)\left(\frac{25}{f_e}\right) = \frac{25}{f_{eff}}$

$\frac{1}{f_{eff}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$; L = distance between 2nd f point and objective image.

Telescopes

Keplerian:

$L = f_o + f_e$ ← telescope length

$M = \frac{\alpha'}{\alpha} = -\frac{f_o}{f_e}$

Galilean:

$L = f_o + f_e$; $f_e < 0$

$M = \frac{\alpha'}{\alpha} = -\frac{f_o}{f_e}$

HPSL: find diagram, optical time eq'n, include sines, set to 0. LTSL: triangle diagram, optical time eq'n, derivative, set to 0.

Chromosphere

Bright gasses above the photosphere.

Spicules: projection in the chromosphere.

Filaments: arcs of plasma that are cooler than the surface.

Corona

Outermost atmosphere of the sun.

1/2 million - 2 million Kelvin.

Magnetic carpet: network of magnetic loops that cover the solar surface.

Solar Wind: atoms, ions that escape the corona and blow outward.

Sunspots

1. cool spots
2. 11-year cycle
3. Zeeman effect indicates strength of sun's magnetic fields
4. Cycle characteristics vary over centuries.
5. Sunspots part of larger magnetic process.

Sun's interior

Helioseismology: the study of the interior of the sun by analysis of its modes of vibration.

Differential Rotation

Sun rotates faster closer to the equator.

Dynamo Effect

Source of sun's magnetic field is in the bottom of the convective zone.

Babcock model: model of sun's magnetic cycle in which the sun's rotation "winds up" the magnetic field.

Chromospheric and Coronal Activity

1. All solar activity is magnetic
2. Releasing of energy in magnetic fields can trigger powerful eruptions.
3. In some regions, the magnetic field doesn't loop back. These produce much of the solar wind.

Star Distances

Stellar parallax: apparent shift of nearby object's position against background due to Earth's motion.

Parsec: 1pc = 206265 AU = 3.26 ly, The distance to a star with 1 second of parallax arc.

Brightness: inversely proportional to the square of distance

Absolute Visual Magnitude, M_V: the intrinsic brightness of a star.

Luminosity: total energy radiated

Star Temperatures

Weak hydrogen Balmer lines:

Cool stars have atoms in the ground state.

Hot stars have excited atoms.

Strong hydrogen Balmer lines:

Indicate intermediate temperatures.

Spectral Classes

	temp (K)	Balmer lines	size
O	40000	weak	largest
B	20000	medium	↑
A	10000	strong	
F	7500	medium	
G	5500	weak	
K	4500	very weak	↓
M	3000	very weak	smallest

New Spectral Types

L dwarfs: cooler, fainter than M's.

T dwarfs: even fainter.

Y dwarfs: even fainter, temp < 500 K.

Hertzsprung-Russel Diagram

Maps star luminosity vs. surface temp. 90% of stars in the main sequence.

Giants, supergiants lie above the main sequence.

Red Dwarfs are at the end of the MS.

White Dwarfs are at the lower left of the HR diagram.

Luminosity Classes

	description	spectral lines
Ia	Luminous Supergiant	thinnest
Ib	Regular Supergiant	↑
II	Bright Giant	
III	Giant	
IV	Subgiant	↓
V	Main sequence	widest

Visual Binary Systems

We can see both stars.

Spectroscopic Binaries

Stars orbiting each other produce spectral lines with doppler shifts.

Eclipsing Binaries

Stars cross in front of each other as seen by the Earth.

Can be recognized by periodicity of brightness.

Mass-Luminosity Relation

$L \propto M$

Density

Giants have very low density, MS stars have relatively low, White Dwarfs have very high.

Cepheid Variable Stars

Pulse with periods of 1 - 60 days.

Period $\propto$ luminosity.

Discovered by Harlow Shapley.

Lie in the "instability strip" on the HR diagram, which intersects the MS at the A and F stars.

Period-Luminosity relation is broken up into three continuous curves: for RR Lyrae, Type II cepheids and Type I cepheids.

Milky Way

Type: barred spiral Diameter: 100-180 kly

Thin stellar disc thickness: 2kly

Number of stars: 100-400 billion

Mass: $0.8 - 1.5 \times 10^{12} M_\odot$

Rotation period: 220-360 Myr.

Semiconductors

$$n = p = n_i = K \cdot \exp\left(\frac{-E_g}{2k_B T}\right)$$

; n = free e^- concentration, p = hole concentration, E_g = band gap, n_i = carrier concentration

$E_g = E_c - E_v$; E_c = conduction band, E_v = valance band

$$K = 2(2\pi k_B T/h^2)^{3/2}(m_e m_h)^{3/4}$$

"Intrinsic Semiconductor": Group 4 pure crystal.

"Doping": adds impurities from Groups 3, 5 to increase conductivity. G5 $\rightarrow e^- \rightarrow$ n-type

G3 $\rightarrow$ holes $\rightarrow$ p-type.

"Intrinsic material" called "n" if no doping.

"Extrinsic material" called "n" when doped.

Thermal generation: generates equal electron-hole pairs if intrinsic mat. for extrinsic, $pn = n_i^2$.

$$P_N = \left(\frac{-N_D + \sqrt{N_D^2 + 4n_i^2}}{2}\right) = \frac{N_D}{2}\left(-1 + \sqrt{1 + 4n_i^2/N_D^2}\right)$$

; P_N = hole concentration due to thermal excitation in n-type semiconductor, N_D = e^- concentration due to doping

If $n_i \ll N_D$: $P_N \approx n_i^2/N_D$, $n_N \approx N_D$

$\therefore (N_D + P_N) P_N = n_i^2$

Majority carriers: e^- if n-type, hole if p-type

Reversed-biased pn junctions $\rightarrow$ photodiode

Forward-biased pn junctions $\rightarrow$ laser diode.

$hf = W_g$, $\lambda = hc/W_g$; for band-gap mat.

If $V_{app} \geq W_g$ in eV, recombination creates photons.

Wavevector changes if indirect band-gap material. ex. Si.

LEDs

G3 $\rightarrow$ Al, Ga, In G5 $\rightarrow$ P, As, Sb

Ternary combinations: $Ga_{1-x}Al_xAs$ (800nm - 900nm)

Quarternary: $In_{1-x}Ga_xAs_yP_{1-y}$ (1μm - 1.7μm)

$E = h\nu = \frac{hc}{\lambda}$, $\lambda(\mu m) = \left(\frac{1.240}{E_g(eV)}\right)$

Bandgap energy for Ternary alloys:

$$E_g = 1.424 + 1.266x + 0.266x^2$$

For Quarternary alloys:

$$E_g = 1.35 - 0.72y + 0.12y^2$$

w/ current-density J, the rate of recombination is:

$$\frac{dn}{dt} = \frac{J}{qd} - \frac{n}{\tau}$$

; q = e^- charge, d = recombination region thickness, thermal generation rate $= \left(\frac{n}{\tau}\right)$, $J = A/cm^2$, τ = recombination lifetime

Equilibrium condition: $n = \frac{J\tau}{qd}$, $\frac{dn}{dt} = 0$

When $J \rightarrow 0$, excess carrier density decays exponentially:

$$n = n_0 e^{-t/\tau}$$

Internal quantum efficiency:

$$\eta_{int} = \frac{R_r}{R_r + R_{nr}} = \frac{\tau_{nr}}{\tau_r + \tau_{nr}} = \frac{\tau}{\tau_r}$$

$$R_r = n/\tau_r, \quad R_{nr} = n/\tau_{nr}$$

$$\frac{1}{\tau} = \frac{1}{\tau_r} + \frac{1}{\tau_{nr}}$$

; R_r = photons generated per second (radiative recombination rate), τ_r = radiative recombination lifetime

R_{nr} = non-radiative recombination rate

τ_{nr} = non-radiative recombination lifetime

τ = bulk recombination lifetime

$$P_{int} = \eta_{int}\left(\frac{I}{q}\right)h\nu = \eta_{int}\left(\frac{hcI}{q\lambda}\right)$$

External quantum Efficiency:

$$\eta_{ext} = \left(\frac{1}{4\pi}\right)\int_0^{\phi_c} T(\phi) \cdot (2\pi \sin(\phi)) \cdot d\phi$$

; ϕ = incident angle, $\phi_c = \frac{\pi}{2} - \Theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

$T(\phi)$ = Fresnel transmissivity

$$T(0) = \left(\frac{4n_1 n_2}{(n_1 + n_2)^2}\right)$$

For typical LED w/ $n = 3.5$:

$$\eta_{ext} \approx \frac{1}{n(n+1)^2}$$

$$P_{ext} = \eta_{ext} P_{int} = \frac{P_{int}}{n(n+1)^2}$$

P_{out} with modulated drive current:

$$P(\omega) = \left(\frac{P_0}{\sqrt{1 + (\omega\tau_i)^2}}\right)$$

; P_0 = power emitted at DC, ω = mod frequency, τ_i = carrier lifetime

3dB optical bandwidth:

$$\frac{P(\omega_{3dB})}{P(0)} = \frac{1}{2}$$

Optical power:

$$\frac{P(\omega)}{P(0)} = \frac{I(\omega)}{I(0)}$$

Electrical Power:

$$p(\omega) = I^2(\omega)R$$

Electrical Bandwidth:

$$BW = 10\log\left(\frac{p(\omega)}{p(0)}\right) = 20\log\left(\frac{I(\omega)}{I(0)}\right)$$

$\therefore$ optical loss = $\frac{1}{2}$ electrical loss

Laser Diodes

If population of excited e^- exceeds population of non-excited e^-, stimulated emission will exceed absorption.

Fabry-Perot Laser Cavity

$FSR = \frac{c}{2nL}$ for f

$FSR = \frac{\lambda^2}{2nL}$ for λ

; L is cavity length, n is refractive index

Longitudinal Optical Field Intensity:

$$I(z,t) = e^{(\Gamma g(h\nu) - \bar{\alpha}(h\nu))z}$$

; I = intensity, ω = radian f, β = propagation constant, Γ = confinement factor, g = gain coefficient, $\bar{\alpha}$ = material absorption

Amplitude during 1 round trip:

$$I(2L) = I(0) R_a R_b e^{2(\Gamma g - \bar{\alpha})h\nu}$$

; $R_a R_b$ = mirror Fresnel reflection

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

Phase Condition:

$$e^{-j2\beta L} = 1$$

Amplitude Condition:

$$I(2L) = I(0)$$

$$\therefore \Gamma g_{th} = \bar{\alpha} + \frac{1}{2L}\ln\left(\frac{1}{R_a R_b}\right) = \bar{\alpha} + \alpha_{end}$$

; α_{end} = cavity mirror loss.

g_{th} = cavity gain threshold

Laser Condition:
gain exceeds optical loss during each round trip

Relationships between power and drive current:

$$\frac{d\Phi}{dt} = Cn\Phi + R_{sp} - \frac{\Phi}{\tau_{ph}}$$

; Φ = photon density, $Cn\Phi$ = stimulated emission, R_{sp} = spontaneous emission rate, τ_{ph} = photon lifetime

$n \geq n_{th} = \frac{1}{C\tau_{ph}}$ is required for photon density to increase.

$$\frac{dn}{dt} = \frac{J}{qd} - \frac{n}{\tau_{sp}} - Cn\Phi$$

; n = e^- density, τ_{sp} = spontaneous recombination lifetime

$$\frac{n_{th}}{\tau_{sp}} = \frac{J_{th}}{qd}$$ If $n = n_{th}$, electrons are absorbed without photon emission.

Steady State: $\Phi_s = \frac{\tau_{ph}}{qd}(J - J_{th}) + \tau_{ph}R_{sp}$

Photons per radiative electron-hole recombination above threshold:

$$\eta_{ext} = \eta_{int}\left(1 - \frac{\alpha}{g_{th}}\right)$$

; $\eta_{int} \approx 0.6-0.7$ at room temperature

$$\eta_{ext} = \frac{q}{E_g}\cdot\frac{dP}{dI} = 0.8065\lambda(\mu m)\frac{dP(mW)}{dI(mA)}$$

; $\lambda(\mu m)$ = emitted wavelength

Phase Condition: $v = m\left(\frac{c}{2nL}\right)$

Spectral Gain Profile: $g(\lambda) = g(0)e^{-\left(\frac{(\lambda-\lambda_0)^2}{2\sigma^2}\right)}$

; $g(0)$ = max gain, λ_0 = λ at spectral center, σ = spectral width of gain.

Single Mode Lasers: $\lambda = \lambda_B \pm \left(\frac{\lambda_B^2}{2n_eL_e}\right)(m+\frac{1}{2})$

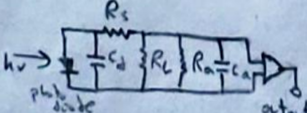
$\lambda_B = \frac{2n_e\Lambda}{k}$, k = order of grating, n_e = mode effective refractive index, Λ = period of corrugation.

Direct modulation capacity, upper limit:

$$\tau_{ph}^{-1} = \frac{c}{n}\left(\alpha + \left(\frac{1}{2L}\right)\ln\left(\frac{1}{R_1R_2}\right)\right)$$

Modulation frequency limit:

$$f = \left(\frac{1}{2\pi}\right)\left(\frac{1}{\sqrt{\tau_{sp}\tau_{ph}}}\right)\cdot\left(\sqrt{\frac{I}{I_{th}}-1}\right)$$



Photodetectors

PIN photodetectors:

upper cutoff $\lambda_c(\mu m) = \frac{hc}{E_g} = \frac{1.24}{E_g(eV)}$

quantum efficiency $\eta = \frac{\text{generated pairs}}{\text{incident photons}} = \frac{I_p/q}{P_0/h\nu}$

responsivity $\mathcal{R} = \frac{I_p}{P_0} = \frac{\eta q}{h\nu}$

Avalanche Photodiodes:

magnification $M = \frac{\text{multiplied photocurrent}}{\text{primary unmultiplied photocurrent}} = \frac{I_M}{I_p}$

responsivity $\mathcal{R}_{APD} = \left(\frac{\eta q}{h\nu}\right)M$

Noise Sources:

Signal Power: $I_p = \mathcal{R}P_0 = \left(\frac{\eta q}{h\nu}\right)P_0$

Noise Power:
- Quantum noise (shot noise)
- Dark current (bulk, surface)
- Thermal noise (due to load resistor)

Quantum Noise:

PIN: $\langle i_Q^2\rangle = \sigma_Q^2 = 2qI_pB$

APD: $\langle i_Q^2\rangle = \sigma_Q^2 = 2qI_pBM^2F(M)$

; I_p = avg photocurrent, M = avalanche magnification, $F(M)$ = noise figure, B = receiver bandwidth

Dark Current:

Bulk:

PIN: $\langle i_{DB}^2\rangle = \sigma_{DB}^2 = 2qI_DB$

APD: $\langle i_{DB}^2\rangle = \sigma_{DB}^2 = 2qI_DBM^2F(M)$

Surface: $\langle i_{DS}^2\rangle = \sigma_{DS}^2 = 2qI_LB$

Thermal Noise: (relevant in PIN only)

$$\langle i_T^2\rangle = \sigma_T^2 = \left(\frac{4k_BT}{R_L}\right)B$$

; k_B = Boltzmann's constant, T = temperature, R_L = load resistance

$k_B = 1.38\times10^{-23}\ m^2kgs^{-2}K^{-1}$

Signal/Noise Ratio:

APD: $$\frac{I_p^2M^2}{2q(I_p+I_D)BM^2F(M)+2qI_LB+4k_BTB/R_L}$$

PIN: $$\frac{I_p^2}{2q(I_p+I_D)B+2qI_LB+4k_BTB/R}$$

Response time depends on:

Transit time $t_d = \frac{W}{v_d}$

; W = depletion layer width, v_d = carrier drift velocity

Diffusion time $t_{diff} = \left(\frac{l^2}{2D_e}\right)$

; l = length of p^+ side, D_e = diffusion coefficient

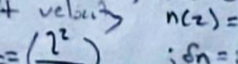
RC time constant $\tau = R_TC_T$

$C_T = C_a // C_d$, $R_T = R_L // R_a$

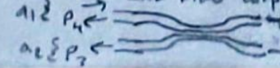
Wavelength Division Multiplexing

C band: 1524nm - 1560nm
L band: 1570nm - 1610nm
f ref: 193.1 THz (1552.524nm)
Spacing: 200GHz, 100GHz, 50GHz, 25GHz

$|\Delta v| = \left(\frac{c}{\lambda^2}\right)|\Delta\lambda|$



2x2 Fiber Coupler:



P_0 & P_3 = IN, P_1 = throughput, P_2 = coupled
P_2, P_3, P_4 = crosstalk

Ideal coupler: $P_2 = P_0\sin^2(Kz)$
$P_1 = P_0\cos^2(Kz)$; K = coupling coefficient, z = coupler draw length

coupling ratio = $\left(\frac{P_2}{P_1+P_2}\right)100\%$

excess loss = $10\log\left(\frac{P_0}{P_1+P_2}\right)$

Insertion loss = $10\log\left(\frac{P_i}{P_j}\right)$

crosstalk loss = $10\log(P_3/P_0)$

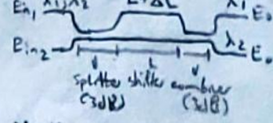
$\begin{bmatrix}b_1\\b_2\end{bmatrix} = S\cdot\begin{bmatrix}a_1\\a_2\end{bmatrix}$; (b_1,b_2) = output field, (a_1,a_2) = input field, S = complex scattering matrix.

$S = \begin{bmatrix}\sqrt{1-\epsilon} & j\sqrt{\epsilon}\\ j\sqrt{\epsilon} & \sqrt{1-\epsilon}\end{bmatrix}$; ϵ = coupling ratio from a_1 to b_2

complete power transfer conditions:

$L = \frac{\pi}{2K}(2m+1)$; L = tapered region length, $m\in\mathbb{Z}$.

Mach-Zehnder Interferometer Multiplexers:



$M = M_{splitter}\cdot M_{shifter}\cdot M_{combiner}$

$M_{shifter} = M_{\Delta\phi} = \begin{bmatrix} e^{jk\Delta L/2} & 0\\ 0 & e^{-jk\Delta L/2}\end{bmatrix}$

$\begin{bmatrix}E_{out_1}\\E_{out_2}\end{bmatrix} = M\begin{bmatrix}E_{in_1}\\E_{in_2}\end{bmatrix} = \begin{bmatrix}M_{11}&M_{12}\\M_{21}&M_{22}\end{bmatrix}\cdot\begin{bmatrix}E_{in_1}\\E_{in_2}\end{bmatrix}$

$P_{out_1} = |E_{out_1}|^2$, $P_{out_2} = |E_{out_2}|^2$

$\Delta\phi = \left(\frac{2\pi n_1}{\lambda}\right)L - \left(\frac{2\pi n_2}{\lambda}\right)(L+\Delta L)$

If same source: $\Delta\phi = K\Delta L$

$M_{coupler} = \begin{bmatrix}S_{11}&S_{12}\\S_{21}&S_{22}\end{bmatrix} = \begin{bmatrix}\cos Kd & j\sin Kd\\ j\sin Kd & \cos Kd\end{bmatrix}$

If 3dB: $M_{coupler} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&j\\j&1\end{bmatrix}$

$M = j\begin{bmatrix}\sin(k\Delta L/2) & \cos(k\Delta L/2)\\ \cos(k\Delta L/2) & -\sin(k\Delta L/2)\end{bmatrix}$

$\Delta f = \left(\frac{c}{2n_{eff}\Delta L}\right)$

4x4 MZIs:

$\Delta L_1 = \Delta L_2 = \frac{c}{2n_{eff}(2\Delta v)}$

$\Delta L_3 = \frac{c}{2n_{eff}(\Delta v)} = 2\Delta L_1$

Grating

$\Lambda(\sin\theta_i - \sin\theta_d) = m\lambda$

Fiber Grating Filters:

$n(z) = n_{core} + \delta n\left[1+\cos\left(\frac{2\pi z}{\Lambda}\right)\right]$

; δn = index change

$$R = \frac{(\kappa L)^2\sinh^2(SL)}{(\delta\beta L)^2\sinh^2(SL) + (SL)^2\cosh^2(SL)}$$

If $(\kappa L)^2 < (\delta\beta L)^2$:

$$R = \frac{(\kappa L)^2\sin^2(QL)}{(\delta\beta L)^2 - (\kappa L)^2\cos^2(QL)}$$

; R = reflectivity, L = grating length, $\delta\beta$ = detuning from λ_{Bragg}

$SL = \sqrt{(\kappa L)^2 - (\delta\beta L)^2}$, $QL = \sqrt{(\delta\beta L)^2 - (\kappa L)^2}$

$\lambda_{Bragg} = 2\Lambda n_{eff}$

$R_{max} = \tanh^2(\kappa L)$

$BW = \left(\frac{\lambda^2_{Bragg}}{\pi n_{eff}L}\right)\left[(\kappa L)^2 + \pi^2\right]^{1/2}$

$\kappa = \frac{\pi\eta\delta n}{\lambda_{Bragg}}$; $\eta = 1 - V^{-2}$

Basic Laws

Snell's reflection: $\theta_1 = \theta_2$

Snell's refraction: $n_1 \sin\phi_1 = n_2 \sin\phi_2$

critical angle: $\sin\phi_c = \frac{n_2}{n_1}$

Fresnel Eq'ns

$$r_p = \frac{-n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}{n\cos\theta + \sqrt{n^2-\sin^2\theta}}$$

$$r_s = \frac{\cos\theta - \sqrt{n^2-\sin\theta}}{\cos\theta + \sqrt{n^2-\sin\theta}}$$

; the fraction of incident power reflected, $n = \frac{n_2}{n_1}$

if $\phi_i > \phi_c$:

$$r_p = \frac{-n_2^2\sin\theta + jn_1\sqrt{n_1^2\cos^2\theta - n_2^2}}{n_2^2\sin\theta + jn_1\sqrt{n_1^2\cos^2\theta - n_2^2}}$$

$$r_s = \frac{n_1\sin\theta - j\sqrt{n_1^2\cos^2\theta - n_2^2}}{n_1\sin\theta + j\sqrt{n_1^2\cos^2\theta - n_2^2}}$$

; $n_1 > n_2$, $r = |r| e^{i\delta_{s,p}}$,

$\theta = \frac{\pi}{2} - \phi_i$

$\delta_{s,p} \rightarrow$ phase shift between E_i, E_r

recall: $e^{i\theta} = \cos\theta + i\sin\theta$

Phase Change for Reflection Coefficients

$$\tan\frac{\delta_p}{2} = -\frac{n\sqrt{n^2\cos^2\theta - 1}}{\sin\theta}$$

$$\tan\frac{\delta_s}{2} = -\frac{\sqrt{n^2\cos^2\theta - 1}}{n\sin\theta} \quad ; n = \frac{n_1}{n_2}$$

Fibers

$\Delta = \frac{n_1 - n_2}{n_1}$

a = radius of core

$NA = n\sin\theta_{o,max} = \sqrt{n_1^2 - n_2^2} \approx n_1\sqrt{2\Delta}$

; acceptance angle

Rays and Modes

LP_{01} HE_{11}

$LP_{11} \begin{cases} TE_{01} \\ TM_{01} \\ HE_{21} \end{cases}$

$LP_{21} \begin{cases} EH_{11} \\ HE_{31} \end{cases}$

Mat Dispersion

$\beta = \frac{2\pi n(\lambda)}{\lambda}$ $\quad v_g = \frac{d\omega}{d\beta} = \frac{c}{n - \lambda\frac{dn}{d\lambda}}$

$n_g = n - \lambda\frac{dn}{d\lambda}$ $\quad \tau_{mat} = \frac{L}{v_g}$

$D_{mat}(\lambda) = \frac{\lambda}{c}\left|\frac{d^2n}{d\lambda^2}\right|$

$\sigma_{mat} = \sigma_\lambda L |D_{mat}(\lambda)|$

$D_{PMD}\sqrt{L} = 2|\beta_{1y} - \beta_{1x}|$

$= \sigma\tau_{PMD} = \left|\frac{2}{v_{gx}} - \frac{2}{v_{gy}}\right|$

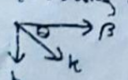
$TE \rightarrow$ s polarization, $E_z = 0$

$TM \rightarrow$ p polarization, $H_z = 0$

Slab Waveguides

n_1 = core index

n_2 = cladding index



$n_{eff} = n_1\cos\theta$

propagation constant $\beta = k\cos\theta = \frac{2\pi n_1}{\lambda}\cos\theta$

Constants

$K_B = 1.38\times10^{-23}$

$q = 1.6\times10^{-19}$

$\varepsilon = 8.85\times10^{-12}$

$c = 2.99\times10^{8}$

$\mu = 1.26\times10^{-6}$

$h = 6.62\times10^{-34}$

dB $\cdot$ const $= 4.343$

Transverse Resonance:

$$\frac{4\pi n_1 d\sin\theta}{\lambda} + 2\delta = 2\pi n$$

physical phase $\quad$ reflection shift

$\Delta = k\sin\theta\, 2d$

; $k = \frac{2\pi n}{\lambda}$

Mode Condition Eq'ns:

$$TM: \tan\left[\frac{\pi n_1 d\sin\theta}{\lambda} - \frac{\pi m}{2}\right] = \frac{n\sqrt{n^2\cos^2\theta - 1}}{\sin\theta}$$

$$TE: \tan\left[\frac{\pi n_1 d\sin\theta}{\lambda} - \frac{\pi m}{2}\right] = \frac{\sqrt{n^2\cos^2\theta - 1}}{n\sin\theta}$$

In normalized form:

$$TM: V\sqrt{1-b} = m\pi + 2\tan^{-1}\left[\frac{n_1^2}{n_2^2}\sqrt{\frac{b}{1-b}}\right]$$

$$TE: V\sqrt{1-b} = m\pi + 2\tan^{-1}\left[\sqrt{\frac{b}{1-b}}\right]$$

; $V = kd\sqrt{n_1^2 - n_2^2}$,

normalized guide index $b = \frac{n_{eff}^2 - n_2^2}{n_1^2 - n_2^2}$

Cutoff condition:

$b = 0$

$V = m\pi$

$\lambda_c = \frac{2d}{m}\sqrt{n_1^2 - n_2^2}$; d = core size

Asymmetrical Waveguides

n_c = cover index

n_f = core index

n_s = substrate index

Transverse Resonance:

$$\frac{4\pi n_f d\sin\theta}{\lambda} + \delta_c + \delta_s = 2\pi m$$

$$;\tan\left[\frac{\delta^{TE}_{s,c}}{2}\right] = -\frac{\sqrt{n_f^2\cos^2\theta - n_{s,c}^2}}{n_f\sin\theta} = -\frac{q_{s,c}}{h}$$

$\beta = \frac{2\pi n_{eff}}{\lambda} = k_0 n_{eff}$

$n_{eff} = n_f\cos\theta$

; $h^2 = k_0^2 n_f^2 - \beta^2$

$q_c^2 = \beta^2 - k_0^2 n_c^2$

$q_s^2 = \beta^2 - k_0^2 n_s^2$

OR $2hd - 2\tan^{-1}(q_c/h) - 2\tan^{-1}(q_s/h) = 2\pi m$

Mode Condition Eq'n:

$$\tan(hd) = \frac{h(q_c + q_s)}{h^2 - q_c q_s}$$

Cutoff Condition:

$$\lambda_c = \frac{2\pi d\sqrt{n_f^2 - n_s^2}}{\tan^{-1}\left[\sqrt{\frac{n_s^2 - n_c^2}{n_f^2 - n_s^2}} + m\pi\right]}$$

; $n_f > n_s \geq n_c$

If Symmetric and $n_c = n_s$

$q_c = q_s = q$

Even mode: $\tan\left(\frac{hd}{2}\right) = \frac{q}{h}$

Odd mode: $\tan\left(\frac{hd}{2}\right) = -\frac{h}{q}$

Even $\quad$ Odd

This is based on the wave eq'ns derived from Maxwell's eq'ns:

$$\left.\begin{aligned} \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{H} &= \vec{J} + \frac{\partial\vec{D}}{\partial t} \\ \nabla\cdot\vec{D} &= \rho_v \\ \nabla\cdot\vec{B} &= 0 \end{aligned}\right\} \text{Maxwell's}$$

$\downarrow$

$$\left.\begin{aligned} \nabla^2\vec{E} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} &= 0 \\ \nabla^2\vec{H} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} &= 0 \end{aligned}\right\} \text{Wave}$$

Helmholtz's Eq'n:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} + k^2 E = 0$$

Circular Waveguides

modes: TE_{vm} TM_{vm} HE_{vm} EH_{vm}

v = cycles in ϕ direction

m = cycles in r direction

Meridional Rays:

$TE_{0m} \rightarrow E_z = 0$

$TM_{0m} \rightarrow H_z = 0$

Weakly-Guide Approximation:

$LP_{0m}: HE_{1m}$

$LP_{1m}: TE_{0m}, TM_{0m}, HE_{2m}$

$LP_{vm}(v \geq 2): HE_{v+1,m}, EH_{v-1,m}$

Fundamental Mode:

$LP_{01}: HE_{11}$

Cutoff Condition:

$\beta = k_1\cos\theta = \frac{2\pi n_1}{\lambda}\cos\theta$

$\beta = k_2 = n_2 k$

Single Mode Condition: $V < 2.405$

Pumping Schemes:
codirectional, counterdirectional, dual

980nm - 1480nm

Sat $\propto$ pump level

Gain $\propto$ length

Dielectric Thin-Film Filters:
Transmission of Ideal etalon:

$$T = \frac{1}{\left[1 + \left(\frac{4R}{(1-R)^2}\right)\sin^2\left(\frac{\phi}{2}\right)\right]}$$

$\phi = \left(\frac{2\pi}{\lambda}\right) 2nL\cos\theta$

$FSR = \frac{\lambda^2}{2nL}$

Finesse $F = \frac{\pi\sqrt{R}}{1-R}$

Tunable Light Sources

$$\frac{\Delta\lambda_{tune}}{\lambda} = \frac{\Delta n_{eff}}{n_{eff}}$$

Δn_{eff} α temperature, injection current
Avoiding crosstalk between channels:
$\Delta\lambda_{channel} \geq 10\Delta\lambda_{signal}$

$N \approx \frac{\Delta\lambda_{tune}}{\Delta\lambda_{channel}}$

; $\Delta\lambda_{signal}$ = spectral width
Multiple-Quantum-Well lasers: an array of tunable lasers, and a combiner.

$\lambda_{Bragg} = 2n_{eff}\Lambda$

Semiconductor Optical Amps
pumping rate:

$$\frac{\partial n(t)}{\partial t} = \frac{J(t)}{qd} - R_{st}(t) - \frac{n(t)}{\tau_r}$$

; $J(t)$ = injection current density, d = active layer thickness, τ_r = time for both spontaneous emission and recombination
Net stimulated emission rate:

$R_{st}(t) = \Gamma a v_g (n - n_{th}) N_{ph} = g v_g N_{ph}$
; v_g = group velocity, a = gain const, n_{th} = threshold carrier density, Γ = confinement factor, g = overall gain per unit length, N_{ph} = photon density

$N_{ph} = \frac{P_s}{v_g(h\nu)(wd)}$; w = active layer width,
P_s = optical power
Steady state:

$$\frac{\partial n(t)}{\partial t} = 0, \quad g = \frac{\frac{J}{qd} - \frac{n_{th}}{\tau_r}}{v_g N_{ph} + 1/(\Gamma a \tau_r)} = \frac{g_0}{1 + N_{ph}/N_{sat}}$$

Saturation photon density:

$$N_{sat} = \frac{1}{\Gamma a v_g \tau_r}$$

Small signal gain per unit length:

$$g_0 = \Gamma a \tau_r \left(\frac{J}{qd} - \frac{n_{th}}{\tau_r}\right)$$

Amplifier gain:

$$G = \frac{P_{s,out}}{P_{s,in}} = e^{g(z)L} = e^{\Gamma(g_m - \alpha)L}$$

$$= 1 + \left(\frac{P_{out}}{P_{s,in}}\right)\ln\left(\frac{G_0}{G}\right)$$

Erbium Doped Fiber Amps
Amplified spontaneous emission:
ASE Power Spectral Density:

$$S_{ASE}(f) = h\nu n_{sp}[G(f) - 1] = \frac{P_{ASE}}{\Delta\nu_{opt}}$$

; P_{ASE} = power in optical bandwidth $\Delta\nu_{opt}$

$$n_{sp} = \frac{n_2}{n_2 - n_1}$$

ASE Noise

$$\langle i^2_{total}\rangle = \sigma^2_{total} = \sigma^2_T + \sigma^2_{shot-s} + \sigma^2_{shot-ASE} + \sigma^2_{s-ASE} + \sigma^2_{ASE-ASE}$$

Thermal noise:

$$\sigma_T^2 = \left(\frac{4k_BT}{R}\right)B$$

Signal shot noise:

$$\sigma^2_{shot-s} = 2qRGP_{s,in}B$$

Signal ASE beat noise:

$$\sigma^2_{s-ASE} = 4(RGP_{s,in})(RS_{ASE}B)$$

ASE ASE beat noise:

$$\sigma^2_{ASE-ASE} = R^2S^2_{ASE}(2\Delta\nu_{opt} - B)B$$

$P_0 = GP_{s,in} + S_{ASE}\Delta\nu_{opt}$

Mean-square receiver photocurrent:

$$\langle i^2_{ph}\rangle = \sigma^2_{ph} = R^2G^2P^2_{s,in}$$

Noise Figure:

$$F = \frac{(S/N)_{in}}{(S/N)_{out}} = \frac{1 + 2\eta n_{sp}(G-1)}{\eta G}$$

$F = 2\eta n_{sp}$ if $G >> 1$.
For perfect Amps: $n_{sp} = 1$,

$\eta = R\frac{h\nu}{q} = 1$

noise figure = 3dB = 2.

Circular Waveguides
Modes: $TE_{\nu m}$, $TM_{\nu m}$, $HE_{\nu m}$, $EH_{\nu m}$
ν = cycles in ϕ, m = cycles in r
Meridional rays:

$TE_{0m} \rightarrow E_z = 0$

$TM_{0m} \rightarrow H_z = 0$

Weakly guided approx:

LP_{0m} : HE_{1m}

LP_{1m} : TE_{0m}, TM_{0m}, HE_{2m}

$LP_{\nu m}$: $HE_{\nu+1\,m}$, $EH_{\nu-1\,m}$

Fundamental Mode:

$LP_{01} = HE_{11}$

Cutoff condition:

$\beta = k_1\cos\theta = \left(\frac{2\pi n_1}{\lambda}\right)\cos\theta$

$\beta = k_2 = n_2 k$

single mode condition:

$V < 2.405$

$$V = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2} = \frac{2\pi a}{\lambda}NA$$

$M \approx \frac{V^2}{2}$; if $V > 10$

Mode Theory

$$\left.\begin{aligned}\nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}\\ \nabla\times\vec{H} &= \frac{\partial\vec{D}}{\partial t}\\ \nabla\cdot\vec{D} &= 0\\ \nabla\cdot\vec{B} &= 0\end{aligned}\right\}\text{Maxwell}$$

$$\left.\begin{aligned}\nabla^2\vec{E} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} &= 0\\ \nabla^2\vec{H} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} &= 0\end{aligned}\right\}\text{General Wave Eq'ns}$$

$$\left.\begin{aligned}\frac{\partial^2E_z}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial E_z}{\partial r} + \frac{1}{r}\cdot\frac{\partial^2E_z}{\partial\phi^2} + q^2E_z &= 0\\ \frac{\partial^2H_z}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial H_z}{\partial r} + \frac{1}{r}\cdot\frac{\partial^2H_z}{\partial\phi^2} + q^2H_z &= 0\end{aligned}\right\}\vec{E},\vec{H}\text{ Wave Eq'ns}$$

; $q^2 = \omega^2\varepsilon\mu - \beta^2 = k^2 - \beta^2$

$$\left.\begin{aligned}E_{z1}(r<a) &= AJ_\nu(ur)e^{j\nu\phi}e^{j(\omega t-\beta z)}\\ H_{z1}(r<a) &= BJ_\nu(ur)e^{j\nu\phi}e^{j(\omega t-\beta z)}\end{aligned}\right\}\text{core Bessel Eq'ns}$$

; $k^2 = k_1^2 - \beta^2$

$$\left.\begin{aligned}E_{z2}(r>a) &= CK_\nu(wr)e^{j\nu\phi}e^{j(\omega t-\beta z)}\\ H_{z2}(r>a) &= DK_\nu(wr)e^{j\nu\phi}e^{j(\omega t-\beta z)}\end{aligned}\right\}\text{cladding Bessel Eq'ns}$$

; $w^2 = \beta^2 - k_2^2$

$$(\mathcal{J}_\nu + \mathcal{K}_\nu)(\mathcal{K}_\nu k_2^2 + k_1^2\mathcal{J}_\nu) = \left(\frac{\beta\nu}{a}\right)^2\left(\frac{1}{u^2} + \frac{1}{w^2}\right)^2$$

; $\mathcal{J}_\nu = \frac{J'_\nu(ua)}{uJ_\nu(ua)}$, $\mathcal{K}_\nu = \frac{K'_\nu(wa)}{wK_\nu(wa)}$

Cutoff for TE_{0m}, TM_{0m}: $J_0(ua) = 0$, $ua = V$

Linearly Polarized Modes
$\Delta << 1$, $k_1 \approx k_2 \approx \beta$

$\mathcal{J}_\nu + \mathcal{K}_\nu = \pm\frac{\nu}{a}\left(\frac{1}{u^2} + \frac{1}{w^2}\right)$

$P_{core} = \frac{1}{2}\int_0^a\int_0^{2\pi}(E_xH_y^* - E_yH_x^*)\,r\,dr\,d\phi$

$P_{clad} = \frac{1}{2}\int_a^\infty\int_0^{2\pi}(E_xH_y^* - E_yH_x^*)\,r\,dr\,d\phi$

poynting vector: $S_z = \frac{1}{2}R_e(\vec{E}\times\vec{H}^*)\cdot\vec{e}_z$

Graded Index Fibers

$$n(r) = \begin{cases} n_1\sqrt{1 - 2(r/a)^\alpha\Delta} & ; r \leq a\\ n_1\sqrt{1-2\Delta} & ; r > a\end{cases}$$

$$NA = \begin{cases}\sqrt{n^2(r) - n_2^2} & ; r \leq a\\ 0 & ; r > a\end{cases}$$

; $\alpha \rightarrow \infty$

$$M = \left(\frac{\alpha}{\alpha+2}\right)a^2k^2n_1^2\Delta \approx \left(\frac{\alpha}{\alpha+2}\right)\left(\frac{V^2}{2}\right)$$

Single Mode Fibers
Mode field diameter: $MFD = 2w_0 = 2\left[\frac{2\int_0^\infty E^2(r)r^3dr}{\int_0^\infty E^2(r)\,r\,dr}\right]^{1/2}$

; $E(r) = E_0e^{-r^2/w_0^2}$. Modes either in y or x direction.

Propagation Modes
Beat length $L_p = \frac{2\pi}{\beta}$ Modal Birefringence $B_f = n_y - n_x$
Birefringence $\beta = k_0(n_y - n_x)$

Signal Degradation
Single Mode Fibers: $\sigma_{wg} = |D_{wg}(\lambda)|\sigma_\lambda L$

$\tau_{wg} = \frac{L}{c}\left[n_2 + n_2\Delta\left(\frac{d(Vb)}{dV}\right)\right]$ $D_{wg}(\lambda) = -\left(\frac{n_2\Delta}{c\lambda}\right)\left[V\frac{d^2(Vb)}{dV^2}\right]$

Waveguide Dispersion: pulse spread $\sigma \approx \left(\frac{\partial\tau}{\partial\lambda}\right)\sigma_\lambda$

normalized prop const: $b = 1 - \left(\frac{ua}{V}\right)^2 = \frac{\beta^2/k^2 - n_2^2}{n_1^2 - n_2^2}$

$\beta = n_2k(b\Delta + 1)$; if Δ is small

Group delay = τ_{wg}

Efficiency $= \frac{P_3}{P_1} = \frac{P_3}{P_2}\cdot\frac{P_2}{P_1}$

$\beta_m(\omega_0) = \frac{\partial^m\beta}{\partial\omega^m}$

Group delay:

$$\tau_g = \frac{L}{V_g} = \frac{L}{d\omega/d\beta}$$

$$= -L\frac{\lambda^2}{2\pi}\cdot\frac{\partial\beta}{\partial\lambda}$$

attenuation $\alpha_p = \frac{1}{z}\ln\left(\frac{P(0)}{P(z)}\right)$

; $P(z) = P(0)e^{-\alpha_pz}$

pulse spreading rms value:

$$\sigma_g = \left|\frac{\partial\tau_g}{\partial\lambda}\right|\cdot\sigma_\lambda$$

; σ_λ = spectral width

Phase Velocity: $V_p = \frac{\omega}{\beta}$

Group Velocity: $V_g = \frac{\partial\omega}{\partial\beta} = \frac{1}{\beta_1}$

Dispersion: $D = \frac{1}{L}\cdot\frac{\partial\tau_g}{\partial\lambda}$

modal delay: $\delta T_{mod} = \frac{Ln_1^2}{cn_2}\Delta$

step-index delay rms: $\sigma_s \approx \frac{L(NA)^2}{4\sqrt{3}n_1c}$

graded-index delay rms: $\sigma_s \approx$ [illegible]

Waves

$f = \frac{v}{\lambda}$

$y(x,t) = A\cos(kx - \omega t)$

$k = \frac{2\pi}{\lambda}$ $\quad \omega = 2\pi f = \frac{2\pi}{T}$

$v = \sqrt{\frac{T}{\mu}} = \lambda f = \frac{\omega}{k}$

Crest Segments

$F = 2T\theta$ (up or down)

$m = R2\theta\mu$

$a = \frac{v^2}{R}$

Power

$P = F \cdot v = \frac{W}{t}$

$P = \mu v \omega^2 A^2 \sin^2(kx - \omega t)$

$P_{ave} = \frac{1}{2}\mu v \omega^2 A^2$

Energy per unit length $= \frac{dE}{dx} = \frac{1}{2}\mu\omega^2 A^2$

Interference

$A\cos(\omega_1 t) + A\cos(\omega_2 t) = 2A\cos(\omega_L t)\cos(\omega_H t)$

$\omega_L = \frac{1}{2}(\omega_1 - \omega_2)$ $\quad \omega_H = \frac{1}{2}(\omega_1 + \omega_2)$

$y(x,t) = \underbrace{2A\cos(\frac{\phi}{2})}_{\text{new } A}\sin(kx - \omega t + \frac{\phi}{2})$

Standing Waves

$y(x,t) = \underbrace{2y_m \sin(kx)}_{A}\cos(\omega t)$

Resonant Frequencies

$f = \frac{v}{\lambda} = n\left(\frac{v}{2L}\right)$ for $n \in \mathbb{Z}$.

or $f = \left(\frac{n}{2L}\right)\sqrt{\frac{T}{\mu}}$ for $n \in \mathbb{Z}$

Accelerating Air

$F = PA - (P + \Delta P)A = -\Delta PA$

$\Delta m = \rho A \Delta x = \rho A v \Delta t$

$a = \frac{\Delta v}{\Delta t}$

$\rho v^2 = -\frac{\Delta P}{\Delta V \div V} = $ Bulk Mod.

$v = \sqrt{\frac{B}{\rho}}$ } speed of sound in fluid w bulk mod. B

Travelling Sound Waves

$s(x,t) = s_m\cos(kx - \omega t)$ ← displacement of particles

$\Delta P(x,t) = \Delta P_m \sin(kx - \omega t)$ ← change in pressure

$\Delta P_m = v\rho\omega s_m$

Phase shift due to Interference

$\phi = \frac{\Delta L}{\lambda}2\pi$ $\quad \Delta L =$ difference in path length.

Intensity

$I = $ Power $\div$ Area $= \frac{Fv}{A}$

$I = \frac{\text{Power}}{4\pi r^2} = \frac{1}{2}\rho v\omega^2 s_m^2$

$I_0 = 10^{-12}$, $\beta = 10\,dB\log\left(\frac{I}{I_0}\right)$

Phasors

magnitude = amplitude

angle = phase shift.

Pipes

Both sides Open

$\lambda = \frac{2L}{n}$, $n \in \mathbb{N}$

One side closed

$\lambda = \frac{4L}{n}$, $n \in \mathbb{N}$

Doppler Effect

$f' = f\left(\frac{v \pm v_D}{v \pm v_s}\right)$

v = speed of sound = 343 m/s

Only works if $v_s < v$

Supersonic Speeds

$\frac{v_s}{v}$ = mach number

$\sin(\theta) = \frac{v}{v_s}$

Gasses and Heat

Thermal Expansion

$L_0 + \Delta L = L_0 + \alpha L_0 \Delta T$

$V_0 + \Delta V = V_0 + \beta V_0 \Delta T$

$\beta = 3\alpha$

$A_0 + \Delta A = A_0 + 2\alpha A_0 \Delta T$

Specific Heat

$Q = \Delta U = mc\Delta T$

cm = heat inertia

Latent Heat

$Q = mL$

$L = J/kg$ needed to change state.

Work by Gasses

$W = P\Delta V \because W = Fd\cos(0) = PAd = P\Delta V$

$W = \int_{V_i}^{V_f} P\,dV$ for varying P.

Thermodynamics Law 1

$\Delta U = Q - W$

$U = \frac{3}{2}PV$

$PV = nRT$ ← ideal gasses

Heat Transfer

Conduction

$P = \frac{\Delta Q}{\Delta t} = \frac{kA\Delta T}{d}$ } one layer

$R = \frac{d}{k} =$ resistance

$T_x = \frac{k_1 L_2 T_C + k_2 L_1 T_H}{k_1 L_2 + k_2 L_1}$ } two layers

$P = \frac{A(T_H - T_C)}{L_1/k_1 + L_2/k_2}$

$P = \frac{A(T_H - T_C)}{\Sigma(L_i/k_i)}$ } n layers

Molar Specific Heat

$Q = nC_V\Delta T$ } constant volume

$C_V = \left(\frac{f}{2}\right)R = 4.16f$

$\Delta E_{int} = nC_V\Delta T$

$\Delta E_{int} = Q - W$

$E_{int} = nC_V\Delta T = \left(\frac{f}{2}\right)nRT$

$Q = nC_p\Delta T$

$C_p = C_V + R$

$W = nR\Delta T = nRT\ln\left(\frac{V_f}{V_i}\right)$

Adiabatic Expansion

$P_i V_i^{\gamma} = P_f V_f^{\gamma}$

$T_i V_i^{\gamma - 1} = T_f V_f^{\gamma - 1}$

Mean Free Path

$\lambda = \frac{1}{\sqrt{2}\pi d^2 (N/V)}$

Ideal Gasses

$PV = nRT$

$W = nRT\ln\left(\frac{V_f}{V_i}\right) = P\Delta V$

Kinetic Theory

Fraction of molecules w $v_1 < v < v_2 = \int_{v_1}^{v_2} P(v)\,dv$

$P(v) = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2} v^2 e^{\frac{-Mv^2}{2RT}}$

T = temp, v = molecular speed, M = molar mass

$v_{rms} = \sqrt{\frac{3RT}{M}}$ $\quad v_p = \sqrt{\frac{2RT}{M}}$

$v_{avg} = \sqrt{\frac{8RT}{\pi M}}$

$v_{avg}^2 = \frac{3RT}{M}$

Energy of Monatomic Gas

$E_{int} = \frac{3}{2}nRT$

trans. $E_{K_{ave}} = \frac{3}{2}kT$

Radiation

$P_{rad} = \sigma \epsilon A T_{env}^4$

$\sigma = 5.6704 \times 10^{-8}$

Absorption

$P_{abs} = \sigma \epsilon A T_{env}^4$

Entropy

$\Delta S = \int_{Q_i}^{Q_f} \frac{1}{T}\,dQ$

Isothermal/Freely Expand

$\Delta S = \frac{Q}{T}$

Engines, Fridges

$\Delta U_{system} = 0$

Heat Engine

$|Q_H| - |Q_C| = |W|$ $\quad T_H \xrightarrow{Q_H} (W) \xrightarrow{Q_C} T_C$

efficiency $= \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H}$

Refrigerator

$|Q_C| - |Q_H| = |W|$ $\quad T_C \xrightarrow{Q_C} (W) \xrightarrow{Q_H} T_H$

$K_r = \frac{Q_C}{W} = \frac{Q_C}{Q_H - Q_C}$

Carnot Cycle

No friction

$\Delta S = 0$

Reversible process

Sound in Gas (Air)

$v_{sound} = \sqrt{\frac{B}{\rho}}$

Degrees of Freedom

degrees = f

monatomic: $f = 3$

diatomic: $f = 5$

polyatomic: $f = 6$

each degree has an energy of $\frac{1}{2}kT$ per molecule associated with it.

Rotation

Torque, Work, Energy

$\tau = r \times F = rF\sin\theta$

$\tau_{Net} = I\alpha$

$W = \Delta E_k = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \int_{\theta_i}^{\theta_f} \tau \, d\theta$

$P = \frac{dw}{dt} = \tau\omega$

$E_k = \frac{1}{2}I\omega^2$

Angular Momentum

$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$

$\tau_{Net_{external}} = \frac{dl}{dt}$

Potential Energy

$\Delta U = W = \int F \, dr$

$U_g = \frac{-GMm}{R}$

For a grav. system, the potential energies between each pair of bodies is added up.

$F(r) = \frac{dU(r)}{dr}$

Fluids at Rest

$\rho = \frac{m}{V}$ $\quad P = \frac{F_\perp}{A}$

normal = 1 atmos = 1.01×10^5 Pa

$P = \rho g h + P_o$

$F = w = mg = \rho V g = \rho g h A$

Every 10.3m of water depth adds 1 atmos.

Every 760mm of mercury adds 1 atmos.

Pressure Measurement

Manometer

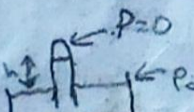
$P_1 = \rho g h + P_o$

$h = \left(\frac{P_1 - P_o}{\rho g}\right)$

Barometer

$P_o = \rho g h + 0$

$h = \left(\frac{P_o}{\rho g}\right)$



Trig Combinations

① $\cos(\omega t) + \cos(\omega t - \phi) = a\cos(\omega t - b)$

$a = 2\cos\left(\frac{\phi}{2}\right)$ $\quad b = \left(\frac{\phi}{2}\right)$

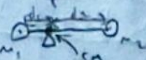
② $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

Escape Speed

$v_i = \sqrt{\frac{2GM}{R_i}}$

Statics

$I_p = I_{cm} + Md^2$



$\frac{d_1}{d_2} = \frac{M_2}{M_1}$

Static Equilibrium

$\vec{F}$ and $\vec{\tau}$ both balanced.

Systems

$\vec{v}_{cm} = \left(\frac{1}{M_T}\right) \cdot \sum_{i=1}^{N} m_i \vec{v}_i$

$\vec{R}_{cm} = \sum_{i=1}^{N} m_i \vec{r}_i \div \sum_{i=1}^{N} m_i$

$I = \sum_{i=1}^{N} m_i r_i^2$

Fluid Dynamics

Volume Flow Rate = Av

$A_1 v_1 = A_2 v_2$

Bernoulli's Equation

$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$

Pressure and Work

$W = (P_1 - P_2)V$

Buoyancy

Archimedes

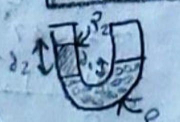
$F_B = (P_2 - P_1)A = W$

Sink or Float?

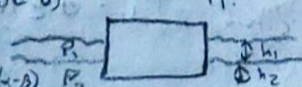
$\frac{V_{displaced\ liquid}}{V_{object}} = \frac{\rho_{object}}{\rho_{liquid}}$

If $\rho_{object} > \rho_{liquid}$, object just sinks.

Two Fluids



$\frac{P_2}{P_1} = \frac{d_1}{d_2}$



$F_B = (h_1 A)\rho_1 g + (h_2 A)\rho_2 g$

Kepler's Laws

① Law of Orbits: orbits are elliptical

② dA/dt = constant.

③ $T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$

T = period, a = semi-major axis

Basic Motion

$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t}$

$\Delta \vec{x} = \frac{\vec{v}_1 - \vec{v}_2}{2}\Delta t$

$\Delta \vec{x} = \vec{v}_1 \Delta t + \frac{\vec{a}\Delta t^2}{2}$

$\Delta \vec{x} = \vec{v}_2 \Delta t - \frac{\vec{a}\Delta t^2}{2}$

$v_2^2 = v_1^2 + 2\vec{a}\Delta\vec{x}$

$W_f = F_f \cdot d = -\mu_k m g d$

Gravitation

$F_{12} = F_{21} = G\left(\frac{m_1 m_2}{R^2}\right)$

$\vec{a}_m = GM \because \vec{F} = m\vec{a}$

Shell Theorems

① A sphere attracts a particle as if its mass were located at its center

② A sphere exerts no net grav. force on a particle located inside it.

Gravity near Earth's Surface

$|F_g| = mG\left(\frac{M_E}{R_E^2}\right)$

Normal Force, Earth's Rotation

$\vec{g} = \vec{a}_g - \omega^2 R$

Work by Grav Force

$W_g = GMm\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$

Is a conservative force.

$W_g = -mg\Delta y$, near surface.

Damped SHM

$x(t) = \underbrace{x_{max} e^{-bt/2m}}_{A} (\cos(\tilde{\omega} t + \phi))$

$\tilde{\omega} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

$E(t) \doteq \frac{1}{2}kx_{max}^2 e^{-bt/m}$

$F_{damping} = -bv$ (v = speed)

$F_d + F_s = ma_x$

Resonance

$\omega_d = \tilde{\omega}$ causes effect on A to be maximized.

Mech. Eng of Earth-Satellite System

$E = U + K = \frac{GMm}{2a}$

Elasticity

linear deformation

$\frac{F_{app}}{Area}$ = Stress $\quad \frac{\Delta L}{L_o}$ = strain

Stress = Y · strain

shear deformation

$\frac{F_{app}}{Area}$ = Stress $\quad \theta \approx \tan\theta = \frac{\Delta x}{h}$ = strain

Stress = S · strain

bulk deformation

$\frac{F_{app}}{Area}$ = Stress $\quad \frac{-\Delta V}{V_o}$ = strain

Stress ≐ B · strain

Simple Harmonic Motion

$U + K$ = constant

$\tau = I\alpha$ $\quad \omega = 2\pi f$

$-V_{max} = \omega A$

Pendulums

$\alpha = -\omega^2\theta$ $\quad \omega = \sqrt{\frac{g}{L}}$

$T = 2\pi\sqrt{\frac{L}{g}}$

Springs

$a = -\omega^2 x$ $\quad \omega = \sqrt{\frac{k}{m}}$

$F_s = -kx$ $\quad T = 2\pi\sqrt{\frac{m}{k}}$

$E_s = \frac{1}{2}kx^2$

Rod Pendulums

$\omega = \sqrt{\frac{3g}{2L}}$

Torsion Pendulums

like spring, but $\tau = -k\theta$ and $\omega = \sqrt{\frac{k}{I}}$.

General Pendulums

$\omega = \sqrt{\frac{Mgh}{I}}$

General SHM

$\theta(t) = \theta_o \cos(\omega t + \phi)$

Constants

$G = 6.67 \times 10^{-11}$

$I_o = 10^{-12}$

$C° = K° + 273$

$c_{water} = 4186$

$R = 8.31$

Avogadro's $= 6.02 \times 10^{23}$

$k = \frac{R}{Avogadro's}$

$c_{water} = 4200$ J/kg K

Pressure

$P = \frac{Force}{Area}$