Homework #5

You should try to solve these problems by yourself. I recommend that you start early and get help in office hours if needed. If you find it helpful to discuss problems with other students, go for it. **You do not need to turn in these problems. The goal is to be ready for the in class quiz that will cover the same or similar problems.**

Problem 1: Odd Cycle Detection

Given an undirected graph $G = (V, E)$, design an algorithm that decides whether or not G contains a cycle of odd length. Prove its correctness. Your algorithm should run in time $O(|V| + |E|)$.

Problem 2: BFS Shortest Path

Suppose that you want to get from vertex s to vertex t in an unweighted graph $G = (V, E)$, but you would like to stop by vertex u if it is possible to do so without increasing the length of your path by more than a factor of α.

Describe an efficient algorithm that would determine an optimal $s-t$ path given your preference for stopping at u along the way if doing so is not prohibitively costly. (It should either return the shortest path from s to t or the shortest path from s to t containing u, depending on the situation.) If it helps, imagine that there are burgers at u.

Problem 3: Directed Graph Cycle Detection

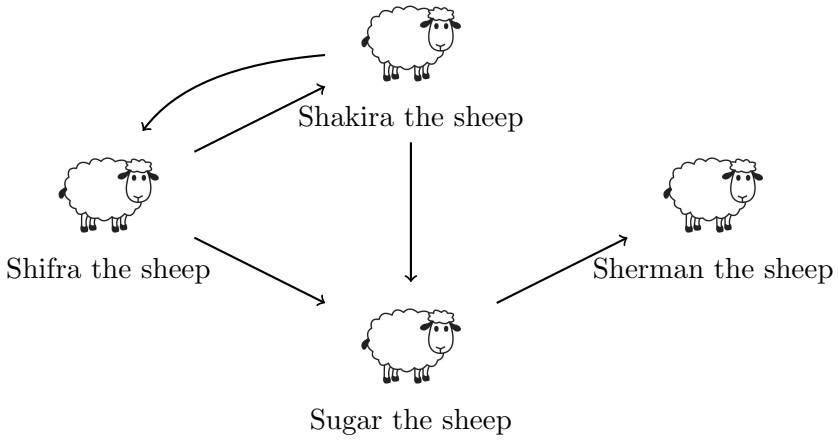
A cycle is a path of edges from a node to itself.

(a) You are given a directed graph $G = (V, E)$, and a special vertex v. Give an algorithm based on BFS that determines in $O(|V|+|E|)$ time whether v is part of a cycle. Justify the runtime of your algoirhtm.

Problem 4: Bah Bah Bah

You arrive on an island with n sheep. The sheep have developed a pretty sophisticated society, and have a social media platform called Baaaahtter (it's like Twitter but for sheep[1]). Some sheep follow other sheep on this platform. Being sheep, they believe and repeat anything that they hear. That is, they will re-post anything that any sheep they are following said. We can represent this by a graph, where $(a) \to (b)$ means that (b) will re-post anything that (a) posted. For example, if the social dynamics on the island were:

[1] Also my new start-up idea



the Sherman the Sheep follows Sugar the Sheep, and Sugar follows both Shakira and Shifra, and so on. This means that Sherman will re-post anything that Sugar posts, Sugar will re-post anything by Shifra and Shikira, and so on. (If there is a cycle then each sheep will only re-post a post once).

For the parts below, let G denote this graph on the n sheep. Let m denote the number of edges in G.

(a) Call a sheep a **source sheep** if anything that they post eventually gets re-posted by every other sheep on the island. In the example above, both Shifra and Shakira are source sheep.

Prove that all source sheep are in the same strongly connected component of G, and every sheep in that component is a source sheep.

(b) Suppose that there is at least one source sheep. Give an algorithm that runs in time $O(n+m)$ and finds a source sheep. You may use any algorithm we have seen in class as a subroutine.

(c) Suppose that you don't know whether or not there is a single source sheep. Give an algorithm that runs in time $O(n+m)$ and either returns a source sheep or returns `no source sheep.` You may use any algorithm we have seen from class as a subroutine.