NP Practice Problems

Problem 1: Collecting Baseball Cards

Consider the problem of trying to collect baseball cards. Baseball cards come in sealed packets; you don't know what you're getting until you open up the package. Consider packets $P_1, \ldots P_m$, each of which constains some subset of this year's available baseball cards. This problem considers whether its possible to collect all of this year's available cards by buying fewer than k packets of cards. Prove that this problem is NP-complete by reducing it to the vertex-cover problem.

The Vertex Cover Problem. *Given a graph G and a number k, does G contain a vertex cover of size at most k? (Recall that a vertex cover $V' \subseteq V$ is a set of vertices such that every edge $e \in E$ has at least one of its endpoints in V'.)*

Problem 2: 2019-colorability

We can extend the definition of 3-**colorability** to the following:

k-**colorability.** *Given a graph G, can the set of vertices be partitioned into k sets such that no two vertices within the same set have an edge between them?*

Show that 2019-**colorability** is NP-complete. *Hint: It is probably easier to think about proving the more general claim: for all $k \geq 3$, k-colorability is NP-complete.*

Problem 3: A Potentially Intractable Problem

A group of hardworking UT students is nearing the end of a particularly arduous semester. They would like to celebrate by renting a boat on Lake Travis for the day. However, they want to make sure everyone enjoys the day, so they don't want to bring along pairs of people who will bicker. As is expected in any group, there are pairs of group members that simply don't get along. Because we think the biggest possible group will be the most fun, our goal is to identify the largest group of students that does not contain any of these pairwise discords. For shorthand, we'll call this problem BoatingParty.

(a) The problem stated above is an optimization problem. State the decision form of the problem.

(b) Prove BoatingParty $\in NP$.

(c) Prove BoatingParty is an NP complete problem by reducing from Clique. As a reminder:

> A **clique** in an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices, each pair of which is connected by an edge in E. The **size** of a clique is the number of vertices it includes.

(d) Truthfully, it's been a really long semester. It turns out that the number of the pairwise feuds is quite large. We assume that every member is involved in *at least* $N - 50$ (no, I didn't tell you what N is, but you can assume that it's greater than 50) feuds. Is this problem an NP complete one? Prove or disprove.