The Two-Squirrel Problem and Its Relatives

Sergey Bereg[1], Manuel Lafond[2], and Binhai Zhu[3]

[1] Department of Computer Science, University of Texas at Dallas, Richardson, TX 75080, USA. besp@utdallas.edu

[2] Department of Computer Science, Université de Sherbrooke, Sherbrooke, Quebec J1K 2R1, Canada. manuel.lafond@usherbrooke.ca

[3] Gianforte School of Computing, Montana State University, Bozeman, MT 59717, USA. bhz@montana.edu

Abstract. In this paper, we start with a variation of the star cover problem called the Two-Squirrel problem. Given a set P of $2n$ points in the plane, and two sites c_1 and c_2, compute two n-stars S_1 and S_2 centered at c_1 and c_2 respectively such that the maximum weight of S_1 and S_2 is minimized. This problem is strongly NP-hard by a reduction from Equal-size Set-Partition with Rational Numbers. Then we consider two variations of the Two-Squirrel problem, namely the Dichotomy Two-Squirrel problem and the Two-MST problem, which are both strongly NP-hard. In terms of approximation algorithms, in fact Two-Squirrel and Dichotomy Two-Squirrel both admit a full PTAS (FPTAS) using the traditional methods. For Two-MST, the scenario is quite different and we are only able to obtain a factor-4.8536 approximation.

Keywords: Minimum star/tree cover · NP-hardness · Set-Partition · Approximation algorithms · Minimum spanning tree (MST)

1 Introduction

Imagine that two squirrels try to fetch and divide $2n$ nuts to their nests. Since each time a squirrel can only carry a nut back, this naturally gives the following problem: they should travel along the edges of an n-star, centered at the corresponding nest, such that each leaf (e.g., nut) is visited exactly once (in and out) and the maximum distance they visit should be minimized (assuming that they travel at the same speed, there is no better way to enforce the fair division under such a circumstance). See Figure 1 for an illustration.

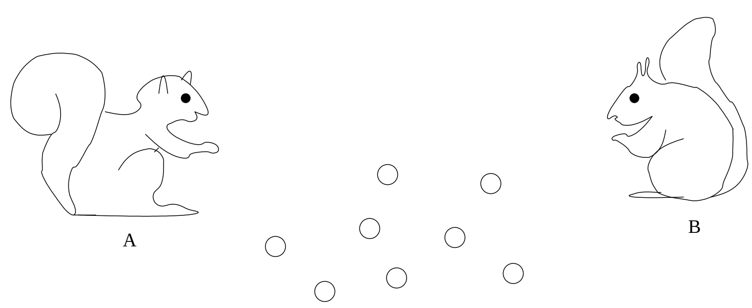

Fig. 1: Two squirrels A and B try to fetch and divide $2n$ nuts.

A star S is a tree where all vertices are leaves except one (which is called the *center* of the star). An n-star is a star with n leaf nodes. When the edges in S carry weights, the weight of S is the sum of weights of all the edges in S. Given two points p, q in the plane, with $p = (x_p, y_p)$ and $q = (x_q, y_q)$, we define the Euclidean distance between p, q as $d(p, q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$.

Formally, the *Two-Squirrel* problem can be defined as: Given a set P of $2n$ points in the plane and two extra point sites c_1 and c_2, compute two n-stars S_1 and S_2 centered at c_1 and c_2 respectively such that each point $p_j \in P$ is a leaf in exactly one of S_1 and S_2; moreover, the maximum weight of S_1 and S_2 is minimized. Here the weight of an edge (c_i, p_j) in S_i is $w(c_i, p_j) = d(c_i, p_j)$ for $i = 1, 2$.

One can certainly consider a variation of the two-squirrel problem where the points are given as pairs (p_{2i-1}, p_{2i}) for $i = 1, ..., n$, and the problem is to split all the pairs (i.e., one to c_1 and the other to c_2) such that maximum weight of the two resulting stars is minimized. We call this version *Dichotomy Two-Squirrel*. A more general version of the problem is when the two squirrels only need to split the $2n$ nuts and each could travel along a Minimum Spanning Tree (MST) of the n points representing the locations of the corresponding nuts, which we call the *Two-MST* problem: Compute a partition of P into n points each, P_1 and P_2, such that the maximum weight of the MST of $P_1 \cup \{c_1\}$ and $P_2 \cup \{c_2\}$, i.e., $\max\{w(P_1 \cup \{c_1\}), w(P_2 \cup \{c_2\})\}$, is minimized.

Covering a (weighted) graph with stars or trees (to minimize the maximum weight of them) is a well-known NP-hard problem in combinatorial optimization [2], for which constant factor approximation is known. Recently, bi-criteria approximations are also reported [3]. In the past, a more restricted version was also investigated on graphs [7]. Our Two-Squirrel problem can be considered a special geometric star cover problem where the two stars are disjoint though are of the same cardinality, and the objective function is also to minimize the maximum weight of them.

It turns out that both Two-Squirrel and Dichotomy Two-Squirrel are strongly NP-hard (under both the Euclidean and L_1 metric, though we focus only on the Euclidean case in this paper). The proofs can be directly from two variations of the famous Set-Partition problem [4, 5], namely, Equal-Size Set-Partition for Rationals and Dichotomy Set-Partition for Rationals, which are both strongly NP-hard with the recent result by Wojtczak [6]. We then show that Dichotomy Set-Partition for Rationals can be reduced to Two-MST in polynomial time, which indicates that Two-MST is also strongly NP-hard.

For the approximation algorithms, both Two-Squirrel and Dichotomy Two-Squirrel admit a FPTAS (note that this does not contradict the known result that a strongly NP-hard problem with an integral objective function cannot be approximated with a FPTAS unless P=NP, simply because our objective functions are not integral). This can be done by first designing a polynomial-time dynamic programming algorithm through scaling and rounding the distances to integers, obtaining the corresponding optimal solutions, and then tracing back to obtain the approximate solutions. The approximation algorithm for Two-MST is more tricky; in fact, with a known lower bound by Chung and Graham related to the famous Steiner Ratio Conjecture [1], we show that a factor 4.8536 approximation can be obtained.

In the next section, we give details for our results for the Two-MST problem. In Section 3, we conclude the paper.

2 Results for the Two-MST Problem

2.1 Preliminaries

In this section, we first define Equal-size Set-Partition for Rationals and Dichotomy Set-Partition for Rationals which are generalizations of Set-Partition [4, 5].

In Dichotomy Set-Partition with Rationals, we are given a set E of $2n$ positive rationals numbers (rationals, for short) with $E = E'_1 \cup E'_2 \cup \cdots E'_n$ such that $E'_i = \{a_{i,1}, a_{i,2}\}$ is a

2-set (or, $E'_i = (a_{i,1}, a_{i,2})$, i.e., as a pair) and the problem is to decide whether E can be partitioned into E_1 and E_2 such that every two elements in E'_i is partitioned into E_1 and E_2 (i.e., one in E_1 and the other in E_2 — clearly $|E_1| = |E_2| = n$) and $\sum_{a\in E_1} a = \sum_{b\in E_2} b$. (Equal-size Set-Partition with Rationals is simply a special case of Dichotomy Set-Partition with Rationals where E is given as a set of $2n$ rationals, i.e., $E = \{a_1, a_2, \cdots, a_{2n}\}$ and E'_i's are not given.)

With integer inputs, both Dichotomy Set-Partition and Equal-size Set-Partition, like their predecessor Set-Partition, can be shown to be weakly NP-complete. Recently, Wojtczak proved that even with rational inputs, Set-Partition is strongly NP-complete [6]. In fact, the proof by Wojtczak implied that Dichotomy Set-Partition and Equal-size Set-Partition are both strongly NP-complete — because in this reduction from a special 3-SAT each pair x_i and $\bar{x}_i$ are associated with two unique rational numbers which must be split in two parts. So we re-state this theorem by Wojtczak.

Theorem 1. *Dichotomy Set-Partition with Rationals and Equal-size Set-Partition with Rationals are both strongly NP-complete.*

2.2 Strong NP-hardness for Two-MST

In this subsection, we prove that the Two-MST problem (2-MST for short), is also strongly NP-hard. Recall that in the 2-MST problem, one is given a set P of $2n$ points in the plane, together with two point sites c_1 and c_2, the objective is to compute two MST T_1 and T_2 each containing n points in P (and c_1 and c_2 respectively) such that the maximum weight of T_1 and T_2, $\max\{w(T_1), w(T_2)\}$, is minimized. (Here the weight of any edge (p_i, p_j) or (p_i, c_k) in $T_k, k = 1..2$, is the Euclidean distance between the two corresponding nodes.) We reduce Dichotomy Set-Partition for Rationals to 2-MST in the following.

Given $E = \cup_{i=1..n} E'_i$, with E'_i containing two rationals $a_{i,1}$ and $a_{i,2}$, i.e., $E'_i = \{a_{i,1}, a_{i,2}\}$, for Dichotomy Set-Partition with Rationals we need to partition each of E'_i into two sets E_1 and E_2 such that the rationals in E_1 and E_2 sum the same, i.e., $t = \sum_{a\in E_1} a = \sum_{b\in E_2} b$. We construct $4n$ points in P as well as 2 points c_1 and c_2 as follows.

First set $c_1 = c_2 = (0, 0)$. Then for $i = 1$ to n, construct 4 points corresponding to $E'_i = \{a_{i,1}, a_{i,2}\}$: $p_{i,1} = (i \cdot t, a_{i,1})$, $p_{i,2} = (i \cdot t, -a_{i,2})$, $q_{i,1} = (i \cdot t, 0)$ and $q_{i,2} = (i \cdot t, 0)$. We loosely call these 4 points forming the i-th cusp C_i. (The sketch of an example is shown in Fig. 2.) Since $t >> a_{i,1}$ or $a_{i,2}$, the optimal MST's must first split the points on the x-axis, i.e., $\{c_1, c_2\} \cup (\cup_{i=1..n}\{q_{i,1}, q_{i,2}\})$, evenly. Secondly, between the neighboring cusps C_i and C_{i+1}, the closest distance is t; therefore, $p_{i,1}$ and $p_{i,2}$ must be connected to exactly one of $q_{i,1}$ and $q_{i,2}$. To make the maximum weight of the resulting MST T_1 and T_2 minimum, it comes to how to connect $p_{i,1}$ and $p_{i,2}$ to $q_{i,1}$ and $q_{i,2}$ such that the weight of T_1 and T_2 is the same, i.e., with a value of $(n+1)t$. It is clear that in this case $|P| = 4n$ and $|T_1| = |T_2| = 2n+1$, due to the addition of c_1 and c_2. Hence, we summarize: Dichotomy Set-Partition with Rationals has a solution iff the 2-MST instance $P \cup \{c_1, c_2\}$ admits a solution with optimal weight of $(n + 1)t$. We therefore have the following theorem.

Theorem 2. *Two-MST is strongly NP-hard.*

We comment that with this proof, a variation of 2-MST, e.g., even if c_1 and c_2 are not given in advance, remains strongly NP-hard.

2.3 A 4.8536-Approximation for Two-MST

First let P_1 be the subset of points closer to c_1, and P_2 the subset of points closer to c_2 (ties are broken arbitrarily). Let T be an MST of $P \cup \{c_1, c_2\}$. T_1 is obtained by removing

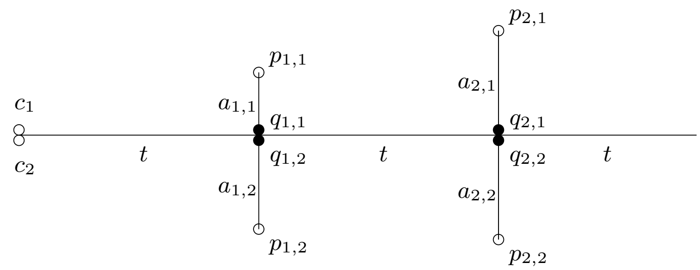

Fig. 2: Illustration for the reduction from Dichotomy Set-Partition with Rationals to 2-MST. Here only the construction for $E_1' = \{a_{1,1}, a_{1,2}\}$ and $E_2' = \{a_{2,1}, a_{2,2}\}$ are shown.

c_2 plus any n points of T. Viewing these removed points as Steiner points, by the bound of Chung and Graham [1], we have $w(T_1) \leq (1/0.82416874) \cdot w(T) \leq 1.2134 \cdot w(T)$. Then, obtain T_2 by taking the points not in T_1. We also have $w(T_2) \leq 1.2134 \cdot w(T)$. Return the best solution among (P_1, P_2) or (T_1, T_2). Note that in either case, the approximate solution APP satisfies $APP \leq 1.2134 \cdot w(T)$.

To obtain the final factor, let M_1 and M_2 be the two MST's of the optimal solution, and let OPT be the maximum weight of M_1 or M_2. By taking the union of M_1 and M_2, and adding an edge between c_1 and c_2, we obtain a spanning tree. Thus, $w(M_1) + w(M_2) + d(c_1, c_2) \geq w(T)$, since T is a minimum spanning tree of $P \cup \{c_1, c_2\}$.

Next we show that $OPT \geq d(c_1, c_2)/2$. If the optimal solution splits P into P_1 and P_2, we just return that. Now assume that the optimal solution does not do that. This means that M_1 has a point of P_2, or M_2 has a point of P_1. Let $p \in M_1 \cap P_2$, then the path from c_1 to p in M_1 shows that $OPT \geq d(c_1, c_2)/2$. The same inequality holds if $p \in M_2 \cap P_1$.

Thus we obtain

$$w(M_1) + w(M_2) + d(c_1, c_2) \leq OPT + OPT + 2 \cdot OPT = 4 \cdot OPT.$$

Combined with the above, this gives $APP \leq 1.2134 \cdot w(T) \leq 1.2134 \cdot (4 \cdot OPT) = 4.8536 \cdot OPT$.

Theorem 3. *Two-MST can be approximated with a factor-4.8536 approximation algorithm which runs in $O(n \log n)$ time.*

3 Concluding Remarks

The obvious question is whether we could improve the approximation factor for 2-MST. Even with the current algorithm, we believe that the actual factor should be around 3.

References

1. F.R.K. Chung and R.L. Graham. A new lower bound for Euclidean Steiner minimal trees. *Annals. NY Academy of Sciences*, 440(1):328-346, 1985.
2. G. Even, N. Garg, J. Koenemann, R. Ravi and A. Sinha. Covering graphs using trees and stars. *Proc. APPROX/RANDOM'03*, pp. 24-35, 2003.
3. B. Gamlath and V. Grinberg. Approximating star cover problems. *Proc. APPROX/RANDOM'20*, pp. 57:1-57:19, 2020.
4. M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*, W.H.Freeman, 1979.
5. R.M. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*, pp. 85-103, Springer US, 1972.
6. D. Wojtczak. On strong NP-completeness of rational numbers. *Proc. CSR'18*, pp. 308-320, 2018.
7. W. Zhao and P. Zhang. Approximation to the Minimum Rooted Star Cover Problem. *Proc. TAMC'07*, pp. 670-679, 2007.