Struct Multidisc Optim (2014) 49:351–366
DOI 10.1007/s00158-013-1002-x

RESEARCH PAPER

Structural optimization using graphic statics

Lauren L. Beghini · Juan Carrion · Alessandro Beghini · Arkadiusz Mazurek · William F. Baker

Received: 24 March 2013 / Revised: 13 September 2013 / Accepted: 13 September 2013 / Published online: 8 November 2013

Abstract This paper presents a method for structural optimization of discrete trusses using Graphic Statics. As opposed to traditional structural optimization techniques, which are typically conducted by manipulating the geometry of the structure (the form diagram), the approach presented in this paper establishes a Graphic Statics solution to the problem, where structural optimization is conducted using design variables in the force domain (force diagram). The proposed approach presents several attractive features compared to traditional approaches. Since it is based on reciprocal graphical relationships between form geometry and forces, member stiffnesses need not be calculated. Additionally, by working on the force diagram, equilibrium of the solution is guaranteed, and no additional methods are required to enforce this condition; for example, there is no need to triangulate the structure or to add small area members. Furthermore, because only solutions that are in equilibrium are permitted, the number of design variables can be reduced. Also, subject to certain relationships, the location of the loads (or reactions) do not need to be set a priori. Through examples, it is shown that the proposed methodology can readily accommodate different tensile and compressive stresses for volume optimization problems and that, through the use of Graphic Statics, other restrictions or constraints on the member forces can easily be incorporated.

J. Carrion · A. Beghini · A. Mazurek · W. F. Baker (✉)
Skidmore, Owings & Merrill, LLP, 224 S.
Michigan Avenue, Chicago, IL 60604, USA
e-mail: william.baker@som.com

L. L. Beghini
Department of Civil and Environmental Engineering,
University of Illinois at Urbana-Champaign,
205 N. Matthews Ave., Urbana, IL 61801, USA

Keywords Structural optimization · Truss optimization · Reciprocal diagrams · Graphic statics

1 Introduction

Structural topology optimization is a powerful and well-established technique to determine the optimal geometry to design efficient systems. Optimization has been successfully used in many fields of science, for example, see (Sigmund 2000; Krog et al. 2004; Sutradhar et al. 2010; Altair Engineering Web Page 2013). In structural engineering, as part of the natural design process, we continuously seek efficient structures and corresponding methodologies to find such structures. These resulting structures must achieve the intended purpose, while typically keeping the cost (and generally the use of natural resources) to a minimum.

There are several methods that have been used for structural optimization, and their utilization depends on the specific project or application considered. These methods include topology optimization, shape optimization, size optimization, and form finding, amongst others. Solutions to these methods can be determined in closed-form for only a limited number of cases, e.g., the Michell trusses (Michell 1904). For structures with more complex loadings and geometries, closed-form solutions are generally not possible; thus, numerical methods must be employed to find the optimal topologies.

1.1 Advantages and shortcomings in structural optimization

Several methodologies have been developed to find efficient structures by means of optimization algorithms. The following is a brief overview of some of these methods and their advantages and shortcomings.

One of the most commonly used approaches is topology optimization by means of a material distribution problem, which is typically based on the Solid Isotropic Material with Penalization (SIMP) model (Rozvany et al. 1992; Zhou and Rozvany 1993; Bendsoe and Sigmund 1999). Using this approach, resulting designs can have any size, shape and connectivity. This solution provides valuable insight about the geometry of the layout of the members of the optimal structure. However, one of the issues associated with such a solution is its associated discretization. To physically realize the resulting design in projects of an architectural scale, the finite element densities must be discretized into line elements representing beams, columns and braces, as part of the design process. This often poses some challenges to the designer. Since the precise locations of the nodes can be subject to interpretation, the resulting discrete structure may be difficult to interpret.

Another popular alternative for structural topology optimization is the ground structures approach, which is conducted using discrete members. In these methods, the design problem typically consists of assuming a base or ground structure with a given layout of members, where the optimization can be conducted as a sizing problem with the cross-sectional areas as the design variables (Achtziger et al. 1992; Ben-Tal and Bendsoe 1993; Oberndorfer et al. 1996; Bendsoe and Sigmund 2002; Sokol 2010; Christensen and Klarbring 2008). In these methods, the final designs are already discretized but often with thousands of members. Limitations include the inability for the design domain, shape or connectivity of the members to change; however, members may be removed throughout the process as they attain zero (or very small) areas. One of the major disadvantages associated with these approaches lies in the large number of design variables (cross-sectional areas) required to solve the problem at hand. Thus, in order to create a practical structure, the structural engineer must interpret a structural layout with a limited number of members.

Other truss optimization techniques include optimizing the structure by changing the position of the elements using their nodal coordinates as design variables, where the element cross-sectional areas can also be included to add or remove material depending on the structural configuration (Hansen and Vanderplaats 1988; Mazurek et al. 2011).

In the design of large structures, a limited number of discrete members is typically used. Often these members are primarily axial force members and have limited flexural stiffness. Unfortunately, many of the existing tools for developing optimal topologies either have a very large number of elements (ground structures) or have members with substantial widths (e.g., density methods). A structure with a large number of members is generally impractical because of the costs associated with fabrication and architectural constraints. The density methods often result in solutions with wide principal members that have flexural stiffness, which is generally not achievable in large scale structures (Stromberg et al. 2012). Figure 1 shows the main structural system of the John Hancock Center building in Chicago, where it can be seen that these members are very narrow relative to the scale of the building and therefore have limited flexural stiffness. This figure also illustrates that large scale braced structures generally have a limited number of axial members. That said, ground structure methods and density methods can provide valuable insight in creating a general arrangement of elements for use in other optimization techniques such as those suggested in this paper. As shall be explained below, Graphic Statics provides a useful optimization approach for a given connectivity of elements. The initial connectivity can be determined first by analyzing the problem through ground structures or density methods and then by interpreting an appropriate element connectivity. Furthermore, in many practical designs, architectural and practical considerations may influence the number of members to be selected.

Fig. 1 John Hancock Center in Chicago, IL (notice the small scale of the main structural members compared to overall size of the building)

1.2 Motivation for graphic statics

Many optimal design problems, such as a "tied arch", concern primarily axial member structures, where the natural flexural stiffness of the arch and/or the deck provides stability to the final structure. When these types of structures are designed, the arrangement of the axial members (truss members) are the primary concern and the structure may not be fully triangulated, resulting in numerically "unstable" structures that must be triangulated during the optimization process. Optimal structures typically have only the minimum number of elements required to support the applied loads (i.e. they may not be stable as fully triangulated trusses), and they are not always stable under other load cases. This characteristic often creates problems in stiffness and/or finite element methods used during the solution of the optimization problem; these structures can become "unstable" resulting in a singular stiffness matrix that cannot be analyzed (see Fig. 2a).

One approach to circumvent this problem is to include additional members in the structure, which create a triangulated, and therefore stable, structure. Figure 2a shows a basic sixteen-panel structure, which produces a singular stiffness matrix. To alleviate stability issues, additional members can be added to the structure, as shown in Fig. 2b–d.

One question arises immediately, does the layout of these additional members influence the final solution of the optimization process? In theory, these additional members at the end of the solution should have zero area, as they are only required for the method used to determine the solution; however, in finite element methods, members with zero area have no stiffness, and therefore make the stiffness matrix singular. To solve this problem, many solutions have been proposed which require that a minimum element area is defined, or the elements have to be removed from the solution once their area falls below a certain value. Both of these measures have limitations: for the first one (minimum element area), the value selected has to be small enough not to influence the solution, and large enough not to cause numerical problems during the inversion of the stiffness matrix. The second approach, in addition to the problem of determining which value is an adequate threshold for the problem, is inconvenienced by the fact that a member that is actually part of the optimum solution can be eliminated during the optimization process (for example, based on a local optimum).

Another alternative would be to formulate the topology optimization problem using a stress-based formulation as opposed to a displacement-based one similarly to what described by Achtziger (1997). Such formulations would not require the calculation of the stiffness matrix and would circumvent the above mentioned numerical problems. However, a stress-based finite element formulation is not common in engineering practice and the force diagram of Graphic Statics provides the designer with insight into the force distribution in the structure.

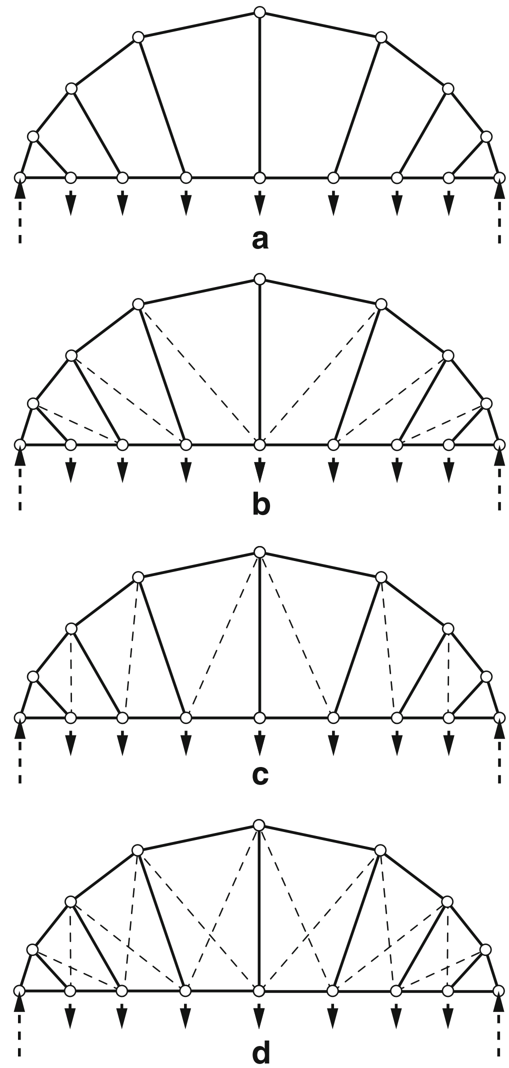


Fig. 2 Triangulated structure for conventional structural optimization (additional members shown with dashed lines): **a** Structure without additional members, **b** Option 1 for additional members, **c** Option 2 for additional members, and **d** Option 3 for additional members

Therefore, the approach to structural optimization proposed in this paper consists of using Graphic Statics to define the variables of the optimization process. The optimization problem is conducted in the force domain using the force diagram, as opposed to previous approaches which conduct the optimization in the geometry domain using the form diagram. Thus, the optimization variables for the proposed methodology consist of the coordinates of the points (nodes) in the force diagram. As described later in more detail, the use of Graphic Statics can provide the following advantages for certain types of problems, as compared to the traditional approach:

- There is no need to triangulate the domain with members of very small areas, which can create numerical difficulties.
- The results are always feasible if the members form closed polygons in the force domain (i.e. equilibrium is guaranteed).
- Using Graphic Statics, there is no need to compute or assemble stiffness matrices; only simple graphical relationships are needed.
- Subject to certain relationships, the final location of the loads (or supports) do not need to be specified a priori but can change as the optimal solution is found.
- Graphic Statics can readily accommodate different tensile and compressive stresses.
- The equilibrium constraints in the force diagram reduces the number of design variables required.

1.3 Paper scope and organization

The remainder of this paper is organized as follows: in the next section, we discuss the history and current state-of-the-art techniques in Graphic Statics and structural optimization of trusses. Then, in Section 3, we introduce the problem statement and propose a new methodology for optimal truss design using Graphic Statics. Finally, we conclude with practical numerical examples in Section 4 and comment on the extensions of this work.

2 Background

Topology design of truss structures in the form of grid-like continua has been studied extensively in the past, starting with the ground-breaking paper by Michell (1904) (Bendsoe and Sigmund 2002; Rozvany et al. 1995). In this section, we give a brief overview of existing truss optimization techniques and highlight the advantages and limitations of incorporating Graphic Statics in the classical layout problem.

2.1 Structural optimization of trusses

The first fundamental properties of optimal truss-like continua were established starting with Michell's seminal paper "The Limits of Economy of Material in Frame-structures" (Michell 1904). This was later studied and expanded upon by Prager (1970), Hemp (1973), Rozvany (1976), Prager (1978), and Rozvany (1989), which has become the well-known modern layout optimization theory. Other contributions to the field were the early numerical methods developed by Fleron (1964) and Dorn et al. (1964).

The so-called ground structures method is commonly used today, in which the layout of a truss can be found by creating a set of connections between a fixed set of nodal points as potential or vanishing structural members (Bendsoe and Sigmund 2002). Though these methods provide valuable insight to the optimal design problem, they are often limited in the sense that the fixed nodal points and solutions are highly dependent on the choice of the initial ground structure. Furthermore, the optimization process typically consists of a large number of members that cannot be achieved in a practical structure. Moreover, these problems are typically formulated in terms of any cross-sectional area, which may not always be feasible to practicing structural engineers.

Continuum methods have been proposed by Bendsoe and Kikuchi (1988), Bendsoe (1989), Zhou and Rozvany (1991), Rozvany et al. (1992), and Bendsoe and Sigmund (2002), in which each finite element can be compared to a "pixel" in a black-white raster of an image. These have become highly popular due to the fact that the physical size, shape and connectivity does not need to be specified *a priori*. However, one of the major disadvantages of continuum topology optimization using numerical methods lies in the interpretation of the final results. Often the final design contains thick regions of solid ("black") material, and therefore, finding the optimal nodal locations is subjective when discretizing the final design into beams and columns. These thick regions of material present in such results also give unrealistic flexural stiffness that may bias the solution.

In this work, we use Graphic Statics to formulate an alternative methodology to those listed above for practicing engineers, as described in the next section.

2.2 Graphic statics

Maxwell established that for certain trusses, the nodes and polygons that represent the geometry of the truss have reciprocal polygons and nodes in the force domain (Maxwell 1864, 1870). Every node in the geometry domain maps into a polygon in the force domain, every polygon in the geometry domain maps into a node in the force domain and every line representing the line of action of each truss member maps into a reciprocal line in the force domain. The mapping used by Maxwell resulted in the reciprocal lines being perpendicular to each other. The use of a different mapping (a hyperbola rather than a paraboloid of revolution) by Cremona (1872) results in the reciprocal lines being parallel. Because these two figures (the form diagram and the force diagram) are reciprocal, the mapping can also go from the force domain to the geometry (form) domain. The length of each of the lines in the force domain are proportional to the axial force in the reciprocal line representing the truss member. The creation of a force diagram from a form diagram (with its external applied forces) is called Graphic Statics. The process of Graphic Statics used to be a standard method of analyzing trusses and could be done using simple drafting tools. Graphic Statics is no longer in common usage for analysis, having been replaced by more mathematical tools, but it still can be used as a design tool and, as discussed below, as a tool in creating optimal structures.

As discussed in Baker et al. (2013), Graphic Statics provides the information needed for minimizing the load path (or volume) of a truss with specified connectivity. The form diagram provides the length of each member and the length of each line in the force diagram provides the force in each member. The reader is referred to Baker et al. (2013) for a brief introduction to Graphic Statics and to textbooks such as Zalewski and Allen (1998). Because of the reciprocal relationships observed by Maxwell, only simple graphical techniques are needed to determine the length of each member and its axial force; there is no need to compute a stiffness matrix or to solve a large system of equations.

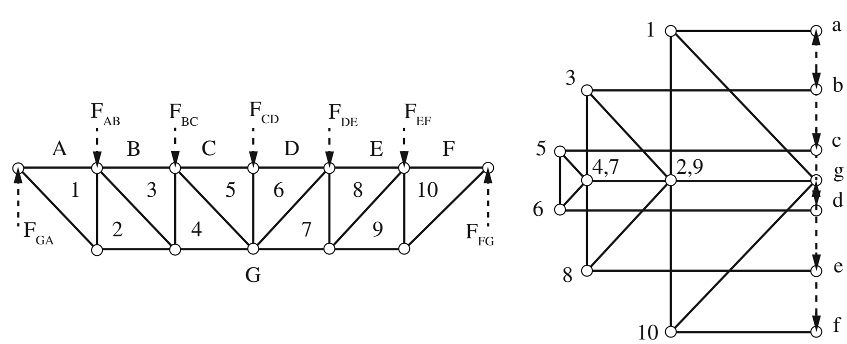
Figure 3 is used to illustrate the basic principles associated with the solution of a problem using Graphic Statics for a six-panel structure. The notation shown here is the interval notation adapted from Bow (1873), given also in the textbook by Zalewski and Allen (1998). On the left, the connectivity, initial geometry and boundary conditions are shown for a given structure, with its corresponding force diagram on the right. It can be easily seen from these two diagrams that all of the information is given to describe the total load path: the "path" or lengths of the members, L_i, are given from the form diagram in Fig. 3(left) and the "load" or internal forces in the members, P_i, are shown in the force diagram in Fig. 3(right). For a detailed description on how to construct the reciprocal diagrams, one can refer to the textbooks by Wolfe (1921) and Zalewski and Allen (1998).

The information about the internal forces can be interpreted from the form and force diagrams as follows: The capital letters, *A,B,C...,* are sequentially placed clockwise in the intervals between external forces (open polygons) in the form diagram and the numbers, 1,2,3,..., are placed in the internal spaces (closed polygons) between members. Each line in the form diagram is bordered by two polygons. Thus, a member may be called using the corresponding letter or number of the adjacent polygons, e.g. *B*-3 or 6-7, and a joint called with a series of letters and numbers, e.g. *A*-*B*-3-2-1-*A*. Similarly, the external forces are called using the adjacent open polygons, for example F_{AB}. The open polygons denoted by capital letters in the form diagram correspond to points (nodes) on the load line of the force diagram, denoted by the lowercase letters, *a,b,c,...*. The numbers denoting the closed polygons in the form diagram also have corresponding nodes in the force diagram.

The axial force in a truss member can then be determined by measuring the length of the reciprocal line in the force diagram. The relative scale of the force diagram is set by drawing the load line representing the external forces to a scale. For example, the force in member *A*-1 in the form diagram of Fig. 3 is proportional to the length of the line between points *a* and 1 in the corresponding force diagram. Similarly, the force in the member between polygons 2 and 3 is proportional to the length of the line between points 2 and 3 of the force diagram. The remaining forces in the other members can be computed likewise. Thus, the forces acting on a node in the form diagram correspond to a polygon in the force diagram, where each force is a side of the polygon. For example, at node *A*-*B*-3-2-1-*A*, the polygon of forces is given by points *a*-*b*-3-2-1-*a*. Reading clockwise around joint *A*-*B*-3-2-1-*A* in the form diagram, we can determine if members *A*-1 and 2-3 are in tension or compression. If we read from 1 to *a* on polygon *a*-*b*-3-2-1-*a*, we move from the left to the right, towards the joint *A*-*B*-3-2-1-*A* of the form diagram. Thus, member *A*-1 is in compression. Likewise, moving from 3 to 2 on the force polygon goes from the upper left to the lower right, or away from the joint in the form diagram, so member 3-2 is in tension. The remaining forces can be interpreted likewise.

Moreover, in Graphic Statics, the member lengths and forces can be determined solely using geometry. These diagrams can be constructed with simple drafting tools (straight edges, triangles, and a scale) or by the use of simple equations for lines and the intersection of lines required to solve a truss.

Fig. 3 Reciprocal member (form) and force diagrams for six-panel truss



2.3 Form-finding using graphic statics

Previously, the reciprocal form and force diagrams in Graphic Statics have been used for structural design, as described in Chapter 14 of the book by Zalewski and Allen (1998). The methodology for form-finding of trusses graphically solves for the nodal locations that give the desired force properties in a structure, such as a constant chord-force truss. For example, consider the simple six-panel truss shown in Fig. 3. If the desired force property is to have a constant force in the bottom chord, the geometry of the truss is as shown in Fig. 4. This can be accomplished by manipulating the force diagram so that the lengths of lines *g*-1, *g*-2, *g*-4, *g*-7, *g*-9 and *g*-10 of the force diagram are the same (representing equal forces), while the points of applied loads in the top chord remain equally spaced horizontally. After the force diagram is modified to achieve the desired properties, one can then work backwards to find the reciprocal form diagram, resulting in the desired geometry. Note that the forces in members 2-3, 4-5, 6-7 and 8-9 are zero because the nodes are overlaid in the force diagram on the right. These members can be eliminated from the structure if the chords have sufficient strength and flexural stiffness to satisfy the demand of non-uniform load cases and stability requirements. Several examples of structures designed using Graphic Statics can be found in the literature (Zastavni 2008; Fivet and Zastavni 2012). More recently, the authors applied Graphic Statics and Rankine's Theorem for the design of a canopy of a high-rise building (Beghini et al. 2013) and for the design of the long-span roof trusses of a convention center as indicated in Section 4.5.

3 A method for truss optimization using graphic statics

An alternative to the methods presented in Section 2.1 for minimizing the value of a structure is manipulation of the force diagram of a Graphic Statics analysis until a minimum total load path is achieved. By manipulating the force diagram rather than the form diagram (as other methods typically do) one can always be assured that the resulting solution is in equilibrium, since the force polygons are always closed. It can also be noted that, because the solution is automatically constrained to be in equilibrium, there are also fewer independent variables than if one tried to manipulate the form diagram. This methodology is described in further detail in this section.

3.1 Problem statement and methodology

The well-known minimum compliance problem for truss topology optimization for a given volume, V, can be written as follows:

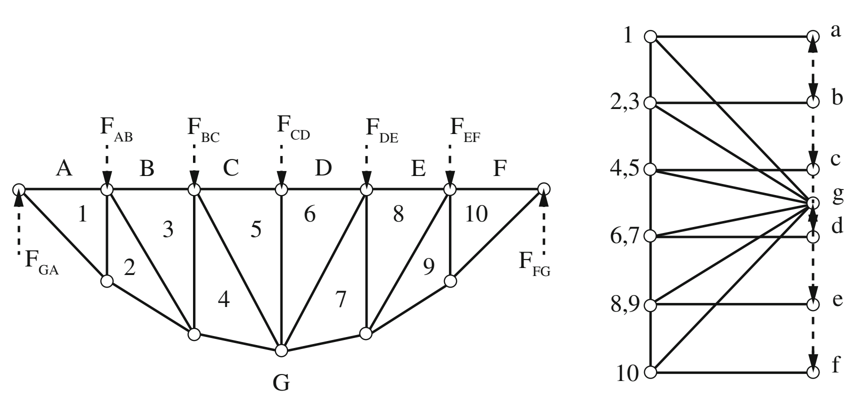
$$\begin{aligned} &\min_{\mathbf{u},\mathbf{t}} \ \mathbf{F}^T\mathbf{u} \\ &s.t. \ \sum_{i=1}^{m} t_i \mathbf{K}_i \mathbf{u} = \mathbf{F} \\ &\qquad \sum_{i=1}^{m} t_i = V \\ &\qquad t_i \geq 0, \ i = 1, \ldots, m \end{aligned} \tag{1}$$

where $\mathbf{F}$ is the global vector of applied forces, $\mathbf{u}$ is the global vector of nodal displacements, $\mathbf{K}_i$ is the element stiffness matrix and t_i is the volume of the ith member and m is the total number of members. Classical truss topology optimization methods based on (1), therefore, require the assembly of a stiffness matrix associated with a chosen ground structure as given below:

$$\mathbf{K}\left(\mathbf{t}\right) = \sum_{i=1}^{m} t_i \mathbf{K}_i \tag{2}$$

One of the major disadvantages of this formulation is that many optimal designs have a singular stiffness matrix when described as part of the full ground structure, so the solution of the equilibrium equation, $\mathbf{K}(\mathbf{t})\mathbf{u} = \mathbf{F}$, typically becomes computationally difficult. By using Graphic Statics to perform the structural analysis, it is no longer necessary to compute (2) and the singularity issue can easily be

Fig. 4 Resulting geometry from form-finding of a constant-force truss (Zalewski and Allen 1998)



circumvented. Alternatively, the topology optimization problem described by (1) can be formulated using one of the dual (stress-based) formulations proposed by Achtziger (1997), which would not require the computation of the stiffness matrix. However, the graphics statics approach is preferred by the authors because it provides more insight in the problem since the structural forces are visualized. This enables the engineer to easily select specific force-based or stiffness-based design criteria similarly to what described in details in Beghini et al. (2013) for the design of a canopy.

Similar to Hansen and Vanderplaats (1988) and Rahami et al. (2008), the authors of this work aim to minimize the total volume, or equivalently the total load path (Baker et al. 2013), of the structure in terms of the locations of the nodes in the force diagram, $\mathbf{x}$, which can be written as follows (assuming a constant state of stress at the optimum Stromberg et al. 2012):

$$\min_{\mathbf{x}} V = \min_{\mathbf{x}} \frac{1}{\sigma} \sum |P_i| \cdot L_i \tag{3}$$

where V represents the total volume of the structure, σ is the value of stress, P_i is the internal force and L_i is the length of the ith member, respectively. Equation (3) illustrates one of the advantages of using graphics statics in the optimization process. The geometry of the structure analyzed and the areas of the cross sectional members are not independent variables. The length of the members is a function of the nodal coordinates in the form diagram and the areas are function of the loads in the members (described by the force diagram) and the stress level. The reciprocal diagrams (form and force) are related in a way that any geometry change in the form diagram affects the force diagram and, vice versa, any force change causes a geometry change. Therefore, the optimization can be conducted on either the form or the force diagram. This eliminates the numerical difficulties associated with the treatment of the nodal coordinates geometry and member areas as independent variables (Achtziger 1997).

Based on (3), the sensitivities of the design variables (i.e. nodal locations) with respect to the objective function can be approximated using the finite difference method below:

$$\frac{\partial V}{\partial x_i} = \frac{V(\mathbf{x}^*) - V(\mathbf{x})}{\partial x_i} \tag{4}$$

where $\mathbf{x}$ represents the vector of the nodal locations, and $\mathbf{x}^*$ is the vector of nodal locations where the ith variable is modified as $x_i^* = x_i + \partial x_i$. We note that, in the proposed methodology, the number of design variables is relatively small, making this option computationally feasible.

In terms of the optimization algorithm used to update the design variables, a large number of algorithms are available, including the optimality criteria (Zhou and Rozvany 1991), interior point algorithms, the method of moving asymptotes (Svanberg 1987), branch and bound methods, genetic algorithms, etc. Though any of these methods can be selected, the user must be careful to avoid local minima in the final solutions. For the optimization algorithm of this work, we employ the Method of Moving Asymptotes (MMA), proposed by Svanberg (1987). The Method of Moving Asymptotes is particularly suited for the proposed algorithm provided in the next section due to its flexibility (in terms of type of design variables, constraints, etc.), stability and speed of convergence. Moreover, the method is fairly easy for the engineer to implement and use; educational codes are also available through the author (Svanberg 2013).

3.2 Graphic statics for truss analysis

An advantage of using Graphic Statics for optimization is that the number of design variables is reduced; therefore, the computational size of the optimization problem is decreased accordingly. The complexity of optimization process grows exponentially with the number of variables, therefore, by reducing the number of design variables, the type and size of problems that can be solved using computational structural optimization can be expanded.

In Graphic Statics, the member i with length L_i has a reciprocal line in the force diagram with length $\hat{L}_i$ that is proportional to the force P_i in the member. Thus, the objective function can be rewritten as follows:

$$\min_{\mathbf{x}} V = \min_{\mathbf{x}} \frac{1}{\sigma} \sum L_i \cdot \hat{L}_i \tag{5}$$

which is equal to the summation of the products of the lengths of the members in the form diagram and the lengths of their reciprocal members in the force diagram. This value can easily be calculated based solely on the geometry of the two diagrams. Therefore, the optimization framework using Graphic Statics can be summarized as follows:

1. Given a specified general geometry and connectivity of a structure (form diagram), draw the corresponding reciprocal force diagram. Determine which node degrees of freedom in the force diagram are restrained by reciprocal relationships with the form diagram.
2. Assign design variables to each node degree of freedom in the force diagram that is not restrained by reciprocal relationships.
3. Compute the sensitivities of the design variables (if necessary) and update the design variables using a suitable optimization algorithm (gradient-based, or other).
4. Update the reciprocal force diagram, and use this to construct a new form diagram (truss geometry).
5. Calculate the length of the lines in both diagrams.

6. Calculate the objective function based on the line lengths and repeat until convergence is achieved.

Some of the advantages of conducting the optimization on the force diagram include the following: (i) Because the force diagram represents an equilibrium configuration, the structure resulting from the optimization process will always satisfy equilibrium. (ii) No additional members are needed to stabilize the structure or satisfy equilibrium. Recall that in traditional structural optimization approaches, it is necessary to add additional members to triangulate the structure to make the numerical solution viable. These zero or low force members at the end of the optimization process have a very small area, however, unless some arbitrary means are taken to remove them from the solution, they will end up present in the final solution, with very small area. The value chosen for the minimum area, can actually change the solution of the numerical optimization process. Furthermore, by replacing the analysis using the stiffness method with Graphic Statics as the analysis tool, there is no need to compute and assemble the stiffness matrices; only geometric relationships are required. Depending on the original design problem objective, the location of the applied loads (or supports) can change subject to the restrictions of the reciprocal relationships between the form and force diagrams.

One of the disadvantages of using Graphic Statics in engineering practice is that the structure can be optimized for only one load case. Therefore, the engineer should identify the dominant load case for the problem at hand, optimize for such load and later check the other load cases. A detailed description of a design completed using Graphic Statics and Rankine's Theorem is described in Beghini et al. (2013).

4 Numerical examples

In this section, numerical examples are given to verify the proposed methodology and illustrate the overall design process.

4.1 Two-bar truss optimization

In Fig. 5, we examine the two-bar truss of Rozvany (1996) using the proposed methodology to find the structure of minimum volume, $\frac{1}{\sigma}\sum L_i \cdot \hat{L}_i$. This two-bar structure is self-reciprocal in that the form and force diagrams are the same, up to a scaling factor. The optimization is done by manipulating nodes of the force diagram. In the force diagram, the relative position of points b and c are set by the force vector F_{BC}; the position of the free point, a, is located to achieve the minimum volume.

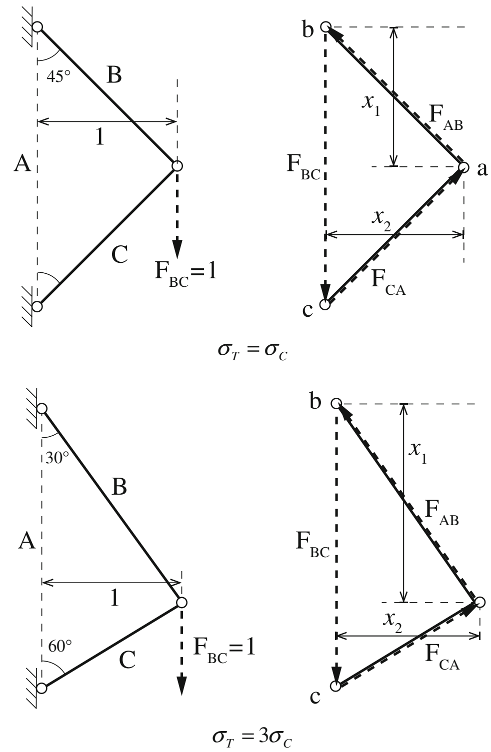


Fig. 5 Solution of two-bar truss optimization problem for minimum volume with equal and unequal stresses: (*left*) form diagram, (*right*) force diagram

For an equal allowable stress in tension and compression, $\sigma_T = \sigma_C$, we observe the expected behavior where each member forms a 45° angle with the supports. On the other hand, if the allowable compressive stress is one-third of the allowable tensile stress, $\sigma_T = 3\sigma_C$, the results show agreement with the example of Rozvany (1996), where the angles are 60° and 30° respectively. It should be noted that the locations of the supports in the form diagram changed depending on permitted stresses. This is an advantage of this method when compared to other techniques where the location of the supports must be set *a priori*.

4.2 Optimal bridge design

As mentioned previously, Graphic Statics can be particularly useful to analyze structures that the stiffness method cannot. For example, the structural connectivity for the problem given in Fig. 6b is an example of an "unstable" structure, which would produce a singular stiffness matrix and thus, could not be analyzed using the stiffness method. To analyze and optimize the structure using Graphic Statics, the results of a continuum topology optimization problem (generated using the educational code provided in Talischi et al. 2012a, b) give insight into what the initial connectivity of the structure is by discretizing the continuum results (see Fig. 6a, b).

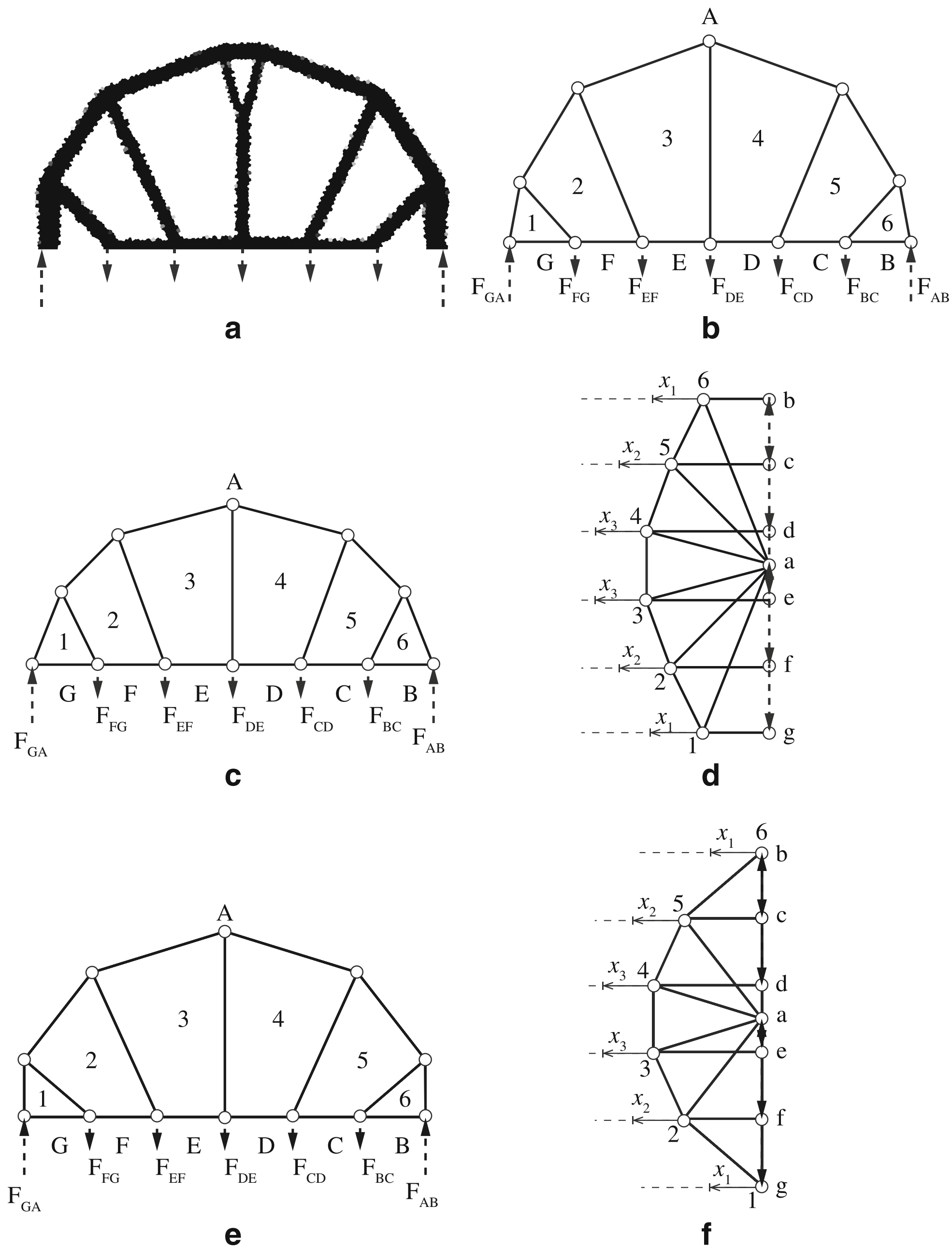


Fig. 6 Example illustrating the solution of an "unstable" bridge: **a** topology optimization results, **b** connectivity interpretation (not necessarily in equilibrium), **c** initial form diagram (in equilibrium), **d** initial (reciprocal) force diagram (modified to be in equilibrium), **e** optimized form diagram (in equilibrium), **f** optimized (reciprocal) force diagram (in equilibrium)

It is important to note that since the initial interpretation does not include diagonal members, it is not guaranteed to be in equilibrium. Thus, before performing the optimization, the initial force diagram must be generated and modified (if necessary) such that each reciprocal polygon is closed (see Fig. 6d). Working backwards, the corresponding initial form diagram can then be drawn using the modified force diagram, such that it is guaranteed to be in equilibrium (see Fig. 6c).

After the initial reciprocal diagrams are in equilibrium, the optimization can be performed on the force diagram of the arch using the *x*-location of points 1 through 6 as the design variables, since the horizontal members along the bottom chord of the form diagram constrain these points to lie along the horizontal lines through them. Because of symmetry, the variables are reduced to x_1, x_2 and x_3. After minimizing the total load path of the structure, the form and force diagrams are shown in Fig. 6e, f. Note that Nodes *g* and 1 overlay (as do Nodes *b* and 6) indicating that members *G*-1 and *B*-6 are zero force members. It should also be noted that Nodes *c*, *d*, *e*, and *f* are not connected to Line *a*-*b* and *a*-*g* in Fig. 6f.

As discussed in Baker et al. (2013), the form and force diagrams are reciprocal; therefore, the force diagram also represents the geometry of a truss (a dual) with its own external loads. Figure 7 shows the dual truss and its loadings. It should be pointed out that the dual truss (Fig. 7 shows the dual truss with its external loadings but Fig. 6d is useful in seeing the truss connectivity more clearly) is statically determinate and has the same total load path as the original problem. This dual truss can be optimized using other tools available for optimizing a statically determinate truss.

For example, the 6-bay bridge problem in Figure 6 has 12 nodes; 3 support degrees of freedom may be assumed; and there are 17 members. This renders the problem underspecified by 4 members ($2 \cdot 12 - 3 - 17 = 4$). On the force diagram there are 13 nodes; the horizontal lines are constrained to fixed y values. These lines (b-6, c-5, d-4, e-3, f-2 and g-1) can be replaced with load vectors. This removes six nodes and six members from the problem. Node "a" can be assumed to be constrained in the x and y-directions without biasing the solution. An additional node, such as Node "6", can be assumed to be constrained in the x-direction to avoid rigid body rotations. There are 11 members remaining in the dual truss. This results in a statically determinate problem ($2 \cdot 7 - 3 - 11 = 0$). In optimizing the geometry for a minimum load path in the dual truss, the y-coordinates of Nodes 1 to 6 are specified; this combined with symmetry leaves only three variables. The load path of the six eliminated members can be added algebraically.

The above observation applies to the Bridge Problem (i.e. a uniformly distributed load between two points of support) regardless of the number of bays. For example, if the Bridge Problem has 100 bays, the geometry is underspecified by 98 members, but the force diagram is statically determinate. It is significant that, in some situations, Graphic Statics can map an optimization problem that is highly underspecified to a dual problem that has the same total load path but is statically determinate.

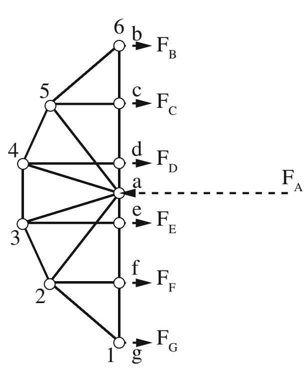


Fig. 7 Dual of optimal "unstable" bridge structure

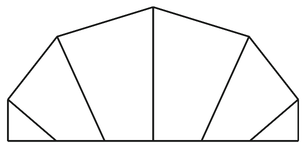
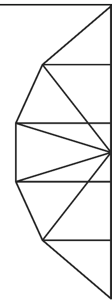
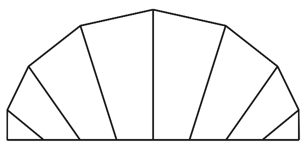
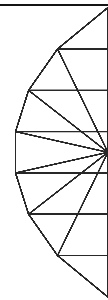
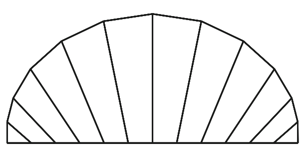
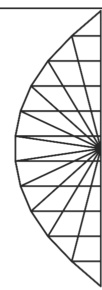
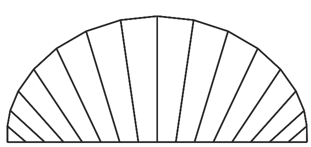
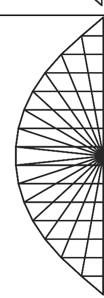
It is also interesting to note that the dual truss is very regular with the chord approaching a parabola. In fact, in Table 1 we show the optimal form and force diagrams for the bridge structure using 6, 8, 12 and 16 bays, along with the total load path. The 16-bay bridge was also studied in Figure 1.5 of Hemp (1973), in which the total optimal load path was computed as 1.067. For the examples given in the table, the coefficient of the equation for a parabola, A, is used as a single design variable, instead of the horizontal coordinates of the force diagram (as was done previously), for optimization of the total load path. When the force diagram is constrained to be parabolic, we notice the resulting optimal load path approximates the actual load path (in which the geometry of the force diagram is unconstrained, similar to Fig. 6d) almost exactly. This approximation provides the designer with a tool to easily compute nodal coordinates that are very close to optimal for a bridge structure using a simple equation.

Notice that the optimum solution for a simply supported truss carrying a uniform load across the span has a more complex geometry as indicated by the studies by McConnel (1974) and Hemp (1974), and more recently by Pichugin et al. (2012). Such geometry would also include spurs at the supports and multiple chords. However, the structure described in Table 1 without spurs and a single chord provides a very good approximation of the more complex geometry and can be used for practical applications for long span simply-supported structures.

4.3 Cantilever benchmark

Here we consider the example of the cantilever in Figure 15 of Rozvany et al. (1995). For this example, a continuum topology optimization model (generated using the educational code of Talischi et al. 2012a, b) is used to provide guidance in determining the overall geometry of the structure (Fig. 8a). This can then be interpreted to create the truss in Fig. 8b. From this interpretation, an initial form diagram and its corresponding reciprocal force diagram can be constructed (Fig. 8c, d). While the geometry of the initial form diagram looks reasonable at first glance, a quick inspection of the force diagram shows obvious irregularities (e.g. the forces in members 6-7, 7-8 and 8-9) suggesting that the structure might not be optimal. From here, the design variables for this structure are selected as the coordinates of the form diagram nodes, which are not constrained by the reciprocity relationships. Figure 8e, f shows the optimal form and force diagrams after a minimum load path is achieved. Once again, it is interesting to note that the force diagram represents the geometry of a dual truss with its own external loads (see Fig. 9).

Table 1 Parabolic approximation of the force diagram for the bridge problem

Number of Bays	Form Diagram	Force Diagram	$\sum FL$ from Optimization	Coefficient A in $A[1-4x^2]$	$\sum FL$ from Parabolic Approximation
6			1.175	0.3383	1.177
8			1.119	0.3237	1.120
12			1.071	0.3106	1.071
16			1.049	0.3046	1.049

4.4 Example with one “force” variable

In Fig. 10a–f, an example is given to illustrate one of the advantages of the proposed methodology. In this example, it is desired to create a minimum load path structure with the same axial force in all the tension and compression chords. Typically, optimization problems are limited in the sense that there are usually two design variables per node, whereas using Graphic Statics, the entire geometry of a structure can be described using only one variable in the reciprocal force diagram. The design process for such an optimal structure is described next.

First, a lenticular truss is shown (from Zalewski and Allen 1998), where the forces in the top and bottom chords have the same magnitude, but different signs. Moreover, by examining the corresponding force diagram, we observe that the diagonal members of the truss are zero force members. If these members are removed, it renders the structure unable to be analyzed with the commonly-used stiffness method. If this “unstable” truss is then modified by manipulating the force diagram (it is guaranteed to be in equilibrium as long as the polygons remain closed) such that the axial forces in all of the members of the outer chords (top and bottom) are equal, the geometry of the form diagram

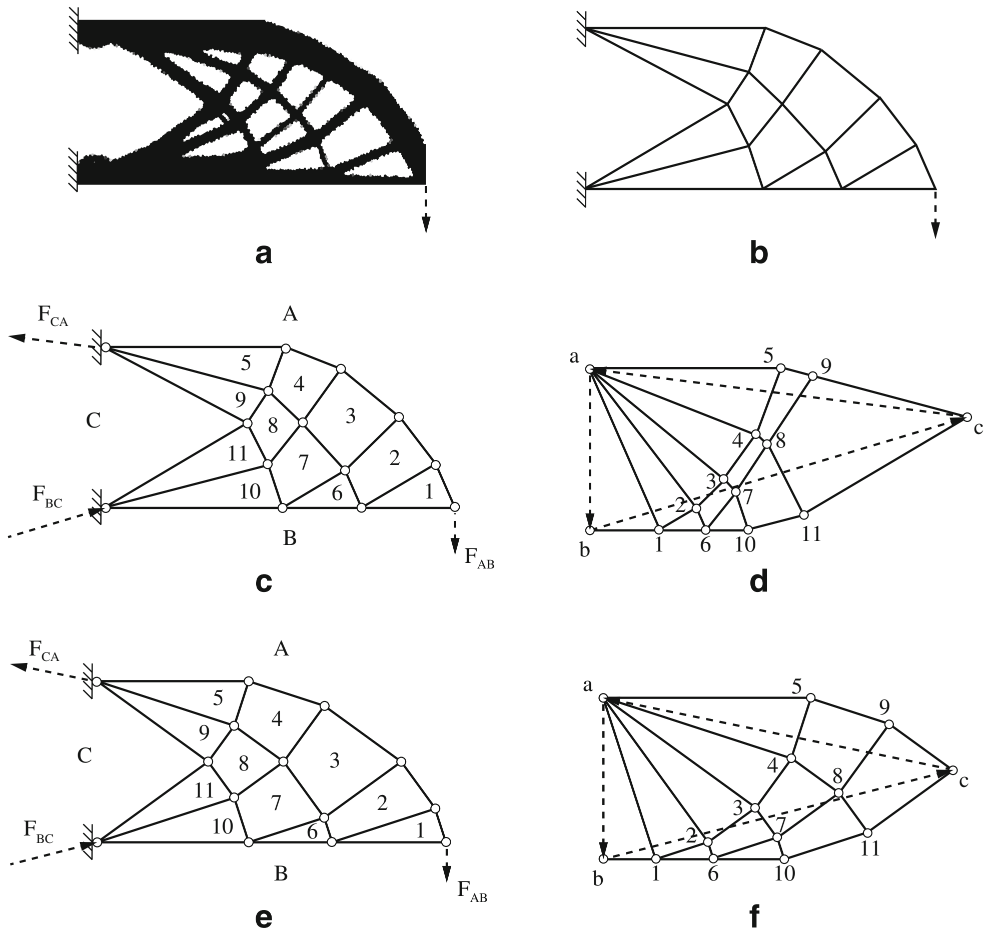


Fig. 8 Solution for the example given in Figure 15 in Rozvany et al. (1995): **a** topology optimization results, **b** connectivity interpretation, **c** initial form diagram, **d** initial (reciprocal) force diagram, **e** optimized form diagram, **f** optimized (reciprocal) force diagram

can be computed as shown in Fig. 10c, d. In this case, the lengths of lines *a*-1, *b*-3, *c*-5, *d*-6, *e*-8, *f*-10, *g*-1, *g*-2, *g*-4, *g*-7, *g*-9, and *g*-10 of the force diagram must be equal. Thus, the entire geometry of the structure can be described using only one variable, representing the total force in these members.

If the geometry of this truss is manipulated directly on the force diagram using the allowable force in the outer members as the sole variable such that $\frac{1}{\sigma} \sum L_i \cdot \hat{L}_i$ is minimized, the resulting optimal structure is shown in Fig. 10e, f. We note here that members 1-2, 2-3, 4-5, 6-7, 8-9 and 9-10 are all zero force members, which were not included in the final form diagram. Furthermore, it has been demonstrated that the proposed methodology is also capable of solving problems where the locations of the applied loads can change, whereas continuum topology optimization or ground structures approaches cannot.

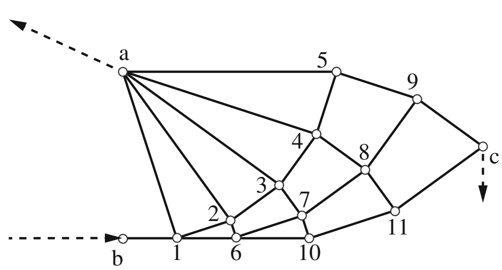


Fig. 9 Dual of optimal cantilever truss

4.5 Application to the design of a long-span roof structure

The optimization methodology based on graphic statics described in the previous sections has been applied for the concept design of the structure for the long-span roof of a convention center (see Fig. 11 for rendering of the structure).

The main structural system for the roof consists of a series of structural steel trusses. Each truss is 162 m long and it is composed of 18 modules of 9 m each. The structure has a large cantilever on the left side (45 m long), and a smaller cantilever on the right side (27 m), which results on a 90 meter center span. The bottom chord of the central span was set to remain horizontal in order to accommodate the functional requirements. The loads were applied on the top chord nodes, based on their tributary area and a uniform roof loading. Different types of truss

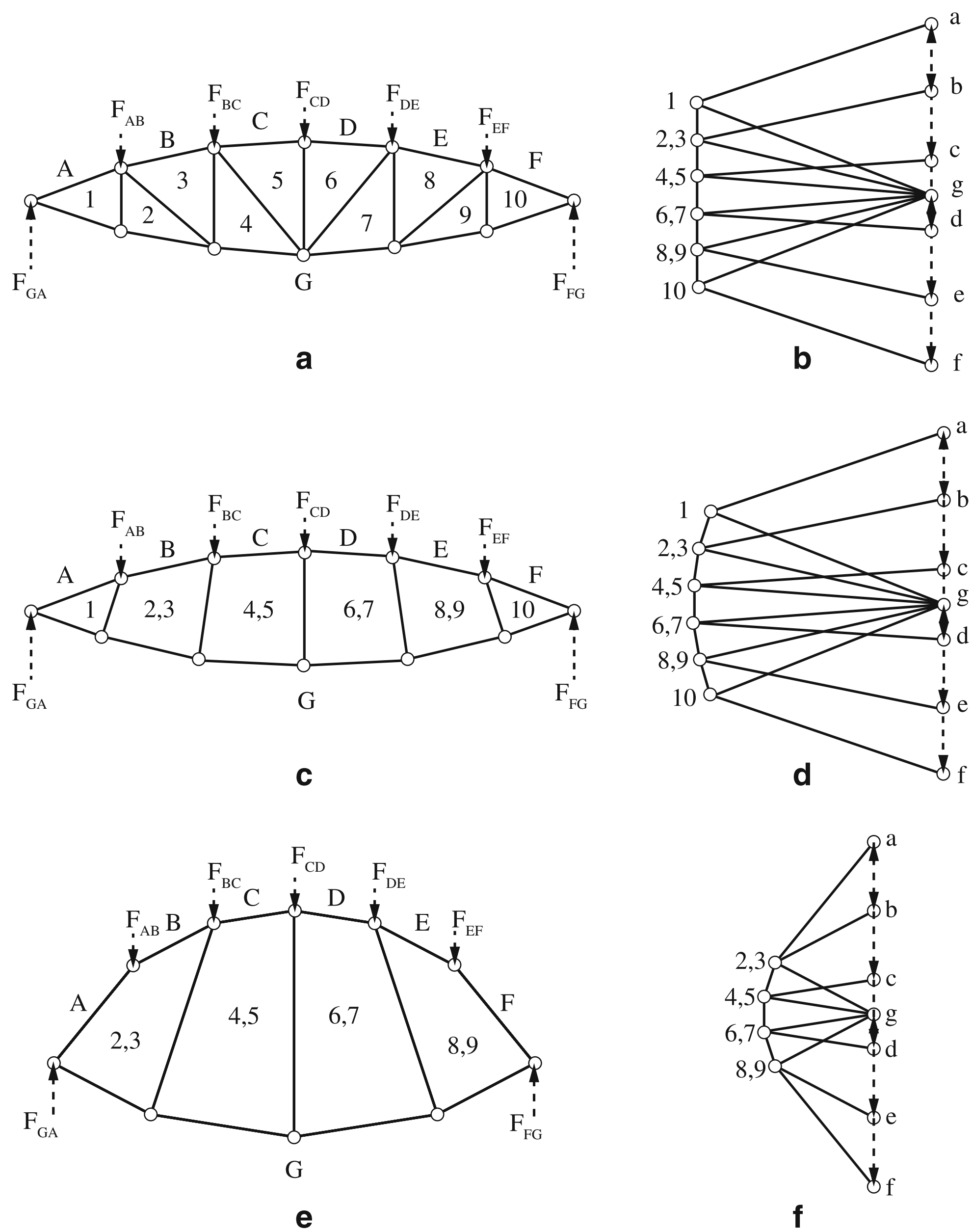


Fig. 10 Optimization of a lenticular truss: **a** initial form diagram (taken from Zalewski and Allen 1998), **b** corresponding initial force diagram (showing diagonals are zero force members), **c–d** modification of truss to achieve a constant force in the outer chords, **e–f** optimized form and force diagrams of the truss. Note that in (**c**)–(**f**), the zero-force members are removed and in (**f**), Nodes 1 and 10 were removed because polygons 1 and 10 in the form diagram collapsed to a line

layouts were initially considered—Pratt, Warren, and X-Bracing. From these preliminary studies, a 9-meter deep, constant depth truss with an X-Bracing layout was selected as starting layout (see case (a) in Table 2). Later, optimization was conducted on this initial truss to determine the most efficient member layout satisfying architectural and functional constraints. The objective function during the optimization process was the minimization of the total steel volume. The truss was initially optimized assuming equal allowable stresses in tension and compression, which resulted in a rather deep truss with large member slenderness. This significantly reduces the allowable compression stress. Therefore, considerations for buckling of compression members were directly included in the optimization process by calculating the slenderness of each member during the iterations and then updating the compressive allowable stress for the member based on the equations of Section E3 of the Specifications in AISC (2010). The optimization was initially run constraining only the lower chord of the truss to be flat as indicated by case (b) in Table 2. As it can be observed, the resulting truss is deeper at the supports and at the center of the middle span, where the moments are large. The normalized total volume for this case is 0.552, which corresponds to a reduction of about 45 % with respect to the reference case (a). Based on architectural, cladding, drainage, and aesthetical considerations,

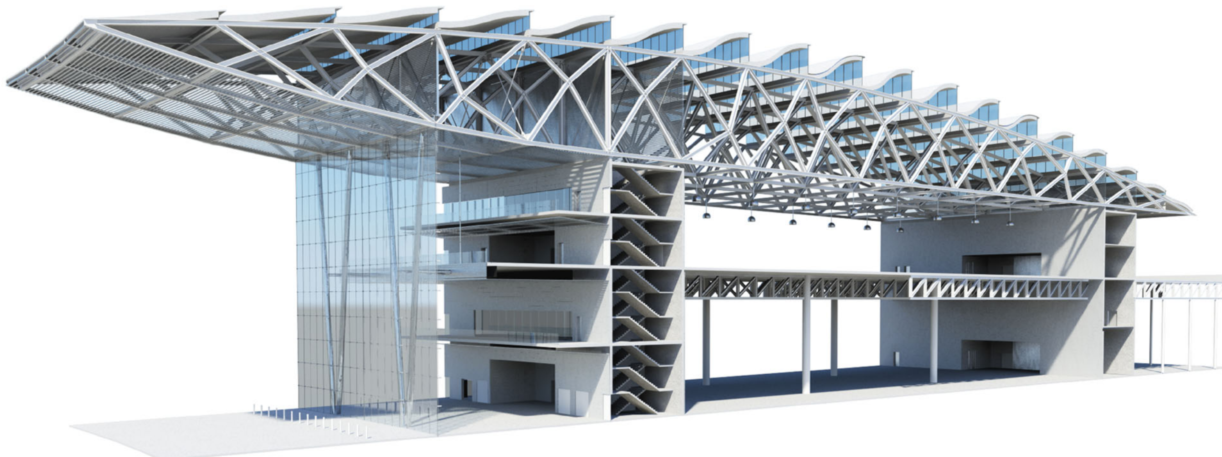


Fig. 11 Rendering of the roof truss for a convention center

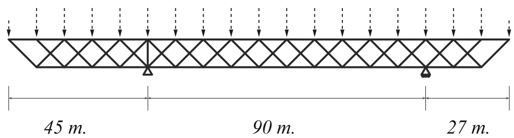
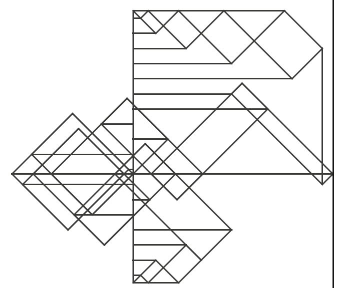
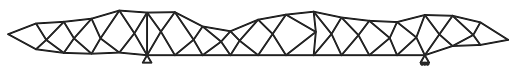
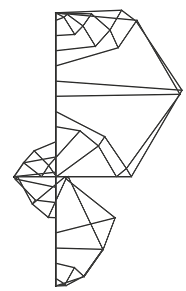
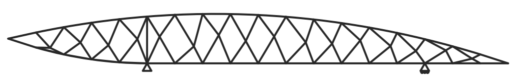
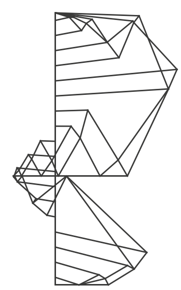
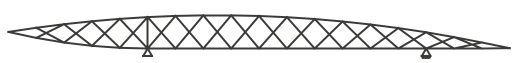
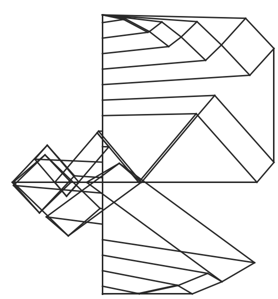
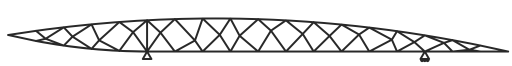
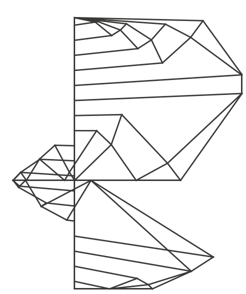
the geometry of case (b) was deemed impractical but it does provide a benchmark for the problem. It was then decided that the top chord should have a constant curvature, which mathematically can be described with a parabola. The bottom chord of the left side cantilever was also constrained to be a parabola, with the additional consideration that it should have a horizontal tangent at the left support, i.e., to be continuous with the horizontal bottom chord at the center span and right side cantilever. Case (c) shows the results of the optimization under these constraints. The total normalized volume for this case is 0.629, which is approximately 14 % higher than the unconstrained case (b). Based on further coordination with the project architects and other disciplines involved, it was determined that the truss depth should be set to 10.7 m. The result of this optimization corresponds to case (d) and (e) in the Table. The optimal truss layout was initially calculated in case (d) with the additional depth constraint assuming straight web members between the top and bottom chord and the resulting normalized volume was 0.852. Next, the location of the work point at the intersection of the web members was optimized as shown in case (e). The total normalized volume for this case was 0.669, which is only 6 % higher than case (c), where the height of the parabola was unconstrained. Therefore, by simply adjusting the work point at the intersection of the web members there was a 21 % improvement on the structural efficiency (as compared to case (d)). Figure 11 shows the architectural rendering of the final scheme adopted, based on case (e) in Table 2. Although conventional stiffness methods could have been used for this design problem, the use of graphic statics and force diagrams provided valuable insight in the force distribution in the structure and the relative importance of the various members. It also helps the designer develop insights into the relationships between form and forces. Such information guided the design engineers in the process of maximizing the efficiency of the structure while satisfying all the functional and architectural constraints.

The design example described in this section was optimized considering a uniform load applied to the top chord. However, as described in details in Beghini et al. (2013), after the preliminary analysis based on the dominant load case in the conceptual phase of the design, the design engineer needs to consider all the possible load combinations that the structure might be subjected to, including pattern (asymmetric) loads. Such additional load combinations will require upsizing some of the members which were relatively small. Additional members may also be required to ensure proper redundancy in the structural load paths. Consequently, the overall volume will increase from the baseline minimum. However, if there is a predominant load case, such increase is generally small, indicating that once the overall geometry of the structure has been optimized for the governing load case, the other load cases do not have a major impact on the structural efficiency, both in terms of material volume and deflections.

5 Summary, conclusions and future work

Graphic Statics provides another tool in determining discrete minimal load path structures. The method optimizes structure by using design variables in the force domain rather than manipulating the geometry of the structure, as is done in other methods. It has great advantages in the optimization of potentially “unstable” discrete trusses because the solutions are constrained to be in equilibrium by the fact that the force diagrams have closed polygons. The method often reduces the number of design variables because of the restrictions of equilibrium and reciprocity with the form diagram. The method can also find minimum load path structures where the point of application of the

Table 2 Optimization of the roof truss

Description	Form Diagram	Force Diagram	Normalized Volume
(a) Initial truss connectivity			1.000
(b) Benchmark truss; top chord, cantilevers and web members unconstrained			0.552
(c) Chord profiles constrained for architectural and functional reasons			0.629
(d) Depth constrained, straight web members (X-diagonals)			0.852
(e) Depth constrained			0.669

loads or the supports can change as the optimal solution is determined.

The authors are currently exploring other applications where the design variables are in the force domain.

References

Altair Engineering Web Page (2013) www.altair.com

Achtziger W (1997) Topology optimization of discrete structures: an introduction in view of computational and nonsmooth aspects. In: Topology optimization in structural mechanics, pp 57–100

Achtziger W, Bendsoe MP, Ben-Tal A, Zowe J (1992) Equivalent displacement based formulations for maximum strength truss topology design. IMPACT Comput Sci Eng 4(4):315–345

AISC (2010) ANSI/AISC 360-10 specification for structural steel buildings, pp 16.1-33

Baker WF, Beghini LL, Mazurek A, Carrion J, Beghini A (2013) Maxwell's reciprocal diagrams and discrete Michell frames. Struct Multidisc Optim. doi:10.1007/s00158-013-0910-0

Beghini A, Beghini L, Carrion J, Mazurek A, Baker WF (2013) Rankine's theorem for the design of cable structures. Struct Multidisc Optim. doi:10.1007/s00158-013-0945-2

Ben-Tal A, Bendsoe MP (1993) A new method for optimal truss topology design. SIAM J Optim 3(2):322–358

Bendsoe MP (1989) Optimal shape design as a material distribution problem. Struct Optim 1(4):193–202

Bendsoe MP, Kikuchi N (1988) Generating optimal topologies in structural design using a homogenization method. Comput Methods Appl Mech Eng 71(2):197–224

Bendsoe MP, Sigmund O (1999) Material interpolation schemes in topology optimization. Arch Appl Mech 69(9–10):635–654

Bendsoe MP, Sigmund O (2002) Topology optimization: theory, methods and applications. Springer, Berlin

Bow R (1873) Economics of construction in relation to framed structures. ICE Publishing, London

Christensen P, Klarbring A (2008) An introduction to structural optimization. Springer, New York. ISBN 978-1402086656

Cremona L (1872) Le figure reciproche nella grafica statica. Tipografia Giuseppe Bernardoni, Milano

Dorn W, Gomory R, Greenberg M (1964) Automatic design of optimal structures. J Mech 3:25–52

Fivet C, Zastavni D (2012) Robert Maillart's key methods from the Salginatobel bridge design process (1928). J IASS 53(1):39–47

Fleron P (1964) The minimum weight of trusses. Bygningsstatiske Meddelelser 35:81–96

Hansen S, Vanderplaats G (1988) An approximation method for configuration optimization of trusses. AIAA J 28(1):161–168

Hemp WS (1973) Optimum structures. Clarendon Press, Oxford

Hemp W (1974) Michell framework for uniform load between fixed supports. Eng Optim 1(1):61–69

Krog L, Tucker A, Kemp M, Boyd R (2004) Topology optimization of aircraft wing box ribs. In: 10th AIAA/ISSMO multidisciplinary analysis and optimization conference, pp 2004–4481

Maxwell JC (1864) On reciprocal figures and diagrams of forces. Phil Mag 26:250–261

Maxwell JC (1870) On reciprocal figures, frames, and diagrams of forces. Edinb Roy Soc Proc 7:160–208

Mazurek A, Baker WF, Tort C (2011) Geometrical aspects of optimum truss like structures. Struct Multidisc Optim 43(2):231–242

McConnel RE (1974) Least weight frameworks for loads across a span. J Eng Mech Div 100(5):885–891

Michell AGM (1904) The limits of economy of material in frame-structures. Phil Mag 8(47):589–597

Oberndorfer JM, Achtziger W, Hörnlein H (1996) Two approaches for truss topology optimization: a comparison for practical use. Struct Multidisc Optim 11(3):137–144

Pichugin AV, Tyas A, Gilbert M (2012) On the optimality of Hemps arch with vertical hangers. Struct Multidiscipl Optim 46(1):17–25. doi:10.1007/s00158-012-0769-5

Prager W (1970) Optimization of structural design. J Optim Theory Appl 6(I):1–21

Prager W (1978) Optimal layout of trusses with finite numbers of joints. J Mech Phys Solids 26:241–250

Rahami H, Kaveh A, Gholipour Y (2008) Sizing, geometry and topology optimization of trusses via force method and genetic algorithm. Eng Struct 30(9):2360–2369

Rozvany GIN (1976) Optimal design of flexural systems. Pergamon Press, Oxford

Rozvany GIN (1989) Structural design via optimality criteria. Kluwer Academic Publishers Group, Dordrecht

Rozvany GIN (1996) Some shortcomings in Michell's truss theory. Struct Multidisc Optim 12(4):244–250

Rozvany GIN, Zhou M, Birker T (1992) Generalized shape optimization without homogenization. Struct Multidisc Optim 4(3):250–252

Rozvany GIN, Bendsoe MP, Kirsch U (1995) Layout optimization of structures. Appl Mech Rev 48(2):41–119

Sigmund O (2000) Topology optimization: a tool for the tailoring of structures and materials. Philos T Roy Soc A 358(1765):211–227

Sokol T (2010) A 99 line code for discretized Michell truss optimization written in Mathematica. Struct Multidisc Optim 43(2):181–190

Stromberg LL, Beghini A, Baker WF, Paulino GH (2012) Topology optimization for braced frames: combining continuum and beam/column elements. Eng Struct 37:106–124

Sutradhar A, Paulino GH, Miller MJ, Nguyen TH (2010) Topological optimization for designing patient-specific large craniofacial segmental bone replacements. Proc Natl Acad Sci USA 107(30):13,222–13,227

Svanberg K (1987) The method of moving asymptotes—a new method for structural optimization. Int J Numer Methods Eng 24:359–373

Svanberg K (2013) Personal webpage of Krister Svanberg, Professor. http://www.math.kth.se/krille/

Talischi C, Paulino GH, Pereira A, Menezes IFM (2012a) PolyMesher: a general-purpose mesh generator for polygonal elements written in Matlab. Struct Multidisc Optim 45(3):309–328

Talischi C, Paulino GH, Pereira A, Menezes IFM (2012b) PolyTop: a Matlab implementation of a general topology optimization framework using unstructured polygonal finite element meshes. Struct Multidisc Optim 45(3):329–357

Wolfe W (1921) Graphical analysis. McGraw-Hill Book Co. Inc., New York

Zalewski W, Allen E (1998) Shaping structures: statics. Wiley, New York

Zastavni D (2008) The structural design of Maillart's Chiasso Shed (1924): a graphic procedure. Struct Eng Int J Int Assoc Bridge Struct Eng 18(3):247–252

Zhou M, Rozvany GIN (1991) The COC algorithm, Part II: topological, geometrical and generalized shape optimization. Comput Methods Appl Mech Eng 89(1–3):309–336

Zhou M, Rozvany GIN (1993) DCOC: an optimality criteria method for large systems Part II: algorithm. Struct Multidisc Optim 6(4):250–262