Finite Element Analysis of 2 D Heat Equation

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For Computational Biomechanics Course

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In[1]:=
```
ClearAll["Global`*"];

(* Basic FE model data *)
nnp = 9;
numel = 4;
ndm = 2;
ndf = 1 ;
nel = 4;

(* Shape functions for Q4 element *)
N1 = (1 - ξ) (1 - η) / 4 ;
N2 = (1 + ξ) (1 - η) / 4 ;
N3 = (1 + ξ) (1 + η) / 4 ;
N4 = (1 - ξ) (1 + η) / 4 ;

(* Materail parameters *)
 kappa = 2.0 ;

(* Geometrical data *)
e1xy = {{0, 0}, {1, 0}, {2, 0}, {0, 1}, {1, 1}, {2, 1}, {0, 2}, {1, 2}, {2, 2}};
ld = {{1, 2, 5, 4} , {2, 3, 6, 5} , {4, 5, 8, 7} , {5, 6, 9, 8}} ;
elh = 1.0 ;
elh2 = elh / 2.0 ;

(* Boundary conditions *)
ubce = {1 , 3};
ubcn = {1 , 4, 7};
fbce = {2, 4};
fbcn = {3, 6, 9};
qhat = 5;
uhat = 20;
uhatn = {uhat, uhat, uhat};


(* Initialization of element and global stiffness matrices *)
ke = ConstantArray[0.0, {nel * ndf, nel * ndf, numel}] ;
Ke = ConstantArray[0.0, {nnp * ndf, nnp * ndf, numel}] ;
Kmat = ConstantArray[0.0, {nnp * ndf, nnp * ndf}] ;
Kmr = ConstantArray[0.0, {nnp * ndf - Length@ubcn * ndf, nnp * ndf - Length@ubcn * ndf}] ;

 (* Initialization of matrices for load vector *)
fed = ConstantArray[0.0, {nel * ndf, numel}] ;
feb = ConstantArray[0.0, {nel * ndf, numel}] ;
Fe  = ConstantArray[0.0, {nnp * ndf,  numel}] ;
Fmat = ConstantArray[0.0, {nnp * ndf}] ;
Fmr = ConstantArray[0.0, {nnp * ndf - Length@ubcn * ndf}] ;

 (* Initialization of solution matrices *)
Usol = ConstantArray[0.0, {nnp * ndf,  ndm + 1}] ;
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noubcn = Complement[Range[Length[Usol]], ubcn] ;


Print["*****************************************************************"] ;
Print["                              Program Output                           "] ;
Print["*****************************************************************"] ;

(* Loop through all the elements to compute the element stiffness matrix *)
Do[
  e = ii;

  x1 = e1xy[[ld[[e, 1]], 1]];  y1 = e1xy[[ld[[e, 1]], 2]];
  x2 = e1xy[[ld[[e, 2]], 1]];  y2 = e1xy[[ld[[e, 2]], 2]];
  x3 = e1xy[[ld[[e, 3]], 1]];  y3 = e1xy[[ld[[e, 3]], 2]];
  x4 = e1xy[[ld[[e, 4]], 1]];  y4 = e1xy[[ld[[e, 4]], 2]];

  Print["---------------------------"] ;
  Print["Processing element ", e, ":"] ;
  Print["---------------------------"] ;

  Print@ListLinePlot[{{x1, y1}, {x2, y2}, {x3, y3}, {x4, y4}, {x1, y1}},
    Filling → Bottom, AspectRatio → 1, PlotMarkers → Automatic,
    PlotRange → {{0, 2.1}, {0, 2.1}}, ImageSize → Small] ;

  x = Simplify[N1 * x1 + N2 * x2 + N3 * x3 + N4 * x4] ;
  y = Simplify[N1 * y1 + N2 * y2 + N3 * y3 + N4 * y4] ;

  Jmat = {{D[x, ξ], D[x, η]}, {D[y, ξ], D[y, η]}} ;

  Jac = Det[Jmat] ;
  Jinv = Inverse[Jmat] ;

  dξdx = Jinv[[1, 1]];
  dξdy = Jinv[[1, 2]];
  dηdx = Jinv[[2, 1]];
  dηdy = Jinv[[2, 2]];

  dN1dx = D[N1, ξ] dξdx + D[N1, η] dηdx;
  dN1dy = D[N1, ξ] dξdy + D[N1, η] dηdy;

  dN2dx = D[N2, ξ] dξdx + D[N2, η] dηdx;
  dN2dy = D[N2, ξ] dξdy + D[N2, η] dηdy;

  dN3dx = D[N3, ξ] dξdx + D[N3, η] dηdx;
  dN3dy = D[N3, ξ] dξdy + D[N3, η] dηdy;

  dN4dx = D[N4, ξ] dξdx + D[N4, η] dηdx;
  dN4dy = D[N4, ξ] dξdy + D[N4, η] dηdy;


  Bmat = {{dN1dx, dN2dx, dN3dx, dN4dx}, {dN1dy, dN2dy, dN3dy, dN4dy}} ;

  Print["Creating the element B matrix... \n", (Bmat) // MatrixForm] ;

  intg = (Transpose[Bmat].Bmat) * Jac ;
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gp1 = -1/Sqrt[3];
gp2 = 1/Sqrt[3];
w1 = 1.0;
w2 = 1.0;
w3 = 1.0;
w4 = 1.0;

(* Computing the element stiffness matrix ke(4×4) *)

ke[[All, All, e]] =
 kappa * (intg * w1 /. {ξ → gp1, η → gp1}) + kappa * (intg * w2 /. {ξ → gp2, η → gp1}) +
  kappa * (intg * w3 /. {ξ → gp2, η → gp2}) + kappa * (intg * w4 /. {ξ → gp1, η → gp2});

Print["Creating the element stiffness matrix ke(4×4)... \n",
 (ke[[All, All, e]] ) // MatrixForm] ;

(* Computing matrices for load vector *)
intfed = (2.0 * x + y^2) * {N1, N2, N3, N4} * Jac ;

fed[[All, e]] = (intfed * w1 /. {ξ → gp1, η → gp1}) + (intfed * w2 /. {ξ → gp2, η → gp1}) +
  (intfed * w3 /. {ξ → gp2, η → gp2}) + (intfed * w4 /. {ξ → gp1, η → gp2});

intfeb = {N2, N3};
feb[[2 ;; 3, e]] =
 If[fbce[[1]] == e || fbce[[2]] == e, qhat * elh2 * (intfeb * w1 /. {ξ → 1.0, η → gp1}) +
   qhat * elh2 * (intfeb * w2 /. {ξ → 1.0, η → gp2}), 0.0];



Do[
 Do[
  Ke[[ld[[e, i]], ld[[e, j]], e]] = Ke[[ld[[e, i]], ld[[e, j]], e]] + ke[[i, j, e]],
  {j, Length@ke}];
 Fe[[ld[[e, i]], e]] = Fe[[ld[[e, i]], e]] + fed[[i, e]] - feb[[i, e]],
 {i, Length@ke}] ;

Print["Convert to the element stiffness matrix Ke(9×9) \n",
 (Ke[[All, All, e]] ) // MatrixForm] ;
Print["Element load vector Fe(9×1) \n", (Fe[[All, e]] ) // MatrixForm] ,

{ii, 1, numel}];


Print["-----------------------------------------"] ;
Print["Assemble the Global stiffness matrix K(9×9): "] ;
Print["-----------------------------------------"] ;

Do[
 Kmat[[All, All]] = Kmat[[All, All]] + Ke[[All, All, i]],
 {i, 1, numel}] ;

Print[ (Kmat) // MatrixForm]

Print["-----------------------------------------"] ;
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Print["Assemble the load vector F(9×1): "];
Print["------------------------------------------"];

Do[
  Fmat[[All]] = Fmat[[All]] + Fe[[All, i]],
  {i, 1, numel}];

Print[(Fmat) // MatrixForm]

(*adding the inhomogenous Dirichlet BC : *)
Do[
  Fmat[[noubcn[[i]]]] = Fmat[[noubcn[[i]]]] - {Kmat[[noubcn[[i]], ubcn[[1]]]],
      Kmat[[noubcn[[i]], ubcn[[2]]]], Kmat[[noubcn[[i]], ubcn[[3]]]]}.uhatn,
  {i, 1, Length@noubcn}];


Print["-----------------------------------------------"];
Print["Reduction of stiffness matrix: "];
Print["-----------------------------------------------"];

removem[a_?MatrixQ, pos_List] := Module[{tmp, length = Length[a]},
   tmp = Complement[Range[length], pos];
   a[[tmp, tmp]]];

Kmr = removem[Kmat, ubcn];
Print[(Kmr) // MatrixForm]

Print["-----------------------------------------------"];
Print["Reduction of the load vector: "];
Print["-----------------------------------------------"];

removev[a_?VectorQ, pos_List] := Module[{tmp, length = Length[a]},
   tmp = Complement[Range[length], pos];
   a[[tmp]]];

Fmr = removev[Fmat, ubcn];

Print[(Fmr) // MatrixForm]

Print["-----------------------------------------------------"];
Print["Solving the linear system of equations to find u:  "];
Print["-----------------------------------------------------"];

Umat = LinearSolve[Kmr, Fmr];

Do[
  Usol[[i, 1 ;; 2]] = Usol[[i, 1 ;; 2]] + e1xy[[i, All]],
  {i, 1, nnp * ndf}];

(* add the computed u to the full-length solution array Usol *)
noubcn = Complement[Range[Length[Usol]], ubcn];

Do[
  Usol[[noubcn[[i]], 3]] = Usol[[noubcn[[i]], 3]] + Umat[[i]],
  {i, 1, Length@noubcn}];

(* add the BC to the solution array *)
Do[
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  Usol[[ubcn[[i]], 3]] = Usol[[ubcn[[i]], 3]] + uhatn[[i]],
  {i, 1, Length@ubcn}] ;

Print[(Usol) // MatrixForm]

fig2 = ListDensityPlot[Usol, InterpolationOrder → 1, ImageSize → Medium,
   ColorFunction → "Rainbow", Mesh → 11, MeshFunctions -> {#3 &},
   BaseStyle → {FontWeight → "Normal", FontSize → 18}, PlotLegends →
    BarLegend[Automatic, LabelStyle → {Black, FontFamily → "STIX", FontSize → 18}],
   LabelStyle → Directive[Black, FontSize → 18, FontFamily → "STIX"],
   PlotRangePadding → Scaled[.0]] ;

Show[fig2, PlotRange → {{0, 2}, {0, 2}}, PlotRangePadding → Scaled[.00]]

Print["-------------------- END OF THE PROGRAM OUTPUT --------------------\n"] ;
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*********************************************************************
                          Program Output
*********************************************************************
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Processing element 1:
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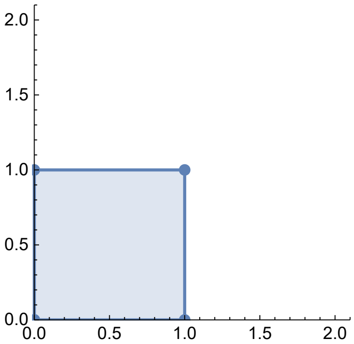


Creating the element B matrix...

$$\begin{pmatrix} \frac{1}{2}(-1+\eta) & \frac{1-\eta}{2} & \frac{1+\eta}{2} & \frac{1}{2}(-1-\eta) \\ \frac{1}{2}(-1+\xi) & \frac{1}{2}(-1-\xi) & \frac{1+\xi}{2} & \frac{1-\xi}{2} \end{pmatrix}$$

Creating the element stiffness matrix ke(4×4)...

$$\begin{pmatrix} 1.33333 & -0.333333 & -0.666667 & -0.333333 \\ -0.333333 & 1.33333 & -0.333333 & -0.666667 \\ -0.666667 & -0.333333 & 1.33333 & -0.333333 \\ -0.333333 & -0.666667 & -0.333333 & 1.33333 \end{pmatrix}$$

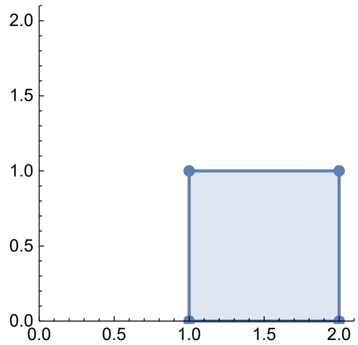
Convert to the element stiffness matrix Ke(9×9)

$$\begin{pmatrix} 1.33333 & -0.333333 & 0. & -0.333333 & -0.666667 & 0. & 0. & 0. & 0. \\ -0.333333 & 1.33333 & 0. & -0.666667 & -0.333333 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ -0.333333 & -0.666667 & 0. & 1.33333 & -0.333333 & 0. & 0. & 0. & 0. \\ -0.666667 & -0.333333 & 0. & -0.333333 & 1.33333 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}$$

Element load vector Fe(9×1)

$$\begin{pmatrix} 0.208333 \\ 0.375 \\ 0. \\ 0.291667 \\ 0.458333 \\ 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

Processing element 2:



Creating the element B matrix...

$$\begin{pmatrix} \frac{1}{2}(-1+\eta) & \frac{1-\eta}{2} & \frac{1+\eta}{2} & \frac{1}{2}(-1-\eta) \\ \frac{1}{2}(-1+\xi) & \frac{1}{2}(-1-\xi) & \frac{1+\xi}{2} & \frac{1-\xi}{2} \end{pmatrix}$$

Creating the element stiffness matrix ke(4×4)...

$$\begin{pmatrix} 1.33333 & -0.333333 & -0.666667 & -0.333333 \\ -0.333333 & 1.33333 & -0.333333 & -0.666667 \\ -0.666667 & -0.333333 & 1.33333 & -0.333333 \\ -0.333333 & -0.666667 & -0.333333 & 1.33333 \end{pmatrix}$$

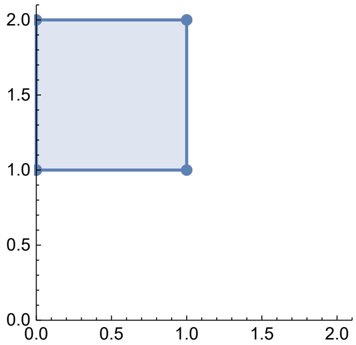
Convert to the element stiffness matrix Ke(9×9)

$$\begin{pmatrix} 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 1.33333 & -0.333333 & 0. & -0.333333 & -0.666667 & 0. & 0. & 0. \\ 0. & -0.333333 & 1.33333 & 0. & -0.666667 & -0.333333 & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & -0.333333 & -0.666667 & 0. & 1.33333 & -0.333333 & 0. & 0. & 0. \\ 0. & -0.666667 & -0.333333 & 0. & -0.333333 & 1.33333 & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}$$

Element load vector Fe(9×1)

$$\begin{pmatrix} 0. \\ 0.708333 \\ -1.625 \\ 0. \\ 0.791667 \\ -1.54167 \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

Processing element 3:



Creating the element B matrix...

$$\begin{pmatrix} \frac{1}{2}(-1+\eta) & \frac{1-\eta}{2} & \frac{1+\eta}{2} & \frac{1}{2}(-1-\eta) \\ \frac{1}{2}(-1+\xi) & \frac{1}{2}(-1-\xi) & \frac{1+\xi}{2} & \frac{1-\xi}{2} \end{pmatrix}$$

Creating the element stiffness matrix ke(4×4)...

$$\begin{pmatrix} 1.33333 & -0.333333 & -0.666667 & -0.333333 \\ -0.333333 & 1.33333 & -0.333333 & -0.666667 \\ -0.666667 & -0.333333 & 1.33333 & -0.333333 \\ -0.333333 & -0.666667 & -0.333333 & 1.33333 \end{pmatrix}$$

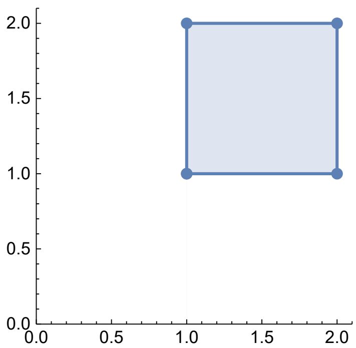
Convert to the element stiffness matrix Ke(9×9)

$$\begin{pmatrix} 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 1.33333 & -0.333333 & 0. & -0.333333 & -0.666667 & 0. \\ 0. & 0. & 0. & -0.333333 & 1.33333 & 0. & -0.666667 & -0.333333 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & -0.333333 & -0.666667 & 0. & 1.33333 & -0.333333 & 0. \\ 0. & 0. & 0. & -0.666667 & -0.333333 & 0. & -0.333333 & 1.33333 & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}$$

Element load vector Fe(9×1)

$$\begin{pmatrix} 0. \\ 0. \\ 0. \\ 0.625 \\ 0.791667 \\ 0. \\ 0.875 \\ 1.04167 \\ 0. \end{pmatrix}$$

Processing element 4:



Creating the element B matrix...

$$\begin{pmatrix} \frac{1}{2}(-1+\eta) & \frac{1-\eta}{2} & \frac{1+\eta}{2} & \frac{1}{2}(-1-\eta) \\ \frac{1}{2}(-1+\xi) & \frac{1}{2}(-1-\xi) & \frac{1+\xi}{2} & \frac{1-\xi}{2} \end{pmatrix}$$

Creating the element stiffness matrix ke(4×4)...

$$\begin{pmatrix} 1.33333 & -0.333333 & -0.666667 & -0.333333 \\ -0.333333 & 1.33333 & -0.333333 & -0.666667 \\ -0.666667 & -0.333333 & 1.33333 & -0.333333 \\ -0.333333 & -0.666667 & -0.333333 & 1.33333 \end{pmatrix}$$

Convert to the element stiffness matrix Ke(9×9)

$$\begin{pmatrix} 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 1.33333 & -0.333333 & 0. & -0.333333 & -0.666667 \\ 0. & 0. & 0. & 0. & -0.333333 & 1.33333 & 0. & -0.666667 & -0.333333 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & -0.333333 & -0.666667 & 0. & 1.33333 & -0.333333 \\ 0. & 0. & 0. & 0. & -0.666667 & -0.333333 & 0. & -0.333333 & 1.33333 \end{pmatrix}$$

Element load vector Fe(9×1)

$$\begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 1.125 \\ -1.20833 \\ 0. \\ 1.375 \\ -0.958333 \end{pmatrix}$$

Assemble the Global stiffness matrix K(9×9):

$$\begin{pmatrix} 1.33333 & -0.333333 & 0. & -0.333333 & -0.666667 & 0. & 0. & 0. & 0. \\ -0.333333 & 2.66667 & -0.333333 & -0.666667 & -0.666667 & -0.666667 & 0. & 0. & 0. \\ 0. & -0.333333 & 1.33333 & 0. & -0.666667 & -0.333333 & 0. & 0. & 0. \\ -0.333333 & -0.666667 & 0. & 2.66667 & -0.666667 & 0. & -0.333333 & -0.666667 & 0. \\ -0.666667 & -0.666667 & -0.666667 & -0.666667 & 5.33333 & -0.666667 & -0.666667 & -0.666667 & -0.666667 \\ 0. & -0.666667 & -0.333333 & 0. & -0.666667 & 2.66667 & 0. & -0.666667 & -0.333333 \\ 0. & 0. & 0. & -0.333333 & -0.666667 & 0. & 1.33333 & -0.333333 & 0. \\ 0. & 0. & 0. & -0.666667 & -0.666667 & -0.666667 & -0.333333 & 2.66667 & -0.333333 \\ 0. & 0. & 0. & 0. & -0.666667 & -0.333333 & 0. & -0.333333 & 1.33333 \end{pmatrix}$$

Assemble the load vector F(9×1):

$$\begin{pmatrix} 0.208333 \\ 1.08333 \\ -1.625 \\ 0.916667 \\ 3.16667 \\ -2.75 \\ 0.875 \\ 2.41667 \\ -0.958333 \end{pmatrix}$$

Reduction of stiffness matrix:

$$\begin{pmatrix} 2.66667 & -0.333333 & -0.666667 & -0.666667 & 0. & 0. \\ -0.333333 & 1.33333 & -0.666667 & -0.333333 & 0. & 0. \\ -0.666667 & -0.666667 & 5.33333 & -0.666667 & -0.666667 & -0.666667 \\ -0.666667 & -0.333333 & -0.666667 & 2.66667 & -0.666667 & -0.333333 \\ 0. & 0. & -0.666667 & -0.666667 & 2.66667 & -0.333333 \\ 0. & 0. & -0.666667 & -0.333333 & -0.333333 & 1.33333 \end{pmatrix}$$

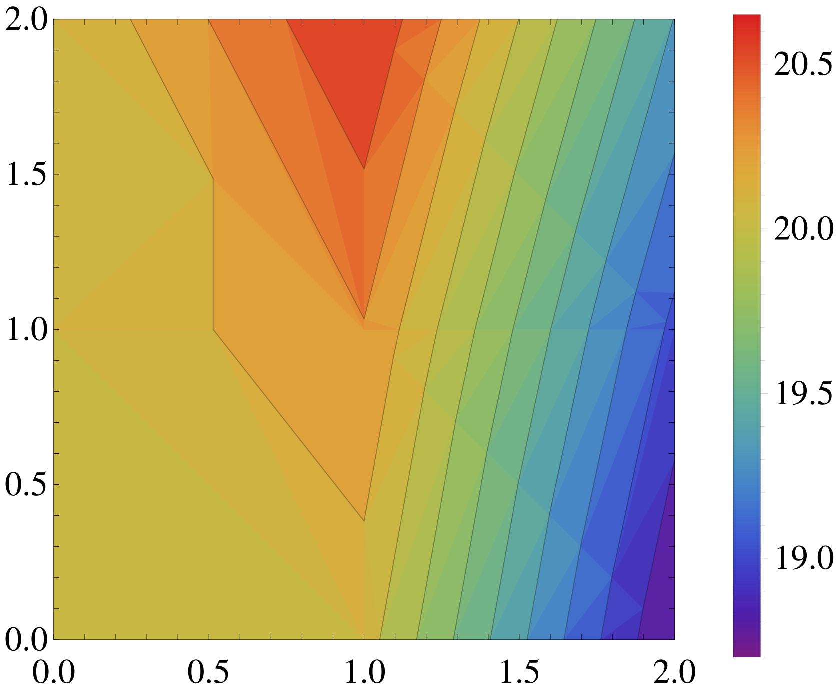
Reduction of the load vector:

$$\begin{pmatrix} 21.0833 \\ -1.625 \\ 43.1667 \\ -2.75 \\ 22.4167 \\ -0.958333 \end{pmatrix}$$

Solving the linear system of equations to find u:

$$\begin{pmatrix} 0. & 0. & 20. \\ 1. & 0. & 20.066 \\ 2. & 0. & 18.6978 \\ 0. & 1. & 20. \\ 1. & 1. & 20.3104 \\ 2. & 1. & 18.9796 \\ 0. & 2. & 20. \\ 1. & 2. & 20.6466 \\ 2. & 2. & 19.343 \end{pmatrix}$$

Out[71]=



-------------------- END OF THE PROGRAM OUTPUT --------------------