Gaussian White Noise Model Estimation

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Estimating Parameters of GWN model

Parameters of GWN Model

$$\mu_i = E[R_{it}]$$
$$\sigma_i^2 = \mathrm{var}(R_{it})$$
$$\sigma_{ij} = \mathrm{cov}(R_{it}, R_{jt})$$
$$\rho_{ij} = \mathrm{cor}(R_{it}, R_{jt})$$

are not known with certainty

First Econometric Task

- Estimate $\mu_i,\ \sigma_i^2,\ \sigma_{ij},\ \rho_{ij}$ using observed sample of historical monthly returns

Estimators and Estimates

Definition: An estimator is a rule or algorithm (mathematical formula) for computing an *ex ante* estimate of a parameter based on a random sample.

Example: Sample mean as estimator of $E[R_{it}] = \mu_i$

$$\begin{aligned}\{R_{i1}, \ldots, R_{iT}\} &= \text{ covariance stationary time series} \\ &= \text{ collection of random variables} \\ \hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} R_{it} &= \text{ sample mean} \\ &= \text{ random variable}\end{aligned}$$

Estimators and Estimates

Definition: An estimate of a parameter is simply the *ex post* value (numerical value) of an estimator based on observed data

Example: Sample mean from an observed sample

$$
\begin{aligned}
\{R_{i1} = .02, R_{i2} = .01, R_{i3} = -.01, \ldots, R_{iT} = .03\} &= \text{observed sample} \\
\hat{\mu}_i &= \frac{1}{T}(.02 + .01 - .01 + \cdots + .03) \\
&= \text{ number } = 0.01 \text{ (say)}
\end{aligned}
$$

Estimators of GWN Model Parameters

Plug-in principle: Estimate model parameters using appropriate sample statistics

$$\mu_i = E[R_{it}] : \hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} R_{it}$$

$$\sigma_i^2 = E[(R_{it} - \mu_i)^2] : \hat{\sigma}_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}(R_{it} - \hat{\mu}_i)^2$$

$$\sigma_i = \sqrt{\sigma_i^2} : \hat{\sigma}_i = \sqrt{\hat{\sigma}_i^2}$$

$$\sigma_{ij} = E[(R_{it} - \mu_i)(R_{jt} - \mu_j)] : \hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(R_{it} - \hat{\mu}_i)(R_{jt} - \hat{\mu}_j)$$

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} : \hat{\rho}_{ij} = \frac{\hat{\sigma}_{ij}}{\hat{\sigma}_i \cdot \hat{\sigma}_j}$$

Example: GWN Model Estimates for MSFT, SBUX and SP500

Example data: monthly cc returns on MSFT, SBUX and SP500 from Jan 1998 - May 2012

```
head(gwnRetS, n=3)
```

```
##             MSFT   SBUX   SP500
## Feb 1998 0.13640 0.0825 0.07045
## Mar 1998 0.05570 0.1438 0.04995
## Apr 1998 0.00691 0.0629 0.00908
```

Example: GWN Model Estimates for MSFT, SBUX and SP500

Estimate of μ

```
muhat = apply(gwnRetC,2,mean)
muhat
```

```
##    MSFT    SBUX   SP500
## 0.00413 0.01466 0.00169
```

Estimate of σ^2

```
sigma2hat = apply(gwnRetC,2,var)
sigma2hat
```

```
##    MSFT    SBUX   SP500
## 0.01004 0.01246 0.00235
```

Example: GWN Model Estimates for MSFT, SBUX and SP500

Estimate of σ

```
sigmahat = apply(gwnRetC,2,sd)
sigmahat
```

```
##   MSFT   SBUX  SP500
## 0.1002 0.1116 0.0485
```

Estimate of Σ

```
covmat = var(gwnRetC)
covmat
```

```
##          MSFT    SBUX   SP500
## MSFT  0.01004 0.00381 0.00300
## SBUX  0.00381 0.01246 0.00248
## SP500 0.00300 0.00248 0.00235
```

Example: GWN Model Estimates for MSFT, SBUX and SP500

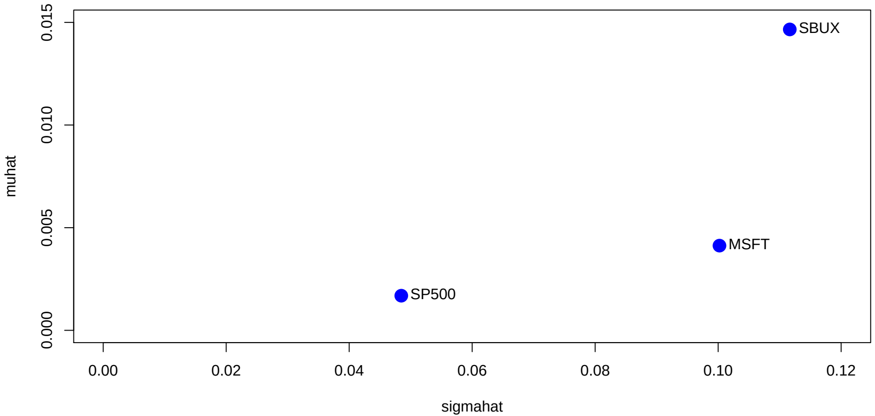
Estimate of **R**

```
cormat = cor(gwnRetC)
cormat
```

```
##        MSFT  SBUX SP500
## MSFT  1.000 0.341 0.617
## SBUX  0.341 1.000 0.457
## SP500 0.617 0.457 1.000
```

Example: GWN Model Estimates for MSFT, SBUX and SP500

Estimated mean-volatility tradeoff



Properties of Estimators

$$\theta = \text{parameter to be estimated}$$

$$\hat{\theta} = \text{estimator of } \theta \text{ from random sample}$$

- $\hat{\theta}$ is a random variable – its value depends on realized values of random sample
- $f(\hat{\theta}) =$ pdf of $\hat{\theta}$ - depends on pdf of random variables in random sample
- Properties of $\hat{\theta}$ can be derived analytically (using probability theory) or by using Monte Carlo simulation

Estimation Error

$$\mathit{error}(\hat{\theta}, \theta) = \hat{\theta} - \theta$$

Bias

$$\mathrm{bias}(\hat{\theta}, \theta) = E\left[\mathit{error}(\hat{\theta}, \theta)\right] = E\left[\hat{\theta}\right] - \theta$$

$$\hat{\theta} \text{ is unbiased if } E[\hat{\theta}] = \theta \Rightarrow \mathrm{bias}(\hat{\theta}, \theta) = 0$$

Remark: An unbiased estimator is *on average* correct, where *on average* means over many hypothetical samples. It most surely will not be exactly correct for the sample at hand!

Precision

$$
\begin{aligned}
mse(\hat{\theta},\theta) &= E\left[error(\hat{\theta},\theta)^2\right] = E\left[\left(\hat{\theta}-\theta\right)^2\right] \\
&= \mathrm{bias}(\hat{\theta},\theta)^2 + \mathrm{var}(\hat{\theta}) \\
\mathrm{var}(\hat{\theta}) &= E[(\hat{\theta}-E[\hat{\theta}])^2
\end{aligned}
$$

Remark: If $\mathrm{bias}(\hat{\theta},\theta) \approx 0$ then precision is typically measured by the *standard error* of $\hat{\theta}$ defined by

$$
\begin{aligned}
\mathrm{SE}(\hat{\theta}) &= \text{ standard error of } \hat{\theta} \\
&= \sqrt{\mathrm{var}(\hat{\theta})} = \sqrt{E[(\hat{\theta}-E[\hat{\theta}])^2]} \\
&= \sigma_{\hat{\theta}}
\end{aligned}
$$

Bias of GWN Model Estimates

- $\hat{\mu}_i, \hat{\sigma}_i^2$ and $\hat{\sigma}_{ij}$ are unbiased estimators:

$$
\begin{aligned}
E\left[\hat{\mu}_i\right] &= \mu_i \Rightarrow \mathrm{bias}(\hat{\mu}_i, \mu_i) = 0 \\
E\left[\hat{\sigma}_i^2\right] &= \sigma_i^2 \Rightarrow \mathrm{bias}(\hat{\sigma}_i^2, \sigma_i^2) = 0 \\
E\left[\hat{\sigma}_{ij}\right] &= \sigma_{ij} \Rightarrow \mathrm{bias}(\hat{\sigma}_{ij}, \sigma_{ij}) = 0
\end{aligned}
$$

- $\hat{\sigma}_i$ and $\hat{\rho}_{ij}$ are biased estimators

$$
\begin{aligned}
E[\hat{\sigma}_i] &\neq \sigma_i \Rightarrow \mathrm{bias}(\hat{\sigma}_i, \sigma_i) \neq 0 \\
E[\hat{\rho}_{ij}] &\neq \rho_{ij} \Rightarrow \mathrm{bias}(\hat{\rho}_{ij}, \rho_{ij}) \neq 0
\end{aligned}
$$

but bias is very small except for very small samples and disappears as sample size T gets large.

Bias of GWN Model Estimates

Remarks

- *On average* being correct doesn't mean the estimate is any good for your sample!
- The value of $\mathrm{SE}(\hat{\theta})$ will tell you how far from θ the estimate $\hat{\theta}$ typically will be.
- Good estimators $\hat{\theta}$ have small bias and small $\mathrm{SE}(\hat{\theta})$

Proof that $E\left[\hat{\mu}_i\right] = \mu_i$

Standard Error Formulas

$$
\begin{aligned}
\mathrm{SE}(\hat{\mu}_i) &= \frac{\sigma_i}{\sqrt{T}} \\
\mathrm{SE}(\hat{\sigma}_i^2) &\approx \frac{\sigma_i^2}{\sqrt{T/2}} \\
\mathrm{SE}(\hat{\sigma}_i) &\approx \frac{\sigma_i}{\sqrt{2T}} \\
\mathrm{SE}(\hat{\sigma}_{ij}) &: \text{ no easy formula!} \\
\mathrm{SE}(\hat{\rho}_{ij}) &\approx \frac{(1-\rho_{ij}^2)}{\sqrt{T}}
\end{aligned}
$$

Note: "$\approx$" denotes "approximately equal to", where approximation error $\longrightarrow 0$ as $T \longrightarrow \infty$ for normally distributed data.

Standard Error Formulas

Remarks

- Large $\mathrm{SE} \Longrightarrow$ imprecise estimate; Small $\mathrm{SE} \Longrightarrow$ precise estimate
- Precision increases with sample size: $\mathrm{SE} \longrightarrow 0$ as $T \longrightarrow \infty$
- $\hat{\sigma}_i$ is generally a more precise estimate than $\hat{\mu}_i$ or $\hat{\rho}_{ij}$
- SE formulas for $\hat{\sigma}_i$ and $\hat{\rho}_{ij}$ are approximations based on the Central Limit Theorem. Monte Carlo simulation and bootstrapping can be used to get better approximations
- SE formulas depend on unknown values of parameters $\Rightarrow$ formulas are not practically useful

Standard Error Formulas

Practically useful formulas replace unknown values with estimated values:

$$\widehat{\mathrm{SE}}(\hat{\mu}_i) = \frac{\hat{\sigma}_i}{\sqrt{T}},\ \hat{\sigma}_i \text{ replaces } \sigma_i$$

$$\widehat{\mathrm{SE}}(\hat{\sigma}_i^2) \approx \frac{\hat{\sigma}_i^2}{\sqrt{T/2}},\ \hat{\sigma}_i^2 \text{ replaces } \sigma_i^2$$

$$\widehat{\mathrm{SE}}(\hat{\sigma}_i) \approx \frac{\hat{\sigma}_i}{\sqrt{2T}},\ \hat{\sigma}_i \text{ replaces } \sigma_i$$

$$\widehat{\mathrm{SE}}(\hat{\rho}_{ij}) \approx \frac{(1-\hat{\rho}_{ij}^2)}{\sqrt{T}},\ \hat{\rho}_{ij} \text{ replaces } \rho_{ij}$$

Deriving $\mathrm{SE}(\hat{\mu}_i)$

Example: Estimated Standard Errors

Compute $\widehat{\mathrm{SE}}(\hat{\mu}_i)$

```
n.obs = nrow(gwnRetC)
seMuhat = sigmahat/sqrt(n.obs)
cbind(muhat, seMuhat)
```

```
##          muhat seMuhat
## MSFT  0.00413 0.00764
## SBUX  0.01466 0.00851
## SP500 0.00169 0.00370
```

- $\widehat{\mathrm{SE}}(\hat{\mu}_i)$ values are large compared to $\hat{\mu}_i$ values. Do not estimate μ values very well.

Example: Estimated Standard Errors

Compute $\widehat{\mathrm{SE}}(\hat{\sigma}_i^2)$ and $\widehat{\mathrm{SE}}(\hat{\sigma}_i)$

```
seSigma2hat = sigma2hat/sqrt(n.obs/2)
seSigmahat = sigmahat/sqrt(2*n.obs)
cbind(sigma2hat, seSigma2hat, sigmahat, seSigmahat)
```

```
##       sigma2hat seSigma2hat sigmahat seSigmahat
## MSFT    0.01004    0.001083   0.1002    0.00540
## SBUX    0.01246    0.001344   0.1116    0.00602
## SP500   0.00235    0.000253   0.0485    0.00261
```

- Estimated SE values for volatility are small compared to estimates. Estimate volatility well.

Example: Estimated Standard Errors

Compute $\widehat{\mathrm{SE}}(\hat{\rho}_{ij})$

```
rhohat = cormat[lower.tri(cormat)]
names(rhohat) = c("msft,sbux","msft,sp500","sbux,sp500")
seRhohat = (1-rhohat^2)/sqrt(n.obs)
cbind(rhohat, seRhohat)
```

```
##             rhohat seRhohat
## msft,sbux    0.341   0.0674
## msft,sp500  0.617   0.0472
## sbux,sp500  0.457   0.0603
```

- Estimated standard errors for correlations are somewhat small compared to estimates. Notice how standard error is smallest for the largest correlation estimate.

Asymptotic Properties of Estimators

- Asymptotic properties are properties that are true for an infinitely large sample (i.e., $T \to \infty$)
- We never have an infinitely large sample so asymptotic properties are properties that are approximately true if the sample size is reasonably large
- Asymptotic properties are derived using theorems called *Laws of Large Numbers* and *Central Limit Theorems*
- Asymptotic properties usually give simple formulas

Consistency

Definition: An estimator $\hat{\theta}$ is consistent for θ (converges in probability to θ) if for any $\varepsilon > 0$

$$\lim_{T \to \infty} \Pr(|\hat{\theta} - \theta| > \varepsilon) = 0$$

- Intuitively, as we get enough data then $\hat{\theta}$ will eventually equal θ.
- Consistency is an asymptotic property - it holds when we have an infinitely large sample. In the real world we only have a finite amount of data so the result is only an approximation for a large sample.

Consistency

Result: An estimator $\hat{\theta}$ is consistent for θ if

- $\mathrm{bias}(\hat{\theta}, \theta) = 0$ as $T \to \infty$
- $\mathrm{SE}\left(\hat{\theta}\right) = 0$ as $T \to \infty$

Result: In the GWN model, the estimators $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$ are consistent.

Consistency

Distribution of GWN Model Estimators

$$\theta = \text{parameter to be estimated}$$
$$\hat{\theta} = \text{estimator of } \theta \text{ from random sample}$$

KEY POINTS

- $\hat{\theta}$ is a random variable – its value depends on realized values of random sample
- $f(\hat{\theta}) =$ pdf of $\hat{\theta}$ - depends on pdf of random variables in random sample
- Properties of $\hat{\theta}$ can be derived analytically (using probability theory), by using Monte Carlo simulation, and by bootstrapping

Distribution of $\hat{\mu}$ in GWN Model

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} R_{it},\ R_{it} = \mu_i + \epsilon_{it},\ \epsilon_{it} \sim \text{iid } N(0, \sigma_i^2)$$

Here, $\hat{\mu}_i$ is $\frac{1}{T}$ times the sum of T normally distributed random variables $\Rightarrow \hat{\mu}_i$ is also normally distributed with

$$E[\hat{\mu}_i] = \mu_i,\ \text{var}(\hat{\mu}_i) = \frac{\sigma_i^2}{T}$$

Distribution of $\hat{\mu}$ in GWN Model

That is,

$$\hat{\mu}_i \sim N\left(\mu_i, \frac{\sigma_i^2}{T}\right)$$

$$f(\hat{\mu}_i) = (2\pi\sigma_i^2/T)^{-1/2} \exp\left\{-\frac{1}{2\sigma_i^2/T}(\hat{\mu}_i - \mu_i)^2\right\}$$

- This is an *exact finite sample* distribution: it holds for any value of T

Distribution of $\hat{\sigma}_i, \ \hat{\sigma}_{ij}, \ $ and$\hat{\rho}_{ij}$

- The pdfs of $\hat{\sigma}_i, \ \hat{\sigma}_{ij}, \ $ and $\hat{\rho}_{ij}$ for a finite sample size T are not normal.
- However, as the sample size T gets large the pdfs of $\hat{\sigma}_i, \ \hat{\sigma}_{ij}, \ $ and $\hat{\rho}_{ij}$ get closer and closer to a normal distribution. This is the due to the famous **Central Limit Theorem**.

Central Limit Theorem (CLT)

Let $X_1, \ldots, X_T$ be a iid random variables with $E[X_t] = \mu$ and $\mathrm{var}(X_t) = \sigma^2$. Then

$$\frac{\bar{X} - \mu}{\mathrm{SE}(\bar{X})} = \frac{\bar{X} - \mu}{\sigma/\sqrt{T}} = \sqrt{T}\left(\frac{\bar{X} - \mu}{\sigma}\right) \sim N(0,1) \text{ as } T \to \infty$$

Equivalently,

$$\bar{X} \sim N\left(\mu, \mathrm{SE}(\bar{X})^2\right) \sim N\left(\mu, \frac{\sigma^2}{T}\right)$$

for large enough T

We say that $\bar{X}$ is asymptotically normally distributed with mean μ and variance $\mathrm{SE}(\bar{X})^2$.

Asymptotic Normality

Definition: An estimator $\hat{\theta}$ is asymptotically normally distributed if

$$\hat{\theta} \sim N(\theta, \mathrm{SE}(\hat{\theta})^2)$$
$$\text{for large enough } T$$

Result: An implication of the CLT is that the estimators $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$ and $\hat{\rho}_{ij}$ are asymptotically normally distributed under the GWN model.

Confidence Intervals

$$\begin{aligned}\hat{\theta} &= \text{estimate of } \theta \\ &= \text{best guess for unknown value of } \theta\end{aligned}$$

- A confidence interval for θ is an interval estimate of θ that covers θ with a stated probability
- Think of a confidence interval like a game of *horse shoes*. For a given sample, there is stated probability that the confidence interval (horse shoe thrown at θ) will cover θ.

Confidence Intervals

Result: Let $\hat{\theta}$ be an asymptotically normal estimator for θ. Then

- An approximate 95% confidence interval for θ is an interval estimate of the form

$$\left[\hat{\theta} - 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right),\ \hat{\theta} + 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)\right]$$
$$\hat{\theta} \pm 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)$$

that covers θ with probability approximately equal to 0.95. That is

$$\Pr\left\{\hat{\theta} - 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right) \leq \theta \leq \hat{\theta} + 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)\right\} \approx 0.95$$

Confidence Intervals

- An approximate 99% confidence interval for θ is an interval estimate of the form

$$\left[\hat{\theta} - 3 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right),\ \hat{\theta} + 3 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)\right]$$
$$\hat{\theta} \pm 3 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)$$

that covers θ with probability approximately equal to 0.99.

- 99% confidence intervals are wider than 95% confidence intervals
- For a given confidence level the width of a confidence interval dependson the size of $\widehat{\mathrm{SE}}(\hat{\theta})$

Confidence Intervals

In the GWN model, 95% Confidence Intervals for $\mu_i,\ \sigma_i,$ and ρ_{ij} are:

$$\hat{\mu}_i \pm 2 \cdot \frac{\hat{\sigma}_i}{\sqrt{T}}$$
$$\hat{\sigma}_i \pm 2 \cdot \frac{\hat{\sigma}_i}{\sqrt{2T}}$$
$$\hat{\rho}_{ij} \pm 2 \cdot \frac{(1 - \hat{\rho}_{ij}^2)}{\sqrt{T}}$$

Example: 95% Confidence Intervals

Approximate 95% CI for μ_i

```
lowerMu = muhat - 2*seMuhat
upperMu = muhat + 2*seMuhat
widthMu = upperMu - lowerMu
cbind(lowerMu, upperMu, widthMu)
```

```
##        lowerMu upperMu widthMu
## MSFT  -0.01116 0.01941  0.0306
## SBUX  -0.00237 0.03168  0.0341
## SP500 -0.00570 0.00908  0.0148
```

- Wide 95% confidence intervals for the mean implies imprecise estimates.
- Notice that the confidence interval width is smallest for the S&P 500 index (why?)

Example: 95% Confidence Intervals

Approximate 95% CI for σ_i

```
lowerSigma = sigmahat - 2*seSigmahat
upperSigma = sigmahat + 2*seSigmahat
widthSigma = upperSigma - lowerSigma
cbind(lowerSigma, upperSigma, widthSigma)
```

```
##       lowerSigma upperSigma widthSigma
## MSFT      0.0894     0.1110     0.0216
## SBUX      0.0996     0.1237     0.0241
## SP500     0.0432     0.0537     0.0105
```

- Confidence intervals for σ are narrow. Estimates are precise.

Example: 95% Confidence Intervals

Approximate 95% CI for ρ_{ij}

```
lowerRho = rhohat - 2*seRhohat
upperRho = rhohat + 2*seRhohat
widthRho = upperRho - lowerRho
cbind(lowerRho, upperRho, widthRho)
```

```
##              lowerRho upperRho widthRho
## msft,sbux       0.206    0.476    0.270
## msft,sp500      0.523    0.712    0.189
## sbux,sp500      0.337    0.578    0.241
```

- Confidence intervals are not too wide and contain all positive values. Hence, estimates are moderately precise.

Stylized Facts for the Estimation of GWN Model Parameters

- The expected return, μ_i is not estimated very precisely for most assets
  - Large standard errors relative to size of mean estimates
  - 95% confidence intervals often contain both negative and positive values
- Volatility, σ_i, and correlations, ρ_{ij}, are estimated more precisely than the expected return, μ_i
  - Small standard errors relative to size of estimates
- Implication: Using historical data we know more about asset *risk* than we do about *expected return*

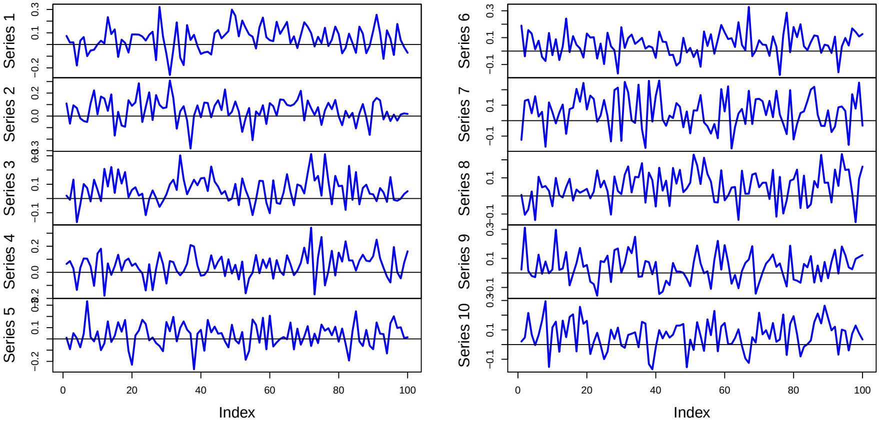
Using Monte Carlo Simulation to Evaluate Bias, Standard Error and Confidence Interval Coverage

- Create many simulated samples from GWN model
- Compute parameter estimates for each simulated sample
- Compute $\hat{\mu}_i$, $\hat{\sigma}_i$, and $\hat{\rho}_{ij}$ for each simulated sample
- Compute 95% confidence intervals for μ_i, σ_i and ρ_{ij} for each sample
- Compute empirical distribution of estimates
- Count fraction of confidence intervals that contain true parameters

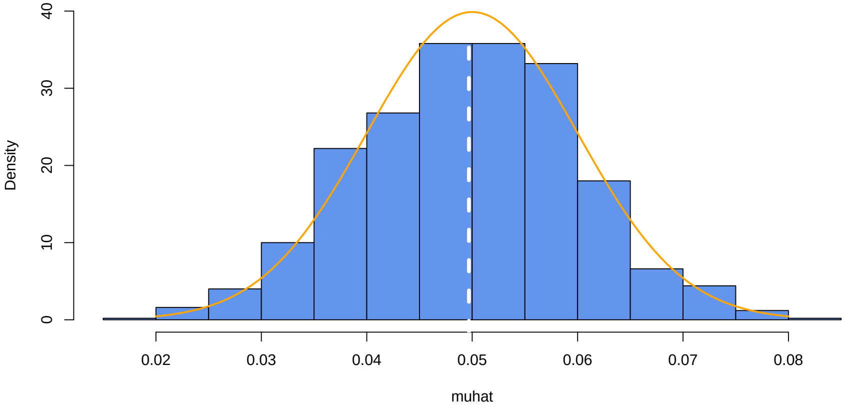
Example: Monte Carlo Simulation in GWN Model

```
mu = 0.05
sigma = 0.10
n.obs = 100
n.sim = 1000
set.seed(111)
sim.means = rep(0,n.sim)
mu.lower = rep(0,n.sim)
mu.upper = rep(0,n.sim)
for (sim in 1:n.sim) {
    sim.ret = rnorm(n.obs,mean=mu,sd=sigma)
    sim.means[sim] = mean(sim.ret)
    se.muhat = sd(sim.ret)/sqrt(n.obs)
    mu.lower[sim] = sim.means[sim]-2*se.muhat
    mu.upper[sim] = sim.means[sim]+2*se.muhat
}
```

Example: Monte Carlo Simulation in GWN Model



Histogram of 1000 Monte Carlo Estimates of $\hat{\mu}$



Properties of Monte Carlo Estimates

Approximate $E[\hat{\mu}]$ using sample mean of MC estimates:

```
mean(sim.means) # true value is 0.05
```

```
## [1] 0.0497
```

Approximate $\widehat{\mathrm{SE}}(\hat{\mu})$ using sample SD of MC estimates:

```
sd(sim.means)
```

```
## [1] 0.0104
```

Compare to analytic formula for $\widehat{\mathrm{SE}}(\hat{\mu})$:

```
sigma/sqrt(n.obs)
```

```
## [1] 0.01
```

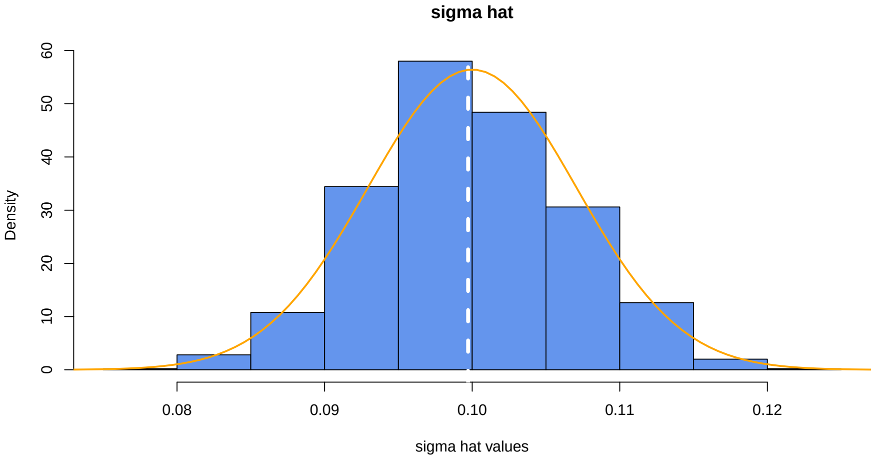
Properties of Monte Carlo Estimates

Approximate 95% CI coverage probability by counting how many times the Monte Carlo confidence intervals contain $\mu = 0.05$

```
in.interval = mu >= mu.lower & mu <= mu.upper
sum(in.interval)/n.sim
```

```
## [1] 0.934
```

Histograms of 1000 Monte Carlo estimates of $\hat{\sigma}$



Properties of Monte Carlo Estimates

Approximate $E[\hat{\sigma}]$ using sample mean of MC estimates:

```
mean(sim.sigma) # true value is 0.10
```

```
## [1] 0.0997
```

Approximate $\widehat{\mathrm{SE}}(\hat{\sigma})$ using sample SD of MC estimates:

```
sd(sim.sigma)
```

```
## [1] 0.00676
```

Compare to approximate analytic formula for $\widehat{\mathrm{SE}}(\hat{\sigma})$:

```
sigma/sqrt(2*n.obs)
```

```
## [1] 0.00707
```

Properties of Monte Carlo Estimates

Approximate 95% CI coverage probability by counting how many times the Monte Carlo confidence intervals contain $\sigma = 0.1$

```
in.interval = sigma >= sigma.lower & sigma <= sigma.upper
sum(in.interval)/n.sim
```

```
## [1] 0.963
```

Estimating Quantiles and VaR from GWN Model

- In the GWN Model, the $100 \times \alpha\%$ quantile of the cc return R_{it} is $q_{\alpha}^{R_i} = \mu_i + \sigma_i \times q_{\alpha}^{Z}$, where q_{α}^{Z} is the $\alpha-$ quantile of a standard Normal rv.
- The estimated $100 \times \alpha\%$ quantile is

$$\hat{q}_{\alpha}^{R_i} = \hat{\mu}_i + \hat{\sigma}_i q_{\alpha}^{Z}$$

- The estimated monthly Value-at-Risk of an initial $\$W_0$ investment is

$$\widehat{\mathrm{VaR}}_{\alpha} = (\exp(\hat{q}_{\alpha}^{R}) - 1) \times W_0$$

Estimating Quantiles and VaR from GWN Model

Result: Under the assumptions of the GWN model, the estimates $\hat{\mu}_i$ and $\hat{\sigma}_i$ are jointly asymptotically normally distributed. That is, as $T \to \infty$

$$\begin{pmatrix} \hat{\mu}_i \\ \hat{\sigma}_i \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_i \\ \sigma_i \end{pmatrix}, \begin{pmatrix} \mathrm{SE}(\hat{\mu}_i) & 0 \\ 0 & \mathrm{SE}(\hat{\sigma}_i) \end{pmatrix} \right)$$

- The result implies that $\mathrm{cov}(\hat{\mu}_i, \hat{\sigma}_i) \approx 0$ for large enough T

Statistical Properties of $\hat{q}_{\alpha}^{R_i}$

- $\hat{q}_{\alpha}^{R_i}$ is approximately unbiased

$$E[\hat{q}_{\alpha}^{R_i}] = E[\hat{\mu}_i] + E[\hat{\sigma}_i] \times q_{\alpha}^{Z} \approx \mu_i + \sigma_i \times q_{\alpha}^{Z}$$

- To compute $\mathrm{SE}(\hat{q}_{\alpha}^{R})$, we first compute

$$\begin{aligned}
\mathrm{var}(\hat{q}_{\alpha}^{R}) &= \mathrm{var}\left(\hat{\mu}_i + \hat{\sigma}_i q_{\alpha}^{Z}\right) \\
&= \mathrm{var}(\hat{\mu}_i) + \left(q_{\alpha}^{Z}\right)^2 \mathrm{var}(\hat{\sigma}_i) + 2 q_{\alpha}^{Z} \mathrm{cov}(\hat{\mu}_i, \hat{\sigma}_i) \\
&\approx \mathrm{var}(\hat{\mu}_i) + \left(q_{\alpha}^{Z}\right)^2 \mathrm{var}(\hat{\sigma}_i), \text{ since } \mathrm{cov}(\hat{\mu}_i, \hat{\sigma}_i) \approx 0 \\
&\approx \frac{\sigma_i^2}{T} + \left(q_{\alpha}^{Z}\right)^2 \times \frac{\sigma_i^2}{2T} = \frac{\sigma_i^2}{T}\left[1 + \frac{1}{2}\left(q_{\alpha}^{Z}\right)^2\right]
\end{aligned}$$

Statistical Properties of $\hat{q}_{\alpha}^{R_i}$

- Then $\mathrm{SE}(\hat{q}_{\alpha}^{R})$ is computed as

$$\mathrm{SE}(\hat{q}_{\alpha}^{R}) = \sqrt{\mathrm{var}(\hat{q}_{\alpha}^{R})} \tag{1}$$

$$\approx \frac{\sigma_i}{\sqrt{T}}\sqrt{1+\frac{1}{2}\left(q_{\alpha}^{Z}\right)^2} \tag{2}$$

- We estimate $\mathrm{SE}(\hat{q}_{\alpha}^{R})$ using

$$\widehat{\mathrm{SE}}(\hat{q}_{\alpha}^{R}) \approx \frac{\hat{\sigma}_i}{\sqrt{T}}\sqrt{1+\frac{1}{2}\left(q_{\alpha}^{Z}\right)^2}$$

Example: Estimating CC Return Quantiles in the GWN Model

```
qhat.05 = muhat + sigmahat*qnorm(0.05)
qhat.01 = muhat + sigmahat*qnorm(0.01)
seQhat.05 = (sigmahat/sqrt(n.obs))*sqrt(1 + 0.5*qnorm(0.05)^2
seQhat.01 = (sigmahat/sqrt(n.obs))*sqrt(1 + 0.5*qnorm(0.01)^2
cbind(qhat.05, seQhat.05, qhat.01, seQhat.01)
```

```
##       qhat.05 seQhat.05 qhat.01 seQhat.01
## MSFT   -0.161   0.01537  -0.229   0.01929
## SBUX   -0.169   0.01712  -0.245   0.02149
## SP500  -0.078   0.00743  -0.111   0.00933
```

SE values for 1% quantiles and bigger than SE values for 5% quantiles

Statistical Properties of $\widehat{\text{VaR}}_\alpha$

Statistical Properties of $\widehat{\text{VaR}}_\alpha$ are not straightforward:

- $\widehat{\text{VaR}}_\alpha$ is biased since

$$E\left[\widehat{\text{VaR}}_\alpha\right] = E\left[(\exp(\hat{q}_\alpha^R) - 1) \times W_0\right] \neq (\exp(E[\hat{q}_\alpha^R]) - 1) \times W_0$$

- Also, computing $\text{SE}(\widehat{\text{VaR}}_\alpha)$ is not easy since

$$\text{var}\left(\widehat{\text{VaR}}_\alpha\right) = \text{var}\left((\exp(\hat{q}_\alpha^R) - 1) \cdot W_0\right) = ??$$

- However, we can use the *bootstrap* instead of analytical calculations to numerically compute $E\left[\widehat{\text{VaR}}_\alpha\right]$ and $\text{SE}(\widehat{\text{VaR}}_\alpha)$