Return Calculations

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The Time Value of Money

- $\$V$ invested for n years at simple interest rate R per year
- Compounding of interest occurs at end of year
- The future value of V after n years with R percent per year is

$$FV_n = \$V \cdot (1 + R)^n,$$

where FV_n is future value after n years

Example

Consider putting \$1000 in an interest checking account that pays a simple annual percentage rate of 3 . The future value after $n = 1, 5$ and 10 years is, respectively,

$$FV_1 = \$1000 \cdot (1.03)^1 = \$1030,$$
$$FV_5 = \$1000 \cdot (1.03)^5 = \$1159.27,$$
$$FV_{10} = \$1000 \cdot (1.03)^{10} = \$1343.92.$$

FV function

FV function is a relationship between four variables: FV_n, V, R, n. Given three variables, you can solve for the fourth:

- **Present value**:

$$V = \frac{FV_n}{(1+R)^n}.$$

- **Compound annual return**:

$$R = \left(\frac{FV_n}{V}\right)^{1/n} - 1.$$

- **Investment horizon**:

$$n = \frac{\ln(FV_n/V)}{\ln(1+R)}.$$

Compounding

- Compounding occurs m times per year

$$
\begin{aligned}
FV_n^m &= \$V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n}, \\
\frac{R}{m} &= \text{periodic interest rate.}
\end{aligned}
$$

- Continuous compounding

$$
\begin{aligned}
FV_n^\infty &= \lim_{m \to \infty} \$V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n} = \$Ve^{R \cdot n}, \\
e^1 &= 2.71828.
\end{aligned}
$$

Example

If the simple annual percentage rate is 10% then the value of \$1000 at the end of one year ($n = 1$) for different values of m is given in the table below.

Compounding Frequency	Value of \$1000 at end of 1 year ($R = 10\%$)
Annually ($m = 1$)	1100.00
Quarterly ($m = 4$)	1103.81
Weekly ($m = 52$)	1105.06
Daily ($m = 365$)	1105.16
Continuously ($m = \infty$)	1105.17

Effective Annual Rate

Annual rate R_A that equates FV_n^m with FV_n; i.e.,

$$\$V\left(1+\frac{R}{m}\right)^{m\cdot n}=\$V(1+R_A)^n.$$

Solving for R_A

$$\left(1+\frac{R}{m}\right)^{m}=1+R_A \Rightarrow R_A=\left(1+\frac{R}{m}\right)^{m}-1.$$

Continuous compounding

$$
\begin{aligned}
\$Ve^{R\cdot n} &= \$V(1+R_A)^n \\
&\Rightarrow e^R = (1+R_A) \\
&\Rightarrow R_A = e^R - 1.
\end{aligned}
$$

Example

Let's compute effective annual rate with semi-annual compounding

The effective annual rate associated with an investment with a simple annual rate $R = 10\%$ and semi-annual compounding ($m = 2$) is determined by solving

$$
\begin{aligned}
(1 + R_A) &= \left(1 + \frac{0.10}{2}\right)^2 \\
&\Rightarrow R_A = \left(1 + \frac{0.10}{2}\right)^2 - 1 = 0.1025.
\end{aligned}
$$

Example cont'd

Compounding Frequency	Value of \$1000 at end of 1 year ($R = 10\%$)	R_A
Annually ($m = 1$)	1100.00	10%
Quarterly ($m = 4$)	1103.81	10.38%
Weekly ($m = 52$)	1105.06	10.51%
Daily ($m = 365$)	1105.16	10.52%
Continuously ($m = \infty$)	1105.17	10.52%

Asset Return Calculations: Simple Returns

- $P_t =$ price at the end of month t on an asset that pays no dividends
- $P_{t-1} =$ price at the end of month $t-1$

Then

$$
\begin{aligned}
R_t &= \frac{P_t - P_{t-1}}{P_{t-1}} = \triangle\, P_t = \text{net return over month } t, \\
1 + R_t &= \frac{P_t}{P_{t-1}} = \text{gross return over month } t, \\
P_{t-1}(1 + R_t) &= P_t \Longrightarrow R_t = \text{simple return over month } t.
\end{aligned}
$$

Example: One month investment in Microsoft stock

Buy stock at end of month $t-1$ at $P_{t-1} = \$85$ and sell stock at end of next month for $P_t = \$90$. Assuming that Microsoft does not pay a dividend between months $t-1$ and t, the one-month simple net and gross returns are

$$\begin{aligned} R_t &= \frac{\$90 - \$85}{\$85} = \frac{\$90}{\$85} - 1 = 1.0588 - 1 = 0.0588, \\ 1 + R_t &= 1.0588. \end{aligned}$$

The one month investment in Microsoft yielded a 5.88% per month return.

Multi-period Returns: Simple two-month return

$$
\begin{aligned}
R_t(2) &= \frac{P_t - P_{t-2}}{P_{t-2}} \\
&= \frac{P_t}{P_{t-2}} - 1.
\end{aligned}
$$

Relationship to one month returns:

$$
\begin{aligned}
R_t(2) &= \frac{P_t}{P_{t-2}} - 1 = \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} - 1 \\
&= (1 + R_t) \cdot (1 + R_{t-1}) - 1.
\end{aligned}
$$

Multi-period Returns: Simple two-month return

Here

$$
\begin{aligned}
1 + R_t &= \text{one-month gross return over month } t, \\
1 + R_{t-1} &= \text{one-month gross return over month } t-1, \\
&\Longrightarrow 1 + R_t(2) = (1 + R_t) \cdot (1 + R_{t-1}).
\end{aligned}
$$

two-month gross return = the product of two one-month gross returns

Note: two-month returns are not additive:

$$
\begin{aligned}
R_t(2) &= R_t + R_{t-1} + R_t \cdot R_{t-1} \\
&\approx R_t + R_{t-1} \text{ if } R_t \text{ and } R_{t-1} \text{ are small}
\end{aligned}
$$

Example: Two-month return on Microsoft

Suppose that the price of Microsoft in month $t-2$ is \$80 and no dividend is paid between months $t-2$ and t. The two-month net return is

$$R_t(2) = \frac{\$90 - \$80}{\$80} = \frac{\$90}{\$80} - 1 = 1.1250 - 1 = 0.1250,$$

or 12.50% per two months. The two one-month returns are

$$\begin{aligned} R_{t-1} &= \frac{\$85 - \$80}{\$80} = 1.0625 - 1 = 0.0625 \\ R_t &= \frac{\$90 - 85}{\$85} = 1.0588 - 1 = 0.0588, \end{aligned}$$

and the geometric average of the two one-month gross returns is

$$1 + R_t(2) = 1.0625 \times 1.0588 = 1.1250.$$

Simple k-month Return

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1$$

$$1 + R_t(k) = (1 + R_t) \cdot (1 + R_{t-1}) \cdot \cdots \cdot (1 + R_{t-k+1})$$

$$= \prod_{j=0}^{k-1} (1 + R_{t-j})$$

Note

$$R_t(k) \neq \sum_{j=0}^{k-1} R_{t-j}$$

Portfolio Returns

- Invest \$$V$ in two assets: A and B for 1 period
- $x_A =$ share of \$$V$ invested in A; \$$V \times x_A =$ \$ amount
- $x_B =$ share of \$$V$ invested in B; \$$V \times x_B =$ \$ amount
- Assume $x_A + x_B = 1$
- Portfolio is defined by investment shares x_A and x_B

Portfolio Returns

At the end of the period, the investments in A and B grow to

$$\begin{aligned}
\$V(1+R_{p,t}) &= \$V\left[x_A(1+R_{A,t}) + x_B(1+R_{B,t})\right] \\
&= \$V\left[x_A + x_B + x_A R_{A,t} + x_B R_{B,t}\right] \\
&= \$V\left[1 + x_A R_{A,t} + x_B R_{B,t}\right] \\
&\Rightarrow R_{p,t} = x_A R_{A,t} + x_B R_{B,t}
\end{aligned}$$

The simple portfolio return is a share weighted average of the simple returns on the individual assets.

Example: Portfolio of Microsoft and Starbucks stock

Purchase ten shares of each stock at the end of month $t-1$ at prices

$$P_{msft,t-1} = \$85,\ P_{sbux,t-1} = \$30,$$

The initial value of the portfolio is

$$V_{t-1} = 10 \times \$85 + 10 \times \$30 = \$1,150.$$

The portfolio shares are

$$x_{msft} = 850/1150 = 0.7391,\ x_{sbux} = 300/1150 = 0.2609.$$

The end of month t prices are $P_{msft,t} = \$90$ and $P_{sbux,t} = \$28$.

Example cont'd

Assuming Microsoft and Starbucks do not pay a dividend between periods $t-1$ and t, the one-period returns are

$$
\begin{aligned}
R_{msft,t} &= \frac{\$90 - \$85}{\$85} = 0.0588 \\
R_{sbux,t} &= \frac{\$28 - \$30}{\$30} = -0.0667
\end{aligned}
$$

The return on the portfolio is

$$R_{p,t} = (0.7391)(0.0588) + (0.2609)(-0.0667) = 0.02609$$

and the value at the end of month t is

$$V_t = \$1,150 \times (1.02609) = \$1,180$$

Portfolio Returns

In general, for a portfolio of n assets with investment shares x_i such that $x_1 + \cdots + x_n = 1$

$$\begin{aligned} 1 + R_{p,t} &= \sum_{i=1}^{n} x_i (1 + R_{i,t}) \\ R_{p,t} &= \sum_{i=1}^{n} x_i R_{i,t} \\ &= x_1 R_{1t} + \cdots + x_n R_{nt} \end{aligned}$$

Adjusting for Dividends

$$
\begin{aligned}
D_t &= \text{ dividend payment between months } t-1 \text{ and } t \\
R_t^{total} &= \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}} \\
&= \text{capital gain return} + \text{dividend yield (gross)} \\
1 + R_t^{total} &= \frac{P_t + D_t}{P_{t-1}}
\end{aligned}
$$

Example: Total return on Microsoft stock

Buy stock in month $t-1$ at $P_{t-1} = \$85$ and sell the stock the next month for $P_t = \$90$. Assume Microsoft pays a \$1 dividend between months $t-1$ and t. The capital gain, dividend yield and total return are then

$$\begin{aligned}R_t^{total} &= \frac{\$90 + \$1 - \$85}{\$85} = \frac{\$90 - \$85}{\$85} + \frac{\$1}{\$85} \\ &= 0.0588 + 0.0118 \\ &= 0.0707\end{aligned}$$

The one-month investment in Microsoft yields a 7.07% per month total return. The capital gain component is 5.88%, and the dividend yield component is 1.18%.

Adjusting for Inflation

The computation of real returns on an asset is a two step process:

- Deflate the nominal price P_t of the asset by an index of the general price level CPI_t
- Compute returns in the usual way using the deflated prices

$$P_t^{\text{Real}} = \frac{P_t}{CPI_t}$$

$$R_t^{\text{Real}} = \frac{P_t^{\text{Real}} - P_{t-1}^{\text{Real}}}{P_{t-1}^{\text{Real}}} = \frac{\frac{P_t}{CPI_t} - \frac{P_{t-1}}{CPI_{t-1}}}{\frac{P_{t-1}}{CPI_{t-1}}}$$

$$= \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t} - 1$$

Adjusting for Inflation

Alternatively, define inflation as

$$\pi_t = \ \Delta CPI_t = \frac{CPI_t - CPI_{t-1}}{CPI_{t-1}}$$

Then

$$R_t^{\text{Real}} = \frac{1 + R_t}{1 + \pi_t} - 1$$

Example: Compute real return on Microsoft stock

Suppose the CPI in months $t-1$ and t is 1 and 1.01, respectively, representing a 1% monthly growth rate in the overall price level. The real prices of Microsoft stock are

$$P_{t-1}^{\text{Real}} = \frac{\$85}{1} = \$85,\ \ P_t^{\text{Real}} = \frac{\$90}{1.01} = \$89.1089$$

The real monthly return is

$$R_t^{\text{Real}} = \frac{\$89.10891 - \$85}{\$85} = 0.0483$$

Example cont'd

The nominal return and inflation over the month are

$$R_t = \frac{\$90 - \$85}{\$85} = 0.0588,\ \pi_t = \frac{1.01 - 1}{1} = 0.01$$

Then the real return is

$$R_t^{\text{Real}} = \frac{1.0588}{1.01} - 1 = 0.0483$$

Notice that simple real return is almost, but not quite, equal to the simple nominal return minus the inflation rate

$$R_t^{\text{Real}} \approx R_t - \pi_t = 0.0588 - 0.01 = 0.0488$$

Annualizing Returns

Returns over different horizons are often converted to an annual return to establish a standard for comparison

Example: Assume same monthly return R_m for 12 months

- Compound annual gross return (CAGR)=

$$1 + R_A = 1 + R_t(12) = (1 + R_m)^{12}$$

- Compound annual net return

$$R_A = (1 + R_m)^{12} - 1$$

Note: We don't use $R_A = 12R_m$ because this ignores compounding.

Example: Annualized return on Microsoft

Suppose the one-month return, R_t, on Microsoft stock is 5.88%. If we assume that we can get this return for 12 months then the compounded annualized return is

$$R_A = (1.0588)^{12} - 1 = 1.9850 - 1 = 0.9850$$

or 98.50% per year. Pretty good!

Note: $12 \times R_t = 12 \times 0.0588 = 0.7056$.

Average Returns

For investments over a given horizon, it is often of interest to compute a measure of average return over the horizon

Consider a sequence of monthly investments over the year with monthly returns

$$R_1, R_2, \ldots, R_{12}$$

The annual return is

$$R_A = R(12) = (1 + R_1)(1 + R_2) \cdots (1 + R_{12}) - 1$$

Q: What is the average monthly return?

Average Returns

Two possibilities

- Arithmetic average (can be misleading)

$$\bar{R} = \frac{1}{12}(R_1 + \cdots + R_{12})$$

- Geometric average (better measure of average return)

$$\begin{aligned}
(1 + \bar{R})^{12} &= (1 + R_A) = (1 + R_1)(1 + R_2) \cdots (1 + R_{12}) \\
&\Rightarrow \bar{R} = (1 + R_A)^{1/12} - 1 \\
&= \left[(1 + R_1)(1 + R_2) \cdots (1 + R_{12})\right]^{1/12} - 1
\end{aligned}$$

Example: Consider a two period invesment with returns

$$R_1 = 0.5,\ \ R_2 = -0.5$$

\$1 invested over two periods grows to

$$FV = \$1 \times (1 + R_1)(1 + R_2) = (1.5)(0.5) = \$0.75$$

for a 2-period return of

$$R(2) = 0.75 - 1 = -0.25$$

Hence, the 2-period investment loses 25%

Example cont'd

The arithmetic average return is

$$\bar{R} = \frac{1}{2}(0.5 + -0.5) = 0$$

This is misleading becuase the actual invesment lost money over the 2 period horizon. The compound 2-period return based on the arithmetic average is

$$(1 + \bar{R})^2 - 1 = 1^2 - 1 = 0$$

The geometric average is

$$[(1.5)(0.5)]^{1/2} - 1 = (0.75)^{1/2} - 1 = -0.1340$$

This is a better measure because it indicates that the investment eventually lost money. The compound 2-period return is

$$(1 + \bar{R})^2 - 1 = (0.867)^2 - 1 = -0.25$$

Contnuously Compounded (cc) Returns

$$r_t = \ln(1 + R_t) = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

$$\ln(\cdot) = \text{natural log function}$$

Note:

$$\ln(1 + R_t) = r_t : \text{ given } R_t \text{ we can solve for } r_t$$

$$R_t = e^{r_t} - 1 : \text{ given } r_t \text{ we can solve for } R_t$$

$$r_t \text{ is always smaller than } R_t$$

Digression on natural log and exponential functions

- $\ln(0) = -\infty$, $\ln(1) = 0$
- $e^{-\infty} = 0$, $e^0 = 1$, $e^1 = 2.7183$
- $\frac{d\ln(x)}{dx} = \frac{1}{x}$, $\frac{de^x}{dx} = e^x$
- $\ln(e^x) = x$, $e^{\ln(x)} = x$
- $\ln(x \cdot y) = \ln(x) + \ln(y)$; $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$
- $\ln(x^y) = y\ln(x)$
- $e^x e^y = e^{x+y}$, $e^x e^{-y} = e^{x-y}$
- $(e^x)^y = e^{xy}$

Contnuously Compounded (cc) Returns: Intuition

$$
\begin{aligned}
e^{r_t} &= e^{\ln(1+R_t)} = e^{\ln(P_t/P_{t-1})} \\
&= \frac{P_t}{P_{t-1}} \\
&\implies P_{t-1} \cdot e^{r_t} = P_t
\end{aligned}
$$

$\implies r_t =$ cc growth rate in prices between months $t-1$ and t

Relationship between Simple Returns and cc Returns

If R_t is small then

$$r_t = \ln(1 + R_t) \approx R_t$$

Proof. For a function $f(x)$, a first order Taylor series expansion about $x = x_0$ is

$$f(x) = f(x_0) + \frac{d}{dx}f(x_0)(x - x_0) + \text{ remainder}$$

Let $f(x) = \ln(1 + x)$ and $x_0 = 0$. Note that

$$\frac{d}{dx}\ln(1 + x) = \frac{1}{1 + x},\ \frac{d}{dx}\ln(1 + x_0) = 1$$

Then

$$\ln(1 + x) \approx \ln(1) + 1 \cdot x = 0 + x = x$$

CC Returns: Computational Trick

$$
\begin{aligned}
r_t &= \ln\left(\frac{P_t}{P_{t-1}}\right) \\
&= \ln(P_t) - \ln(P_{t-1}) \\
&= p_t - p_{t-1} \\
&= \text{difference in log prices}
\end{aligned}
$$

where

$$p_t = \ln(P_t)$$

Example: Compute cc return

Let $P_{t-1} = \$85, P_t = \$90$ and $R_t = 0.0588$. Then the cc monthly return can be computed in two ways:

$$r_t = \ln(1.0588) = 0.0571$$
$$r_t = \ln(90) - \ln(85) = 4.4998 - 4.4427 = 0.0571.$$

Notice that r_t is slightly smaller than R_t.

Multi-period CC Returns

$$\begin{aligned} r_t(2) &= \ln(1 + R_t(2)) \\ &= \ln\left(\frac{P_t}{P_{t-2}}\right) \\ &= p_t - p_{t-2} \end{aligned}$$

Note that

$$\begin{aligned} e^{r_t(2)} &= e^{\ln(P_t/P_{t-2})} \\ &\Rightarrow P_{t-2}e^{r_t(2)} = P_t \end{aligned}$$

$\Longrightarrow r_t(2) =$ cc growth rate in prices between months $t - 2$ and t

cc returns are additive

$$
\begin{aligned}
r_t(2) &= \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}}\right) \\
&= \ln\left(\frac{P_t}{P_{t-1}}\right) + \ln\left(\frac{P_{t-1}}{P_{t-2}}\right) \\
&= r_t + r_{t-1}
\end{aligned}
$$

where $r_t =$ cc return between months $t-1$ and t, $r_{t-1} =$ cc return between months $t-2$ and $t-1$

Example. Compute cc two-month return

Suppose $P_{t-2} = \$80, P_{t-1} = \$85$ and $P_t = \$90$. The cc two-month return can be computed in two equivalent ways: (1) take difference in log prices

$$r_t(2) = \ln(90) - \ln(80) = 4.4998 - 4.3820 = 0.1178.$$

(2) sum the two cc one-month returns

$$\begin{aligned} r_t &= \ln(90) - \ln(85) = 0.0571 \\ r_{t-1} &= \ln(85) - \ln(80) = 0.0607 \\ r_t(2) &= 0.0571 + 0.0607 = 0.1178. \end{aligned}$$

Notice that $r_t(2) = 0.1178 < R_t(2) = 0.1250$.

General Result

$$
\begin{aligned}
r_t(k) &= \ln(1 + R_t(k)) = \ln(\frac{P_t}{P_{t-k}}) \\
&= \sum_{j=0}^{k-1} r_{t-j} \\
&= r_t + r_{t-1} + \cdots + r_{t-k+1}
\end{aligned}
$$

CC Returns for a Portfolio

$$R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t}$$

$$r_{p,t} = \ln(1 + R_{p,t}) = \ln(1 + \sum_{i=1}^{n} x_i R_{i,t}) \neq \sum_{i=1}^{n} x_i r_{i,t}$$

$$\Rightarrow \text{ portfolio returns are not additive}$$

Note: If $R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t}$ is not too large, then $r_{p,t} \approx R_{p,t}$ otherwise, $R_{p,t} > r_{p,t}$.

Example: Compute cc return on portfolio

Consider a portfolio of Microsoft and Starbucks stock with

$$
\begin{aligned}
x_{msft} &= 0.25, x_{sbux} = 0.75, \\
R_{msft,t} &= 0.0588, R_{sbux,t} = -0.0503 \\
R_{p,t} &= x_{msft} R_{msft,t} + x_{sbux,t} R_{sbux,t} = -0.02302
\end{aligned}
$$

The cc portfolio return is

$$r_{p,t} = \ln(1 - 0.02302) = \ln(0.977) = -0.02329$$

Note

$$
\begin{aligned}
r_{msft,t} &= \ln(1 + 0.0588) = 0.05714 \\
r_{sbux,t} &= \ln(1 - 0.0503) = -0.05161 \\
x_{msft} r_{msft} + x_{sbux} r_{sbux} &= -0.02442 \neq r_{p,t}
\end{aligned}
$$

Adjusting for Inflation

The cc one period real return is

$$
\begin{aligned}
r_t^{\text{Real}} &= \ln(1 + R_t^{\text{Real}}) \\
1 + R_t^{\text{Real}} &= \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
r_t^{\text{Real}} &= \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t}\right) = \ln\left(\frac{P_t}{P_{t-1}}\right) + \ln\left(\frac{CPI_{t-1}}{CPI_t}\right) \\
&= \ln(P_t) - \ln(P_{t-1}) + \ln(CPI_{t-1}) - \ln(CPI_t) \\
&= r_t - \pi_t^{cc}
\end{aligned}
$$

where

$$
\begin{aligned}
r_t &= \ln(P_t) - \ln(P_{t-1}) = \text{nominal cc return} \\
\pi_t^{cc} &= \ln(CPI_t) - \ln(CPI_{t-1}) = \text{ cc inflation}
\end{aligned}
$$

Example: Compute cc real return

Suppose:

$$
\begin{aligned}
R_t &= 0.0588 \\
\pi_t &= 0.01 \\
R_t^{\text{Real}} &= 0.0483
\end{aligned}
$$

The real cc return is

$$r_t^{\text{Real}} = \ln(1 + R_t^{\text{Real}}) = \ln(1.0483) = 0.047.$$

Equivalently,

$$r_t^{\text{Real}} = r_t - \pi_t^{cc} = \ln(1.0588) - \ln(1.01) = 0.047$$