Single Index Model

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Implications from Portfolio Theory

- Individual assets have *idiosyncratic risk* that can be diversified in a portfolio.
- Portfolios have *systematic risk* that cannot be diversified away.
- The marginal contribution of risk *MCR* of an asset, can be represented as:

$$\mathrm{MCR}_i^{\sigma} = \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \beta_i \sigma_p(\mathbf{x})$$

- Riskiness of an asset should be judged in a portfolio context - relative to the portfolio (beta)

$$\beta_i = \frac{\mathrm{cov}(R_i(\mathbf{x}), R_p(\mathbf{x}))}{\mathrm{var}(R_p(\mathbf{x}))}$$

Sharpe's Single Index Model

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it} \\
& i = 1, \ldots, N; \; t = 1, \ldots, T
\end{aligned}
$$

where

$$
\begin{aligned}
&\alpha_i, \; \beta_i \text{ are constant over time} \\
&R_{Mt} = \text{ return on diversified market index portfolio} \\
&\varepsilon_{it} = \text{ random error term unrelated to } R_{Mt}
\end{aligned}
$$

Assumptions:

- $\mathrm{cov}(R_{Mt}, \varepsilon_{is}) = 0$ for all t, s; $\mathrm{cov}(\varepsilon_{is}, \varepsilon_{jt}) = 0$ for all $i \neq j$, t and s
- $\varepsilon_{it} \sim \text{iid } N(0, \sigma^2_{\varepsilon,i})$; $R_{M,t} \sim \text{iid } N(\mu_M, \sigma^2_M)$

Interpretation of β_i

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$

$$\beta_i = \frac{\mathrm{cov}(R_{it}, R_{Mt})}{\mathrm{var}(R_{Mt})} = \frac{\sigma_{iM}}{\sigma_M^2}$$

β_i captures the contribution of asset i to the volatility of the market index (recall risk budgeting calculations). Sensitivity to market portfolio.

Derivation:

$$\begin{aligned}
\mathrm{cov}(R_{it}, R_{Mt}) &= \mathrm{cov}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}, R_{Mt}) \\
&= \mathrm{cov}(\beta_i R_{Mt}, R_{Mt}) + \mathrm{cov}(\varepsilon_{it}, R_{Mt}) \\
&= \beta_i \mathrm{var}(R_{Mt}) \quad \text{since } \mathrm{cov}(\varepsilon_{it}, R_{Mt}) = 0 \\
\Rightarrow \beta_i &= \frac{\mathrm{cov}(R_{it}, R_{Mt})}{\mathrm{var}(R_{Mt})}
\end{aligned}$$

Two Sources of News

Interpretation of R_{Mt} and ε_{it}

$$\varepsilon_{it} = R_{it} - \alpha_i - \beta_i R_{Mt}$$

- Return on market index, R_{Mt}, captures common "market-wide" news.
- β_i measures sensitivity to "market-wide" news
- Random error term ε_{it} captures "firm specific" news unrelated to market-wide news.
- Returns are correlated only through their exposures to common "market-wide" news captured by β_i.

Relationship of SI to GWN Model

Remark:

- The GWN model is a special case of Single Index (SI) Model where $\beta_i = 0$ for all $i = 1, \dots, N$.

$$R_{it} = \alpha_i + \varepsilon_{it}$$

  In this case, $\alpha_i = E[R_i] = \mu_i$

- In the GWN model there is only one source of news
- In the Single Index model there are two sources of news: market news and asset specific news

SI in Matrix Notation

Single Index Model with Matrix Algebra

$$\begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \beta_1 R_{Mt} \\ \vdots \\ \beta_N R_{Mt} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Nt} \end{pmatrix}$$

or

$$\mathbf{R}_t = \alpha + \beta \times R_{Mt} + \varepsilon_t$$

where

$$\mathbf{R}_t = \begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix},\ \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix},\ \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix},\ \varepsilon_t = \begin{pmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Nt} \end{pmatrix}$$

Statistical Properties of the SI Model (Unconditional)

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$

- $\mu_i = E[R_{it}] = \alpha_i + \beta_i \mu_M$
- $\sigma_i^2 = \mathrm{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2$
- $\sigma_{ij} = \mathrm{cov}(R_{it}, R_{jt}) = \sigma_M^2 \beta_i \beta_j$
- $R_{it} \sim N(\mu_i, \sigma_i^2) = N(\alpha_i + \beta_i \mu_M, \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2)$

Two Sources of Risk for Asset i

Derivations:

$$
\begin{aligned}
\mathrm{var}(R_{it}) &= \mathrm{var}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}) \\
&= \beta_i^2 \mathrm{var}(R_{Mt}) + \mathrm{var}(\varepsilon_{it}) + 2\beta_i \mathrm{cov}(R_{Mt}, \varepsilon_{it}) \\
&= \beta_i^2 \mathrm{var}(R_{Mt}) + \mathrm{var}(\varepsilon_{it}) \text{ (assume } \mathrm{cov}(R_{Mt}, \varepsilon_{it}) = 0\text{)} \\
&= \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_i^2 \sigma_M^2 &= \text{variance due to market news} \\
\sigma_{\varepsilon,i}^2 &= \text{variance due to non-market news}
\end{aligned}
$$

Relationship between Asset i and j

$$
\begin{aligned}
\sigma_{ij} &= \mathrm{cov}(R_{it}, R_{jt}) \\
&= \mathrm{cov}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}, \alpha_j + \beta_j R_{Mt} + \varepsilon_{jt}) \\
&= \mathrm{cov}(\beta_i R_{Mt}, \beta_j R_{Mt}) + \mathrm{cov}(\beta_i R_{Mt}, \varepsilon_{jt}) + \mathrm{cov}(\beta_j R_{Mt}, \varepsilon_{it}) + \mathrm{cov}(\varepsilon_{it}, \varepsilon_{jt}) \\
&= \beta_i \beta_j \mathrm{cov}(R_{Mt}, R_{Mt}) \\
&= \sigma_M^2 \beta_i \beta_j
\end{aligned}
$$

Relationship between Asset i and j

Implications:

$$\begin{aligned}\sigma_{ij} &= \mathrm{cov}(R_{it}, R_{jt}) \\ &= \sigma_M^2 \beta_i \beta_j\end{aligned}$$

- $\sigma_{ij} = 0$ if $\beta_i = 0$ or $\beta_j = 0$ (asset i or asset j do not respond to market news)
- $\sigma_{ij} > 0$ if $\beta_i, \beta_j > 0$ or $\beta_i, \beta_j < 0$ (asset i and j respond to market news in the same direction)
- $\sigma_{ij} < 0$ if $\beta_i > 0$ and $\beta_j < 0$ or if $\beta_i < 0$ and $\beta_j > 0$ (asset i and j respond to market news in opposite direction)

Statistical Properties of the SI Model (Conditional on $R_{Mt} = r_{Mt}$)

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$

Given that we observe, $R_{Mt} = r_{Mt}$

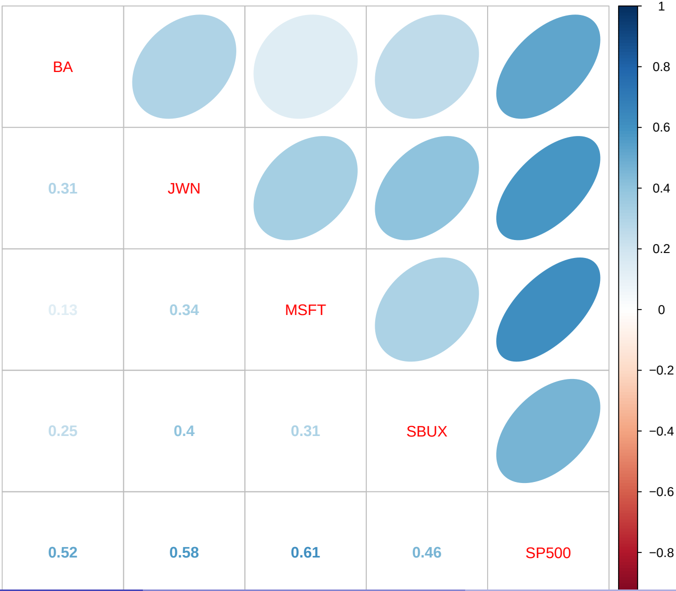
- $E[R_{it}|R_{Mt} = r_{Mt}] = \alpha_i + \beta_i r_{Mt}$
- $\mathrm{var}(R_{it}|R_{Mt} = r_{Mt}) = \sigma^2_{\varepsilon,i}$
- $\mathrm{cov}(R_{it}, R_{jt}|R_{Mt} = r_{Mt}) = 0$
- $R_{it}|R_{Mt} = r_{Mt} \sim N(\alpha_i + \beta_i r_{Mt}, \sigma^2_{\varepsilon,i})$

Get data from package IntroCompfinR

```
data(baDailyPrices, jwnDailyPrices, msftDailyPrices, sbuxDaily
baPrices = to.monthly(baDailyPrices, OHLC=FALSE)
jwnPrices = to.monthly(jwnDailyPrices, OHLC=FALSE)
msftPrices = to.monthly(msftDailyPrices, OHLC=FALSE)
sbuxPrices = to.monthly(sbuxDailyPrices, OHLC=FALSE)
sp500Prices = to.monthly(sp500DailyPrices, OHLC=FALSE)
siPrices = merge(baPrices, jwnPrices, msftPrices, sbuxPrices,
smpl = "1998-01::2012-05"
siPrices = siPrices[smpl]
siRetS = na.omit(Return.calculate(siPrices, method="simple"))
head(siRetS, n=3)
```

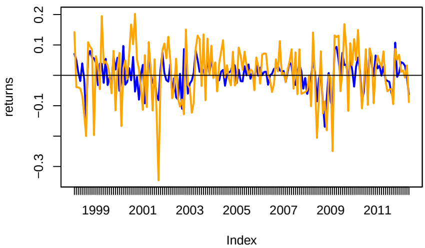
```
##                BA    JWN    MSFT   SBUX   SP500
## Feb 1998  0.1425 0.1298 0.13640 0.0825 0.07045
## Mar 1998 -0.0393 0.1121 0.05570 0.1438 0.04995
## Apr 1998 -0.0396 0.0262 0.00691 0.0629 0.00908
```

Sample covariance and correlation matrix

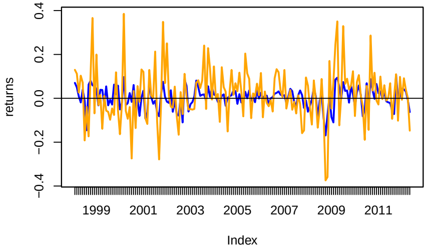


Timeplots with S&P 500 returns

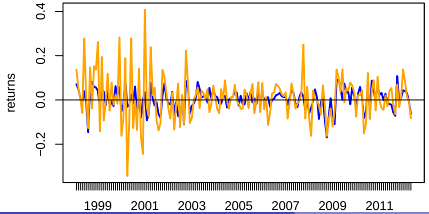
S&P 500 and Boeing



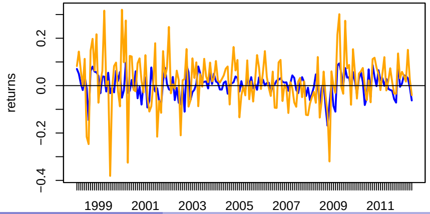
S&P 500 and Nordstom



S&P 500 and Microsoft



S&P 500 and Starbucks



Estimations of SI model parameters

```r
# sample statistics
assetNames = colnames(siRetS)[1:4]
muhat = colMeans(siRetS)
sig2hat = diag(covmatHat)
covAssetsSp500 = covmatHat[assetNames, "SP500"]
```

```r
# compute betahat
betaHat = covAssetsSp500/sig2hat["SP500"]
betaHat
```

```
##    BA   JWN  MSFT  SBUX
## 0.978 1.485 1.303 1.057
```

Estimations of SI model parameters Cont.

```
# compute alphahat
alphaHat = muhat[assetNames] - betaHat*muhat["SP500"]
alphaHat
```

```
##       BA     JWN    MSFT    SBUX
## 0.00516 0.01231 0.00544 0.01785
```

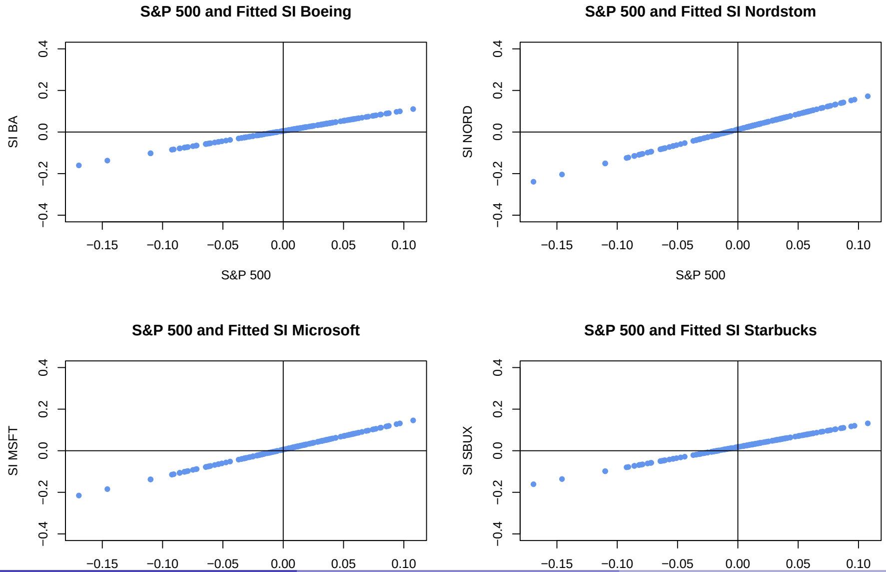
SI Model

The SI Model estimates are given by:

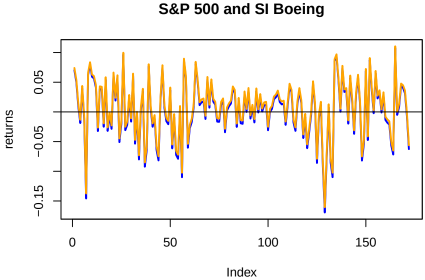
$$
\begin{aligned}
R_{BA} &= 0.00516 + 0.9758 * R_M \\
R_{JWN} &= 0.01231 + 1.485 * R_M \\
R_{MSFT} &= 0.00544 + 1.303 * R_M \\
R_{SBUX} &= 0.01785 + 1.057 * R_M
\end{aligned}
$$

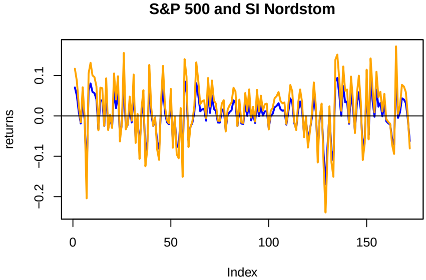
```
si.BA.fitted=alphaHat[1]+betaHat[1]*siRetS[,5]
si.JWN.fitted=alphaHat[2]+betaHat[2]*siRetS[,5]
si.MSFT.fitted=alphaHat[3]+betaHat[3]*siRetS[,5]
si.SBUX.fitted=alphaHat[4]+betaHat[4]*siRetS[,5]
```

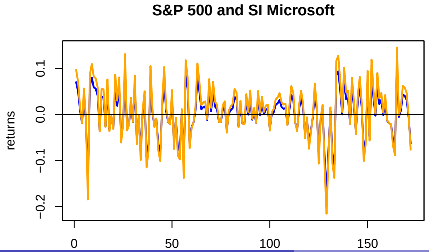
Scatterplots against S&P 500 returns

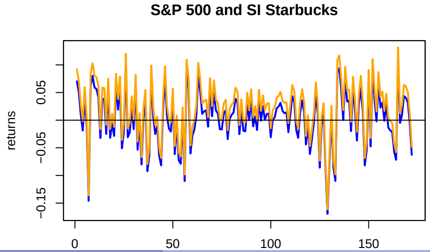


Fitted SI Timeplots with S&P 500 returns









Decomposition of Total Variance

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it} \\
\sigma_i^2 &= \mathrm{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2 \\
&\text{total variance} = \text{market variance} + \text{non-market variance}
\end{aligned}
$$

Divide both sides by σ_i^2

$$
\begin{aligned}
1 &= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} + \frac{\sigma_{\varepsilon,i}^2}{\sigma_i^2} \\
&= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} + \frac{\sigma_i^2 - \beta_i^2 \sigma_M^2}{\sigma_i^2} \\
&= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} + (1 - \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2})
\end{aligned}
$$

Decomposition of Total Variance

$$
\begin{aligned}
1 &= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} + \frac{\sigma_{\varepsilon,i}^2}{\sigma_i^2} \\
&= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} + \left(1 - \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2}\right) \\
&= R_i^2 + (1 - R_i^2)
\end{aligned}
$$

where

$$
\begin{aligned}
R_i^2 &= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} = \text{proportion of market variance} \\
1 - R_i^2 &= \text{proportion of non-market variance}
\end{aligned}
$$

Sharpe's Rule of Thumb: A typical stock has $R_i^2 = 30\%$; i.e., proportion of market variance in typical stock is 30% of total variance.

Return Covariance Matrix

Three asset example

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ \ i = 1, 2, 3 \\
\sigma_i^2 &= \mathrm{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2 \\
\sigma_{ij} &= \mathrm{cov}(R_{it}, R_{jt}) = \sigma_M^2 \beta_i \beta_j
\end{aligned}
$$

Covariance matrix

$$
\begin{aligned}
\Sigma &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \\
&= \begin{pmatrix} \beta_1^2 \sigma_M^2 + \sigma_{\varepsilon,1}^2 & \sigma_M^2 \beta_1 \beta_2 & \sigma_M^2 \beta_1 \beta_3 \\ \sigma_M^2 \beta_1 \beta_2 & \beta_2^2 \sigma_M^2 + \sigma_{\varepsilon,2}^2 & \sigma_M^2 \beta_2 \beta_3 \\ \sigma_M^2 \beta_1 \beta_3 & \sigma_M^2 \beta_2 \beta_3 & \beta_3^2 \sigma_M^2 + \sigma_{\varepsilon,3}^2 \end{pmatrix} \\
&= \sigma_M^2 \begin{pmatrix} \beta_1^2 & \beta_1 \beta_2 & \beta_1 \beta_3 \\ \beta_1 \beta_2 & \beta_2^2 & \beta_2 \beta_3 \\ \beta_1 \beta_3 & \beta_2 \beta_3 & \beta_3^2 \end{pmatrix} + \begin{pmatrix} \sigma_{\varepsilon,1}^2 & 0 & 0 \\ 0 & \sigma_{\varepsilon,2}^2 & 0 \\ 0 & 0 & \sigma_{\varepsilon,3}^2 \end{pmatrix}
\end{aligned}
$$

Return Covariance Matrix Cont.

$$\mathbf{R}_t = \alpha + \beta R_{Mt} + \varepsilon_t$$

$$\mathbf{R}_t = \begin{pmatrix} R_{1t} \\ R_{2t} \\ R_{3t} \end{pmatrix},\ \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},\ \varepsilon_t = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}$$

Then

$$\begin{aligned}\Sigma &= var(\mathbf{R}) = \beta \mathrm{var}(R_{Mt})\beta' + var(\varepsilon_t) \\ &= \sigma_M^2 \cdot \beta\beta' + \mathbf{D}\end{aligned}$$

where

$$\begin{aligned}\sigma_M^2 \cdot \beta\beta' &= \text{ covariance due to market} \\ \mathbf{D} &= \mathrm{diag}(\sigma_{\varepsilon,1}^2, \sigma_{\varepsilon,2}^2, \sigma_{\varepsilon,3}^2) = \text{asset specific variances}\end{aligned}$$

SI Model and Portfolios

Two Asset Example

$$\begin{aligned} R_{1t} &= \alpha_1 + \beta_1 R_{Mt} + \varepsilon_{1t} \\ R_{2t} &= \alpha_2 + \beta_2 R_{Mt} + \varepsilon_{2t} \\ x_1, x_2 &= \text{share invested in asset 1, 2} \\ x_1 + x_2 = 1 \end{aligned}$$

Portfolio return

$$\begin{aligned} R_{p,t} &= x_1 R_{1t} + x_2 R_{2t} \\ &= x_1(\alpha_1 + \beta_1 R_{Mt} + \varepsilon_{1t}) + x_2(\alpha_2 + \beta_2 R_{Mt} + \varepsilon_{2t}) \\ &= (x_1\alpha_1 + x_2\alpha_2) + (x_1\beta_1 + x_2\beta_2) R_{Mt} + (x_1\varepsilon_{1t} + x_2\varepsilon_{2t}) \\ &= \alpha_p + \beta_p R_{Mt} + \varepsilon_{p,t} \end{aligned}$$

where

$$\begin{aligned} \alpha_p &= x_1\alpha_1 + x_2\alpha_2 \\ \beta_p &= x_1\beta_1 + x_2\beta_2 \end{aligned}$$

SI Model with Large Portfolios (n Assets)

$$
\begin{aligned}
i &= 1, \ldots, N \text{ assets (e.g. } N = 500) \\
x_i &= \frac{1}{N} = \text{equal investment shares} \\
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}
\end{aligned}
$$

Portfolio return

$$
\begin{aligned}
R_{p,t} &= \sum_{i=1}^{N} x_i R_{it} = \sum_{i=1}^{N} x_i \left(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}\right) \\
&= \sum_{i=1}^{N} x_i \alpha_i + \left(\sum_{i=1}^{N} x_i \beta_i\right) R_{Mt} + \sum_{i=1}^{N} x_i \varepsilon_{it} \\
&= \frac{1}{N} \sum_{i=1}^{N} \alpha_i + \left(\frac{1}{N} \sum_{i=1}^{N} \beta_i\right) R_{Mt} + \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{it} \\
&= \bar{\alpha} + \bar{\beta} R_{Mt} + \bar{\varepsilon}_t
\end{aligned}
$$

SI Model with Large Portfolios (n Assets) Cont.

$$\bar{\alpha} = \frac{1}{N}\sum_{i=1}^{N}\alpha_i$$

$$\bar{\beta} = \frac{1}{N}\sum_{i=1}^{N}\beta_i$$

$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}$$

Result: For large N,

$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it} \approx E[\varepsilon_{it}] = 0$$

because $\varepsilon_{it} \sim \text{iid } N(0, \sigma^2_{\varepsilon,i})$.

Implications

In a large well diversified portfolio, the following results hold:

- $R_{p,t} \approx \bar{\alpha} + \bar{\beta} R_{Mt}$: all non-market risk is diversified away
- $\mathrm{var}(R_{p,t}) = \bar{\beta}^2 \mathrm{var}(R_{Mt})$: Magnitude of portfolio variance is proportional to market variance. Magnitude of portfolio variance is determined by portfolio beta $\bar{\beta}$
- $R_p^2 \approx 1$: Approximately 100% of portfolio variance is due to market variance

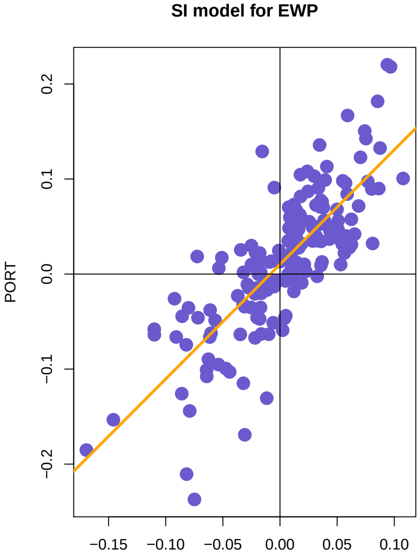
SI Model for an Equally Weighted Portfolio

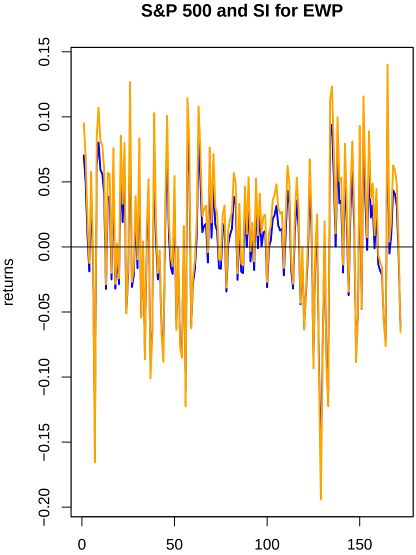
```
PORT = (siRetSmat[, "MSFT"] + siRetSmat[, "SBUX"] + siRetSmat[
# estimate SI model for equally weighted portfolio
PORT.fit = lm(PORT~siRetSmat[, "SP500"])
PORT.fit$coefficients
```

```
##          (Intercept) siRetSmat[, "SP500"]
##               0.0102               1.2056
```

$$R_p = 0.01019 + 1.2056 * R_M$$

SI Model for an Equally Weighted Portfolio Cont.





Estimating the Single Index Model

Sharpe's Single (SI) model:

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ t = 1, \ldots, T \\
\varepsilon_{it} &\sim \text{iid } N(0, \sigma^2_{\varepsilon,i}),\ R_{M,t} \sim \text{iid } N(\mu_M, \sigma^2_M) \\
\mathrm{cov}(R_{Mt}, \varepsilon_{is}) &= 0 \text{ for } t, s
\end{aligned}
$$

$$
\begin{aligned}
E[R_{it}] &= \mu_i = \alpha_i + \beta_i \mu_M \\
\mathrm{var}(R_{it}) &= \beta_i^2 \sigma^2_M + \sigma^2_{\varepsilon,i} \\
\alpha_i &= \mu_i - \beta_i \mu_M \\
\beta_i &= \frac{\mathrm{cov}(R_{it}, R_{Mt})}{\mathrm{var}(R_{Mt})} = \frac{\sigma_{iM}}{\sigma^2_M}
\end{aligned}
$$

Parameters to estimate: α_i, β_i and $\sigma^2_{\varepsilon,i}$

Plug-in Principle Estimators

Estimate model parameters using sample statistics

$$
\begin{aligned}
\hat{\beta}_i &= \frac{\hat{\sigma}_{iM}}{\hat{\sigma}_M^2} \\
\hat{\sigma}_{iM} &= \frac{1}{T-1}\sum_{t=1}^{T}(R_{it}-\hat{\mu}_i)(R_{Mt}-\hat{\mu}_M) \\
\hat{\sigma}_M^2 &= \frac{1}{T-1}\sum_{t=1}^{T}(R_{Mt}-\hat{\mu}_M)^2 \\
\hat{\mu}_i &= \frac{1}{T}\sum_{t=1}^{T}R_{it}, \\
\hat{\mu}_M &= \frac{1}{T}\sum_{t=1}^{T}R_{Mt}
\end{aligned}
$$

Plug-in Principle Estimators Cont.

Plug-in principle estimator for $\alpha_i = \mu_i - \beta_i \mu_M$:

$$\hat{\alpha}_i = \hat{\mu}_i - \hat{\beta}_i \hat{\mu}_M$$

Plug-in principle estimator of ε_{it} :

$$\hat{\varepsilon}_{it} = R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt}$$

Plug-in principle estimator for $\sigma^2_{\varepsilon,i} = \mathrm{var}(\varepsilon_{it})$:

$$\begin{aligned}
\hat{\sigma}^2_{\varepsilon,i} &= \frac{1}{T-2} \sum_{t=1}^{T} \hat{\varepsilon}_t^2 \\
&= \frac{1}{T-2} \sum_{t=1}^{T} \left(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt} \right)^2
\end{aligned}$$

Least Squares Estimation of SI Model Parameters

SI model postulates a linear relationship between R_{it} and R_{Mt} with intercept α_i and slope β_i :

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$

Estimate α_i and β_i by finding the "best fitting line" to the scatterplot of data

- Problem: How to define the best fitting line?
- Least Squares solution: minimize the sum of squared residuals (errors)

Least Squares Algorithm

$$
\begin{aligned}
\hat{\alpha}_i &= \text{initial guess for } \alpha_i \\
\hat{\beta}_i &= \text{initial guess for } \beta_i \\
\hat{R}_{it} &= \hat{\alpha}_i + \hat{\beta}_i R_{Mt} = \text{fitted line} \\
\hat{\varepsilon}_{it} &= R_{it} - \hat{R}_{it} \\
&= R_{it} - (\hat{\alpha}_i + \hat{\beta}_i R_{Mt}) = \text{residual}
\end{aligned}
$$

Determine the best fitting line by minimizing the *Sum of Squared Residuals* (SSR)

$$
\begin{aligned}
\mathrm{SSR}(\hat{\alpha}_i, \hat{\beta}_i) &= \sum_{t=1}^{T} \hat{\varepsilon}_{it}^2 \\
&= \sum_{t=1}^{T} \left(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt} \right)^2
\end{aligned}
$$

Least Squares Algorithm Cont.

That is, the least squares estimates solve

$$\min_{\hat{\alpha}_i, \hat{\beta}_i} \mathrm{SSR}(\hat{\alpha}_i, \hat{\beta}_i) = \sum_{t=1}^{T} \left(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt} \right)^2$$

Note: Because $\mathrm{SSR}(\hat{\alpha}_i, \hat{\beta}_i)$ is a quadratic function in $\hat{\alpha}_i, \hat{\beta}_i$, the first order conditions for a minimum give two linear equations in two unknowns and so there is an analytic solution to the minimization problem that we can find using calculus.

Calculus Solution

The first order conditions for a minimum are

$$0 = \frac{\partial \mathrm{SSR}(\hat{\alpha}_i, \hat{\beta}_i)}{\partial \hat{\alpha}_i} = -2\sum_{t=1}^{T}(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt}) = -2\sum_{t=1}^{T}\hat{\varepsilon}_{it}$$

$$0 = \frac{\partial \mathrm{SSR}(\hat{\alpha}_i, \hat{\beta}_i)}{\partial \hat{\beta}_i} = -2\sum_{t=1}^{T}(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt})R_{Mt} = -2\sum_{t=1}^{T}\hat{\varepsilon}_{it}R_{Mt}$$

These are two linear equations in two unknowns. Solving for $\hat{\alpha}_i$ and $\hat{\beta}_i$ gives

$$\hat{\alpha}_i = \hat{\mu}_i - \hat{\beta}_i \hat{\mu}_M$$

$$\hat{\beta}_i = \frac{\hat{\sigma}_{iM}}{\hat{\sigma}_M^2}$$

which are exactly the plug-in principle estimators!

Calculus Solution Cont.

Estimators for $\sigma^2_{\varepsilon,i}$ and $R - squared$

Utilize plug-in principle

$$
\begin{aligned}
\hat{\varepsilon}_{it} &= R_{it} - \hat{\alpha} - \hat{\beta}_i R_{Mt} \\
\hat{\sigma}^2_{\varepsilon,i} &= \frac{1}{T-2}\sum_{t=1}^{T} \hat{\varepsilon}^2_{it} \\
\hat{\sigma}_{\varepsilon,i} &= \sqrt{\hat{\sigma}^2_{\varepsilon,i}} = \text{SER} \\
&= \text{ standard error of regression}
\end{aligned}
$$

- $\hat{\sigma}_{\varepsilon,i}$ typical magnitude of residual = standard error of regression (SER)
- Divide by $T-2$ to get unbiased estimate of $\sigma^2_{\varepsilon,i}$
- $T-2$ = degrees of freedom = sample size - number of estimated parameters (α_i and β_i)

R-squared of Least Squares

Recall

$$\begin{aligned} R_i^2 &= \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2} \\ &= 1 - \frac{\sigma_{\varepsilon,i}^2}{\sigma_i^2} \\ &= \text{\% of variability due to market} \end{aligned}$$

Estimate using plug-in principle

$$\begin{aligned} \hat{R}_i^2 &= \frac{\hat{\beta}_i^2 \hat{\sigma}_M^2}{\hat{\sigma}_i^2} \\ &= 1 - \frac{\hat{\sigma}_{\varepsilon,i}^2}{\hat{\sigma}_i^2} \end{aligned}$$

Statistical Properties of Least Squares Estimates

Assuming the SI model generates the observed data, the estimators ($\hat{\alpha}_i$, $\hat{\beta}_i$ and $\hat{\sigma}^2_{\varepsilon,i}$) are random variables.

Properties:

- $\hat{\alpha}_i$, $\hat{\beta}_i$ and $\hat{\sigma}^2_{\varepsilon,i}$ are unbiased estimators

$$
\begin{aligned}
E[\hat{\alpha}_i] &= \alpha_i \\
E[\hat{\beta}_i] &= \beta_i \\
E[\hat{\sigma}^2_{\varepsilon,i}] &= \sigma^2_{\varepsilon,i}
\end{aligned}
$$

Statistical Properties of Least Squares Estimates Cont.

- Analytic standard errors are available for $\widehat{\mathrm{SE}}(\hat{\alpha}_i)$ and $\widehat{\mathrm{SE}}(\hat{\beta}_i)$

$$\widehat{\mathrm{SE}}(\hat{\alpha}_i) = \frac{\hat{\sigma}_{\varepsilon,i}}{\sqrt{T \cdot \hat{\sigma}_M^2}} \cdot \sqrt{\frac{1}{T}\sum_{t=1}^{T} R_{Mt}^2}$$

$$\widehat{\mathrm{SE}}(\hat{\beta}_i) = \frac{\hat{\sigma}_{\varepsilon,i}}{\sqrt{T \cdot \hat{\sigma}_M^2}}$$

- $\widehat{\mathrm{SE}}(\hat{\alpha}_i)$ and $\widehat{\mathrm{SE}}(\hat{\beta}_i) \to 0$ as T gets large $\Rightarrow$ $\hat{\alpha}_i$ and $\hat{\beta}_i$ are consistent estimators

Statistical Properties of Least Squares Estimates Cont.

- Standard errors for $\hat{\sigma}^2_{\varepsilon,i}$, $\hat{\sigma}_{\varepsilon,i}$ or $R-$square can be computed using the bootstrap
- For T large enough, the central limit theorem (CLT) tells us that

$$\hat{\alpha}_i \sim N(\alpha_i, \widehat{\mathrm{SE}}(\hat{\alpha}_i)^2)$$
$$\hat{\beta}_i \sim N(\beta_i, \widehat{\mathrm{SE}}(\hat{\beta}_i)^2)$$

- Approximate 95% confidence intervals

$$\hat{\alpha}_i \pm 2 \cdot \widehat{\mathrm{SE}}(\hat{\alpha}_i)$$
$$\hat{\beta}_i \pm 2 \cdot \widehat{\mathrm{SE}}(\hat{\beta}_i)$$

SI Model Using Matrix Algebra

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ \ t = 1, \ldots, T$$

Stack over observations $t = 1, \ldots, T$

$$\begin{pmatrix} R_{i1} \\ \vdots \\ R_{iT} \end{pmatrix} = \alpha_i \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \beta_i \begin{pmatrix} R_{M1} \\ \vdots \\ R_{MT} \end{pmatrix} + \begin{pmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} \end{pmatrix}$$

or

$$\mathbf{R}_i = \alpha_i \cdot \mathbf{1} + \beta_i \cdot \mathbf{R}_M + \varepsilon_i = \begin{pmatrix} \mathbf{1} & \mathbf{R}_M \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} + \varepsilon_i$$

$$= \mathbf{X}\gamma_i + \varepsilon_i$$

$$\mathbf{X} = \begin{pmatrix} \mathbf{1} & \mathbf{R}_M \end{pmatrix},\ \gamma_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$$

SI Model Using Matrix Algebra Cont.

Recall the least squares normal equations

$$0 = \frac{\partial \mathrm{SSR}(\hat{\alpha}_i, \hat{\beta}_i)}{\partial \hat{\alpha}_i} = -2\sum_{t=1}^{T}(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt})$$

$$0 = \frac{\partial \mathrm{SSR}(\hat{\alpha}_i, \hat{\beta}_i)}{\partial \hat{\beta}_i} = -2\sum_{t=1}^{T}(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{Mt})R_{Mt}$$

Using matrix algebra these equations are

$$\begin{pmatrix} \sum_{t=1}^{T} R_{it} \\ \sum_{t=1}^{T} R_{it} R_{Mt} \end{pmatrix} = \begin{pmatrix} T & \sum_{t=1}^{T} R_{Mt} \\ \sum_{t=1}^{T} R_{Mt} & \sum_{t=1}^{T} R_{Mt}^2 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_i \\ \hat{\beta}_i \end{pmatrix}$$

SI Model Using Matrix Algebra Cont.

Equivalently,

$$\left(\begin{array}{c}\mathbf{1}'\mathbf{R}_i\\ \mathbf{R}_M'\mathbf{R}_i\end{array}\right)=\left(\begin{array}{cc}\mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{R}_M\\ \mathbf{1}'\mathbf{R}_M & \mathbf{R}_M'\mathbf{R}_M\end{array}\right)\left(\begin{array}{c}\hat{\alpha}_i\\ \hat{\beta}_i\end{array}\right)$$

or

$$\mathbf{X}'\mathbf{R}_i=\mathbf{X}'\mathbf{X}\widehat{\gamma}_i$$

Solving for $\widehat{\gamma}_i$ gives the least squares estimates

$$\widehat{\gamma}_i=\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{R}_i$$

Estimating SI Model Covariance Matrix

Recall, in the SI model

$$\Sigma = \sigma_M^2 \beta\beta' + \mathbf{D}$$

$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \; \mathbf{D} = \begin{pmatrix} \sigma_{\varepsilon,1}^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{\varepsilon,n}^2 \end{pmatrix}$$

Estimate Σ using plug-in principle

$$\hat{\Sigma} = \hat{\sigma}_M^2 \widehat{\beta}\widehat{\beta}' + \widehat{\mathbf{D}}$$

where

$$\widehat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \; \widehat{\mathbf{D}} = \begin{pmatrix} \hat{\sigma}_{\varepsilon,1}^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \hat{\sigma}_{\varepsilon,n}^2 \end{pmatrix}$$

Single Index Model and Portfolio Theory

Idea: Use estimated SI model covariance matrix instead of sample covariance matrix in forming minimum variance portfolios:

$$
\begin{aligned}
\min_{x} \; & \mathbf{x}'\hat{\Sigma}\mathbf{x} \\
\text{s.t. } & \mathbf{x}'\widehat{\mu} = \mu_{p,0} \\
& \mathbf{x}'\mathbf{1} = 1
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\Sigma} &= \hat{\sigma}_M^2 \widehat{\beta}\widehat{\beta}' + \widehat{\mathbf{D}} \\
\widehat{\mu} &= \text{sample means}
\end{aligned}
$$