CFRM 462/Econ 424
Hypothesis Testing in the CER Model

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Hypothesis Testing

1. Specify hypothesis to be tested

$$H_0: \text{ null hypothesis versus. } H_1: \text{ alternative hypothesis}$$

2. Specify significance level of test

$$\text{level} = \Pr(\text{Reject } H_0 | H_0 \text{ is true})$$

3. Construct test statistic, T, from observed data

4. Use test statistic T to evaluate data evidence regarding H_0

$$|T| \text{ is big } \Rightarrow \text{ evidence against } H_0$$
$$|T| \text{ is small } \Rightarrow \text{ evidence in favor of } H_0$$

Decide to reject H_0 at specified significance level if value of T falls in the rejection region

$$T \in \text{ rejection region } \Rightarrow \text{ reject } H_0$$

Usually the rejection region of T is determined by a critical value, cv, such that

$$\begin{aligned} |T| &> cv \Rightarrow \text{reject } H_0 \\ |T| &\leq cv \Rightarrow \text{ do not reject } H_0 \end{aligned}$$

Decision Making and Hypothesis Tests

	Reality	
Decision	H_0 is true	H_0 is false
Reject H_0	Type I error	No error
Do not reject H_0	No error	Type II error

Significance Level of Test

$$\text{level} = \Pr(\text{Type I error})$$
$$\Pr(\text{Reject } H_0 | H_0 \text{ is true})$$

Goal: Constuct test to have a specified small significance level

$$\text{level} = 5\% \text{ or level} = 1\%$$

Power of Test

$$1 - \mathsf{Pr}(\text{Type II error})$$
$$= \mathsf{Pr}(\text{Reject } H_0 | H_0 \text{ is false})$$

Goal: Construct test to have high power

Problem: Impossible to simultaneously have level ≈ 0 and power ≈ 1. As level $\rightarrow 0$ power also $\rightarrow 0$.

Hypothesis Testing in CER Model

$$
\begin{aligned}
r_{it} &= \mu_i + \epsilon_{it} \quad t = 1, \cdots, T; \quad i = 1, \cdots N \\
\epsilon_{it} &\sim \text{iid } N(0, \sigma_i^2) \\
\text{cov}(\epsilon_{it},\ \epsilon_{jt}) &= \sigma_{ij},\ \text{cor}(\epsilon_{it},\ \epsilon_{jt}) = \rho_{ij} \\
\text{cov}(\epsilon_{it},\ \epsilon_{js}) &= 0 \quad t \neq s,\ \text{for all } i,\ j
\end{aligned}
$$

- Test for specific value

$$H_0 : \mu_i = \mu_i^0 \text{ vs. } H_1 : \mu_i \neq \mu_i^0$$
$$H_0 : \sigma_i = \sigma_i^0 \text{ vs. } H_1 : \sigma_i \neq \sigma_i^0$$
$$H_0 : \rho_{ij} = \rho_{ij}^0 \text{ vs. } H_1 : \rho_{ij} \neq \rho_{ij}^0$$

- Test for sign

$$H_0 : \mu_i = 0 \text{ vs. } H_1 : \mu_i > 0 \text{ or } \mu_i < 0$$
$$H_0 : \rho_{ij} = 0 \text{ vs. } H_1 : \rho_{ij} > 0 \text{ or } \rho_{ij} < 0$$

- Test for normal distribution

$$H_0 : r_{it} \sim \text{iid } N(\mu_i, \sigma_i^2)$$
$$H_1 : r_{it} \sim \text{ not normal}$$

- Test for no autocorrelation

$$H_0 : \rho_j = \text{corr}(r_{it}, r_{i,t-j}) = 0,\ j > 1$$
$$H_1 : \rho_j = \text{corr}(r_{it}, r_{i,t-j}) \neq 0 \text{ for some } j$$

- Test of constant parameters (covariance stationarity)

$$H_0 : \mu_i, \sigma_i \text{ and } \rho_{ij} \text{ are constant over entire sample}$$
$$H_1 : \mu_i\ \sigma_i \text{ or } \rho_{ij} \text{ changes in some sub-sample}$$

Tests for Coefficient Value Based on Asymptotic Normality (CLT)

Let $\hat{\theta}$ denote an estimator for θ. In many cases the CLT justifies the asymptotic normal distribution

$$\hat{\theta} \sim N(\theta, \mathsf{SE}(\hat{\theta})^2)$$

Consider testing

$$H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \neq \theta_0$$

Result: Under H_0,the t-statistic

$$t_{\theta=\theta_0} = \frac{\hat{\theta} - \theta^0}{\widehat{\mathsf{SE}}(\hat{\theta})} \sim N(0, 1) = Z$$

for large enough sample sizes T.

Intuition:

- If $t_{\theta=\theta_0} \approx 0$ then $\hat{\theta} \approx \theta_0$, and $H_0 : \theta = \theta_0$ should not be rejected

- If $|t_{\theta=\theta_0}| > 2$, say, then $\hat{\theta}$ is more than 2 values of $\widehat{\mathsf{SE}}(\hat{\theta})$ away from θ_0. This is very unlikely if $\theta = \theta_0$ because $\hat{\theta} \sim N(\theta_0, \mathsf{SE}(\hat{\theta})^2)$, so $H_0 : \theta \neq \theta_0$ should be rejected.

Steps for Hypothesis Test Based on t-statistic

1. Set significance level and determine critical value. A commonly used significance level is

$$\text{Pr(Type I error)} = 5\%$$

Test has two-sided alternative so the test critical value, $cv_{.025}$, is determined using

$$\begin{aligned} &\Pr(|Z| > cv_{.025}) = 0.05 \\ &\quad \Rightarrow cv_{.025} = -q^{Z}_{.025} = q^{Z}_{.975} = 1.96 \approx 2 \end{aligned}$$

2. Rule of thumb decision rule: Reject

$$H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \neq \theta_0$$

at 5% level if

$$|t_{\theta=\theta_0}| = \left| \frac{\hat{\theta} - \theta^0}{\widehat{\mathsf{SE}}(\hat{\theta})} \right| > 2$$

P-Value of two-sided test

$$\begin{aligned}
&\text{significance level at which test is just rejected} \\
&= \mathsf{Pr}(|Z| > t_{\theta=\theta_0}) \\
&= \mathsf{Pr}(Z < -t_{\theta=\theta_0}) + \mathsf{Pr}(t_{T-1} > t_{\theta=\theta_0}) \\
&= 2 \cdot \mathsf{Pr}(Z > |t_{\theta=\theta_0}|) \\
&= 2 \times (1 - \mathsf{Pr}(Z \leq |t_{\theta=\theta_0}|))
\end{aligned}$$

Decision rule based on P-Value

$$\text{Reject } H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \neq \theta_0 \text{ at } 5\% \text{ level if}$$
$$\text{P-Value } < 5\%$$

Relationship Between Hypothesis Tests and Confidence Intervals

$$H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \neq \theta_0$$
$$\text{level } = 5\%$$
$$cv_{.025} = q^Z_{.975} \approx 2$$
$$t_{\theta=\theta_0} = \frac{\hat{\theta} - \theta^0}{\widehat{\mathsf{SE}}(\hat{\theta})}$$
$$\text{Reject at 5\% level if } |t_{\theta=\theta_0}| > 2$$

Approximate 95% confidence interval for θ

$$\begin{aligned}\hat{\theta} &= \pm 2 \cdot \widehat{\mathsf{SE}}(\hat{\theta}) \\ &= [\hat{\theta} - 2 \cdot \widehat{\mathsf{SE}}(\hat{\theta}),\ \hat{\theta} + 2 \cdot \widehat{\mathsf{SE}}(\hat{\theta})]\end{aligned}$$

Decision: Reject $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ at 5% level if θ_0 does not lie in 95% confidence interval.

Test for Coefficient Sign

$$H_0 : \theta = 0 \text{ vs. } H_1 : \theta > 0$$

1. Test statistic

$$t_{\theta=0} = \frac{\hat{\theta}}{\widehat{\mathrm{SE}}(\hat{\theta})}$$

Intuition:

- If $t_{\theta=0} \approx 0$ then $\hat{\theta} \approx 0$, and $H_0 : \theta = 0$ should not be rejected

- If $t_{\theta=0} >> 0$, then this is very unlikely if $\theta = 0$, so $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$ should be rejected.

2. Set significance level and determine critical value. For example,

$$\Pr(\text{Type I error}) = 5\%$$

One-sided critical value $cv_{.05}$ is determined using

$$\begin{aligned} \Pr(Z > cv_{.05}) &= 0.05 \\ \Rightarrow cv_{.05} &= -q^Z_{.05} = q^Z_{.95} = 1.645 \end{aligned}$$

3. Decision rule:

$$\text{Reject } H_0 : \theta = 0 \text{ vs. } H_1 : \theta > 0 \text{ at 5\% level if}$$
$$t_{\theta=0} > 1.645$$

Test for Normal Distribution

$$H_0 : r_t \sim \text{iid } N(\mu, \sigma^2)$$
$$H_1 : r_t \sim \text{ not normal}$$

1. Test statistic (Jarque-Bera statistic)

$$\text{JB} = \frac{T}{6}\left(\widehat{\text{skew}}^2 + \frac{(\widehat{\text{kurt}} - 3)^2}{4}\right)$$

See R package tseries function `jarque.bera.test`

Intuition

- If $r_t \sim$ iid $N(\mu, \sigma^2)$ then $\widehat{\text{skew}}(r_t) \approx 0$ and $\widehat{\text{kurt}}(r_t) \approx 3$ so that JB ≈ 0.

- If r_t is not normally distributed then $\widehat{\text{skew}}(r_t) \neq 0$ and/or $\widehat{\text{kurt}}(r_t) \neq 3$ so that JB $>> 0$

Distribution of JB under H_0

If $H_0 : r_t \sim$ iid $N(\mu, \sigma^2)$ is true then for large enough T (so that CLT holds)

$$\text{JB} \sim \chi^2(2),$$

where $\chi^2(2)$ denotes a chi-square distribution with 2 degrees of freedom (d.f.).

Definition: Chi-square random variable and distribution

Let $Z_1, \ldots, Z_q$ be iid $N(0,1)$ random variables. Define

$$X = Z_1^2 + \cdots + Z_q^2$$

Then

$$\begin{aligned} X &\sim \chi^2(q) \\ q &= \text{degrees of freedom (d.f.)} \end{aligned}$$

Properties of $\chi^2(q)$ distribution

$$\begin{aligned} X &> 0 \\ E[X] &= q \\ \chi^2(q) &\to \text{normal as } q \to \infty \end{aligned}$$

R functions

`rchisq()`: simulate data

`dchisq()`: compute density

`pchisq()`: compute CDF

`qchisq()`: compute quantiles

2. Set significance level and determine critical value

$$\text{Pr(Type I error)} = 5\%$$

Critical value cv is determined using

$$\begin{aligned} \Pr(\chi^2(2) > cv_{.05}) &= 0.05 \\ \Rightarrow cv_{.05} &= q_{.95}^{\chi^2(2)} \approx 6 \end{aligned}$$

where $q_{.95}^{\chi^2(2)} \approx 6 \approx 95\%$ quantile of chi-square distribution with 2 degrees of freedom.

3. Decision rule:

$$\begin{gathered} \text{Reject } H_0 : r_t \sim \text{iid } N(\mu, \sigma^2) \\ \text{at } 5\% \text{ level if JB} > 6 \end{gathered}$$

4. P-Value of test

$$\text{significance level at which test is just rejected}$$
$$= \mathsf{Pr}(\chi^2(2) > \mathsf{JB})$$

Test for No Autocorrelation

Recall, the jth lag autocorrelation for r_t is

$$\begin{aligned}\rho_j &= \text{cor}(r_t, r_{t-j}) \\ &= \frac{\text{cov}(r_t, r_{t-j})}{\text{var}(r_t)}\end{aligned}$$

Hypotheses to be tested

$$\begin{aligned}H_0 &: \rho_j = 0, \text{ for all } j = 1, \ldots, q \\ H_1 &: \rho_j \neq 0 \text{ for some } j\end{aligned}$$

1. Estimate ρ_j using sample autocorrelation

$$\hat{\rho}_j = \frac{\frac{1}{T-1}\sum_{t=j+1}^{T}(r_t - \hat{\mu})(r_{t-j} - \hat{\mu})}{\frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{\mu})^2}$$

Result: Under $H_0 : \rho_j = 0$ for all $j = 1, \ldots, q$, if T is large then

$$\hat{\rho}_j \sim N\left(0, \frac{1}{T}\right) \text{ for all } j \geq 1$$

$$\text{SE}(\hat{\rho}_j) = \frac{1}{\sqrt{T}}$$

2. Test Statistic

$$t_{\rho_{j=0}} = \frac{\hat{\rho}_j}{\text{SE}(\hat{\rho}_j)} = \frac{\hat{\rho}_j}{1/\sqrt{T}} = \sqrt{T}\hat{\rho}_j$$

and 95% confidence interval

$$\hat{\rho}_j \pm 2 \cdot \frac{1}{\sqrt{T}}$$

3. Decision rule

$$\text{Reject } H_0 : \rho_j = 0 \text{ at 5\% level}$$
$$\text{if } |t_{\rho_{j=0}}| = \left|\sqrt{T}\hat{\rho}_j\right| > 2$$

That is, reject if

$$\hat{\rho}_j > \frac{2}{\sqrt{T}} \text{ or } \hat{\rho}_j < \frac{-2}{\sqrt{T}}$$

Remark:

The dotted lines on the sample ACF are at the points $\pm 2 \cdot \frac{1}{\sqrt{T}}$

Diagnostics for Constant Parameters

$$H_0 : \mu_i \text{ is constant over time vs. } H_1 : \mu_i \text{ changes over time}$$
$$H_0 : \sigma_i \text{ is constant over time vs. } H_1 : \sigma_i \text{ changes over time}$$
$$H_0 : \rho_{ij} \text{ is constant over time vs. } H_1 : \rho_{ij} \text{ changes over time}$$

Remarks

- Formal test statistics are available but require advanced statistics
  - See R package **strucchange**
- Informal graphical diagnostics: Rolling estimates of μ_i, σ_i and ρ_{ij}

Rolling Means

Idea: compute estimate of μ_i over rolling windows of length $n < T$

$$\begin{aligned}\hat{\mu}_{it}(n) &= \frac{1}{n}\sum_{j=0}^{n-1} r_{it-j} \\ &= \frac{1}{n}(r_{it} + r_{it-1} + \cdots + r_{it-n+1})\end{aligned}$$

R function (package **zoo**)

```
rollapply()
```

If $H_0 : \mu_i$ is constant is true, then $\hat{\mu}_{it}(n)$ should stay fairly constant over different windows.

If $H_0 : \mu_i$ is constant is false, then $\hat{\mu}_{it}(n)$ should fluctuate across different windows

Rolling Variances and Standard Deviations

Idea: Compute estimates of σ_i^2 and σ_i over rolling windows of length $n < T$

$$\hat{\sigma}_{it}^2(n) = \frac{1}{n-1}\sum_{j=0}^{n-1}(r_{it-j} - \hat{\mu}_{it}(n))^2$$
$$\hat{\sigma}_{it}(n) = \sqrt{\hat{\sigma}_{it}^2(n)}$$

If $H_0 : \sigma_i$ is constant is true, then $\hat{\sigma}_{it}(n)$ should stay fairly constant over different windows.

If $H_0 : \sigma_i$ is constant is false, then $\hat{\sigma}_{it}(n)$ should fluctuate across different windows

Rolling Covariances and Correlations

Idea: Compute estimates of σ_{jk} and ρ_{jk} over rolling windows of length $n < T$

$$\hat{\sigma}_{jk,t}(n) = \frac{1}{n-1}\sum_{i=0}^{n-1}(r_{jt-i} - \hat{\mu}_j(n))(r_{kt-i} - \hat{\mu}_k(n))$$
$$\hat{\rho}_{jk,t}(n) = \frac{\hat{\sigma}_{jk,t}(n)}{\hat{\sigma}_{jt}(n)\hat{\sigma}_{kt}(n)}$$

If $H_0 : \rho_{jk}$ is constant is true, then $\hat{\rho}_{jk,t}(n)$ should stay fairly constant over different windows.

If $H_0 : \rho_{jk}$ is constant is false, then $\hat{\rho}_{jk,t}(n)$ should fluctuate across different windows