Probability Review: Bivariate Random Variables

Eric Zivot

3/28/2021

Example - Two discrete rv's X and Y

Bivariate pdf

		Y		
	%	0	1	Pr(X)
	0	1/8	0	1/8
X	1	2/8	1/8	3/8
	2	1/8	2/8	3/8
	3	0	1/8	1/8
	Pr(Y)	4/8	4/8	1

$$p(x, y) = \Pr(X = x, Y = y) = \text{values in table}$$

$$e.g., \ p(0, 0) = \Pr(X = 0, Y = 0) = \ 1/8$$

Properties of joint pdf $p(x, y)$

$$
\begin{aligned}
S_{XY} &= \{(0,0),\ (0,1),\ (1,0),\ (1,1), \\
&\quad (2,0),\ (2,1),\ (3,0),\ (3,1)\} \\
p(x,y) &\geq 0 \text{ for } x, y \in S_{XY} \\
\sum_{x,y \in S_{XY}} p(x,y) &= 1
\end{aligned}
$$

Marginal pdfs

$$
\begin{aligned}
p(x) = \Pr(X = x) &= \sum_{y \in S_Y} p(x, y) \\
&= \text{sum over columns in joint table}
\end{aligned}
$$

$$
\begin{aligned}
p(y) = \Pr(Y = y) &= \sum_{x \in S_X} p(x, y) \\
&= \text{sum over rows in joint table}
\end{aligned}
$$

Conditional Probability

Suppose we know $Y = 0$. How does this knowledge affect the probability that $X = 0, 1, 2$, or 3? The answer involves conditional probability.

Example

$$\begin{aligned}\Pr(X = 0|Y = 0) &= \frac{\Pr(X = 0, Y = 0)}{\Pr(Y = 0)} \\ &= \frac{\text{joint probability}}{\text{marginal probability}} = \frac{1/8}{4/8} = 1/4\end{aligned}$$

Remark

$$\begin{aligned}&\Pr(X = 0|Y = 0) = 1/4 \neq \Pr(X = 0) = 1/8 \\ &\Longrightarrow X \text{ depends on } Y\end{aligned}$$

Definition - Conditional Probability

- The conditional pdf of X given $Y = y$ is, for all $x \in S_X$,

$$p(x|y) = \Pr(X = x|Y = y) = \frac{\Pr(X = x, Y = y)}{\Pr(Y = y)}$$

- The conditional pdf of Y given $X = x$ is, for all values of $y \in S_Y$

$$p(y|x) = \Pr(Y = y|X = x) = \frac{\Pr(X = x, Y = y)}{\Pr(X = x)}$$

Example: Conditional pdf for X given Y

x	$Pr(X = x)$	$Pr(X\|Y = 0)$	$Pr(X\|Y = 1)$
0	$1/8$	$2/8$	0
1	$3/8$	$4/8$	$2/8$
2	$3/8$	$2/8$	$4/8$
3	$1/8$	0	$2/8$

Example: Conditional pdf for Y given X

y	$Pr(Y = y)$	$Pr(Y\|X = 0)$	$Pr(Y\|X = 1)$	$Pr(Y\|X = 2)$
0	$1/2$	1	$2/3$	$1/3$
1	$1/2$	0	$1/3$	$2/3$

Conditional Mean and Variance

$$\mu_{X|Y=y} = E[X|Y=y] = \sum_{x \in S_X} x \cdot \Pr(X=x|Y=y),$$
$$\mu_{Y|X=x} = E[Y|X=x] = \sum_{y \in S_Y} y \cdot \Pr(Y=y|X=x).$$

$$\sigma^2_{X|Y=y} = \mathrm{var}(X|Y=y) = \sum_{x \in S_X} (x - \mu_{X|Y=y})^2 \cdot \Pr(X=x|Y=y),$$
$$\sigma^2_{Y|X=x} = \mathrm{var}(Y|X=x) = \sum_{y \in S_Y} (y - \mu_{Y|X=x})^2 \cdot \Pr(Y=y|X=x).$$

Example:

$$E[X] = 0 \cdot 1/8 + 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot 1/8 = 3/2$$
$$E[X|Y=0] = 0 \cdot 1/4 + 1 \cdot 1/2 + 2 \cdot 1/4 + 3 \cdot 0 = 1,$$
$$E[X|Y=1] = 0 \cdot 0 + 1 \cdot 1/4 + 2 \cdot 1/2 + 3 \cdot 1/4 = 2,$$

$$\begin{gathered}
\mathrm{var}(X) = (0-3/2)^2 \cdot 1/8 + (1-3/2)^2 \cdot 3/8 \\
+(2-3/2)^2 \cdot 3/8 + (3-3/2)^2 \cdot 1/8 = 3/4, \\
\mathrm{var}(X|Y=0) = (0-1)^2 \cdot 1/4 + (1-1)^2 \cdot 1/2 \\
+(2-1)^2 \cdot 1/2 + (3-1)^2 \cdot 0 = 1/2, \\
\mathrm{var}(X|Y=1) = (0-2)^2 \cdot 0 + (1-2)^2 \cdot 1/4 \\
+(2-2)^2 \cdot 1/2 + (3-2)^2 \cdot 1/4 = 1/2.
\end{gathered}$$

Independence

Let X and Y be discrete rvs with pdfs $p(x)$, $p(y)$, sample spaces S_X, S_Y and joint pdf $p(x,y)$. Then X and Y are independent rv's if and only if

$$p(x,y) = p(x) \cdot p(y)$$

$$\text{for all values of } x \in S_X \text{ and } y \in S_Y$$

Result: If X and Y are independent rv's, then

$$\begin{aligned} p(x|y) &= p(x) \;\; \text{for all } x \in S_X,\ y \in S_Y \\ p(y|x) &= p(y) \;\; \text{for all } x \in S_X,\ y \in S_Y \end{aligned}$$

Intuition

- Knowledge of X does not influence probabilities associated with Y
- Knowledge of Y does not influence probablities associated with X

Bivariate Distributions - Continuous rv's

The joint pdf of X and Y is a non-negative function $f(x, y)$ such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$$

Let $[x_1, x_2]$ and $[y_1, y_2]$ be intervals on the real line. Then

$$\begin{aligned}
&\Pr(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) \\
&\quad = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) dxdy \\
&\quad = \text{volume under probability surface} \\
&\quad \text{over the intersection of the intervals} \\
&\quad [x_1, x_2] \text{ and } [y_1, y_2]
\end{aligned}$$

Marginal Distributions

The marginal pdf of X is found by integrating y out of the joint pdf $f(x, y)$ and the marginal pdf of Y is found by integrating x out of the joint pdf:

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} f(x, y) dy, \\
f(y) &= \int_{-\infty}^{\infty} f(x, y) dx.
\end{aligned}
$$

Conditional Distributions

The conditional pdf of X given that $Y = y$, denoted $f(x|y)$, is computed as

$$f(x|y) = \frac{f(x,y)}{f(y)},$$

and the conditional pdf of Y given that $X = x$ is computed as

$$f(y|x) = \frac{f(x,y)}{f(x)}.$$

Marginal and Conditional Distributions

The conditional means are computed as

$$\mu_{X|Y=y} = E[X|Y=y] = \int x \cdot p(x|y)dx,$$
$$\mu_{Y|X=x} = E[Y|X=x] = \int y \cdot p(y|x)dy$$

and the conditional variances are computed as

$$\sigma^2_{X|Y=y} = \mathrm{var}(X|Y=y) = \int (x - \mu_{X|Y=y})^2 p(x|y)dx,$$
$$\sigma^2_{Y|X=x} = \mathrm{var}(Y|X=x) = \int (y - \mu_{Y|X=x})^2 p(y|x)dy.$$

Independence

Let X and Y be continuous random variables . X and Y are independent iff

$$f(x, y) = f(x)f(y)$$

Result: X and Y are independent iff

$$\begin{aligned} f(x|y) &= f(x), \text{ for } -\infty < x, y < \infty, \\ f(y|x) &= f(y), \text{ for } -\infty < x, y < \infty. \end{aligned}$$

Note: Independence is extremely useful in practice because it gives us an easy way to compute the joint pdf for two independent random variables: we simple compute the product of the marginal distributions.

Example: Bivariate standard normal distribution

Let $X \sim N(0,1)$, $Y \sim N(0,1)$ and let X and Y be independent. Then

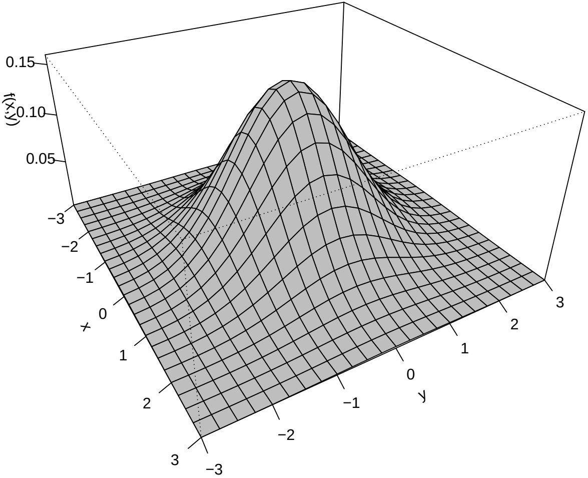
$$
\begin{aligned}
f(x,y) = f(x)f(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}.
\end{aligned}
$$

To find $\Pr(-1 < X < 1, -1 < Y < 1)$ we must solve

$$
\int_{-1}^{1}\int_{-1}^{1} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dxdy
$$

which, unfortunately, does not have an analytical solution. Numerical approximation methods are required to evaluate the above integral. See R package **mvtnorm**.

Bivariate Standard Normal Distribution

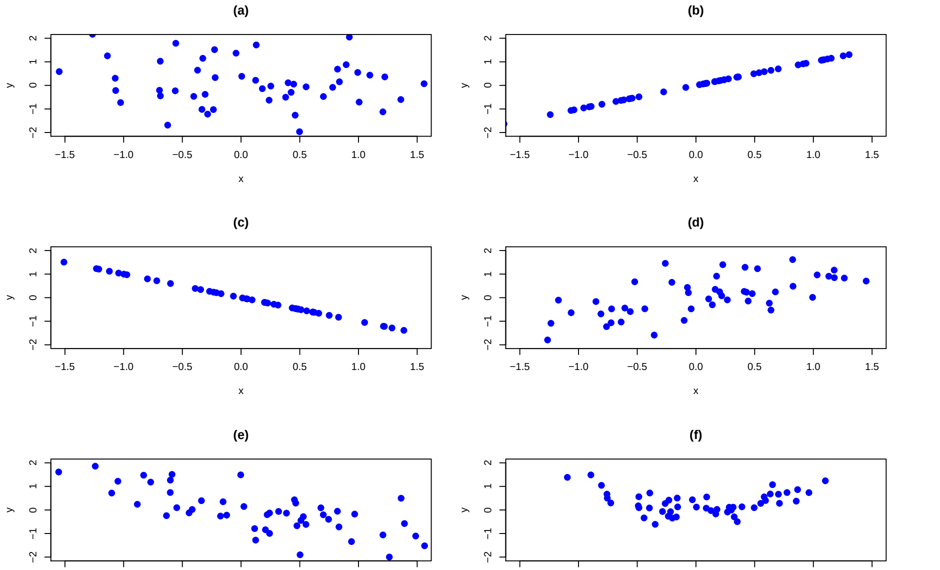


Independence continued

Result: If the random variables X and Y (discrete or continuous) are independent then the random variables $g(X)$ and $h(Y)$ are independent for any functions $g(\cdot)$ and $h(\cdot)$.

For example, if X and Y are independent then X^2 and Y^2 are also independent.

Covariance and Correlation

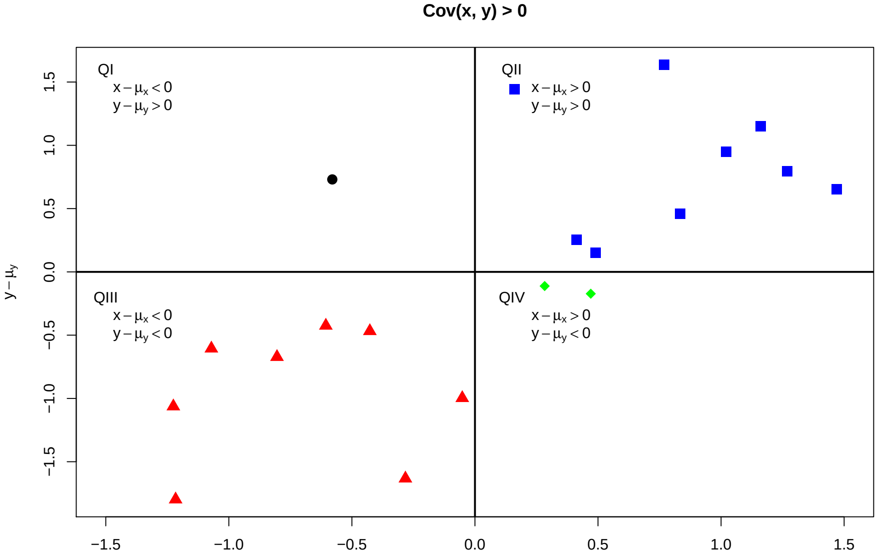


Covariance and Correlation - Measuring linear dependence between two rv's

Covariance measures direction but not strength of linear relationship between 2 rv's

$$
\begin{aligned}
\sigma_{XY} &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= \sum_{x.y \in S_{XY}} (x-\mu_X)(y-\mu_Y)\cdot p(x,y) \text{ (discrete)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f(x,y)dxdy \quad \text{(cts)}
\end{aligned}
$$

Illustrate Covariance



Example: Bivariate Discrete Distribution

Bivariate pdf

		Y		
	%	0	1	Pr(X)
	0	1/8	0	1/8
X	1	2/8	1/8	3/8
	2	1/8	2/8	3/8
	3	0	1/8	1/8
	Pr(Y)	4/8	4/8	1

$$
\begin{aligned}
\sigma_{XY} = \mathrm{Cov}(X, Y) &= (0 - 3/2)(0 - 1/2) \cdot 1/8 \\
&+ (0 - 3/2)(1 - 1/2) \cdot 0 + \cdots \\
&+ (3 - 3/2)(1 - 1/2) \cdot 1/8 = 1/4
\end{aligned}
$$

Properties of Covariance

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathrm{Cov}(Y, X) \\
\mathrm{Cov}(aX, bY) &= a \cdot b \cdot \mathrm{Cov}(X, Y) = a \cdot b \cdot \sigma_{XY} \\
\mathrm{Cov}(X, X) &= \mathrm{Var}(X) \\
X, Y \text{ independent } &\implies \mathrm{Cov}(X, Y) = 0 \\
\mathrm{Cov}(X, Y) &= 0 \nRightarrow X \text{ and } Y \text{ are independent} \\
\mathrm{Cov}(X, Y) &= E[XY] - E[X]E[Y]
\end{aligned}$$

Correlation

Correlation measures direction and strength of linear relationship between 2 rv's

$$\rho_{XY} = \mathrm{Cor}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\mathrm{SD}(X) \cdot \mathrm{SD}(Y)}$$
$$= \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y} = \text{ scaled covariance}$$

Example: Bivariate Discrete Distribution

$$\rho_{XY} = \mathrm{Cor}(X, Y) = \frac{1/4}{\sqrt{(3/4) \cdot (1/2)}} = 0.577$$

Properties of Correlation

$$
\begin{aligned}
-1 \leq \rho_{XY} &\leq 1 \\
\rho_{XY} &= 1 \text{ if } Y = aX + b \text{ and } a > 0 \\
\rho_{XY} &= -1 \text{ if } Y = aX + b \text{ and } a < 0 \\
\rho_{XY} &= 0 \text{ if and only if } \sigma_{XY} = 0 \\
\rho_{XY} &= 0 \nRightarrow \ X \text{ and } Y \text{ are independent in general} \\
\rho_{XY} &= 0 \implies \text{independence if } X \text{ and } Y \text{ are normal}
\end{aligned}
$$

Bivariate normal distribution

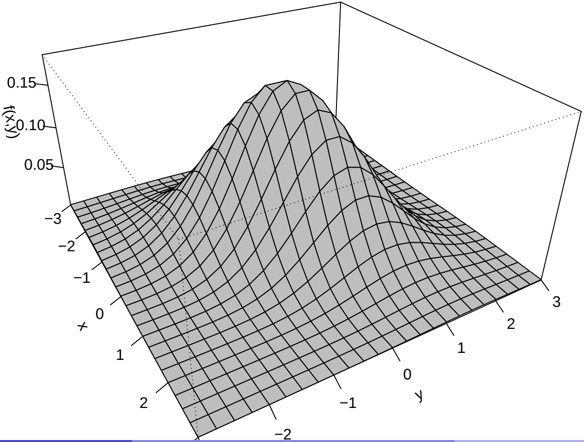
Let X and Y be distributed bivariate normal. The joint pdf is given by

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times$$
$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

where $E[X] = \mu_X$, $E[Y] = \mu_Y$, $\mathrm{SD}(X) = \sigma_X$, $\mathrm{SD}(Y) = \sigma_Y$, and $\rho = \mathrm{cor}(X,Y)$.

Bivariate Normal Distribution

$\mu_X = \mu_Y = 0, \sigma_X = \sigma_Y = 1$ and $\rho = 0.5$.



Linear Combination of 2 rv's

Let X and Y be rv's. Define a new rv Z that is a linear combination of X and Y :

$$Z = aX + bY$$

where a and b are constants. Then

$$\begin{aligned}\mu_Z &= E[Z] = E[aX + bY] \\ &= aE[X] + bE[Y] \\ &= a \cdot \mu_X + b \cdot \mu_Y\end{aligned}$$

Linear Combination of 2 rv's

$$\begin{aligned}
\sigma_Z^2 &= \mathrm{Var}(Z) = \mathrm{Var}(a \cdot X + b \cdot Y) \\
&= a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y) + 2a \cdot b \cdot \mathrm{Cov}(X, Y) \\
&= a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2a \cdot b \cdot \sigma_{XY}
\end{aligned}$$

Linear Combination of 2 rv's

Result: If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ then

$$Z \sim N(\mu_Z, \sigma_Z^2)$$

Example: Portfolio returns

$R_A =$ return on asset A with $E[R_A] = \mu_A$ and $\mathrm{Var}(R_A) = \sigma_A^2$

$R_B =$ return on asset B with $E[R_B] = \mu_B$ and $\mathrm{Var}(R_B) = \sigma_B^2$

$\mathrm{Cov}(R_A, R_B) = \sigma_{AB}$ and $\mathrm{Cor}(R_A,\ R_B) = \rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \cdot \sigma_B}$

Portfolio

$x_A =$ share of wealth invested in asset $A, x_B =$ share of wealth invested in asset B

$x_A + x_B = 1$ (exhaust all wealth in 2 assets)

$R_P = x_A \cdot R_A + x_B \cdot R_B =$ portfolio return

Portfolio Problem

Q: How much wealth should be invested in assets A and B?

Portfolio expected return (gain from investing)

$$
\begin{aligned}
E[R_P] = \mu_P &= E[x_A \cdot R_A + x_B \cdot R_B] \\
&= x_A E[R_A] + x_B E[R_B] \\
&= x_A \mu_A + x_B \mu_B
\end{aligned}
$$

Portfolio Problem

Portfolio variance (risk from investing)

$$
\begin{aligned}
\mathrm{Var}(R_P) &= \sigma_P^2 = \mathrm{Var}(x_A R_A + x_B R_B) \\
&= x_A^2 \mathrm{Var}(R_A) + x_B^2 \mathrm{Var}(R_B) + \\
& 2 \cdot x_A \cdot x_B \cdot \mathrm{Cov}(R_A,\ R_B) \\
&= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \\
\mathrm{SD}(R_P) &= \sqrt{\mathrm{Var}(R_P)} = \sigma_P \\
&= \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \right)^{1/2}
\end{aligned}
$$

Linear Combination of N rv's.

Let $X_1,\ X_2, \cdots, X_N$ be rvs and let $a_1, a_2, \ldots, a_N$ be constants. Define

$$Z = a_1 X_1 + a_2 X_2 + \cdots + a_N X_N = \sum_{i=1}^{N} a_i X_i$$

Then

$$\begin{aligned}
\mu_Z = E[Z] &= a_1 E[X_1] + a_2 E[X_2] + \cdots + a_N E[X_N] \\
&= \sum_{i=1}^{N} a_i E[X_i] = \sum_{i=1}^{N} a_i \mu_i
\end{aligned}$$

Linear Combination of N rv's.

For the variance,

$$
\begin{aligned}
\sigma_Z^2 = \mathrm{Var}(Z) &= a_1^2 \mathrm{Var}(X_1) + \cdots + a_N^2 \mathrm{Var}(X_N) \\
&+ 2a_1 a_2 \mathrm{Cov}(X_1,\ X_2) + 2a_1 a_3 \mathrm{Cov}(X_1,\ X_3) + \cdots \\
&+ 2a_2 a_3 \mathrm{Cov}(X_2,\ X_3) + 2a_2 a_4 \mathrm{Cov}(X_2,\ X_4) + \cdots \\
&+ 2a_{N-1} a_N \mathrm{Cov}(X_{N-1}, X_N)
\end{aligned}
$$

Note: N variance terms and $N(N-1) = N^2 - N$ covariance terms. If $N = 100$, there are $100 \times 99 = 9900$ covariance terms!

Linear Combination of N rv's.

Result: If $X_1,\ X_2, \cdots, X_N$ are each normally distributed random variables then

$$Z = \sum_{i=1}^{N} a_i X_i \sim N(\mu_Z, \sigma_Z^2)$$

Example: Portfolio variance with three assets

R_A, R_B, R_C are simple returns on assets A, B and C

x_A, x_B, x_C are portfolio shares such that $x_A + x_B + x_C = 1$

$R_p = x_A R_A + x_B R_B + x_C R_C$

Portfolio variance

$$\begin{aligned}\sigma_P^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\ &\quad + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC}\end{aligned}$$

Example: Portfolio variance with three assets

Note: Portfolio variance calculation may be simplified using matrix layout

$$\begin{array}{cccc} & x_A & x_B & x_C \\ x_A & \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ x_B & \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ x_C & \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{array}$$

The square-root-of-time rule

$$r_t = \ln(1 + R_t) = \text{ monthly cc return}$$
$$r_t \sim N(\mu,\ \sigma^2) \quad \text{for all } t$$
$$\mathrm{Cov}(r_t, r_s) = 0 \text{ for all } t \neq s$$

Annual return

$$\begin{aligned} r_t(12) &= \sum_{j=0}^{11} r_{t-j} \\ &= r_t + r_{t-1} + \cdots + r_{t-11} \end{aligned}$$

The square-root-of-time rule

Then

$$
\begin{aligned}
E[r_t(12)] &= \sum_{j=0}^{11} E[r_{t-j}] \\
&= \sum_{j=0}^{11} \mu \qquad (E[r_t] = \mu \text{ for all } t) \\
&= 12\mu \quad (\mu = \text{mean of monthly return})
\end{aligned}
$$

The square-root-of-time rule

$$\begin{aligned}
\mathrm{Var}(r_t(12)) &= \mathrm{Var}\left(\sum_{j=0}^{11} r_{t-j}\right) \\
&= \sum_{j=0}^{11} \mathrm{Var}(r_{t-j}) = \sum_{j=0}^{11} \sigma^2 \\
&= 12 \cdot \sigma^2 \quad (\sigma^2 = \text{monthly variance}) \\
\mathrm{SD}(r_t(12)) &= \sqrt{12} \cdot \sigma \text{ (square root of timerule)}
\end{aligned}$$

Result:

$$r_t(12) \sim N(12\mu, 12\sigma^2)$$