Risk Budgeting

Eric Zivot

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Portfolio Risk Budgeting

Idea: Additively decompose a measure of portfolio risk into contributions from the individual assets in the portfolio.

- Show which assets are most responsible for portfolio risk
- Help make decisions about rebalancing the portfolio to alter the risk
- Construct "risk parity" portfolios where assets have equal risk contributions

Example: 2 risky asset portfolio

$$
\begin{aligned}
R_p &= x_1 R_1 + x_2 R_2 \\
\sigma_p^2 &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12} \\
\sigma_p &= \left(x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}\right)^{1/2}
\end{aligned}
$$

Q: How much of σ_p is attributable to each asset?

Example: 2 risky asset portfolio

Case 1: $\sigma_{12} = 0$

$$
\begin{aligned}
\sigma_p^2 &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 = \text{additive decomposition} \\
x_1^2\sigma_1^2 &= \text{portfolio variance contribution of asset 1} \\
x_2^2\sigma_2^2 &= \text{portfolio variance contribution of asset 2} \\
\frac{x_1^2\sigma_1^2}{\sigma_p^2} &= \text{percent variance contribution of asset 1} \\
\frac{x_2^2\sigma_2^2}{\sigma_p^2} &= \text{percent variance contribution of asset 2}
\end{aligned}
$$

Example: 2 risky asset portfolio

Note:

$$\sigma_p = \sqrt{x_1^2\sigma_1^2 + x_2^2\sigma_2^2} \neq x_1\sigma_1 + x_2\sigma_2.$$

To get an additive decomposition we use

$$\begin{aligned}
\frac{x_1^2\sigma_1^2}{\sigma_p} + \frac{x_2^2\sigma_2^2}{\sigma_p} &= \frac{\sigma_p^2}{\sigma_p} = \sigma_p \\
\frac{x_1^2\sigma_1^2}{\sigma_p} &= \text{portfolio sdcontribution of asset 1} \\
\frac{x_2^2\sigma_2^2}{\sigma_p} &= \text{portfolio sdcontribution of asset 2}
\end{aligned}$$

Notice that percent sd contributions are the same as percent variance contributions.

Example: 2 risky asset portfolio

Case 2: $\sigma_{12} \neq 0$

$$\begin{aligned} \sigma_p^2 &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12} \\ &= \left(x_1^2\sigma_1^2 + x_1x_2\sigma_{12}\right) + \left(x_2^2\sigma_2^2 + x_1x_2\sigma_{12}\right). \end{aligned}$$

Here we split the covariance contribution $2x_1x_2\sigma_{12}$ to portfolio variance evenly between the two assets and define

$$\begin{aligned} x_1^2\sigma_1^2 + x_1x_2\sigma_{12} &= \text{variance contribution of asset 1} \\ x_2^2\sigma_2^2 + x_1x_2\sigma_{12} &= \text{variance contribution of asset 2} \end{aligned}$$

Example: 2 risky asset portfolio

We can also define an additive decomposition for σ_p

$$\begin{aligned}
\sigma_p &= \frac{x_1^2\sigma_1^2 + x_1 x_2 \sigma_{12}}{\sigma_p} + \frac{x_2^2\sigma_2^2 + x_1 x_2 \sigma_{12}}{\sigma_p} \\
\frac{x_1^2\sigma_1^2 + x_1 x_2 \sigma_{12}}{\sigma_p} &= \text{sd contribution of asset 1} \\
\frac{x_2^2\sigma_2^2 + x_1 x_2 \sigma_{12}}{\sigma_p} &= \text{sd contribution of asset 2}
\end{aligned}$$

Euler's Theorem and Risk Decompositions

- When we used σ_p^2 or σ_p to measure portfolio risk, we were able to easily derive sensible risk decompositions.
- If we measure portfolio risk by value-at-risk or some other risk measure it is not so obvious how to define individual asset risk contributions.
- For portfolio risk measures that are homogenous functions of degree one in the portfolio weights, *Euler's theorem* provides a general method for additively decomposing risk into asset specific contributions.

Homogenous functions and Euler's theorem

Definition. Let $f(x_1, \ldots, x_n)$ be a continuous and differentiable function of the variables $x_1, \ldots, x_n$. f is *homogeneous of degree one* if for any constant c, $f(c \cdot x_1, \ldots, c \cdot x_n) = c \cdot f(x_1, \ldots, x_n)$.

Note: In matrix notation we have $f(x_1, \ldots, x_n) = f(\mathbf{x})$ where $\mathbf{x} = (x_1, \ldots, x_n)'$. Then f is homogeneous of degree one if $f(c \cdot \mathbf{x}) = c \cdot f(\mathbf{x})$

Examples

Let $f(x_1, x_2) = x_1 + x_2$. Then

$$f(c \cdot x_1, c \cdot x_2) = c \cdot x_1 + c \cdot x_2 = c \cdot (x_1 + x_2) = c \cdot f(x_1, x_2)$$

Let $f(x_1, x_2) = x_1^2 + x_2^2$. Then

$$f(c \cdot x_1, c \cdot x_2) = c^2 x_1^2 + x_2^2 c^2 = c^2 (x_1^2 + x_2^2) \neq c \cdot f(x_1, x_2)$$

Let $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ Then

$$f(c \cdot x_1, c \cdot x_2) = \sqrt{c^2 x_1^2 + c^2 x_2^2} = c\sqrt{(x_1^2 + x_2^2)} = c \cdot f(x_1, x_2)$$

Examples

Define $\mathbf{x} = (x_1, x_2)'$ and $\mathbf{1} = (1, 1)'$.

Let $f(x_1, x_2) = x_1 + x_2 = \mathbf{x}'\mathbf{1} = f(\mathbf{x})$. Then

$$f(c \cdot \mathbf{x}) = (c \cdot \mathbf{x})' \, \mathbf{1} = c \cdot (\mathbf{x}'1) = c \cdot f(\mathbf{x}).$$

Let $f(x_1, x_2) = x_1^2 + x_2^2 = \mathbf{x}'\mathbf{x} = f(\mathbf{x})$. Then

$$f(c \cdot \mathbf{x}) = (c \cdot \mathbf{x})'(c \cdot \mathbf{x}) = c^2 \cdot \mathbf{x}'\mathbf{x} \neq c \cdot f(\mathbf{x}).$$

Let $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} = (\mathbf{x}'\mathbf{x})^{1/2} = f(\mathbf{x})$. Then

$$f(c \cdot \mathbf{x}) = \left((c \cdot \mathbf{x})'(c \cdot \mathbf{x})\right)^{1/2} = c \cdot \left(\mathbf{x}'\mathbf{x}\right)^{1/2} = c \cdot f(\mathbf{x}).$$

Homogeneity of Portfolio Quantities

Consider a portfolio of n assets with weights $\mathbf{x} = (x_1, \ldots, x_n)'$ with

$$\begin{aligned} \mathbf{R} &= (R_1, \ldots, R_n)' \\ E[\mathbf{R}] &= \mu,\ \mathrm{cov}(\mathbf{R}) = \Sigma \end{aligned}$$

Define

$$\begin{aligned} R_p &= R_p(\mathbf{x}) = \mathbf{x}'\mathbf{R}, \\ \mu_p &= \mu_p(\mathbf{x}) = \mathbf{x}'\mu \\ \sigma_p^2 &= \sigma_p^2(\mathbf{x}) = \mathbf{x}'\Sigma\mathbf{x}, \sigma_p = \sigma_p(\mathbf{x}) = (\mathbf{x}'\Sigma\mathbf{x})^{1/2} \end{aligned}$$

Result: Portfolio return $R_p(\mathbf{x})$, expected return $\mu_p(\mathbf{x})$ and standard deviation $\sigma_p(\mathbf{x})$ are homogenous functions of degree one in the portfolio weight vector $\mathbf{x}$.

Homogeneity of Portfolio Quantities

The key result is for volatility $\sigma_p(\mathbf{x}) = (\mathbf{x}'\Sigma\mathbf{x})^{1/2}$:

$$\begin{aligned}
\sigma_p(c \cdot \mathbf{x}) &= ((c \cdot \mathbf{x})'\Sigma(c \cdot \mathbf{x}))^{1/2} \\
&= c \cdot (\mathbf{x}'\Sigma\mathbf{x})^{1/2} \\
&= c \cdot \sigma_p(\mathbf{x})
\end{aligned}$$

Homogeneity of Normal Value-at-Risk

Result. Let W_0 denote the initial value of the portfolio and assume $\mathbf{R} \sim N(\mu, \Sigma)$. Then the $\alpha \times 100\%$ Value-at-Risk

$$\mathrm{VaR}_\alpha(\mathbf{x}) = \left(\mu_p(\mathbf{x}) + \sigma_p(\mathbf{x}) \times q_\alpha^Z\right) \times W_0$$

is homogenous of degree one in the portfolio weight vector $\mathbf{x}$

Euler's theorem

Let $f(x_1, \ldots, x_n) = f(\mathbf{x})$ be a continuous, differentiable and homogenous of degree one function of the variables $\mathbf{x} = (x_1, \ldots, x_n)'$. Then

$$\begin{aligned} f(\mathbf{x}) &= x_1 \cdot \frac{\partial f(\mathbf{x})}{\partial x_1} + x_2 \cdot \frac{\partial f(\mathbf{x})}{\partial x_2} + \cdots + x_n \cdot \frac{\partial f(\mathbf{x})}{\partial x_n} \\ &= \mathbf{x}' \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}, \end{aligned}$$

where

$$\underset{(n \times 1)}{\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}} = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{pmatrix}$$

Verifying Euler's theorem

The function $f(x_1, x_2) = x_1 + x_2 = f(\mathbf{x}) = \mathbf{x}'\mathbf{1}$ is homogenous of degree one, and

$$
\begin{aligned}
\frac{\partial f(\mathbf{x})}{\partial x_1} &= \frac{\partial f(\mathbf{x})}{\partial x_2} = 1 \\
\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} &= \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{1}
\end{aligned}
$$

By Euler's theorem,

$$
\begin{aligned}
f(x) &= x_1 \cdot 1 + x_2 \cdot 1 = x_1 + x_2 \\
&= \mathbf{x}'\mathbf{1}
\end{aligned}
$$

Verifying Euler's theorem

The function $f(x_1, x_2) = (x_1^2 + x_2^2)^{1/2} = f(\mathbf{x}) = (\mathbf{x}'\mathbf{x})^{1/2}$ is homogenous of degree one, and

$$\begin{aligned}
\frac{\partial f(\mathbf{x})}{\partial x_1} &= \frac{1}{2}\left(x_1^2 + x_2^2\right)^{-1/2} 2x_1 = x_1\left(x_1^2 + x_2^2\right)^{-1/2}, \\
\frac{\partial f(\mathbf{x})}{\partial x_2} &= \frac{1}{2}\left(x_1^2 + x_2^2\right)^{-1/2} 2x_2 = x_2\left(x_1^2 + x_2^2\right)^{-1/2}.
\end{aligned}$$

By Euler's theorem

$$\begin{aligned}
f(x) &= x_1 \cdot x_1\left(x_1^2 + x_1^2\right)^{-1/2} + x_2 \cdot x_2\left(x_1^2 + x_2^2\right)^{-1/2} \\
&= \left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2\right)^{-1/2} \\
&= \left(x_1^2 + x_2^2\right)^{1/2}.
\end{aligned}$$

Verifying Euler's theorem

Using matrix algebra we have

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \frac{\partial (\mathbf{x}'\mathbf{x})^{1/2}}{\partial \mathbf{x}} = \frac{1}{2}(\mathbf{x}'\mathbf{x})^{-1/2}\frac{\partial \mathbf{x}'\mathbf{x}}{\partial \mathbf{x}} = \frac{1}{2}(\mathbf{x}'\mathbf{x})^{-1/2}2\mathbf{x} = (\mathbf{x}'\mathbf{x})^{-1/2} \cdot \mathbf{x}$$

so by Euler's theorem

$$f(\mathbf{x}) = \mathbf{x}'\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{x}'(\mathbf{x}'\mathbf{x})^{-1/2} \cdot \mathbf{x} = (\mathbf{x}'\mathbf{x})^{-1/2}\mathbf{x}'\mathbf{x} = (\mathbf{x}'\mathbf{x})^{1/2}$$

Risk decomposition using Euler's theorem

Let $\mathrm{RM}_p(\mathbf{x})$ denote a portfolio risk measure that is a homogenous function of degree one in the portfolio weight vector $\mathbf{x}$. For example,

$$\mathrm{RM}_p(\mathbf{x}) = \sigma_p(\mathbf{x}) = (\mathbf{x}'\Sigma\mathbf{x})^{1/2}$$

Euler's theorem gives the additive risk decomposition

$$\begin{aligned} \mathrm{RM}_p(\mathbf{x}) &= x_1 \frac{\partial \mathrm{RM}_p(\mathbf{x})}{\partial x_1} + x_2 \frac{\partial \mathrm{RM}_p(\mathbf{x})}{\partial x_2} + \cdots + x_n \frac{\partial \mathrm{RM}_p(\mathbf{x})}{\partial x_n} \\ &= \sum_{i=1}^{n} x_i \frac{\partial \mathrm{RM}_p(\mathbf{x})}{\partial x_i} \\ &= \mathbf{x}' \frac{\partial \mathrm{RM}_p(\mathbf{x})}{\partial \mathbf{x}} \end{aligned}$$

Risk decomposition using Euler's theorem

Here, $\frac{\partial \mathrm{RM}_p(\mathbf{x})}{\partial x_i}$ are called *marginal contributions to risk* (MCRs):

$$\mathrm{MCR}_i^{RM} = \frac{\partial \mathrm{RM}_p(\mathbf{x})}{\partial x_i} = \text{ marginal contribution to risk of asset i,}$$

The *contributions to risk* (CRs) are defined as the weighted marginal contributions:

$$\mathrm{CR}_i^{RM} = x_i \cdot \mathrm{MCR}_i^{RM} = \text{ contribution to risk of asset i,}$$

Then

$$\begin{aligned} \mathrm{RM}_p(\mathbf{x}) &= x_1 \cdot \mathrm{MCR}_1^{RM} + x_2 \cdot \mathrm{MCR}_2^{RM} + \cdots + x_n \cdot \mathrm{MCR}_n^{RM} \\ &= \mathrm{CR}_1^{RM} + \mathrm{CR}_2^{RM} + \cdots + \mathrm{CR}_n^{RM} \end{aligned}$$

Risk decomposition using Euler's theorem

If we divide the contributions to risk by $\mathrm{RM}_p(\mathbf{x})$ we get the *percent contributions to risk* (PCRs)

$$1 = \frac{\mathrm{CR}_1^{RM}}{\mathrm{RM}_p(\mathbf{x})} + \cdots + \frac{\mathrm{CR}_n^{RM}}{\mathrm{RM}_p(\mathbf{x})} = \mathrm{PCR}_1^{RM} + \cdots + \mathrm{PCR}_n^{RM},$$

where

$$\mathrm{PCR}_i^{RM} = \frac{\mathrm{CR}_i^{RM}}{\mathrm{RM}_p(\mathbf{x})} = \text{ percent contribution of asset i}$$

Risk Decomposition for Portfolio SD

$$\mathrm{RM}_p(\mathbf{x}) = \sigma_p(\mathbf{x}) = (\mathbf{x}'\Sigma\mathbf{x})^{1/2}$$

Because $\sigma_p(\mathbf{x})$ is homogenous of degree 1 in $\mathbf{x}$, by Euler's theorem

$$\sigma_p(\mathbf{x}) = x_1\frac{\partial\sigma_p(\mathbf{x})}{\partial x_1} + x_2\frac{\partial\sigma_p(\mathbf{x})}{\partial x_2} + \cdots + x_n\frac{\partial\sigma_p(\mathbf{x})}{\partial x_n} = \mathbf{x}'\frac{\partial\sigma_p(\mathbf{x})}{\partial\mathbf{x}}$$

Now

$$\begin{aligned}
\frac{\partial\sigma_p(\mathbf{x})}{\partial\mathbf{x}} &= \frac{\partial(\mathbf{x}'\Sigma\mathbf{x})^{1/2}}{\partial\mathbf{x}} = \frac{1}{2}(\mathbf{x}'\Sigma\mathbf{x})^{-1/2}2\Sigma\mathbf{x} \\
&= \frac{\Sigma\mathbf{x}}{(\mathbf{x}'\Sigma\mathbf{x})^{1/2}} = \frac{\Sigma\mathbf{x}}{\sigma_p(\mathbf{x})} \\
&\Rightarrow \frac{\partial\sigma_p(\mathbf{x})}{\partial x_i} = \mathrm{MCR}_i^{\sigma} = \text{ith row of } \frac{\Sigma\mathbf{x}}{\sigma_p(\mathbf{x})}
\end{aligned}$$

Example: 2 asset portfolio

$$\begin{aligned}
\sigma_p(\mathbf{x}) &= (\mathbf{x}'\Sigma\mathbf{x})^{1/2} = \left(x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}\right)^{1/2} \\
\Sigma\mathbf{x} &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\sigma_1^2 + x_2\sigma_{12} \\ x_2\sigma_2^2 + x_1\sigma_{12} \end{pmatrix} \\
\frac{\Sigma\mathbf{x}}{\sigma_p(\mathbf{x})} &= \begin{pmatrix} \left(x_1\sigma_1^2 + x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\ \left(x_2\sigma_2^2 + x_1\sigma_{12}\right)/\sigma_p(\mathbf{x}) \end{pmatrix}
\end{aligned}$$

so that

$$\begin{aligned}
\mathrm{MCR}_1^{\sigma} &= \left(x_1\sigma_1^2 + x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\
\mathrm{MCR}_2^{\sigma} &= \left(x_2\sigma_2^2 + x_1\sigma_{12}\right)/\sigma_p(\mathbf{x})
\end{aligned}$$

Example: 2 asset portfolio

Then

$$
\begin{aligned}
\mathrm{MCR}_1^{\sigma} &= \left(x_1\sigma_1^2 + x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\
\mathrm{MCR}_2^{\sigma} &= \left(x_2\sigma_2^2 + x_1\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\
\mathrm{CR}_1^{\sigma} &= x_1 \times \left(x_1\sigma_1^2 + x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) = \left(x_1^2\sigma_1^2 + x_1x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\
\mathrm{CR}_2^{\sigma} &= x_2 \times \left(x_2\sigma_2^2 + x_2\sigma_2\right)/\sigma_p(\mathbf{x}) = \left(x_2^2\sigma_2^2 + x_1x_2\sigma_{12}\right)/\sigma_p(\mathbf{x})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{PCR}_1^{\sigma} &= \mathrm{CR}_1^{\sigma}/\sigma_p(\mathbf{x}) = \left(x_1^2\sigma_1^2 + x_1x_2\sigma_{12}\right)/\sigma_p^2(\mathbf{x}) \\
\mathrm{PCR}_2^{\sigma} &= \mathrm{CR}_2^{\sigma}/\sigma_p(\mathbf{x}) = \left(x_2^2\sigma_2^2 + x_1x_2\sigma_{12}\right)/\sigma_p^2(\mathbf{x})
\end{aligned}
$$

How to Interpret and Use MCR_i^{σ}

$$\begin{aligned} \mathrm{MCR}_i^{\sigma} &= \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} \approx \frac{\Delta \sigma_p}{\Delta x_i} \\ &\Rightarrow \Delta \sigma_p \approx \mathrm{MCR}_i^{\sigma} \cdot \Delta x_i \end{aligned}$$

However, in a portfolio of n assets

$$x_1 + x_2 + \cdots + x_n = 1$$

so that increasing or decreasing x_i means that we have to decrease or increase our allocation to one or more other assets. Hence, the formula

$$\Delta \sigma_p \approx \mathrm{MCR}_i^{\sigma} \cdot \Delta x_i$$

ignores this re-allocation effect.

How to Interpret and Use MCR_i^σ

If the increase in allocation to asset i is offset by a decrease in allocation to asset j, then

$$\Delta x_j = -\Delta x_i$$

and the change in portfolio volatility is approximately

$$\begin{aligned}\Delta\sigma_p &\approx \mathrm{MCR}_i^\sigma \cdot \Delta x_i + \mathrm{MCR}_j^\sigma \cdot \Delta x_j \\ &= \mathrm{MCR}_i^\sigma \cdot \Delta x_i - \mathrm{MCR}_j^\sigma \cdot \Delta x_i \\ &= \left(\mathrm{MCR}_i^\sigma - \mathrm{MCR}_j^\sigma\right) \cdot \Delta x_i\end{aligned}$$

Risk Reports

A common portfolio risk report summarizes asset and portfolio risk measures as well as risk budgets

Asset	**\$**d_i	x_i	RM_i	MCR_i^{RM}	CR_i^{RM}	PCR_i^{RM}
Asset 1	\$$\mathrm{d}_1$	x_1	RM_1	MCR_1^{RM}	CR_1^{RM}	PCR_1^{RM}
Asset 2	\$$\mathrm{d}_2$	x_2	RM_2	MCR_2^{RM}	CR_2^{RM}	PCR_2^{RM}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Asset N	\$$\mathrm{d}_N$	x_N	RM_N	MCR_N^{RM}	CR_N^{RM}	PCR_N^{RM}
Portfolio (Sum)	\$$W_0$	1			$\mathrm{RM}(\mathbf{x})$	1

Table 1: Portfolio Risk Report

Three Asset Example

- Consider creating a portfolio volatility and Value-at-Risk report from an equally weighted portfolio of Microsoft, Nordstrom, and Starbucks stock.
- The initial wealth invested in the portfolio is \$100,000. The expected return vector and covariance matrix is based on sample statistics computed over the five-year period January, 1995 through January, 2000.
- We use the same example data that we used for the portfolio theory examples

Three Asset Example

Asset information

```
asset.names <- c("MSFT", "NORD", "SBUX")
mu.vec = c(0.0427, 0.0015, 0.0285)
sigma.mat = matrix(c(0.0100, 0.0018, 0.0011,
                     0.0018, 0.0109, 0.0026,
                     0.0011, 0.0026, 0.0199),
                   nrow=3, ncol=3)
sig.vec = sqrt(diag(sigma.mat))
names(mu.vec) = names(sig.vec) = asset.names
dimnames(sigma.mat) = list(asset.names, asset.names)
```

Three Asset Example

Portfolio information

```
W0 = 100000
x = rep(1/3, 3)
d = x*W0
names(x) = asset.names
mu.px = as.numeric(crossprod(x, mu.vec))
sig.px = as.numeric(sqrt(t(x)%*%sigma.mat%*%x))
```

Example: Risk Budgeting Calculations

Use matrix algebra to compute MCR^{σ}:

$$\mathrm{MCR}^{\sigma} = (\boldsymbol{\Sigma}\mathbf{x})/\sigma_p(\mathbf{x})$$

The other components follow directly.

```
MCR.vol.x = (sigma.mat%*%x)/sig.px
CR.vol.x = x*MCR.vol.x
PCR.vol.x = CR.vol.x/sig.px
```

Example: Volatility Risk Report

```
riskReportVol.px = cbind(d, x, sig.vec, MCR.vol.x, CR.vol.x,
                         PCR.vol.x)
PORT = c(W0, 1, NA, NA, sum(CR.vol.x), sum(PCR.vol.x))
riskReportVol.px = rbind(riskReportVol.px, PORT)
colnames(riskReportVol.px) = c("Dollar", "Weight", "Vol",
                               "MCR", "CR", "PCR")
riskReportVol.px
```

```
##      Dollar Weight   Vol    MCR     CR   PCR
## MSFT  33333  0.333 0.100 0.0567 0.0189 0.249
## NORD  33333  0.333 0.104 0.0672 0.0224 0.295
## SBUX  33333  0.333 0.141 0.1037 0.0346 0.456
## PORT 100000  1.000    NA     NA 0.0759 1.000
```

Change in portfolio volatility due to rebalancing

For the equally weighted portfolio, increase x_{msft} by 0.1 and decrease x_{sbux} by 0.1. The approximate change in portfolio volatility is given by

```
delta.vol.px = (MCR.vol.x["MSFT",] - MCR.vol.x["SBUX",])*0.1
as.numeric(delta.vol.px)
```

```
## [1] -0.0047
```

The new portfolio volatility is

```
sig.px + delta.vol.px
```

```
##   MSFT
## 0.0712
```

Change in portfolio volatility due to rebalancing

The exact change in volatility from rebalancing is

```
x1 = x + c(0.1, 0, -0.1)
sig.px1 = as.numeric(sqrt(t(x1)%*%sigma.mat%*%x1))
sig.px1 - sig.px
```

```
## [1] -0.00293
```

$x - \sigma - \rho$ Decomposition of Portfolio Volatility

Recall,

$$\mathrm{MCR}_i^{\sigma} = \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \text{ith row of } \frac{\Sigma \mathbf{x}}{\sigma_p(\mathbf{x})} = \frac{\mathrm{cov}(R_i, R_p(\mathbf{x}))}{\sigma_p(\mathbf{x})}$$

Using

$$\begin{aligned} \rho_{i,p} &= \mathrm{corr}(R_i, R_p(\mathbf{x})) = \frac{\mathrm{cov}(R_i, R_p(\mathbf{x}))}{\sigma_i \sigma_p(\mathbf{x})} \\ &\Rightarrow \mathrm{cov}(R_i, R_p(\mathbf{x})) = \rho_{i,p} \sigma_i \sigma_p(\mathbf{x}) \end{aligned}$$

$x-\sigma-\rho$ Decomposition of Portfolio Volatility

Then

$$\mathrm{MCR}_i^{\sigma} = \frac{\rho_{i,p}\sigma_i\sigma_p(\mathbf{x})}{\sigma_p(\mathbf{x})} = \rho_{i,p}\sigma_i$$

and

$$\begin{aligned}\mathrm{CR}_i^{\sigma} &= x_i \times \mathrm{MCR}_i^{\sigma} \\ &= x_i \times \sigma_i \times \rho_{i,p} \\ &= \text{allocation} \times \text{standalone risk} \times \text{correlation with portfolio}\end{aligned}$$

$x - \sigma - \rho$ Decomposition of Portfolio Volatility

Remarks:

- $x_i \times \sigma_i =$ standalone contribution to risk (ignores correlation effects with other assets)
- $\mathrm{CR}_i^{\sigma} = x_i \times \sigma_i$ only when $\rho_{i,p} = 1$
- If $\rho_{i,p} < 1$ then $\mathrm{CR}_i^{\sigma} < x_i \times \sigma_i$

Example: Add $x - \sigma - \rho$ decomposition to risk report

You can compute each asset's correlation to the portfolio, $\rho_{i,p}$, from the information in the risk report:

$$\rho_{i,p} = \mathrm{MCR}_i^{\sigma} / \sigma_i$$

```
rho.x = MCR.vol.x/sig.vec
riskReportVol.px = cbind(riskReportVol.px, c(rho.x, 1))
colnames(riskReportVol.px)[7] = "Rho"
```

Example: Add $x - \sigma - \rho$ decomposition to risk report

The new risk report for the equally weighted portfolio is

```
riskReportVol.px
```

```
##      Dollar Weight   Vol    MCR     CR   PCR
## MSFT  33333  0.333 0.100 0.0567 0.0189 0.249
## NORD  33333  0.333 0.104 0.0672 0.0224 0.295
## SBUX  33333  0.333 0.141 0.1037 0.0346 0.456
## PORT 100000  1.000    NA     NA 0.0759 1.000
##        Rho
## MSFT 0.567
## NORD 0.644
## SBUX 0.735
## PORT 1.000
```

Example: Add $x-\sigma-\rho$ decomposition to risk report

Comments:

- SBUX has the highest correlation with the portfolio at 0.735. Its MCR is just smaller than its standalone volatility.
- MSFT and NORD have similar correlations with the portfolo. Their MCR values are much smaller than their standalone volatilies.

Beta as a Measure of Asset Contribution to Portfolio Volatility

For a portfolio of n assets with return

$$R_p(\mathbf{x}) = x_1 R_1 + \cdots + x_n R_n = \mathbf{x}'\mathbf{R}$$

we derived the portfolio volatility decomposition

$$\begin{aligned}
\sigma_p(\mathbf{x}) &= x_1 \frac{\partial \sigma_p(\mathbf{x})}{\partial x_1} + x_2 \frac{\partial \sigma_p(\mathbf{x})}{\partial x_2} + \cdots + x_n \frac{\partial \sigma_p(\mathbf{x})}{\partial x_n} = \mathbf{x}' \frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}} \\
\frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}} &= \frac{\Sigma \mathbf{x}}{\sigma_p(\mathbf{x})}, \; \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \text{ith row of } \frac{\Sigma \mathbf{x}}{\sigma_p(\mathbf{x})}
\end{aligned}$$

Beta as a Measure of Asset Contribution to Portfolio Volatility

With a little bit of algebra we can derive an alternative expression for

$$\mathrm{MCR}_i^\sigma = \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = i\text{th row of } \frac{\Sigma \mathbf{x}}{\sigma_p(\mathbf{x})}$$

First, define the *beta of asset i with respect to the portfolio* as

$$\beta_i = \frac{\mathrm{cov}(R_i, R_p(\mathbf{x}))}{\mathrm{var}(R_p(\mathbf{x}))} = \frac{\mathrm{cov}(R_i, R_p(\mathbf{x}))}{\sigma_p^2(\mathbf{x})}$$

Beta as a Measure of Asset Contribution to Portfolio Volatility

Result: β_i measures asset contribution to $\sigma_p(\mathbf{x})$:

$$\begin{aligned}
\mathrm{MCR}_i^{\sigma} &= \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \beta_i \times \sigma_p(\mathbf{x}) \\
\mathrm{CR}_i^{\sigma} &= x_i \times \beta_i \times \sigma_p(\mathbf{x}) \\
\mathrm{PCR}_i^{\sigma} &= x_i \times \beta_i
\end{aligned}$$

Beta as a Measure of Asset Contribution to Portfolio Volatility

- By construction, the beta of the portfolio is 1

$$\beta_p = \frac{\mathrm{cov}(R_p(\mathbf{x}), R_p(\mathbf{x}))}{\mathrm{var}(R_p(\mathbf{x}))} = \frac{\mathrm{var}(R_p(\mathbf{x}))}{\mathrm{var}(R_p(\mathbf{x}))} = 1$$

- When $\beta_i = 1$ asset i has the same risk as the portfolio:

$$\begin{aligned} \mathrm{MCR}_i^\sigma &= \sigma_p(\mathbf{x}) \\ \mathrm{CR}_i^\sigma &= x_i \sigma_p(\mathbf{x}) \\ \mathrm{PCR}_i^\sigma &= x_i \end{aligned}$$

Beta as a Measure of Asset Contribution to Portfolio Volatility

- When $\beta_i > 1$ asset i is a *portfolio risk enhancer*:

$$\begin{aligned} \mathrm{MCR}_i^\sigma &> \sigma_p(\mathbf{x}) \\ \mathrm{CR}_i^\sigma &> x_i \times \sigma_p(\mathbf{x}) \\ \mathrm{PCR}_i^\sigma &> x_i \end{aligned}$$

- When $\beta_i < 1$ asset i is a *portfolio risk reducer*:

$$\begin{aligned} \mathrm{MCR}_i^\sigma &< \sigma_p(\mathbf{x}) \\ \mathrm{CR}_i^\sigma &< x_i \times \sigma_p(\mathbf{x}) \\ \mathrm{PCR}_i^\sigma &< x_i \end{aligned}$$

Derivation of Result

Recall,

$$\frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}} = \frac{\Sigma \mathbf{x}}{\sigma_p(\mathbf{x})}$$

Now,

$$\Sigma \mathbf{x} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Derivation of Result

The first row of $\Sigma \mathbf{x}$ is

$$x_1\sigma_1^2 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n}$$

Now consider

$$\begin{aligned}
\mathrm{cov}(R_1, R_p) &= \mathrm{cov}(R_1, x_1R_1 + \cdots + x_nR_n) \\
&= \mathrm{cov}(R_1, x_1R_1) + \cdots + \mathrm{cov}(R_1, x_nR_n) \\
&= x_1\sigma_1^2 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n}
\end{aligned}$$

Next, note that

$$\beta_1 = \frac{\mathrm{cov}(R_1, R_p)}{\sigma_p^2(\mathbf{x})} \Rightarrow \mathrm{cov}(R_1, R_p) = \beta_1\sigma_p^2(\mathbf{x})$$

Derivation of Result

Hence, the first row of $\Sigma\mathbf{x}$ is

$$x_1\sigma_1^2 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n} = \beta_1\sigma_p^2(\mathbf{x})$$

and so

$$\begin{aligned} \mathrm{MCR}_1^\sigma &= \frac{\partial\sigma_p(\mathbf{x})}{\partial x_1} = \text{first row of } \frac{\Sigma\mathbf{x}}{\sigma_p(\mathbf{x})} \\ &= \frac{\beta_1\sigma_p^2(\mathbf{x})}{\sigma_p(\mathbf{x})} = \beta_1\sigma_p(\mathbf{x}) \end{aligned}$$

Derivation of Result

In a similar fashion, we have

$$\begin{aligned}\mathrm{MCR}_i^{\sigma} &= \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \text{i'th row of } \frac{\Sigma \mathbf{x}}{\sigma_p(\mathbf{x})} \\ &= \frac{\beta_i \sigma_p^2(\mathbf{x})}{\sigma_p(\mathbf{x})} = \beta_i \sigma_p(\mathbf{x})\end{aligned}$$

Example: Add portfolio beta to risk report

You can compute each asset's portfolio beta, β_i, from the information in the risk report:

$$\beta_i = \mathrm{PCR}_i / x_i$$

```
beta.x = PCR.vol.x/x
riskReportVol.px = cbind(riskReportVol.px, c(beta.x, 1))
colnames(riskReportVol.px)[8] = "Beta"
```

Add portfolio beta to risk report

The new risk report for the equally weighted portfolio is

```
riskReportVol.px
```

```
##       Dollar Weight   Vol    MCR     CR   PCR
## MSFT   33333  0.333 0.100 0.0567 0.0189 0.249
## NORD   33333  0.333 0.104 0.0672 0.0224 0.295
## SBUX   33333  0.333 0.141 0.1037 0.0346 0.456
## PORT 100000  1.000    NA     NA 0.0759 1.000
##         Rho  Beta
## MSFT 0.567 0.747
## NORD 0.644 0.886
## SBUX 0.735 1.367
## PORT 1.000 1.000
```

Add portfolio beta to risk report

Comments:

- SBUX has a portfolo beta bigger than 1. Its PCR is bigger than its allocation weight in the portfolio and is a risk enhanser.
- MSFT and NORD have a portfolo betas less than 1. Their PCRs are smaller than their allocation weights in the portfolio and they are risk reducers.