# Introduction to Portfolio Theory

Yuming Liu and Eric Zivot

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# Portfolios of Two Risky Assets

Consider investing in two risky assets (Amazon and Boeing)

$$
\begin{aligned}
R_A &= \text{ simple return on asset A} \\
R_B &= \text{ simple return on asset B} \\
W_0 &= \text{ initial wealth}
\end{aligned}
$$

Assumptions

- $R_A$ and $R_B$ are described by the GWN model

$$
R_i \sim \text{iid } N(\mu_i, \sigma_i^2),\ i = A, B
$$
$$
\text{cov}(R_A, R_B) = \sigma_{AB},\ \text{cor}(R_A, R_B) = \rho_{AB}
$$

# Portfolios of Two Risky Assets

- Traditionally in portfolio theory, returns are simple and not continuously compounded because the simple return on a portfolio is a share weighted average of the individual asset returns
- Assume that GWN model for continuously compounded returns can also be used for simple returns

# Portfolios of Two Risky Assets

Investor Preferences:

- Investors like high $E[R_i] = \mu_i$
- Investors dislike high $\mathrm{var}(R_i) = \sigma_i^2$
- Investment horizon is one period (e.g., one month or one year)

# Portfolios of Two Risky Assets

**Portfolios**

$$x_A = \text{share of wealth in asset A} = \frac{\$ \text{ in A}}{W_0}$$
$$x_B = \text{share of wealth in asset B} = \frac{\$ \text{ in B}}{W_0}$$

Long position

$$x_A,\ x_B > 0$$

Short position

$$x_A < 0 \text{ or } x_B < 0$$

# Portfolios of Two Risky Assets

Allocate all wealth between assets A and B

$$x_A + x_B = 1$$

Portfolio Return

$$R_p = x_A R_A + x_B R_B$$

# Portfolios of Two Risky Assets

Portfolio Distribution

$$\mu_p = E[R_p] = x_A\mu_A + x_B\mu_B$$
$$\sigma_p^2 = \mathrm{var}(R_p) = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}$$
$$= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\rho_{AB}\sigma_A\sigma_B$$
$$R_p \sim \mathrm{iid}\ N(\mu_p, \sigma_p^2)$$

End of Period Wealth

$$W_1 = W_0(1 + R_p) = W_0(1 + x_AR_A + x_BR_B)$$
$$W_1 \sim N(W_0(1 + \mu_p), \sigma_p^2W_0^2)$$

# Portfolios of Two Risky Assets

**Result**: Portfolio SD is not a weighted average of asset SD unless $\rho_{AB} = 1$ :

$$\begin{aligned}\sigma_p &= \left(x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \rho_{AB}\sigma_A\sigma_B\right)^{1/2} \\ &\neq x_A\sigma_A + x_B\sigma_B \text{ for } \rho_{AB} \neq 1\end{aligned}$$

If $\rho_{AB} = 1$ then

$$\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = \sigma_A\sigma_B$$

and

$$\begin{aligned}\sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \sigma_A\sigma_B \\ &= (x_A\sigma_A + x_B\sigma_B)^2 \\ &\Rightarrow \sigma_p = x_A\sigma_A + x_B\sigma_B\end{aligned}$$

# Portfolios of Two Risky Assets

Example Data (calibrated for annual investment horizon and to create nice graphs)

$$
\begin{aligned}
\mu_A &= 0.175,\ \ \mu_B = 0.055 \\
\sigma_A^2 &= 0.067,\ \ \sigma_B^2 = 0.013 \\
\sigma_A &= 0.258,\ \ \sigma_B = 0.115 \\
\sigma_{AB} &= -0.004875, \\
\rho_{AB} &= \frac{\sigma_{AB}}{\sigma_A \sigma_B} = -0.164
\end{aligned}
$$

Note: Asset A has higher expected return and risk than asset B

# Portfolios of Two Risky Assets

```
mu.A = 0.175
sig.A = 0.258
sig2.A = sig.A^2
mu.B = 0.055
sig.B = 0.115
sig2.B = sig.B^2
rho.AB = -0.164
sig.AB = rho.AB*sig.A*sig.B
```

# Example: Long only two asset portfolio

Consider an equally weighted portfolio with $x_A = x_B = 0.5$. The expected return, variance and volatility are

$$\begin{aligned}
\mu_p &= (0.5)\cdot(0.175) + (0.5)\cdot(0.055) = 0.115 \\
\sigma_p^2 &= (0.5)^2\cdot(0.067) + (0.5)^2\cdot(0.013) \\
&\quad + 2\cdot(0.5)(0.5)(-0.004875) = 0.01751 \\
\sigma_p &= \sqrt{0.01751} = 0.1323
\end{aligned}$$

This portfolio has expected return half-way between the expected returns on assets A and B, but the portfolio standard deviation is less than half-way between the asset standard deviations. This reflects risk reduction via diversification.

# Long-Only Portfolio

```r
x.A.p1 = 0.5
x.B.p1 = 0.5
mu.p1 = x.A.p1*mu.A + x.B.p1*mu.B
sig2.p1 = x.A.p1^2 * sig2.A + x.B.p1^2 * sig2.B +
  2*x.A.p1*x.B.p1*sig.AB
sig.p1 = sqrt(sig2.p1)
```

```
##                      Expected Return Std Dev Variance
## Asset A                       0.1750  0.2580   0.0666
## Asset B                       0.0550  0.1150   0.0132
## Long-Only Portfolio           0.1150  0.1323   0.0175
```

# Example: Long-Short two asset portfolio

Next, consider a long-short portfolio with $x_A = 1.5$ and $x_B = -0.5$. In this portfolio, asset B is sold short and the proceeds of the short sale are used to leverage the investment in asset A. The portfolio characteristics are

$$\begin{aligned}
\mu_p &= (1.5) \cdot (0.175) + (-0.5) \cdot (0.055) = 0.235 \\
\sigma_p^2 &= (1.5)^2 \cdot (0.067) + (-0.5)^2 \cdot (0.013) \\
&\quad + 2 \cdot (1.5)(-0.5)(-0.004875) = 0.1604 \\
\sigma_p &= \sqrt{0.01751} = 0.4005
\end{aligned}$$

This portfolio has both a higher expected return and standard deviation than asset A

# Long-Short two asset portfolio

```r
x.A.p2 = 1.5
x.B.p2 = -0.5
mu.p2 = x.A.p2*mu.A + x.B.p2*mu.B
sig2.p2 = x.A.p2^2 * sig2.A + x.B.p2^2 * sig2.B +
  2*x.A.p2*x.B.p2*sig.AB
sig.p2 = sqrt(sig2.p2)
```

```
##                        Expected Return Std Dev Variance
## Asset A                         0.1750  0.2580   0.0666
## Asset B                         0.0550  0.1150   0.0132
## Long-Short Portfolio            0.2350  0.4005   0.1604
```

# Portfolio Value-at-Risk

- Given an initial investment of $\$W_0$ in the portfolio of assets A and B.
- Given that the simple return $R_p \sim N(\mu_p, \sigma_p^2)$. For $\alpha \in (0,1)$, the $\alpha \times 100\%$ portfolio value-at-risk is

$$\begin{aligned} \mathrm{VaR}_{p,\alpha} &= q_{p,\alpha}^{R} W_0 \\ &= (\mu_p + \sigma_p q_\alpha^z)\, W_0 \end{aligned}$$

where $q_{p,\alpha}^R$ is the $\alpha$ quantile of the distribution of $R_p$ and $q_\alpha^z = \alpha$ quantile of $Z \sim N(0,1)$.

# Relationship between Portfolio VaR and Individual Asset VaR

**Result**: Portfolio VaR is not a weighted average of asset VaR:

$$\mathrm{VaR}_{p,\alpha} \neq x_A \mathrm{VaR}_{A,\alpha} + x_B \mathrm{VaR}_{B,\alpha}$$

unless $\rho_{AB} = 1$.

Asset VaRs for A and B are:

$$\begin{aligned}
\mathrm{VaR}_{A,\alpha} &= q_\alpha^{R_A} W_0 = (\mu_A + \sigma_A q_\alpha^z) W_0 \\
\mathrm{VaR}_{B,\alpha} &= q_\alpha^{R_B} W_0 = (\mu_B + \sigma_B q_\alpha^z) W_0
\end{aligned}$$

Portfolio VaR is:

$$\begin{aligned}
\mathrm{VaR}_{p,\alpha} &= \left(\mu_p + \sigma_p q_\alpha^z\right) W_0 \\
&= \left[ (x_A \mu_A + x_B \mu_B) + \left( x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \right)^{1/2} q_\alpha^z \right] W_0
\end{aligned}$$

# Relationship between Portfolio VaR and Individual Asset VaR Cont.

If $\rho_{AB} \neq 1$, weighted Average of the asset VaR is:

$$
\begin{aligned}
x_A \mathrm{VaR}_{A,\alpha} + x_B \mathrm{VaR}_{B,\alpha} &= x_A(\mu_A + \sigma_A q_\alpha^z) W_0 + x_B(\mu_B + \sigma_B q_\alpha^z) W_0 \\
&= \left[(x_A\mu_A + x_B\mu_B) + (x_A\sigma_A + x_B\sigma_B)\, q_\alpha^z\right] W_0 \\
&\neq \left[(x_A\mu_A + x_B\mu_B) + \left(x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}\right)^{1/2} q_\alpha^z\right] W_0 \\
&= \left(\mu_p + \sigma_p q_\alpha^z\right) W_0 = \mathrm{VaR}_{p,\alpha}
\end{aligned}
$$

If $\rho_{AB} = 1$, then $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = \sigma_A\sigma_B$ and:

$$
\begin{aligned}
\sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_A\sigma_B = (x_A\sigma_A + x_B\sigma_B)^2 \\
&\Rightarrow \sigma_p = x_A\sigma_A + x_B\sigma_B
\end{aligned}
$$

and so,

$$
x_A \mathrm{VaR}_{A,\alpha} + x_B \mathrm{VaR}_{B,\alpha} = \mathrm{VaR}_{p,\alpha}
$$

# Example: Two Risky Asset VaR

For an initial investment of $W_0 =$\$100,000:

The 5% VaRs on assets A and B are:

$$\mathrm{VaR}_{A,0.05} = q_{0.05}^{R_A} W_0 = (0.175 + 0.258(-1.645)) \cdot 100,000 = -24,937,$$
$$\mathrm{VaR}_{B,0.05} = q_{0.05}^{R_B} W_0 = (0.055 + 0.115(-1.645)) \cdot 100,000 = -13,416.$$

The 5% VaR on the equal weighted portfolio with $x_A = x_B = 0.5$ is:

$$\mathrm{VaR}_{p,0.05} = q_{0.05}^{R_p} W_0 = (0.115 + 0.1323(-1.645)) \cdot 100,000 = -10,268,$$

The weighted average of the asset VaRs is:

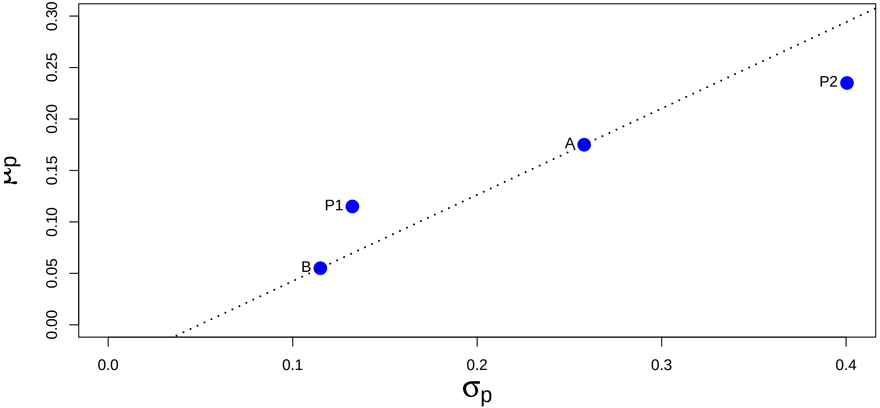
$$x_A \mathrm{VaR}_{A,0.05} + x_B \mathrm{VaR}_{B,0.05} = 0.5(-24,937) + 0.5(-13,416) = -19,177.$$

# Portfolio Frontier

Vary investment shares $x_A$ and $x_B$ and compute resulting values of $\mu_p$ and $\sigma_p^2$. Plot $\mu_p$ against $\sigma_p$ as functions of $x_A$ and $x_B$

- Portfolios with *only* asset A or *only* asset B will be on the frontier.
- Shape of portfolio frontier depends on correlation between assets A and B
- If $\rho_{AB} = -1$ then there exists portfolio shares $x_A$ and $x_B$ such that $\sigma_p^2 = 0$
- If $\rho_{AB} = 1$ then there is no benefit from diversification
- Diversification is beneficial even if $0 < \rho_{AB} < 1$

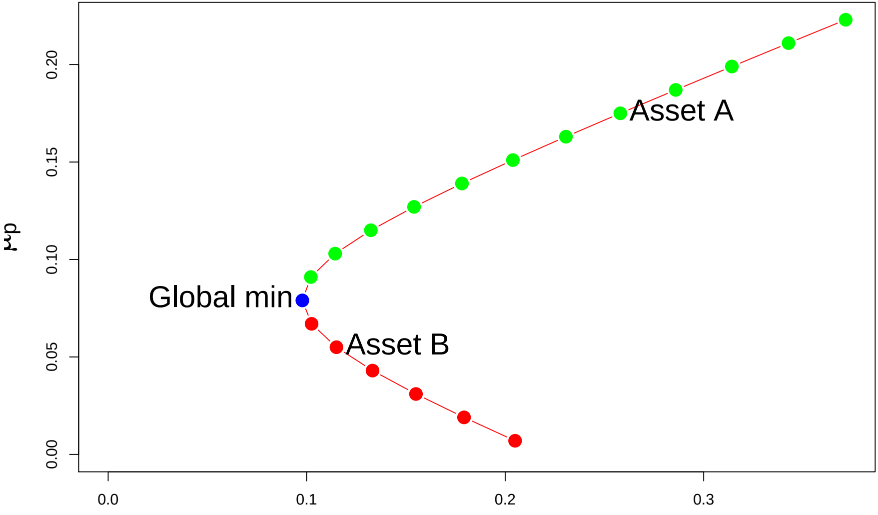
# Risk return characteristics of portfolios



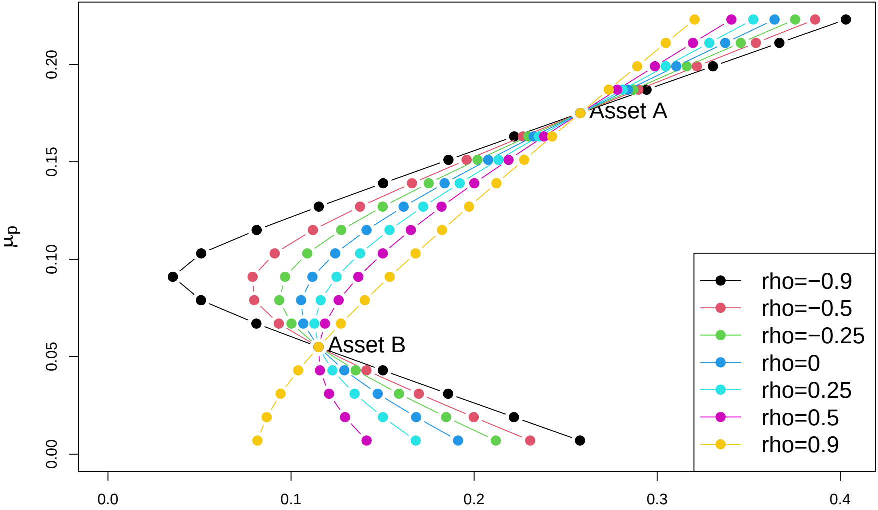
# Create portfolio frontier

```
# create portfolios for frontier plot
x.A = seq(from=-0.4, to=1.4, by=0.1)
x.B = 1 - x.A
mu.p = x.A*mu.A + x.B*mu.B
sig2.p = x.A^2 * sig2.A + x.B^2 * sig2.B + 2*x.A*x.B*sig.AB
sig.p = sqrt(sig2.p)
```

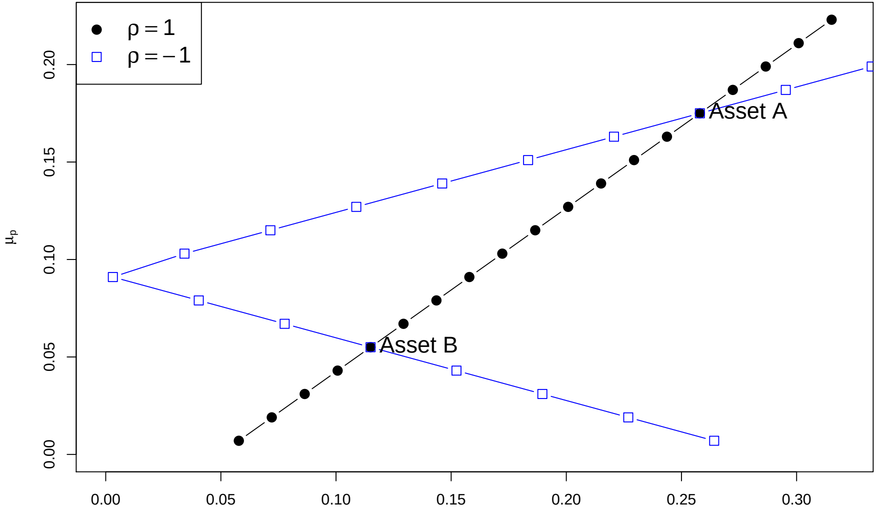
# Plot portfolio frontier



# Portfolio frontier as a function of $\rho_{AB}$



# Portfolio Frontier when $\rho_{AB} = +/-1$

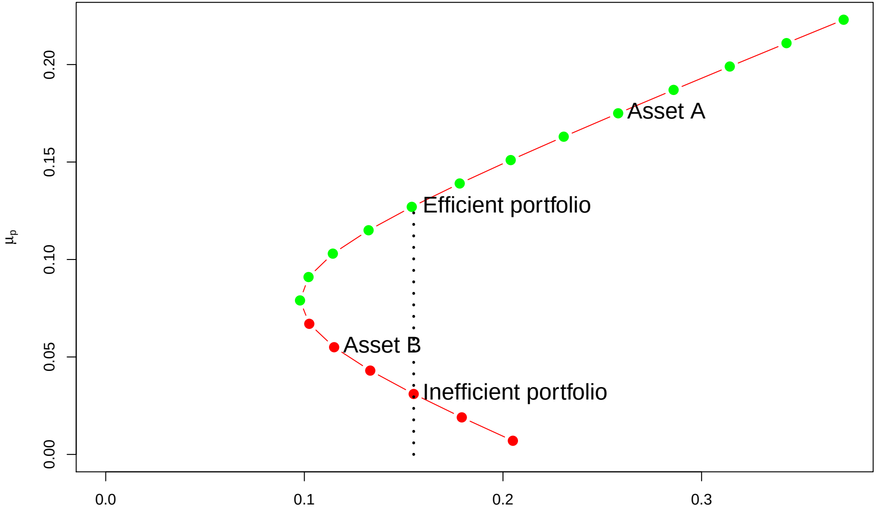


# Efficient Portfolios

**Definition**: Portfolios with the highest expected return for a given level of risk, as measured by portfolio standard deviation, are efficient portfolios.

- If investors like portfolios with high expected returns and dislike portfolios with high return standard deviations then they will want to hold efficient portfolios.
- Which efficient portfolio an investor will hold depends on their risk preferences.
    - Very risk averse investors dislike volatility and will hold portfolios near the global minimum variance portfolio. They sacrifice expected return for the safety of low volatility.
    - Risk tolerant investors (tolerate risk) and will hold portfolios that have high expected returns-high risk trade-offs. They gain expected return by taking on more volatility.

# Efficient Portfolios



# Globabl Minimum Variance Portfolio

- The efficient portfolio with the smallest possible variance is called the **global minimum variance portfolio**.
- This portfolio is chosen by the most risk averse individuals.
- To find this portfolio, one has to solve the following *constrained minimization problem*

$$\min_{x_A, x_B} \ \sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}$$
$$s.t. \ x_A + x_B = 1$$

# Review of Optimization Techniques: Constrained Optimization

**Example**: Find the minimum of a bivariate function subject to a linear constraint

$$
\begin{aligned}
y = f(x, z) &= x^2 + z^2 \\
\min_{x,z}\ y &= f(x, z) \\
s.t.\ x + z &= 1
\end{aligned}
$$

Solution methods:

- Substitution
- Lagrange multipliers

# Method of Substitution

Substitute $z = 1 - x$ in $f(x, z)$ and solve univariate minimization:

$$
\begin{aligned}
y &= f(x, 1-x) = x^2 + (1-x)^2 \\
&\min_x \; f(x, 1-x)
\end{aligned}
$$

First order conditions:

$$
\begin{aligned}
0 &= \frac{d}{dx}(x^2 + (1-x)) = 2x + 2(1-x)(-1) \\
&= 4x - 2 \\
&\Rightarrow x = 0.5
\end{aligned}
$$

Solving for $z$:

$$
z = 1 - 0.5 = 0.5
$$

## Method of Lagrange Multipliers

Augment function to be minimized with extra terms to impose constraints.

1. Put constraints in homogeneous form:

$$x + z = 1 \Rightarrow x + z - 1 = 0$$

2. Form Lagrangian function:

$$L(x, z, \lambda) = x^2 + z^2 + \lambda(x + z - 1)$$
$$\lambda = \text{Lagrange multiplier(Shadow Price)}$$

3. Minimize Lagrangian function:

$$\min_{x,z,\lambda} \; L(x, z, \lambda)$$

# Method of Lagrange Multipliers Cont.

First order conditions:

$$
\begin{aligned}
0 &= \frac{\partial L(x,z,\lambda)}{\partial x} = 2 \cdot x + \lambda \rightarrow 2x = -\lambda \\
0 &= \frac{\partial L(x,z,\lambda)}{\partial z} = 2 \cdot z + \lambda \rightarrow 2z = -\lambda \\
0 &= \frac{\partial L(x,z,\lambda)}{\partial \lambda} = x + z - 1
\end{aligned}
$$

We have three linear equations in three unknowns. Solving gives:

$$
\begin{gathered}
2x = 2z = -\lambda \Rightarrow x = z \\
2z - 1 = 0 \Rightarrow z = 0.5,\ x = 0.5
\end{gathered}
$$

# Example: Finding the Global Minimum Variance Portfolio

Two methods for solution:

- Analytic solution using Calculus
- Numerical solution
  - use the Solver in Excel
  - use R function `solve.QP()` in package **quadprog* for quadratic optimization problems with equality and inequality constraints

# Calculus Solution

Minimization problem:

$$\min_{x_A, x_B} \ \sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \sigma_{AB}$$
$$s.t. \ x_A + x_B = 1$$

Use substitution method with:

$$x_B = 1 - x_A$$

to give the univariate minimization,

$$\min_{x_A} \ \sigma_p^2 = x_A^2\sigma_A^2 + (1 - x_A)^2\sigma_B^2 + 2x_A(1 - x_A)\sigma_{AB}$$

# Calculus Solution Cont.

First order conditions:

$$0 = \frac{d}{dx_A}\sigma_p^2 = \frac{d}{dx_A}\left(x_A^2\sigma_A^2 + (1-x_A)^2\sigma_B^2 + 2x_A(1-x_A)\sigma_{AB}\right)$$

$$= 2x_A\sigma_A^2 - 2(1-x_A)\sigma_B^2 + 2\sigma_{AB}(1-2x_A)$$

$$\Rightarrow x_A^{\min} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}},\ x_B^{\min} = 1 - x_A^{\min}$$

# Excel Solver Solution

The Solver is an Excel add-in, that can be used to numerically solve general linear and nonlinear optimization problems subject to equality or inequality constraints.

- The solver is made by FrontLine Systems and is provided with Excel
- The solver add-in may not be installed in a "default installation" of Excel
    - Tools/Add-Ins and check the Solver Add-In box
    - If Solver Add-In box is not available, the Solver Add-In must be installed from original Excel installation CD

# Computing the Minimum Variance Portfolio

```
# compute mininimum variance portfolio weights
xA.min = (sig2.B - sig.AB)/(sig2.A + sig2.B - 2*sig.AB)
xB.min = 1 - xA.min
xA.min
```

```
## [1] 0.202
```

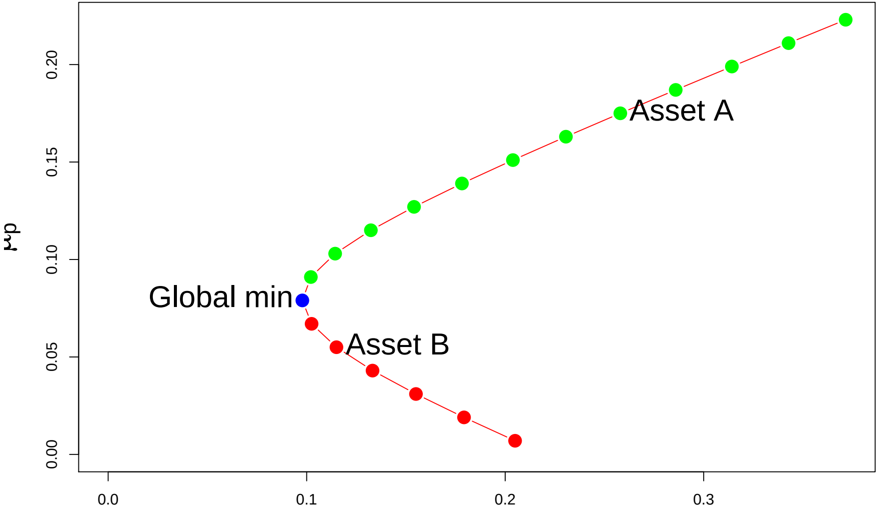
```
xB.min
```

```
## [1] 0.798
```

```
# compute expected return and volatility
mu.p.min = xA.min*mu.A + xB.min*mu.B
sig2.p.min = xA.min^2 * sig2.A + xB.min^2 * sig2.B + 2*xA.min*
sig.p.min = sqrt(sig2.p.min)
mu.p.min
```

```
## [1] 0.0793
```

# Minimum Variance Portfolio



# Portfolios with a Risk Free Asset

Risk Free Asset (The Risk Free Rate):

- Asset with fixed and known rate of return over investment horizon
- Usually U.S. government T-Bill rate (horizons $<1$ year) or T-Note rate (horizon $>1$ year)
- T-Bill or T-Note rate is only nominally risk free.

# Properties of Risk-Free Asset

$$
\begin{aligned}
R_f &= \text{return on risk-free asset} \\
E[R_f] &= r_f = \text{ constant} \\
\text{var}(R_f) &= 0 \\
\text{cov}(R_f, R_i) &= 0,\ R_i = \text{ return on any asset}
\end{aligned}
$$

Portfolios of Risky Asset and Risk Free Asset:

$$
\begin{aligned}
x_f &= \text{ share of wealth in T-Bills} \\
x_B &= \text{ share of wealth in asset B} \\
x_f + x_B &= 1 \\
\Rightarrow x_f &= 1 - x_B
\end{aligned}
$$

# Properties of Risk-Free Asset Cont.

Portfolio return:

$$\begin{aligned} R_p &= x_f r_f + x_B R_B \\ &= (1 - x_B) r_f + x_B R_B \\ &= r_f + x_B (R_B - r_f) \end{aligned}$$

Portfolio excess return:

$$R_p - r_f = x_B (R_B - r_f)$$

Portfolio Distribution:

$$\begin{aligned} \mu_p &= E[R_p] = r_f + x_B(\mu_B - r_f) \\ \sigma_p^2 &= \mathrm{var}(R_p) = x_B^2 \sigma_B^2 \\ \sigma_p &= x_B \sigma_B \\ R_p &\sim N(\mu_p, \sigma_p^2) \end{aligned}$$

# Risk Premium

$$
\begin{aligned}
\mu_B - r_f &= \text{excess expected return on asset B} \\
&= \text{expected return on risky asset over return on safe asset}
\end{aligned}
$$

Portfolio of T-Bills and asset B:

$$
\begin{aligned}
\mu_p - r_f &= x_B(\mu_B - r_f) \\
&= \text{expected portfolio return over T-Bill} \\
&= \text{excess expected return on portfolio}
\end{aligned}
$$

The risk premia is an increasing function of the amount invested in asset B. (willingness to take on more risk)

# Leveraged Investment

**Leverage** - to multiply gains and losses.

Short the Risk-free asset and Long the Risky asset

$$x_f < 0,\ x_B > 1$$

Borrow at T-Bill rate to buy more of asset B.

**Result**: Leverage increases portfolio expected return and risk.

$$\mu_p = r_f + x_B(\mu_B - r_f)$$
$$\sigma_p = x_B \sigma_B$$
$$x_B \uparrow \ \Rightarrow \mu_p \ \& \ \sigma_p \uparrow$$

# Determining Portfolio Frontier

Goal: Plot $\mu_p$ vs. $\sigma_p$.

$$
\begin{aligned}
\sigma_p &= x_B \sigma_B \Rightarrow x_B = \frac{\sigma_p}{\sigma_B} \\
\mu_p &= r_f + x_B(\mu_B - r_f) \\
&= r_f + \frac{\sigma_p}{\sigma_B}(\mu_B - r_f) \\
&= r_f + \left(\frac{\mu_B - r_f}{\sigma_B}\right)\sigma_p
\end{aligned}
$$

where,

$$
\left(\frac{\mu_B - r_f}{\sigma_B}\right) = \mathrm{SR}_B = \text{Asset B Sharpe Ratio}
$$
$$
= \text{excess expected return per unit risk (linear slope)}
$$

# Sharpe Ratio (SR)

The Sharpe Ratio:

$$\mathrm{SR}_B = \left( \frac{\mu_B - r_f}{\sigma_B} \right)$$

**Remarks**:

- William Sharpe - developed while at UW Foster
- The Sharpe Ratio (SR) is commonly used to rank assets.
- Assets with high Sharpe Ratios (steep slope) are preferred to assets with low Sharpe Ratios.

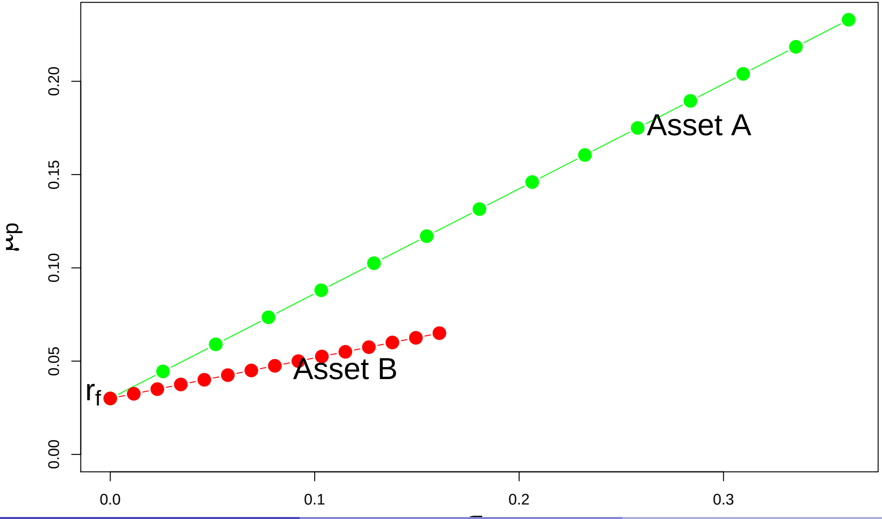
# Portfoliols of T-Bills and 1 risky asset

```r
# Risk-free rate
r.f = 0.03
# T-bills + asset A
x.A = seq(from=0, to=1.4, by=0.1)
mu.p.A = r.f + x.A*(mu.A - r.f)
sig.p.A = x.A*sig.A
sharpe.A = (mu.A - r.f)/sig.A
sharpe.A
```

```
## [1] 0.562
```

```r
# T-bills + asset B
x.B = seq(from=0, to=1.4, by=0.1)
mu.p.B = r.f + x.B*(mu.B - r.f)
sig.p.B = x.B*sig.B
sharpe.B = (mu.B - r.f)/sig.B
sharpe.B
```

# Portfoliols of T-Bills and 1 risky asset



# Efficient Portfolios with 2 Risky Assets and a Risk Free Asset

Investment in 2 Risky Assets and T-Bill:

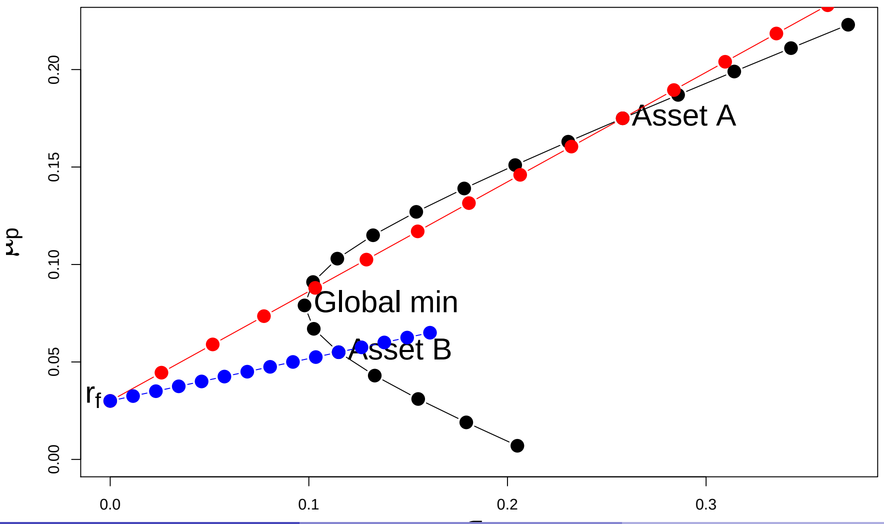
$$
\begin{aligned}
R_A &= \text{ simple return on asset A} \\
R_B &= \text{ simple return on asset B} \\
R_f &= r_f = \text{ return on T-Bill}
\end{aligned}
$$

Assumptions:

- $R_A$ and $R_B$ are described by the GWN model:

$$
\begin{aligned}
&R_i \sim iid\ N(\mu_i, \sigma_i^2),\ i = A, B \\
&\text{cov}(R_A, R_B) = \sigma_{AB},\ \text{corr}(R_A, R_B) = \rho_{AB}
\end{aligned}
$$

# Portfolios of T-Bills and 2 risky assets

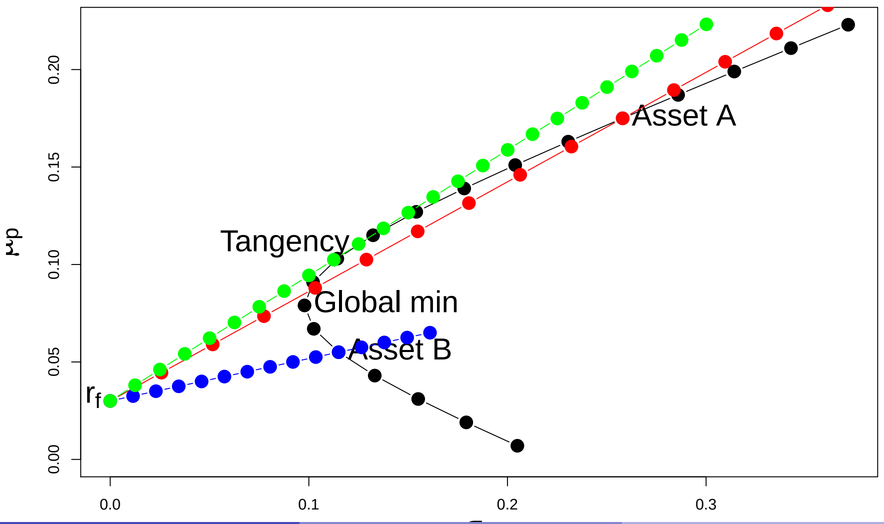


# Efficient Portfolios with 2 Risky Assets and a Risk Free Asset Cont.

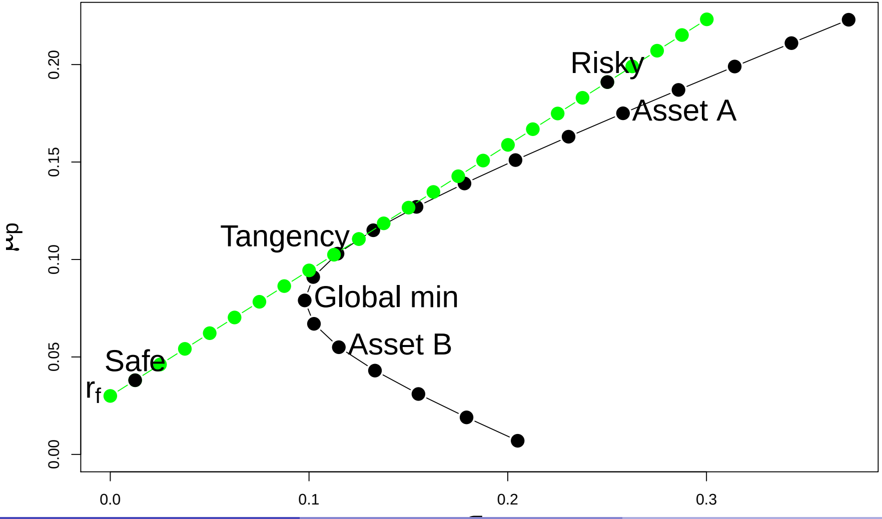
**Results**:

- The "best portfolio" of two risky assets and T-Bills is the one with the highest Sharpe Ratio.
- Graphically, this portfolio occurs at the tangency point of a line drawn from $R_f$ to the risky asset only frontier.
  - $E[R_p^e] = \mu_p^e$ with standard deviation $\sigma_p^e$
- The maximum Sharpe Ratio portfolio is called the "tangency portfolio"

# Portfolios of T-Bills and 2 risky assets



# Efficient portfolios and risk preferences



# Mutual Fund Separation Theorem

Efficient portfolios are combinations of two portfolios (mutual funds):

- T-Bill portfolio
- Tangency portfolio - portfolio of assets A and B that has the maximum Shape ratio

Implication: All investors hold assets A and B according to their proportions in the tangency portfolio regardless of their risk preferences.

# Finding the Tangency Portfolio

The Tangency Portfolio is the portfolio with the maximum Sharpe Ratio.

$$
\begin{aligned}
\max_{x_A,\ x_B} \mathrm{SR}_p &= \frac{\mu_p - r_f}{\sigma_p} \\
\text{subject to } \mu_p &= x_A \mu_A + x_B \mu_B \\
\sigma_p^2 &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \\
1 &= x_A + x_B
\end{aligned}
$$

Solution can be found analytically or numerically (e.g. using solver in Excel).

# Finding the Tangency Portfolio Cont.

Using the substitution method it can be shown that:

$$x_A^{\tan} = \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}$$

$$x_B^{\tan} = 1 - x_A^{\tan}$$

Portfolio characteristics:

$$\mu_p^{\tan} = x_A^{\tan}\mu_A + x_B^{\tan}\mu_B$$
$$\left(\sigma_p^{\tan}\right)^2 = \left(x_A^{\tan}\right)^2 \sigma_A^2 + \left(x_B^{\tan}\right)^2 \sigma_B^2 + 2x_A^{\tan}x_B^{\tan}\sigma_{AB}$$

# Efficient Portfolios: Tangency Portfolio plus T-Bills

Portfolio with the Risk-Free Asset + Tangency Portfolio

$$
\begin{aligned}
x_{\mathsf{tan}} &= \text{ share of wealth in tangency portfolio} \\
x_f &= \text{share of wealth in T-bills} \\
x_{\mathsf{tan}} + x_f &= 1 \\
\mu_p^e &= r_f + x_{\mathsf{tan}}(\mu_p^{\mathsf{tan}} - r_f) \\
\sigma_p^e &= x_{\mathsf{tan}}\sigma_p^{\mathsf{tan}}
\end{aligned}
$$

**Result**: The weights $x_{\mathsf{tan}}$ and $x_f$ are determined by an investor's risk preferences

- Risk averse investors hold mostly T-Bills
- Risk tolerant investors hold mostly tangency portfolio

## Example:Tangent Portfolio and Efficient Portfolio Distribution

For the two asset example, the tangency portfolio is:

$$\begin{aligned}
x_A^{\tan} &= .46,\ x_B^{\tan} = 0.54 \\
\mu_p^{\tan} &= (.46)(.175) + (.54)(.055) = 0.11 \\
\left(\sigma_p^{\tan}\right)^2 &= (.46)^2(.067) + (.54)^2(.013) + 2(.46)(.54)(-.005) \\
&= 0.015 \\
\sigma_p^{\tan} &= \sqrt{.015} = 0.124
\end{aligned}$$

Efficient portfolios have the following characteristics:

$$\begin{aligned}
\mu_p^e &= r_f + x_{\tan}(\mu_p^{\tan} - r_f) \\
&= 0.03 + x_{\tan}(0.11 - 0.03) \\
\sigma_p^e &= x_{\tan}\sigma_p^{\tan} \\
&= x_{\tan}(0.124)
\end{aligned}$$

# Example: Efficient Portfolio with Targeted Risk

What is the efficient portfolio that has the same risk $(\sigma_p^e)$ as asset B? That is, determine $x_{\tan}$ and $x_f$ such that

$$\sigma_p^e = \sigma_B = 0.114 = \text{target risk}.$$

Note: The efficient portfolio will have a higher expected return than asset B.

# Solution: Efficient Portfolio with Targeted Risk

Since the portfolio variance is: $\sigma_p^{2,e} = x_{tan}^2 \sigma_p^{2,tan}$

$$
\begin{aligned}
.114 = \sigma_p^e &= x_{\text{tan}} \sigma_p^{\text{tan}} \\
&= x_{\text{tan}}(.124) \\
&\Rightarrow x_{\text{tan}} = \frac{.114}{.124} = .92 \\
x_f &= 1 - x_{\text{tan}} = .08
\end{aligned}
$$

Efficient portfolio with same risk as asset B has:

$$
\begin{aligned}
(.92)(.46) &= .42 \text{ in asset A} \\
(.92)(.54) &= .50 \text{ in asset B} \\
& .08 \text{ in T-Bills}
\end{aligned}
$$

If $r_f = 0.03$, then expected Return on efficient portfolio is:

$$\mu_p^e = .03 + (.92)(.11 - 0.03) = .104.$$

# Example: Efficient Portfolio with Targeted Expected Return

For $r_f = 0.03$, find the efficient portfolio that has the same expected return as asset B. That is, determine $x_{\text{tan}}$ and $x_f$ such that:

$$\mu_p^e = \mu_B = 0.055 = \text{target expected return.}$$

Note: The efficient portfolio will have a lower SD than asset B.

# Solution: Efficient Portfolio with Targeted Expected Return

Since the expected portfolio return:

$$\begin{aligned} \mu_p^e &= r_f + x_{\tan}(\mu_p^{\tan} - r_f) \\ 0.055 = \mu_p^e &= 0.03 + x_{\tan}(.11 - .03) \\ x_{\tan} &= \frac{0.055 - 0.03}{.11 - .03} = .31 \\ x_f &= 1 - x_{\tan} = .69 \end{aligned}$$

Efficient portfolio with same expected return as asset B has:

$$\begin{aligned} (.31)(.46) &= .14 \text{ in asset A} \\ (.31)(.54) &= .17 \text{ in asset B} \\ &\phantom{=}.69 \text{ in T-Bills} \end{aligned}$$

The standard deviation of the efficient portfolio is:

$$\sigma_p^e = .31(.124) = .038.$$

# Interpreting efficient portfolios