Seminar: Computational Behavioral and Social Science **Spring 2020**

Lecture 5: Introduction to Differential Equations

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5.1 Complex Systems Science: A Very Brief Overview

A complex system, roughly speaking, is one with many parts, whose behaviors are both highly variable and strongly dependent on the behavior of the other parts.

"Complex systems science" is the field whose ambition is to understand complex systems.

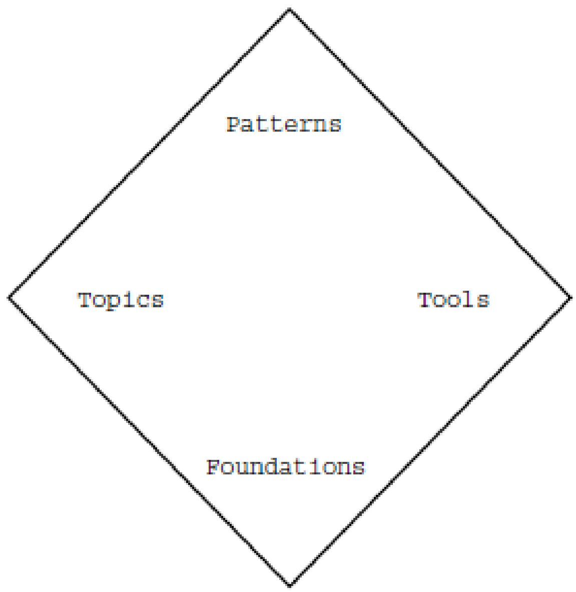

Figure 5.1: The quadrangle of complex systems.

1. A pattern is a recurring theme in the analysis of many different systems, a cross-systemic regularity.

Example Bacterial chemotaxis can be thought of as a way of resolving the tension between the exploitation of known resources, and costly exploration for new, potentially more valuable, resources. Should the bacterium (center) exploit the currently-available patch of food, or explore, in hopes of finding richer patches elsewhere?

Many species solve this problem by performing a random walk (jagged line), tumbling randomly every so often. The frequency of tumbling increases when the concentration of nutrients is high, making the bacterium take long steps in resource-poor regions, and persist in resource-rich ones.

This same tension is present in a vast range of adaptive systems. The pattern "trade-off between exploitation and exploration" serves to orient us to broad features of novel situations: artificial agents, human decision-makers or colonial organisms.

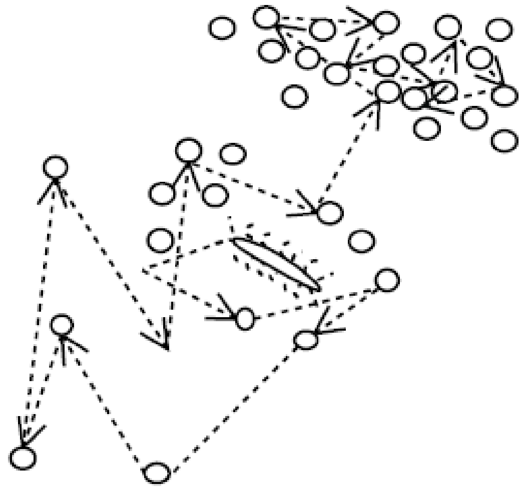

Figure 5.2: Bacterial chemotaxis.

There are many other such patterns in complex systems science:

- stability through hierarchically structured interactions
- positive feedback leading to highly skewed outcomes
- local inhibition and long-rate activation create spatial patterns

2. "Foundations" means attempts to build a basic, mathematical science concerned with such topics as the measurement of complexity, the nature of organization, the relationship between physical processes and information and computation, and the origins of complexity in nature and its increase (or decrease) over time.

3. "Topics" are canonical complex systems, the particular systems, natural, artificial and fictional, which complex systems science has traditionally and habitually sought to understand.

A sample list of topics: networks, turbulence, physio-chemical pattern formation and biological morphogenesis, genetic algorithms, evolutionary dynamics, spin glasses, neuronal networks, the immune system, social insects, ant-like robotic systems, the evolution of cooperation, evolutionary economics.

4. "Tools" refers to procedures for analyzing data, constructing and evaluating models, and measuring the complexity of data or models.

- Statistical learning and data mining
- Time series analysis
- Nonlinear dynamics
- Cellular automata

- Agent-based models
- Complexity measures

5.2 Modeling Using Differential Equations

Mathematical modeling is the process of describing a phenomenon from everyday life in mathematical formalism. We build a mathematical model by identifying the quantities to be studied and the mathematical relationships among them.

The steps in building a model involve:

1. Stating the underlying assumptions.
2. Identifying the relevant variables and parameters.
3. Using the assumptions in the first step to formulate equations relating the variables in the second step.

Mathematical models that are expressed as differential equations involve assumptions about derivatives. In other words, the assumptions often include phrases such as " the rate of change of . . . " or "the rate of increase of . . . ."

Example Modeling the growth of a population.

Year	t	Actual	$P(t) = 3.9e^{0.03067t}$	Year	t	Actual	$P(t) = 3.9e^{0.03067t}$
1790	0	3.9	3.9	1930	140	123	286
1800	10	5.3	5.3	1940	150	132	388
1810	20	7.2	7.2	1950	160	151	528
1820	30	9.6	9.8	1960	170	179	717
1830	40	13	13	1970	180	203	975
1840	50	17	18	1980	190	227	1,320
1850	60	23	25	1990	200	249	1,800
1860	70	31	33	2000	210	281	2,450
1870	80	39	45	2010	220		3,320
1880	90	50	62	2020	230		4,520
1890	100	63	84	2030	240		6,140
1900	110	76	114	2040	250		8,340
1910	120	91	155	2050	260		11,300
1920	130	106	210				

Figure 5.3: U.S. census figures, in millions of people (www.census.gov)

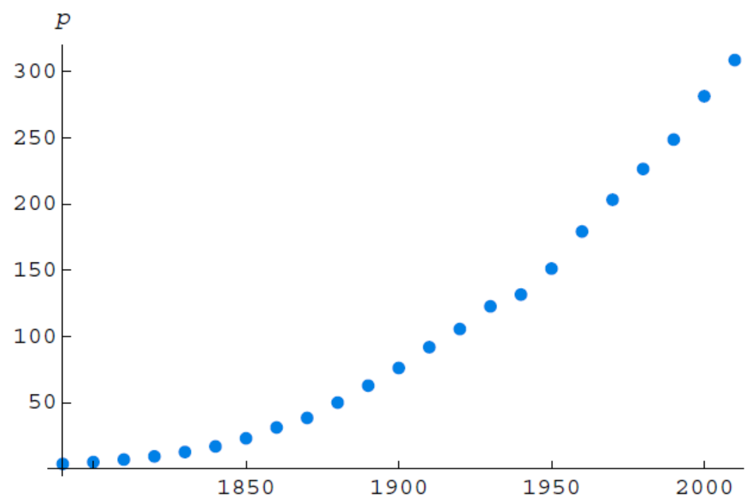

Figure 5.4: The dots represent actual census data.

Malthusian Growth Model: “Population, when unchecked, increases in a geometrical ratio,” quote from Thomas Malthus, *An Essay on the Principle of Population*, 1798.

Assumption: Growth rate of the population is proportional to the population.

$$\frac{dp}{dt} = kp$$

The solution of the differential equation is

$$P(t) = P_0 e^{kt}$$

$P_0 = P(0) = 3.9$, $P(10) = P_0 e^{10t} = 5.3$, $k = 0.3067$.

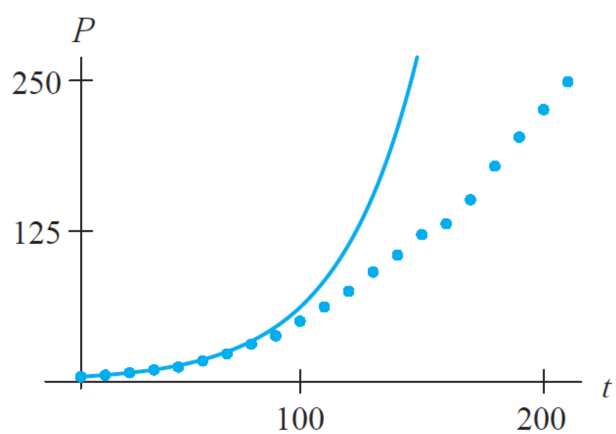

Figure 5.5: The solid line is the solution of the exponential growth model.

To adjust the exponential growth model to account for a limited environment and limited resources, we modify the assumptions of the exponential growth model:

- If the population is small, its growth rate is proportional to the size of the population.

- As the population increases, its *relative growth rate* decreases.

Relative growth rate can be formalized as $\frac{dp}{dt}/p = \frac{1}{p}\frac{dp}{dt}$. Constant relative growth rate means exponential growth.

The logistic population model assumes that the relative growth rate decreases *linearly* as the population increases.

$$\frac{1}{p}\frac{dp}{dt} = k + mp = k - \frac{k}{N}p$$

which leads to the Logistic Population Model:

$$\frac{dp}{dt} = kp(1 - \frac{p}{N})$$

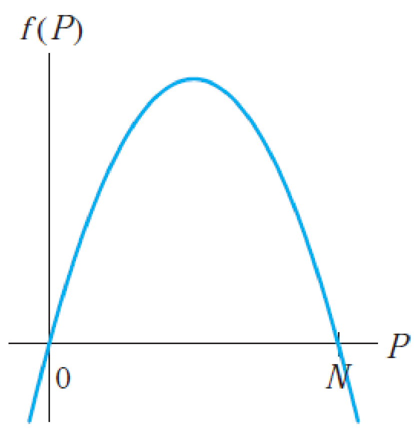

Figure 5.6: The logistic differential equation: qualitative analysis.

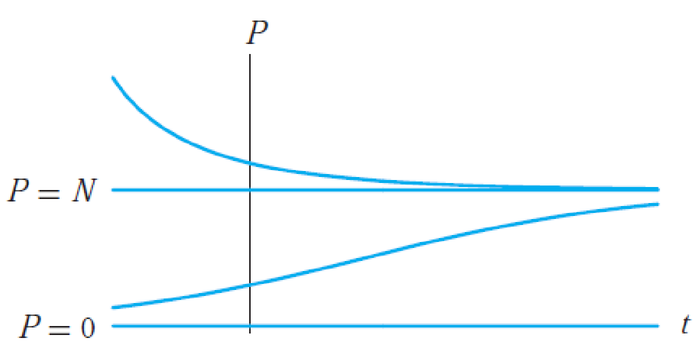

Figure 5.7: The logistic differential equation: qualitative analysis.

If we want to study this model numerically, we need estimates for the parameters k and N. We can approximate $\frac{dp}{dt}$ by $\frac{p_{i+1} - p_i}{p_i}$.

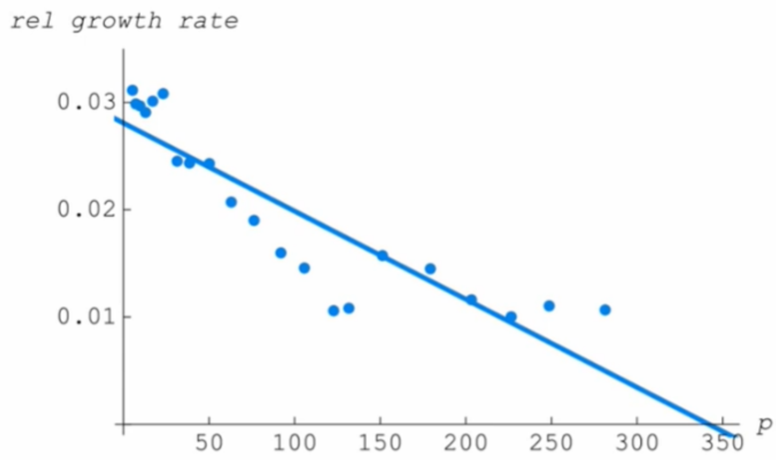

Figure 5.8: Relative growth rate versus population

Using this statistical analysis, we obtain the differential equation

$$\frac{1}{p}\frac{dp}{dt} = k(1 - \frac{p}{N}) = 0.028 - 0.000082p$$

we can calculate $k = 0.028$, $N \approx 342$.

5.3 What is a Defferential Equation

1. A first order differential equation is of the form

$$\frac{dy}{dt} = f(t, y(t))$$

the purpose is to find function $y(t)$.

2. A initial-value problem is of the form

$$\frac{dy}{dt} = f(t, y(t)), \quad y(t_0) = y_0$$

A solution to the initial-value problem is a differentiable function $y(t)$ defined on some interval $a < t < b$ containing t_0 such that

   1. $\frac{dy}{dt} = f(t, y(t))$ for all t in the interval $a < t < b$ and
   2. $y(t_0) = y_0$

3. There is a substantial distinction between the independent and the dependent variables.

Example $\frac{dy}{dt} = 2t$

The solutions to this equation are

$$y(t) = t_2 + c,$$

where c is an arbitrary constant.

Example $\dfrac{dy}{dt} = 2y$

The solutions to this equation are

$$y(t) = y_0 e^{2t},$$

where y_0 is an arbitrary constant.

4. The general solution to a differential equation is "all possible solutions to all possible initial-value problems". See the above two examples.

5. The "no-wrong-answer" principle: by differentiating, we can verify statements such as "All functions of the form $y(t) = y_0 e^{2t}$, where y_0 is an arbitrary constant, are solutions to the differential equation $\dfrac{dy}{dt} = 2y$."

6. Even some relatively simple looking differential equations have solutions that cannot be expressed in terms of the functions that we are familiar with: polynomial, rational, trigonometry, exponential, logs, roots.

Example $\dfrac{dy}{dt} = y^3 + t^2, \quad y(0) = 0.$
The solutions to this equation cannot be expressed in a clean format.

5.4 Systems of Differential Equations

Let's first consider an example of 2nd order differential equation.

Example Simple mass-spring system. We model the motion of a mass-spring system using Newton's second law $F = ma$ and Hooke's law.

Hooke's law asserts that the restoring force of a spring is linearly proportional to its displacement from its rest position.

$$m\frac{d^2y}{dx^2} + ky = 0$$

Figure 5.9: Simple mass-spring system

In general, a second-order autonomous equation has

- one independent variable (often t) and
- one dependent variable (often x or y).

It has the format

$$\frac{d^2y}{dt^2} = f(y, \frac{dy}{dt})$$

Example van der Pol

$$\frac{d^2y}{dt^2} - (1 - x^2)\frac{dy}{dt} + x = 0$$

Example Duffing

$$\frac{d^2y}{dt^2} + b\frac{dx}{dt} + kx + x^3 = 0$$

The mass-spring system can also be written as a first-order system by introducing the variable $v = \dfrac{dy}{dt}$:

$$\begin{cases} \dfrac{dy}{dx} = v \\ \\ \dfrac{dv}{dt} = -\dfrac{k}{m}y \end{cases}$$

with an initial condition $(y_0; v_0)$.

For a more typical system of differential equation, consider the most basic predator-prey system.

Example Predator prey system

The prey are often referred to as "rabbits" and the predators are often referred to as "foxes." The most basic predator-prey system is based on the following four assumptions:

1. If no foxes are present, the rabbits reproduce at a rate proportional to their population, and they are not affected by overcrowding.
2. The foxes eat the rabbits, and the rate at which the rabbits are eaten is proportional to the rate at which the foxes and rabbits interact.
3. Without rabbits to eat, the fox population declines at a rate proportional to itself.
4. The rate at which foxes are born is proportional to the number of rabbits eaten by foxes which, by the second assumption, is proportional to the rate at which the foxes and rabbits interact.

$$\begin{cases} \dfrac{dR}{dt} = aR - bRF \\ \\ \dfrac{dF}{dt} = -cF + dRF \end{cases}$$

where a, b, c, and d are positive parameters.

In general, a first-order autonomous system has

1. one independent variable (often t) and
2. two dependent variables.

$$\begin{cases} \dfrac{dx}{dt} = f(x, y) \\ \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

with *initial condition*: (x_0, y_0), which is a pair of numbers.

Solution to an initial-value problem: a pair of functions $(x(t), y(t))$ that satisfy the system and the initial conditions.

Equilibrium solutions: Solutions for which both component functions are constant for all t.

5.5 Analytical Method: Separation of Variables

A differential equation is separable if it can be written in the form

$$\frac{dy}{dt} = (f_1(t))(f_2(t)).$$

A separable differential equation can be solved using the method of substitution by separating variables

$$\frac{1}{f_2(t)} \frac{dy}{dt} = f_1(t).$$

Example $\dfrac{dy}{dt} = -2ty^2$

The general solution is

$$\begin{cases} y(t) = \dfrac{1}{t^2 + k} & \text{for any constant k;} \\ \\ y(t) = 0 & \text{for all t;} \end{cases}$$

For initial condition $y(0) = -\dfrac{1}{2}$, the solution is $y(t) = \dfrac{1}{t^2 - 2}$, with domain $-\sqrt{2} < t < \sqrt{2}$.

5.6 Slope Fields and Vector Fields

For a first-order differential equation $\dfrac{dy}{dt} = f(x, y)$, $f(x, y)$ defines a slope field, and the graph of a solution must be everywhere tangent to it.

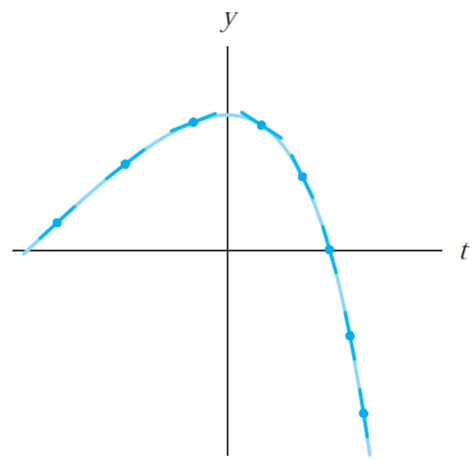

Figure 5.10: If $y = y(t)$ is a solution, then the slope of any tangent must equal $f(t, y)$.

Examples $\dfrac{dy}{dt} = f(x)$

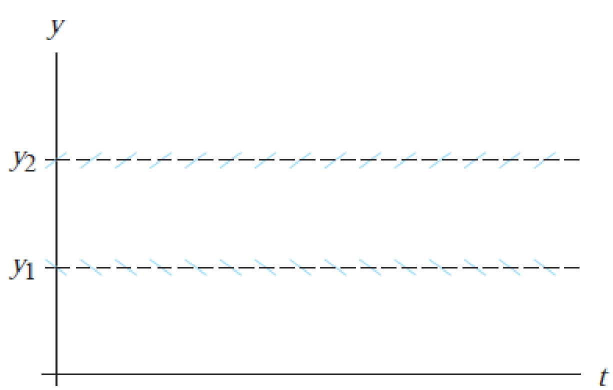

Figure 5.11: Slope field example.

Examples $\dfrac{dy}{dt} = y - t$

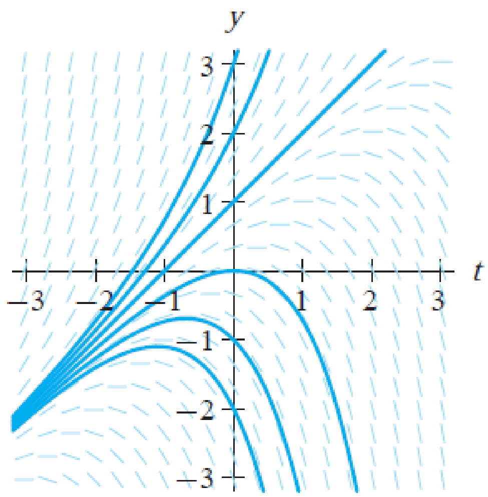

Figure 5.12: Slope field example.

Examples $\dfrac{dy}{dt} = 4y(1-y)$

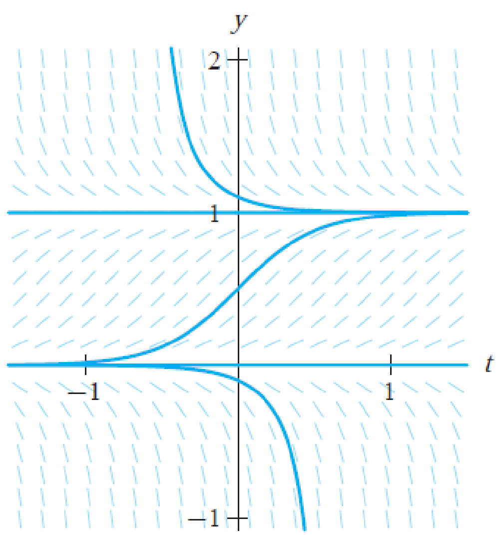

Figure 5.13: Slope field example.

Consider the system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

We use the right-hand side of this system to form a vector field in the xy-plain:

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

Let

$$\mathbf{Y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Then the (scalar) system of differential equations can be rewritten as one vector differential equation.

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

We can also define component graphs as graphs of $x(t)$ and $y(t)$; solution curve as the parametrized curve of $(x(t), y(t))$; phase plane as xy-plane; phase portrait as multiple solution curves.

Example Mass spring system again, $\mathbf{F}(\mathbf{y}, \mathbf{v})^{\mathbf{T}} = (\mathbf{v}, -\mathbf{y})^{\mathbf{T}}$

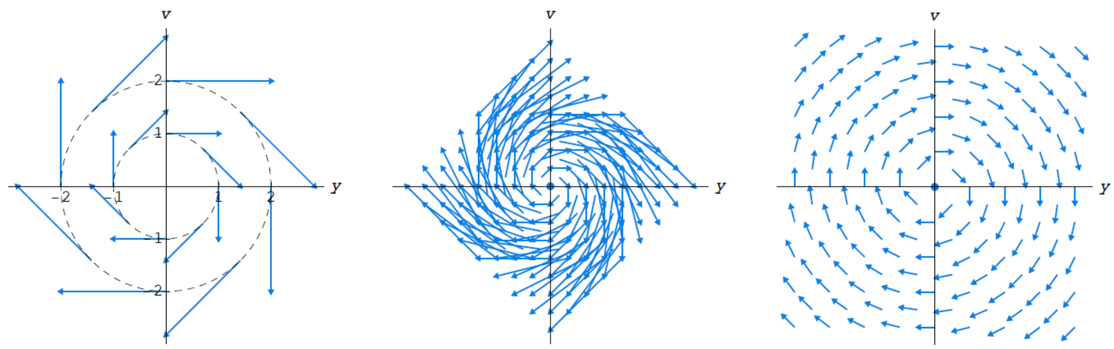

Figure 5.14: Vector field example.

Example Predator prey system again

$$\begin{cases} \dfrac{dR}{dt} = 2R - 1.2RF \\ \\ \dfrac{dF}{dt} = -1F + 0.9RF \end{cases}$$

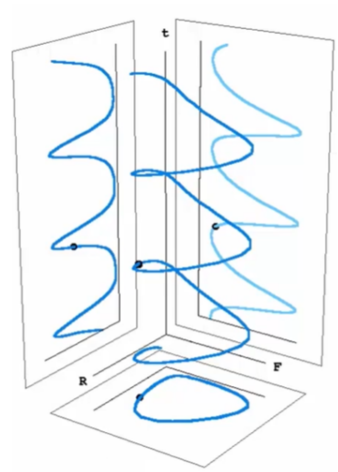

Figure 5.15: Phase portrait example.

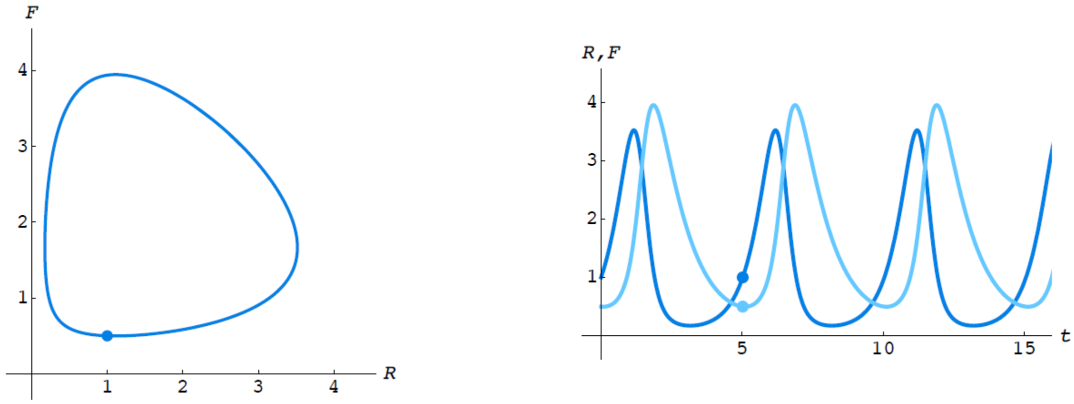

Figure 5.16: Solution curve and phase portrait example.

See the Matlab Demo.

5.7 Euler's method

For first order differential equations, we can draw the solution curve using Euler's method. The method determines the point (t_{k+1}, y_{k+1}) by assuming that it lies on the line through (t_k, y_k) with slope $f(t_k, y_k)$.

Example $\dfrac{dy}{dt} = 2y - 1$, $y(0) = 1$, with $\Delta t = 0.1$

k	t_k	y_k	$f(t_k, y_k)$
0	0	1	1
1	0.1	1.100	1.20
2	0.2	1.220	1.44
3	0.3	1.364	1.73
4	0.4	1.537	2.07
5	0.5	1.744	2.49
6	0.6	1.993	2.98
7	0.7	2.292	3.58
8	0.8	2.650	4.30
9	0.9	3.080	5.16
10	1.0	3.596	

Figure 5.17: Euler's method

Euler's method is the most basic fixed-step-size algorithm for the numerical approximation of solutions.

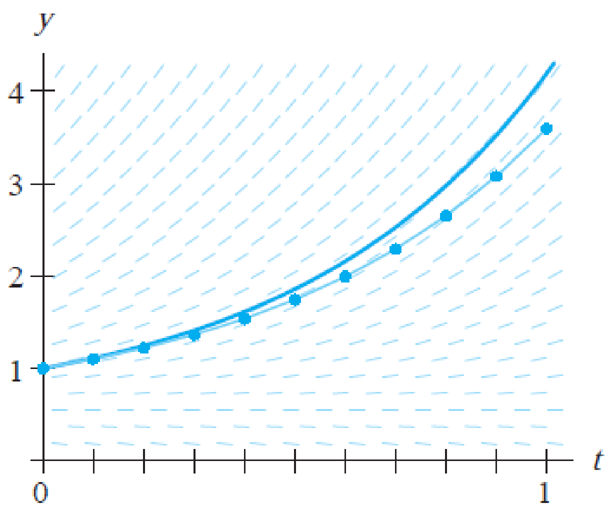

Figure 5.18: Euler's method

We say that it is a first-order algorithm because the error in the approximation is proportional to the first power of the step size. More precisely, we have

$$error \leq C \cdot (\Delta t)^1$$

where C is a constant that depends only on the right-hand side of the differential equation and the length of the interval over which the solution is approximated.

In practice, if we halve the step size (and double the number of steps), we typically halve the error. For numerical work where a high degree of accuracy is essential, Euler's method is not the algorithm of choice. There are algorithms that usually are more accurate and that require fewer calculations to attain that accuracy.

The higher-order Runge-Kutta methods are usually more efficient and more accurate than Euler's method. They use the right-hand side of the differential equation evaluated at more than one point to compute the approximate value of the solution at the next step. For example, the classical Runge-Kutta method (also referred to as RK4) uses a weighted average of four values to calculate the approximation at the next step. RK4 has order 4. That is, if we double the number of steps, the error usually improves by a factor of 16 (= 24).

There are initial-value problems that are not amenable to any fixed-step-size algorithm.

Example $\dfrac{dy}{dt} = e^t siny,\ y(0) = 5$

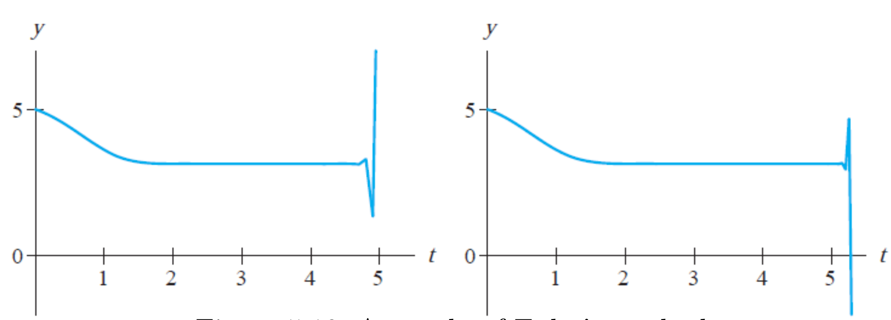

Figure 5.19: Anomaly of Euler's method

5.8 Bifurcation

We have already introduced the logistic model of population growth

$$\frac{dp}{dt} = kp(1 - \frac{p}{N})$$

where k is a growth-rate parameter and N is the carrying capacity. Now let's add a controllable constant rate of death C.

$$\frac{dp}{dt} = kp(1 - \frac{p}{N}) - C$$

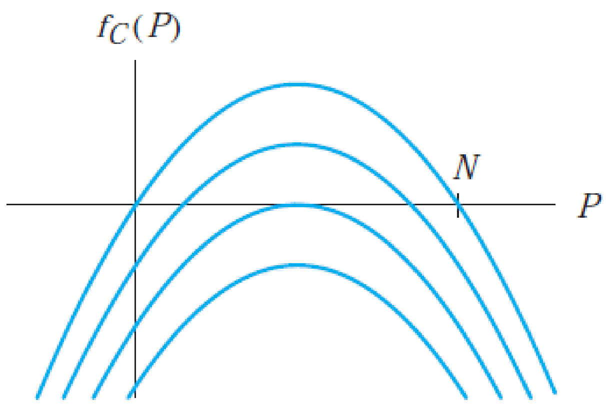

Figure 5.20: Graph of $f_C(P)$.

We can summarize the behavior of this one-parameter family of differential equations using a bifurcation diagram.

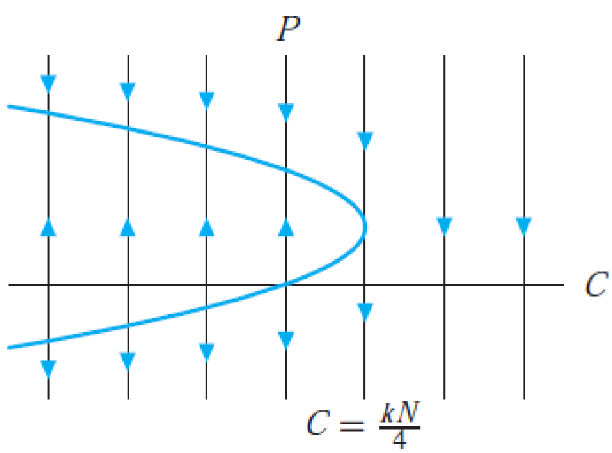

Figure 5.21: Bifurcation diagram and phase lines.

For the one-parameter family of differential equations

$$\frac{dp}{dt} = kp(1 - \frac{p}{N}) - C,$$

we say that

- All of the systems (differential equations and their phase lines) with $C < kN/4$ are qualitatively equivalent, and
- all of the systems with $C > kN/4$ are qualitatively equivalent.
- The value $C = kN/4$ is a bifurcation value for this one-parameter family

Some one-parameter families of differential equations have more than one bifurcation value.

This is an example of a concrete dynamic system implementation of the causal relation between death rate C and the equilibrium points. The modeling of these micro-macro relations should be the focus of the study rather than the pseudo problem of the metaphysics of causation.

References

[1] C. R. Shalizi, "Methods and Techniques of Complex Systems Science: An Overview," in T. S. Deisboeck (Ed.) *Complex Systems Science in Biomedicine*, 2006, pp.33–114.

[2] P. Blanchard, R.L. Devaney and G. R. Hall, *Differential Equations*, Cengage Learning; 4th edition, 2012.