**Seminar: Computational Behavioral and Social Science** **Spring 2020**

# Lecture 8: Dynamical Systems: Chaos

*Lecturer: Renjie Yang*

## 8.1 Chaotic Behavior

We talked about two major kinds of attractor:

1. Point attractors, or stable equilibrium points, represent behavior that is either static or approaching it.
2. Limit cycle attractors, or stable closed orbits, represent behavior that is periodic (or approaching it).

For a long time, scientists and mathematicians thought that those were the only two kinds of attractor that could exist, either mathematically or physically. It turns out they were wrong, both mathematically and in real systems (Hilborn 2000).

Figure 8.1: The plume from this candle flame goes from laminar to turbulent.

| Behavior | Mathematical model |
|---|---|
| equilibrium | stable equilibrium point ("point attractor") |
| oscillation | limit cycle attractor |

Figure 8.2: Forms of motions and their math models.

**Example** The Hastings and Powell Food Chain Model

Remember the simple predator-prey model:

$$\begin{cases} \dfrac{dN}{dt} = N - NP \\[2ex] \dfrac{dP}{dt} = NP - P \end{cases}$$

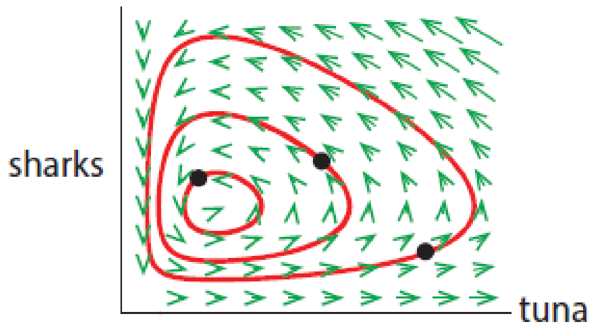


Figure 8.3: A typical predator-pray system.

An improvement of the classic model is the Holling–Tanner model:

$$\begin{cases} \dfrac{dN}{dt} = r_1 N(1 - \dfrac{N}{k}) - \dfrac{wN}{d+N} P \\ \\ \dfrac{dP}{dt} = r_2 P(1 - \dfrac{jP}{N}) \end{cases}$$

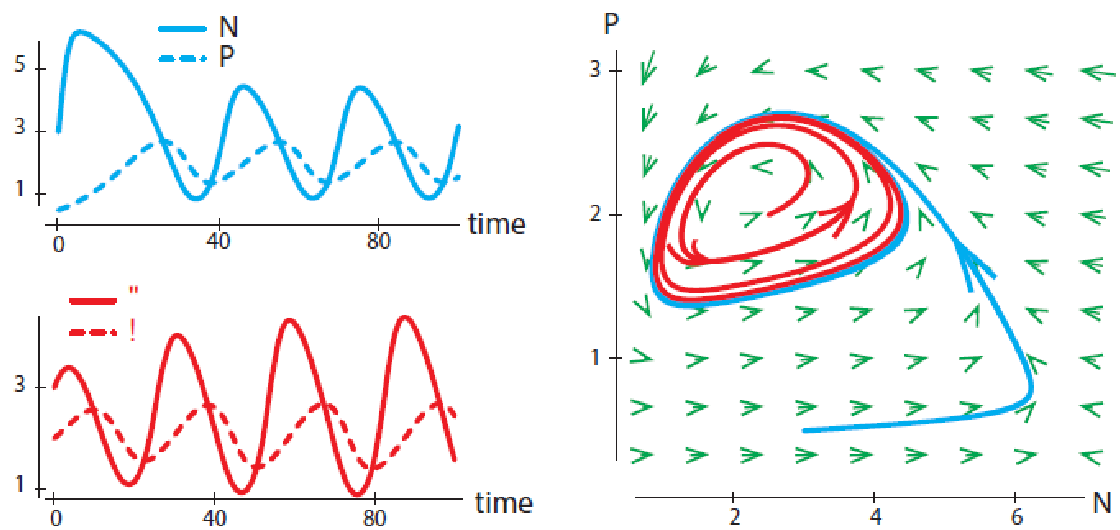


Figure 8.4: Two simulations of the Holling–Tanner model with w = 1 starting from different initial conditions

We can extend this model further to a food chain that contains three species:

$$\begin{cases} \dfrac{dX}{dt} = rX(1 - \dfrac{X}{K}) - \dfrac{a_1 X}{1 + b_1 X} Y \\ \\ \dfrac{dY}{dt} = c_1 \dfrac{a_1 X}{1 + b_1 X} Y - d_1 Y - \dfrac{a_2 Y}{1 + b_2 Y} Z \\ \\ \dfrac{dZ}{dt} = c_2 \dfrac{a_2 Y}{1 + b_2 Y} Z - d_2 Z \end{cases}$$

Simulate the model in Matlab:

```
% Aim: to solve the Hasting equations
a1=5; b1=3; a2=0.1; b2=2; d1=0.4; d2=0.01 % Parameters
t(1)=0.0; % Initial t
x(1)=1; y(1)=0.5; z(1)=8; % Initial x,y,z
dt=0.005; % Time step
nn=300000; % Number of time steps
for k=1:nn % Time loop
    fx = x(k) * (1 - x(k)) - (a1 * x(k)/(1 + b1 * x(k))) * y(k); % RHS of x equation
    fy = (a1 * x(k)/(1 + b1 * x(k))) * y(k) - (a2 * y(k)/(1 + b2 * y(k))) * z(k) - d1 * y(k);
        % RHS of y equation
    fz = (a2 * y(k)/(1 + b2 * y(k))) * z(k) - d2 * z(k); % RHS of z equation
    x(k + 1) = x(k) + dt * fx; % Find new x
    y(k + 1) = y(k) + dt * fy; % Find new y
    z(k + 1) = z(k) + dt * fz; % Find new z
    t(k + 1) = t(k) + dt; % Find new t
end % Close time loop
plot(t,x) % Plot x vs t
title('Hastings') % Title
xlabel('t'); ylabel('xyz'); % Label axes
hold on
plot(t,y)
hold on
plot(t,z)
```

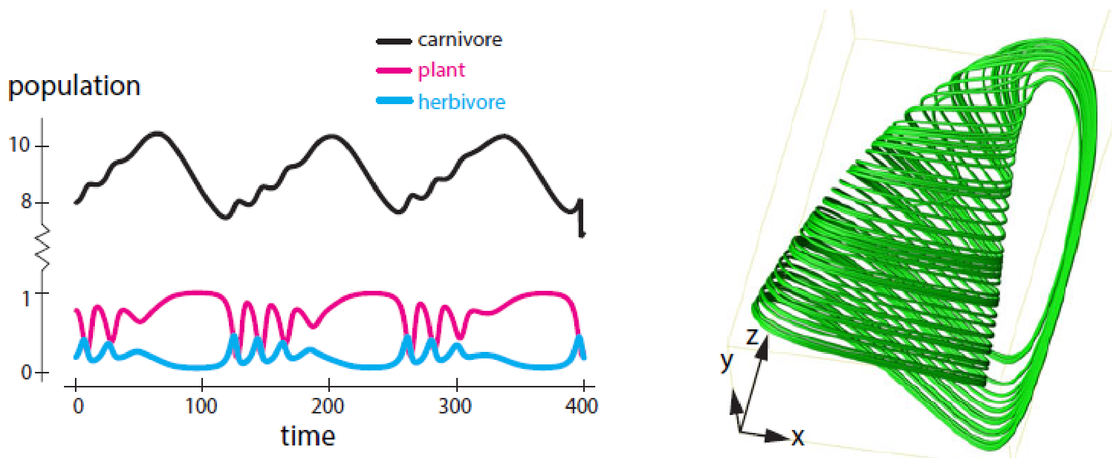


Figure 8.5: Left: a simulation of the three-species food chain model described in the text with $a_1 = 5, b_1 = 3, a_2 = 0.1, b_2 = 2, d_1 = 0.4$, and $d_2 = 0.01$. Right: a typical trajectory of the three-species model.

The main point is not chaos, although the phenomenon is called chaos: aperiodic, bounded behavior exhibited by a deterministic differential equation. This kind of attractor is first called "strange attractor". Now it is called chaotic attractor.

Note that chaos is not a kind of system. It is a kind of behavior that a system may exhibit or not exhibit.

| Behavior | Mathematical model |
|---|---|
| equilibrium | stable equilibrium point (“point attractor”) |
| oscillation | limit cycle attractor |
| chaos | chaotic attractor |

Figure 8.6: Forms of motions and their math models.

We also have an alternative type of bifurcation from oscillation to chaos behavior.

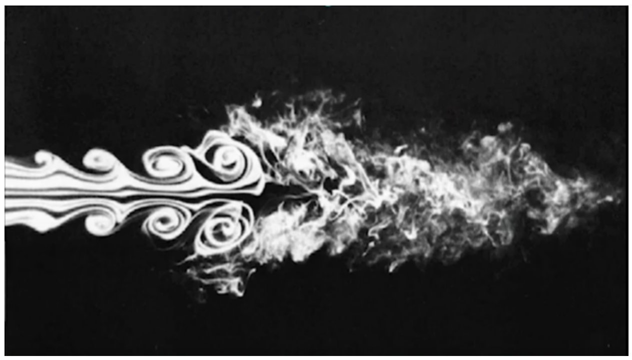

Figure 8.7: Transition from equilibrium to oscillation to chaos.

## 8.2 Characteristics of Chaos

Chaos is dynamical behavior that is deterministic, bounded in state space, irregular, and, most intriguingly, extremely sensitive to initial conditions.

**1. Determinism**

To say that a system is deterministic means that each state is completely determined by the previous state.

In the food chain model, just as in all the other models we have studied, there are no unmodeled outside influences or chance events.

**2. Boundedness**

Boundedness means that the system does not go off to infinity. Rather it stays within a certain region of state space.

**3. Irregularity**

Chaotic behavior is irregular, or aperiodic. Aperiodic behavior never exactly repeats.

**4. Attractor**

An attractor of a dynamical system is a subset A of state space that has the following property:

For every X in some neighbourhood of A, $X \to A$ as $t \to \infty$

"Point attractor" is stable equilibrium point, while "limit cycle attractor" is stable oscillation. Chaotic behaviors have attractors occasionally.

### 5. Sensitive Dependence on Initial Conditions

Suppose $M_0$ and $N_0$ are two different initial conditions for the food chain model. Let's define $d(M_0, N_0)$ as the distance between $M_0$ and $N_0$. Since $M_0$ and $N_0$ are points in 3-dimensional $(X, Y, Z)$ space, the distance between them is the Euclidean distance

$$d(M,N) = \sqrt{(X_M - X_N)^2) + (Y_M - Y_N)^2) + (Z_M - Z_N)^2)}$$

After a time $t$, the two points $M_0$ and $N_0$ have evolved to $M_t$ and $N_t$. Sensitive dependence says that the distance $d(M_t, N_t)$ grows exponentially with time for some $\lambda$

$$d(M_t, N_t) = e^{\lambda t} \cdot d(M_0 - N_0)$$

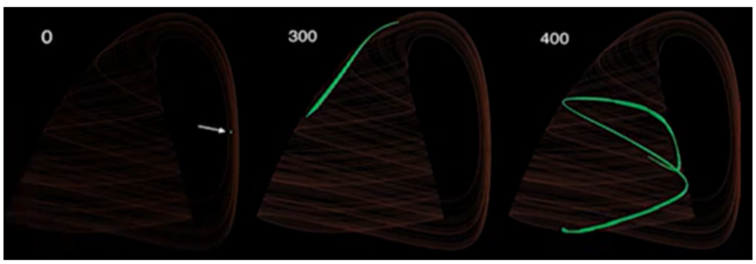


Figure 8.8: The evolution of one million initial conditions (Figure 5.32, p252).

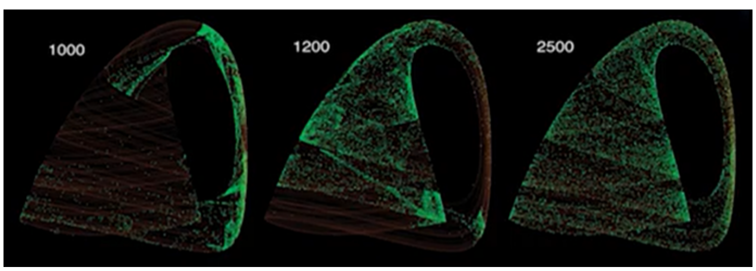


Figure 8.9: The evolution of one million initial conditions(continued).

**Example**: The double pendulum.

**Example**: Lorenz system.

$$\begin{cases} \dfrac{dX}{dt} = \sigma(y - x) \\[2ex] \dfrac{dY}{dt} = x(\rho - z) - y \\[2ex] \dfrac{dZ}{dt} = xy - \beta z \end{cases}$$

## 8.3 Discrete Time Dynamical Systems

Discrete time dynamical systems have the following format:

$$X_{n+1} = f(X_n)$$

Linear discrete time dynamical system takes the form $X_{n+1} = kX_n$. Depending on the value of $k$, $X_n$ will grow or decrease exponentially. The system is stable if $k < 1$ and unstable otherwise.

The logistic map was popularized in a 1976 paper by the biologist Robert May. The title of the paper is "Simple Mathematical Models with Very Complicated Dynamics."

$$X_{n+1} = bX_n(1 - \frac{X_n}{K})$$

Let $K$ scale to 1, then $X_n \in [0, 1]$.

$$X_{n+1} = rX_n(1 - X_n)$$

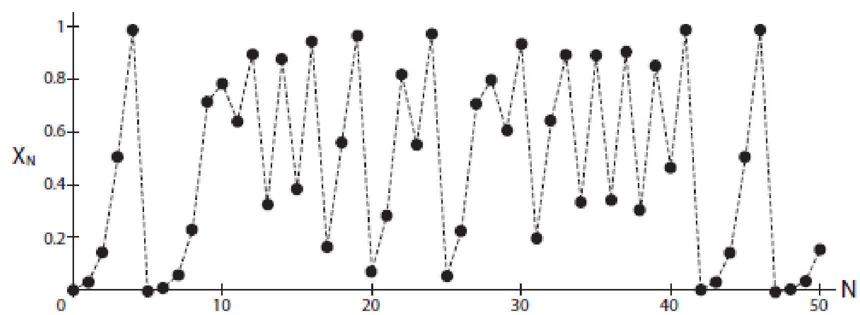


Figure 8.10: A simulation of the discrete logistic model with $r = 4$ and $X_0 = 0.01$.

The system is aperiodic, bounded, deterministic. It is also sensitive to initial conditions:

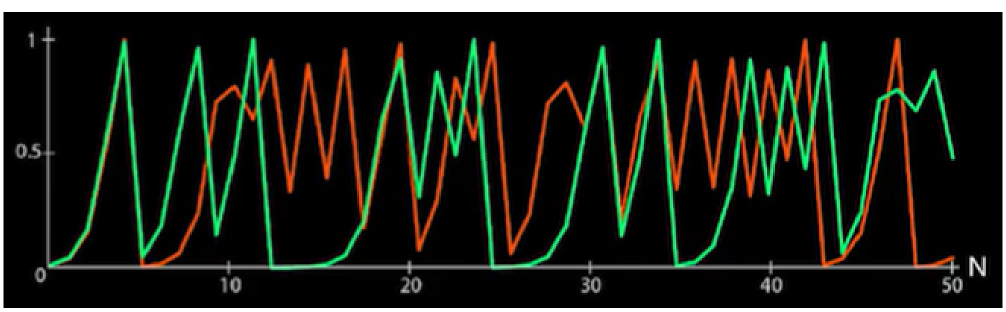


Figure 8.11: $X_0 = 2$ vs. $X_0 = 2.01$ .

**Tricky Exercise**: Let $X_0 = 0.74$. Compute $X_50$ for the system $X_{n+1} = 4X_n(1 - X_n)$.

Although we cannot predict the system behavior precisely, we may be able to find the causes of the chaos.

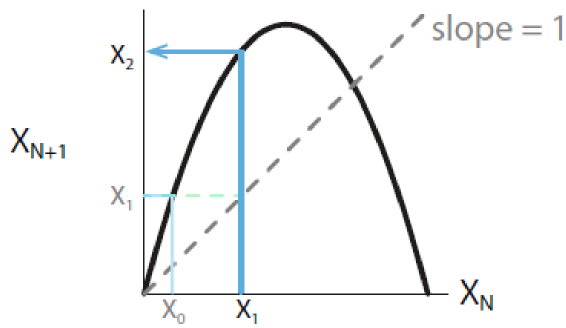


Figure 8.12: The dynamics of the discrete logistic model.

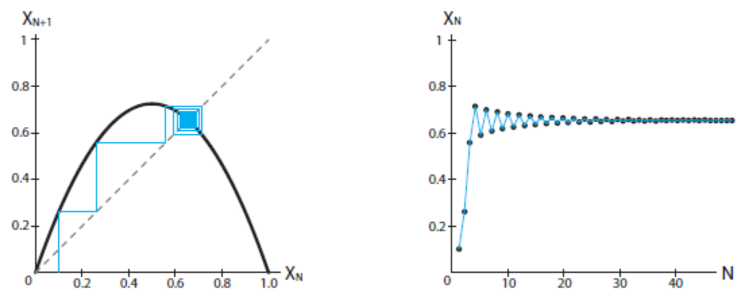

Figure 8.13: Discrete logistic model with r = 2.9.

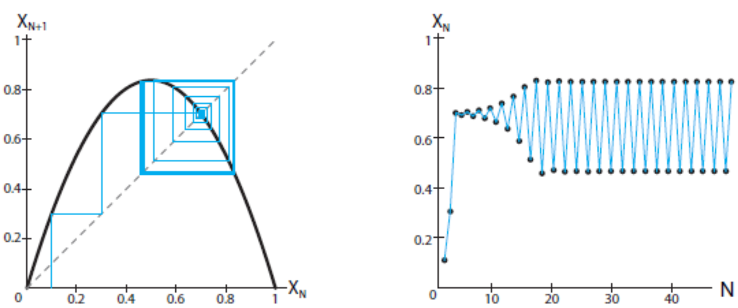

Figure 8.14: Discrete logistic model with r = 3.35.

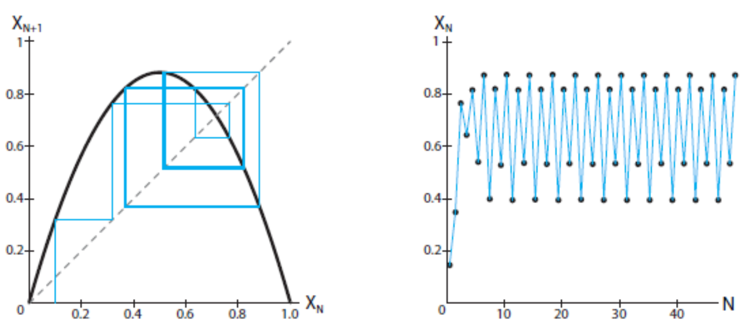

Figure 8.15: Discrete logistic model with r = 3.53.

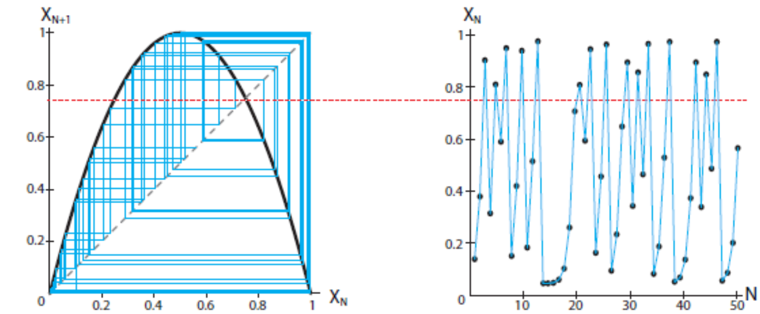


Figure 8.16: Discrete logistic model with r = 4.

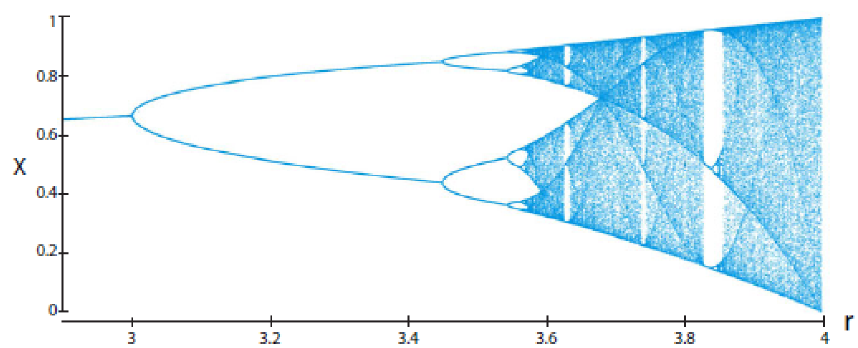


Figure 8.17: Bifurcation diagram for the discrete logistic model.

**Route to Chaos**: A sequence of bifurcations from equilibrium points to simple oscillations to complex oscillations to chaos.

This information about the dynamical system could be used to make diagnoses (by lowering r).

**Example**: The relationship interpretation of the dynamic logistic model.

The difference between chaos and random noise is if the variability is internal.

**Example** What is the cause of the movements of the candle flame?

**Example** Locus of control in psychiatry.

## 8.4 Examples

**Example** Route to Chaos in the Three-Species Food Chain Model.

The key value is $b_1$.

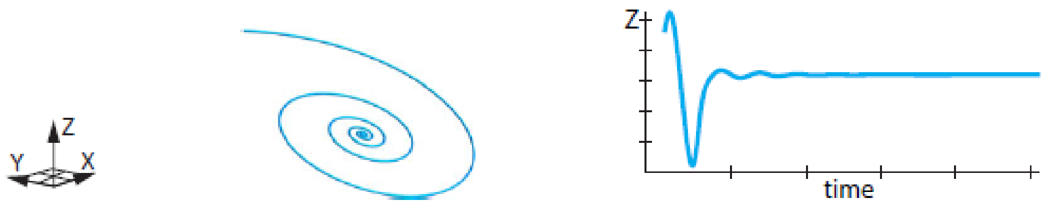


Figure 8.18: For low values of $b_1$, the system has a stable equilibrium point of spiral type.

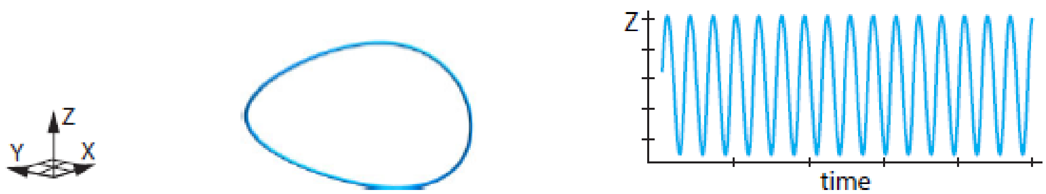


Figure 8.19: For slightly higher values of $b_1$, the system exhibits a stable oscillation.

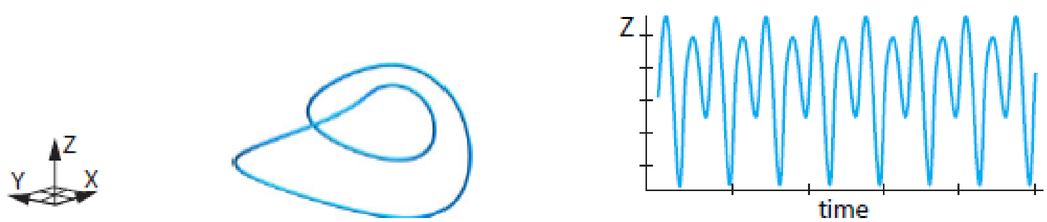


Figure 8.20: As $b_1$ further increases, a period-doubling bifurcation occurs, and the rhythm becomes more complex.


Figure 8.21: Still further increases in $b_1$ cause a second period-doubling bifurcation, to an even more complex periodic rhythm.

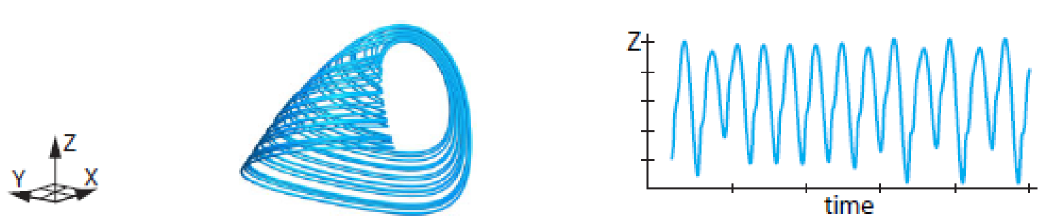


Figure 8.22: Increasing $b_1$ even more produces a bifurcation to a chaotic attractor.

**Example:** Dripping Faucet

Dripping faucet is a process that has two characteristic time intervals in it:

1. the drop formation process has a characteristic time interval that is set by the flow rate, controlled by the handle. For slow dripping, this is $\approx 1$ sec.
2. the snap-back of the unseparated part of the drop is faster, at $\approx 0.1$ sec.

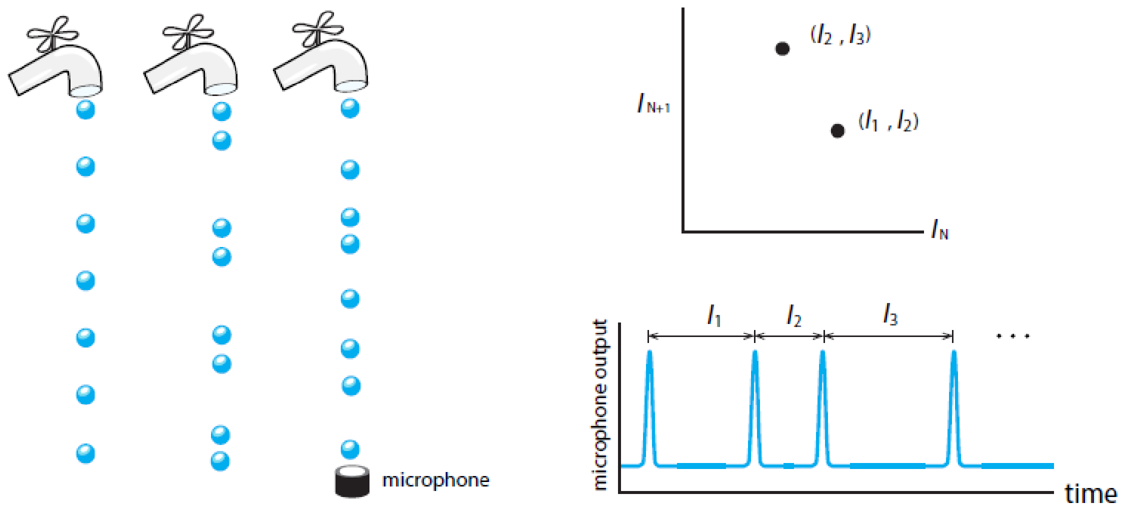


Figure 8.23: Apparatus of the dripping faucet experiment (Crutchfield et al., 1986).

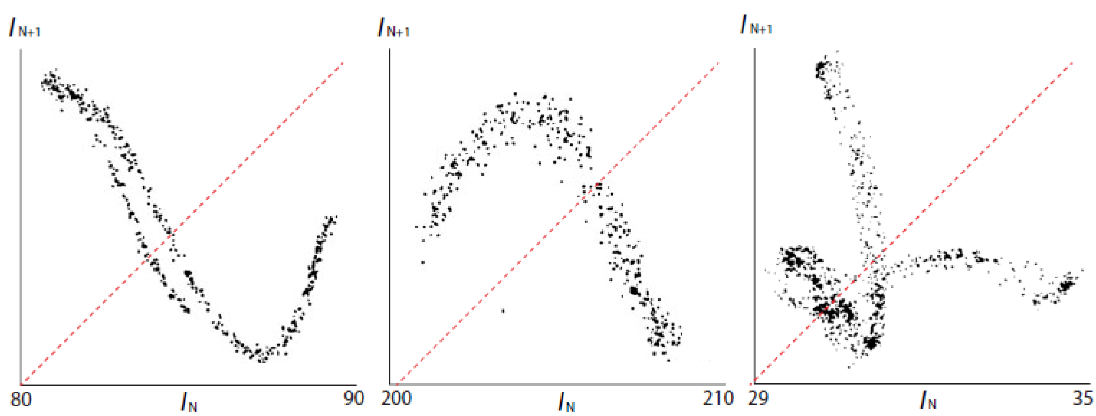


Figure 8.24: Three examples of Poincaré plots of interdrip intervals in the dripping faucet(P. Martien et al., 1985).

## References


[1] A. Garfinkel, J. Shevtsov and Y. Guo, *Modeling Life: The Mathematics of Biological Systems*, Springer International Publishing; 4th edition, 2017.

[2] P. Blanchard, R.L. Devaney and G. R. Hall, *Differential Equations*, Cengage Learning; 4th edition, 2012.