Seminar: Computational Behavioral and Social Science | Spring 2020

Lecture 7: Dynamical Systems: Oscillation

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7.1 Oscillation

We have all seen scientific concepts of equilibrium playing a fundamental role in many scientific theories.

- Chemistry. We are told that chemical substances placed in a box will quickly go to equilibrium, called "chemical equilibrium."
- Ecology. Older theories were often phrased in terms of equilibrium concepts such as "carrying capacity" and "climatic climax." The population rises or falls until it reaches the ecosystem's carrying capacity, or the community composition changes until it reaches a state determined by climate and soil, at which point the system is in "ecological equilibrium."
- Economics. We are told that a free market with many small traders will reach an equilibrium price where supply meets demand, called "economic equilibrium."
- Physiology. We are taught the doctrine of homeostasis, which says that the body regulates all physiological variables, such as temperature and hormone levels, to remain in "physiological equilibrium."

Let's consider the physiological equilibrium. Think about the concept of "homeostasis". The mechanism of homeostasis is the negative feedback loops. Here are two examples of homeostasis:

Example Homeostasis and temperature control.

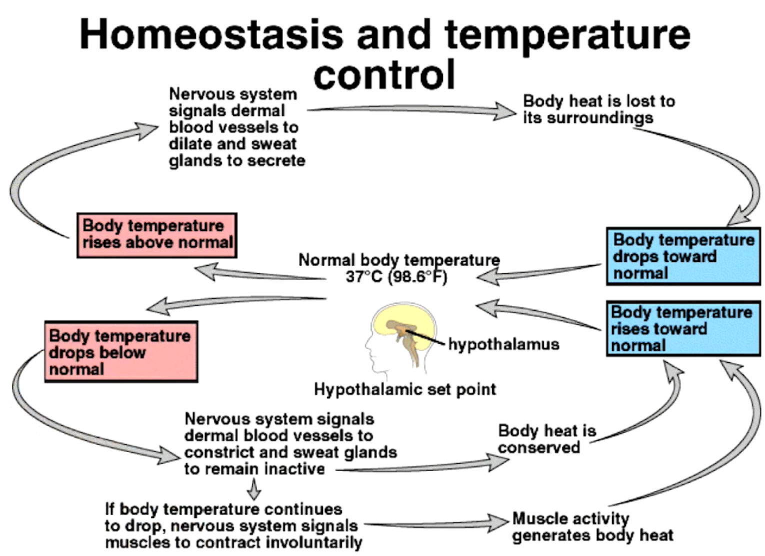


Figure 7.1: Homeostasis and temperature control.

Example Homeostasis and hormone levels.

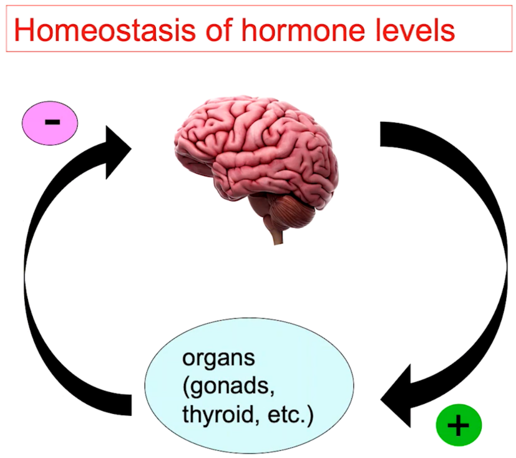
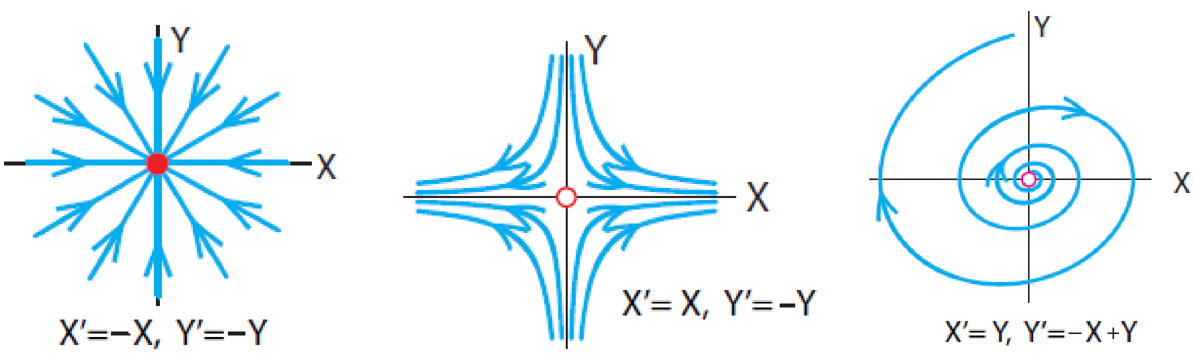


Figure 7.2: Homeostasis and hormone levels.

We have examined the math theories of equilibrium in Lecture 6. We get equilibrium when the derivative function equals zero; while the negative feedback loop guarantees that the equilibrium point being stable.



(a) (b) (c)

Figure 7.3: The math of equilibrium and stability.

Are these equilibrium theories right as science? Are natural processes really governed by equilibrium dynamics?

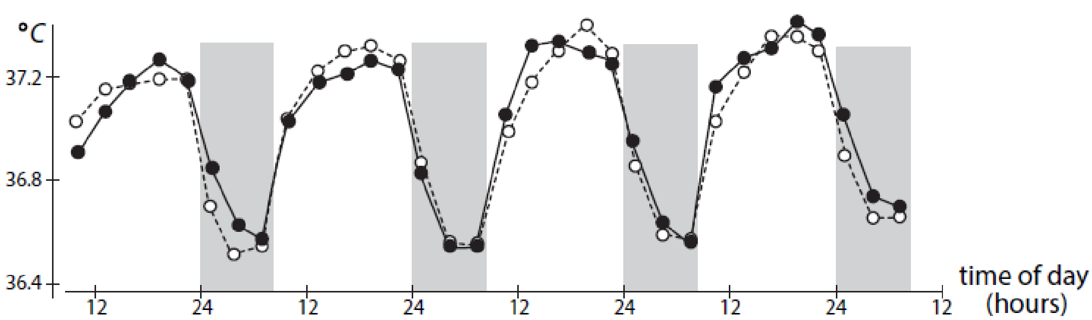


Figure 7.4: Four days of core body temperature (measured rectally) in human subjects. “Human circadian rhythms in continuous darkness: entrainment by social cues,” by J. Aschoff, M. Fatranska, H. Giedke, P. Doerr, D. Stamm, and H. Wisser, (1971), Science 171(3967):213–15

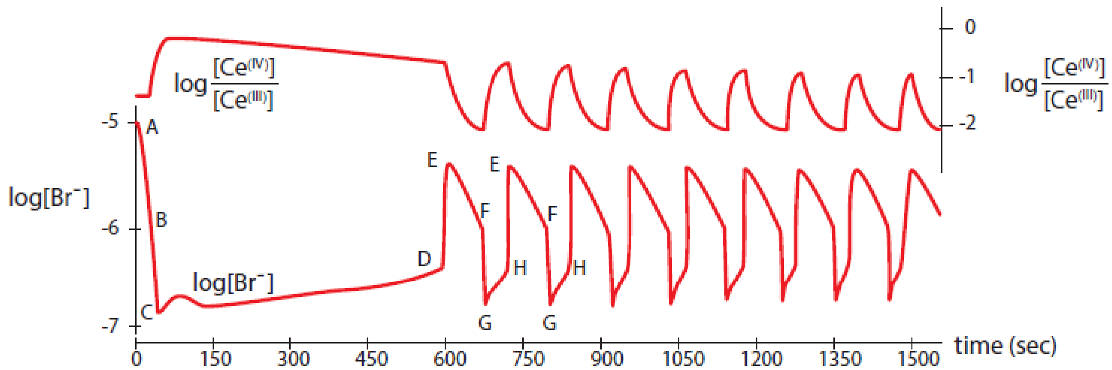


Figure 7.5: Oscillations in reaction products in the Belousov reaction. “Oscillations in chemical systems II: Thorough analysis of temporal oscillation in the bromate–cerium–malonic acid system,” by R.J. Field, E. Koros, and R.M. Noyes, (1972), Journal of the American Chemical Society 94(25):8649–8664

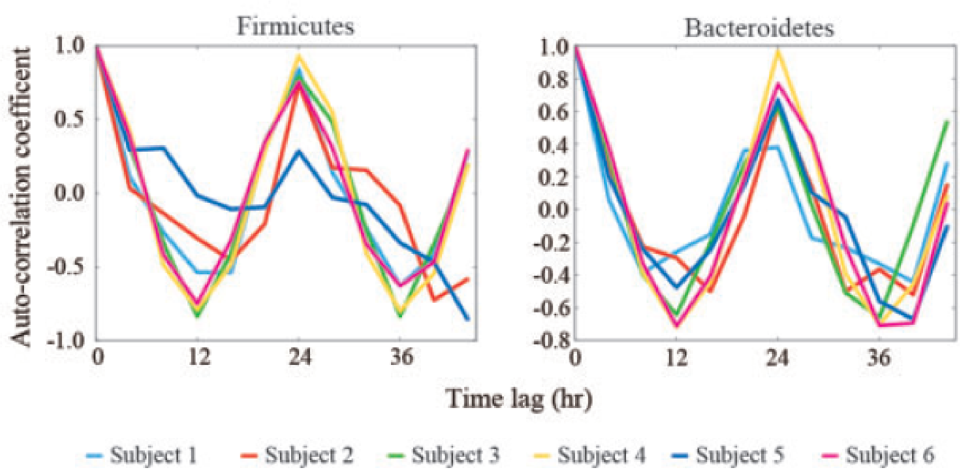


Figure 7.6: Autocorrelation coefficients of the relative abundance of Firmicutes, Bacteroidetes in all the subjects. The horizontal axis represents time lag used for autocorrelation calculation. “Circadian oscillations of microbial and functional composition in the human salivary microbiome”, by Lena Takayasu et al., DNA Research, 2017, 24(3), 261–270

Consider the predator-prey system, where the oscillation is not stable. The instability is not what biological systems need. We need to combine the ideas of oscillation and stability. So how do we model stable oscillation?

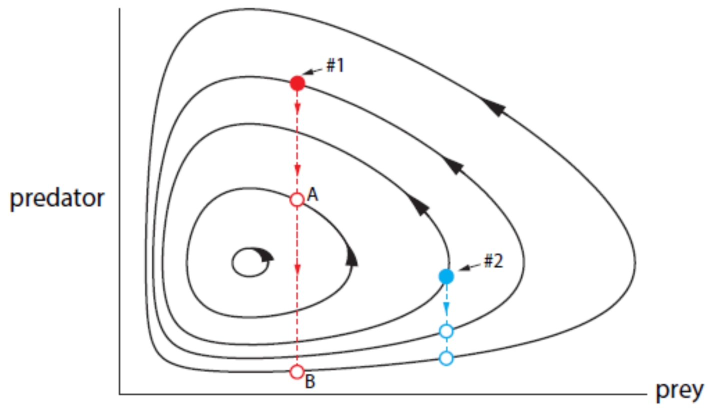
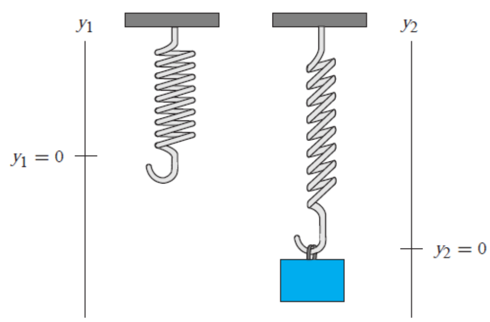


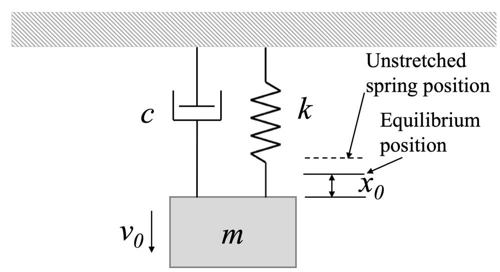
Figure 7.7: Response of the predator-prey system to perturbations depends on the strength and timing of the perturbation. The outcome of a perturbation is to place the state point on a new trajectory, whose amplitudes may be higher or lower than before.

7.2 Linear Systems of Differential Equation

The classic example of oscillation systems in mechanical engineering is the damping spring system. Let's analyze the system first.



(a) The spring system.



(b) The damping spring system

Figure 7.8: Vertical damping spring system.

See the wiki url `https://en.wikipedia.org/wiki/Damping_ratio` for an animation.

The differential equation of the system is

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

Transform the second order differential equation to a first order system of differential equation:

$$\begin{cases} \dfrac{dy}{dt} = v \\[2ex] \dfrac{dv}{dt} = -\dfrac{k}{m}y - \dfrac{c}{m}v \end{cases}$$

This is a linear system of first order equations. For linear systems of equations, there is a complete analytic method for the solutions using linear algebra. Let's introduce the method here. Without loss of generality, consider the case of two dependent variables.

In general, a linear system with two dependent variables has the general form of the following:

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

We can write the linear system in vector notation

$$\frac{d\mathbf{X}}{dt} = \mathbf{AX}$$

The geometry of the above linear system is a vector field, and the graph of a solution must be everywhere tangent to it. For example, consider the mass spring system again, $\mathbf{F}(\mathbf{y}, \mathbf{v})^{\mathbf{T}} = (\mathbf{v}, -\mathbf{y})^{\mathbf{T}}$

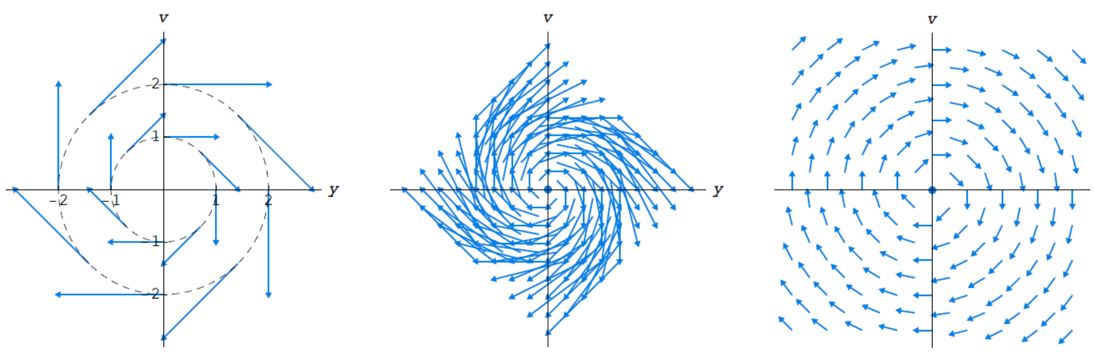


Figure 7.9: Vector field example.

A side note: the differential function $\mathbf{F}$ is a map or a function of vectors. It can be represented as a matrix when $\mathbf{F}$ is a linear function (map, operator) $\mathbf{L}$. The representation convention is defined as follows:

$$\mathbf{L}(\mathbf{c_1 e_1} + \mathbf{c_2 e_2} + \cdots + \mathbf{c_n e_n}) = [\epsilon_\mathbf{1}, \epsilon_\mathbf{2}, \cdots, \epsilon_\mathbf{m}] \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}}_{A_{ij}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The Linearity Principle for the linear system $\dfrac{d\mathbf{X}}{dt} = \mathbf{AX}$:

1. If $\mathbf{X}(\mathbf{t})$ is a solution of this system and k is any constant, then $k\mathbf{X}(\mathbf{t})$ is also a solution.
2. If $\mathbf{X_1}(\mathbf{t})$ and $\mathbf{X_2}(\mathbf{t})$ are two solutions of this system, then $\mathbf{X_1}(\mathbf{t}) + \mathbf{X_2}(\mathbf{t})$ is also a solution.

This principle gives us a more general way to find solutions of linear systems. Once we find two linear independent solutions, we can infer that any linear combinations of them is also a solution. Therefore the solution space of the linear dynamic system is a linear space. For systems with two dependent variables, the dimension of the linear space is two.

For an arbitrary linear system $\frac{d\mathbf{X}}{dt} = \mathbf{AX}$, how many solutions do we need to solve every initial-value problem? For systems with two dependent variables, we need two linearly independent solutions. That is, we need two solutions whose initial conditions do not lie on the same line through the origin. In general, for n dependent variables we need n linearly independent solutions.

Since $\mathbf{A}$ is a linear map from a vector space to a vector field, there are special cases where the image vector is aligned with the original vector that is projected.

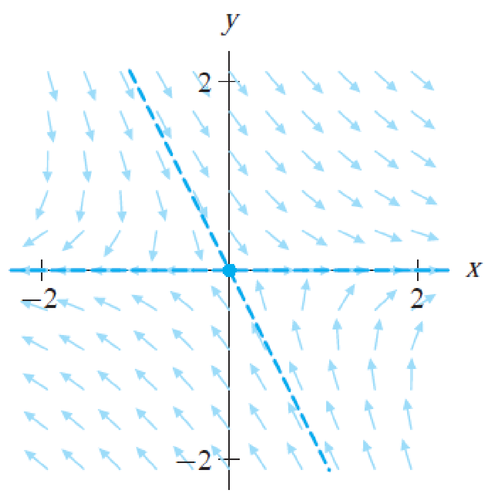


Figure 7.10: Line solution example.

For example, the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

has an aligned result when it operate on vector $(2, 1)^T$.

This special phenomenon is characterized by the eigenvalue and eigenvectors of the matrix $\mathbf{A}$:

$$\mathbf{AX_0} = \lambda \mathbf{X_0}$$

Singular matrix: The matrix equation $\mathbf{BY} = \mathbf{0}$ has non-trivial solutions $\mathbf{Y}$ if and only if $det(\mathbf{B}) = \mathbf{0}$. Most matrices are non-singular.

To find eigenvalues and eigenvectors, we need to solve the Characteristic polynomial of the matrix:

$$det(\mathbf{A} - \lambda \mathbf{I}) = \mathbf{0}$$

The previous matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

has eigenvalues 2 and -1, with eigenvectors being the line through the origin of slope 0.5 and 2.

Basic facts about eigenvalues and eigenvectors:

- For a two by two matrix, its characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.
- Given an eigenvector $\mathbf{X_0}$ associated to an eigenvalue λ, any nonzero scalar multiple $\mathbf{X_0}$ is also an eigenvector associated with λ.
- Eigenvectors associated to distinct eigenvalues are linear independent.

Let's go back to the damped harmonic oscillator equation:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

Transform the second order differential equation to a first order system of differential equation:

$$\begin{cases} \dfrac{dy}{dt} = v \\ \\ \dfrac{dv}{dt} = -2y - 3v \end{cases}$$

The eigenvalues are -2 and -1, with eigenvectors being with slope -2 and -1. The solution of the differential equation is

$$\mathbf{X(t)} = \mathbf{k_1 e^{-2t}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \mathbf{k_2 e^{-t}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The previous matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$$

The eigenvalues are complex numbers: $-2 \pm i$. Eigenvector for $-2+i$ is $(2, 1+i)^T$.

We would like "straight-line" solutions just as we have if the eigenvalues are real numbers. That is,

$$\mathbf{X_C(t)} = \mathbf{e^{(-2+i)t}} \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

But two questions come to mind immediately.

1. What does this formula mean?
2. What good is a complex-valued solution given that we are interested in real-valued solutions to our linear systems?

An illustration of Eular's formula: the tylor series of the exponential function is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

let $x = bi$, then we have

$$e^{bi} = cos(b) + i \cdot sin(b)$$

Apply Eular's formula to the complex-valued function:

$$e^{(a+bi)t} = e^{at}(cos(bt) + i \cdot sin(bt))$$

Therefore the solution of the differential equation can be written as

$$\mathbf{X}_C(t) = e^{-2t}(cos(t) + i \cdot sin(t)) \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

Theorem 7.1 *Consider the linear system*

$$\frac{d\mathbf{X}}{dt} = \mathbf{AX}$$

where $\mathbf{A}$ *is a matrix with real entries. If* $\mathbf{X_c(t)}$ *is a complex-valued solution, then both* $Re\mathbf{X_c(t)}$ *and* $Im\mathbf{X_c(t)}$*, are real-valued solutions, and they are linearly independent.*

We can derive the general solution by applying the theorem to

$$\mathbf{X_C(t)} = \mathbf{e^{(-2+i)t}} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = \mathbf{e^{-2t}(cos(t) + i \cdot sin(t))} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = \mathbf{e^{-2t}} \begin{bmatrix} 2cos(t) \\ cos(t) - sin(t) \end{bmatrix} + \mathbf{ie^{-2t}} \cdot \begin{bmatrix} 2sin(t) \\ cos(t) + sin(t) \end{bmatrix}$$

If we change the damped harmonic oscillator to a smaller damping factor, then the solution becomes oscillation. For example we can solve

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 0$$

The solution is similar to $\mathbf{X_C(t)}$ with factor $e^{-1/2t}$ and $e^{-\sqrt{7}/2it}$.

7.3 Phase Portrait

For real number eigenvalues, the key feature is the sign of the eigenvalues.
Consider matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}$$

Its eigenvalues are $\lambda_1 = \dfrac{-3-\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{-3+\sqrt{5}}{2}$, the corresponding eigenvectors are $y_1 = \dfrac{2}{3+\sqrt{5}}x_1$, $y_2 = \dfrac{2}{3-\sqrt{5}}x_2$

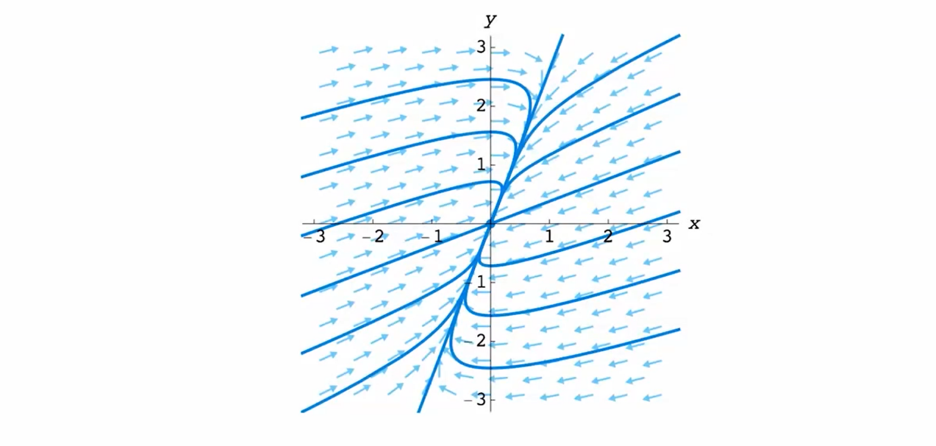


Figure 7.11: Real eigenvalue, $\lambda_1 < \lambda_2 < 0$.

$$\mathbf{A} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix}$$

Its eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 6$, the corresponding eigenvectors are $y_1 = x_1$, $y_2 = -\dfrac{2}{5}x_2$

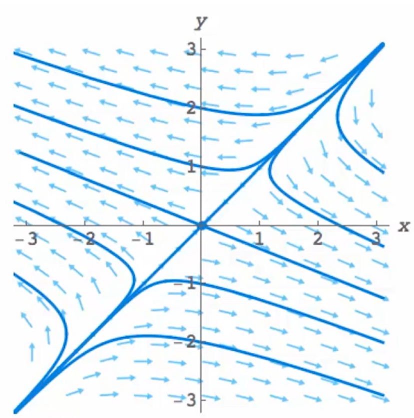


Figure 7.12: Real eigenvalue, $\lambda_1 < 0 < \lambda_2$.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$

Its eigenvalues are $\lambda_1 = \dfrac{3-\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{3+\sqrt{5}}{2}$, the corresponding eigenvectors are $y_1 = \dfrac{-2}{3+\sqrt{5}}x_1$, $y_2 = \dfrac{-2}{3-\sqrt{5}}x_2$

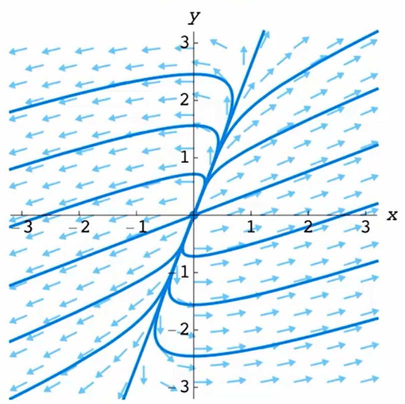


Figure 7.13: Real eigenvalue, $0 < \lambda_1 < \lambda_2$.

For linear systems of differential equations with complex eigenvalues $\lambda = a \pm bi$, the key feature is the sign of the real component of the eigenvalue a.
For $a = 0$, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}$$

Its eigenvalues are $\lambda_1 = 2i$ and $\lambda_2 = -2i$, one corresponding eigenvector is $y_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$.

The general solution is

$$y_t = k_1 \begin{pmatrix} cos2t - sin2t \\ 2cos2t \end{pmatrix} + k_2 \begin{pmatrix} cos2t + sin2t \\ 2sin2t \end{pmatrix}$$

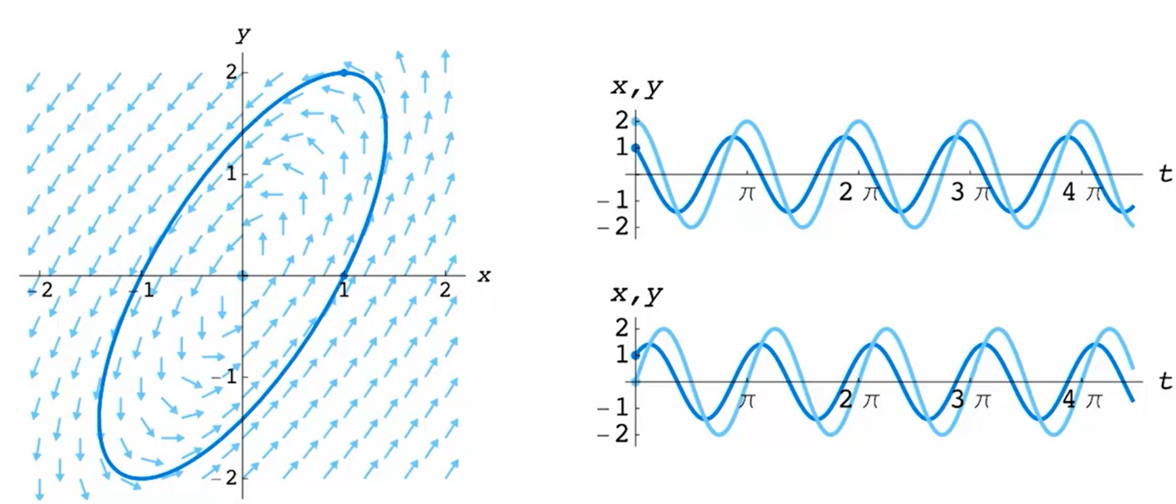


Figure 7.14: Complex eigenvalue, $a = 0$.

For $a < 0$, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1.9 & -2 \\ 4 & -2.1 \end{pmatrix}$$

Its eigenvalues are $\lambda_1 = -0.1+2i$ and $\lambda_2 = -0.1-2i$, one corresponding eigenvector is $y_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$.

The general solution is

$$y_t = k_1 e^{-t/10} \begin{pmatrix} cos2t - sin2t \\ 2cos2t \end{pmatrix} + k_2 e^{-t/10} \begin{pmatrix} cos2t + sin2t \\ 2sin2t \end{pmatrix}$$

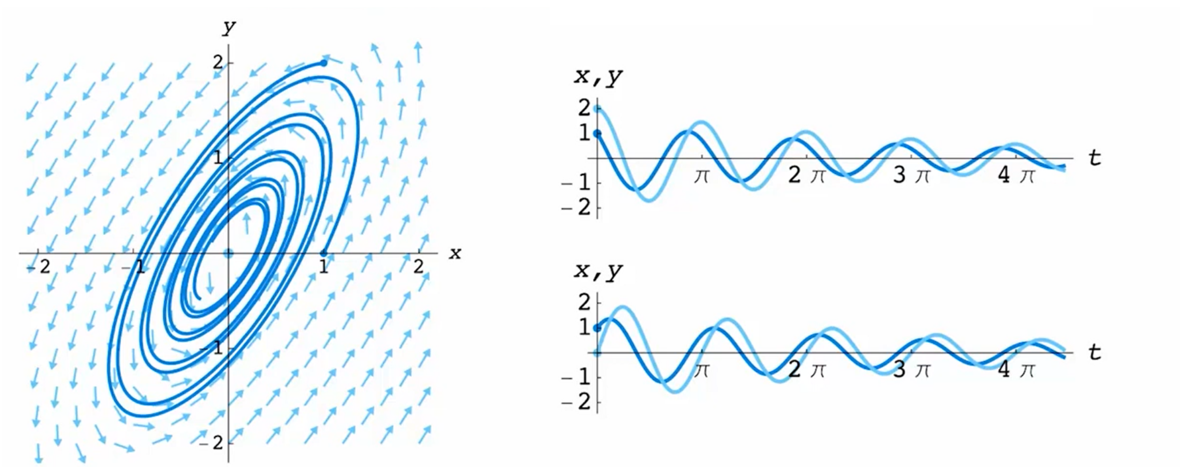


Figure 7.15: Complex eigenvalue, $a < 0$.

For $a > 0$, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix}$$

Its eigenvalues are $\lambda_1 = 1+i\sqrt{5}$ and $\lambda_2 = 1-i\sqrt{5}$, one corresponding eigenvector is $y_0 = \begin{pmatrix} 2 \\ 1+i\sqrt{5} \end{pmatrix}$.

The general solution is

$$y_t = k_1 e^t \begin{pmatrix} 2cos(\sqrt{5}t) \\ cos(\sqrt{5}t) - \sqrt{5}sin(\sqrt{5}t) \end{pmatrix} + k_2 e^t \begin{pmatrix} 2sin(\sqrt{5}t) \\ \sqrt{5}cos(\sqrt{5}t) + sin(\sqrt{5}t) \end{pmatrix}$$

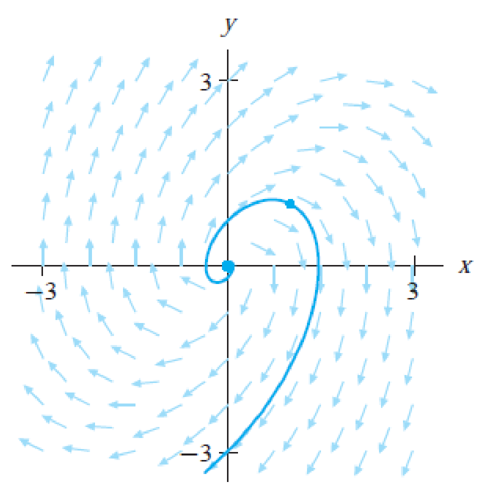


Figure 7.16: Complex eigenvalue, $a > 0$.

7.4 Rayleigh's Clarinet

A beautiful set of examples of stable limit cycles can be found in the pioneering work by Lord Rayleigh (1842–1919) on the physics behind musical instruments.

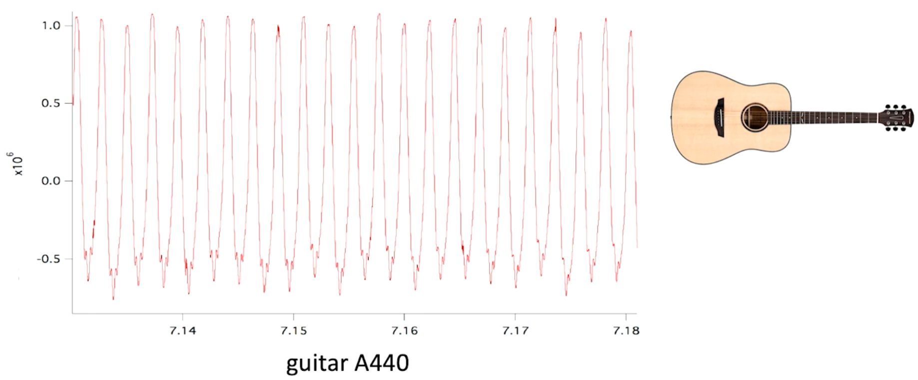


Figure 7.17: A cord record of a guitar.

The frequency of the instrument cannot be dependent on how hard you strike the cord, which is measured by magnitude of the oscillation.
Rayleigh thinks about the clarinet. The reed of the clarinet is like a spring with friction produced by the air resistance. The key idea is the following:

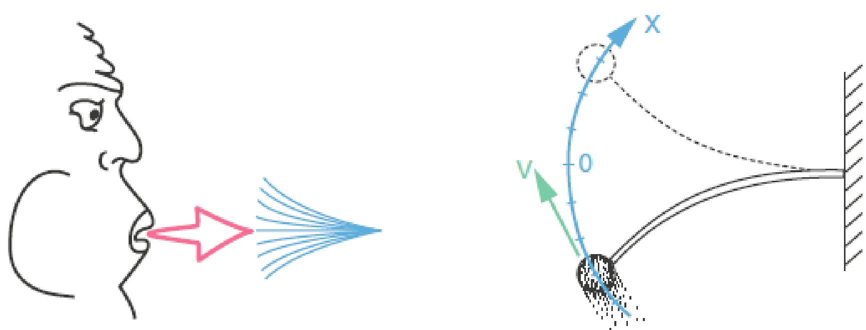


Figure 7.18: The reed of the clarinet is like a spring with friction.

- The clarinetist produces a negative friction at low velocity, driving the system away from the equilibrium.
- At high velocities, air resistance becomes the dominant force, the friction becomes positive.

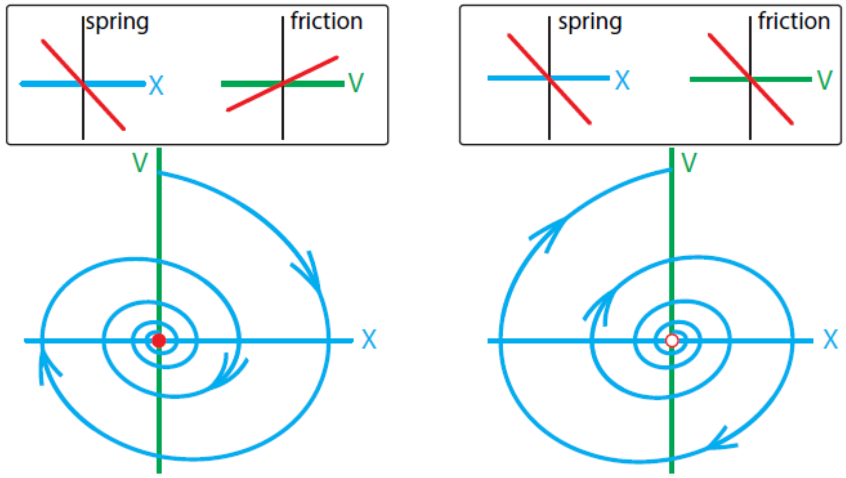


Figure 7.19: When the friction force is positive, the system has a point attractor of spiral type at (0, 0). Right: When the friction is negative, the origin becomes a spiral type unstable equilibrium.

So we need a function that is negative with low velocity and positive with high velocity. Reyleigh's idea is to choose a simple function

$$F_f = v^3 - v$$

The new friction function gives us a new system of differential equations:

$$\begin{cases} \dfrac{dx}{dt} = v \\ \\ \dfrac{dv}{dt} = -x - v^3 - v \end{cases}$$

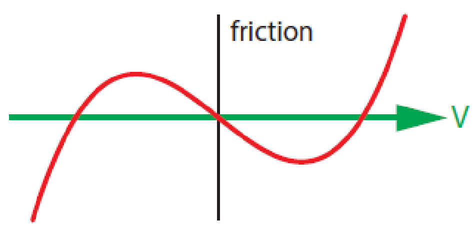


Figure 7.20: A hypothetical nonlinear friction force.

If you choose an initial condition that is not on the red loop, the ensuing trajectory will get closer and closer to the red loop, and will approach it as t $\rightarrow \infty$. This is true whether you are inside the red loop or outside it; all trajectories, with the exception of the one point at (0, 0), approach the red loop arbitrarily closely.

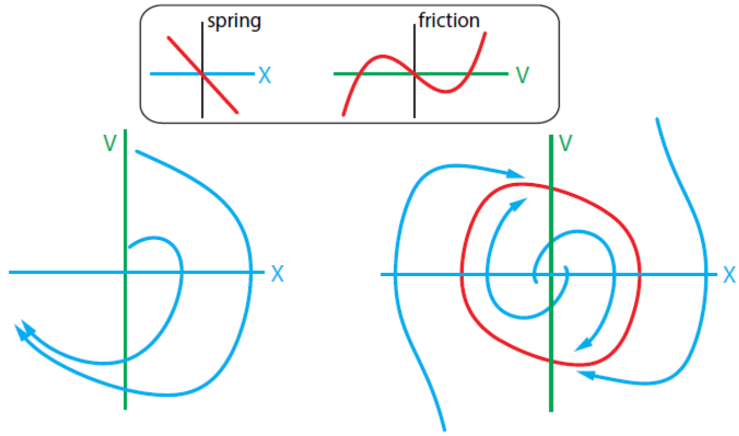


Figure 7.21: Upper: spring force and friction force for the Rayleigh clarinet model. Lower Left: Two representative trajectories for this model. Lower Right: All trajectories, from any initial condition except (0, 0), approach the red loop asymptotically.

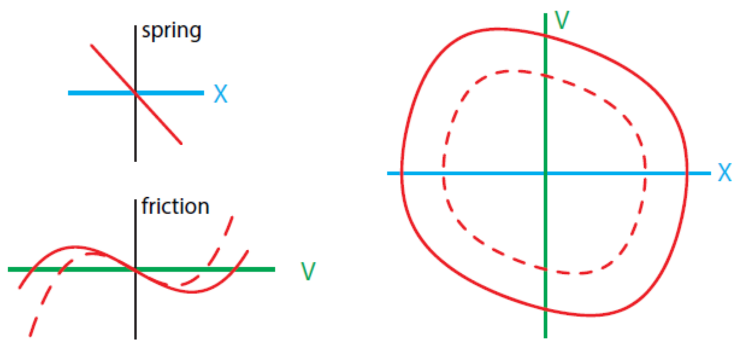


Figure 7.22: Blowing harder. Left: The solid lines show the forces in the Rayleigh clarinet model, under the “blowing harder” condition. The dotted line represents the model without blowing harder. Right: Limit cycle attractors for the two models. Note similarity of shape.

7.5 Predator-Prey Model

Remember the simple predator-prey model

$$\begin{cases} \dfrac{dN}{dt} = N - NP \\[2ex] \dfrac{dP}{dt} = NP - P \end{cases}$$

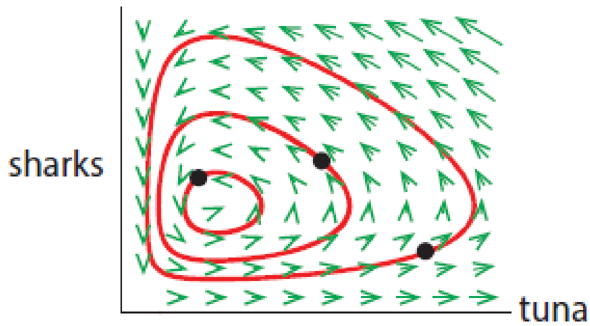


Figure 7.23: A typical predator-pray system.

An improvement of the classic model is the Holling–Tanner model.

$$
\begin{cases}
\dfrac{dN}{dt} = r_1 N(1 - \dfrac{N}{k}) - \dfrac{wN}{d+N} P \\[2ex]
\dfrac{dP}{dt} = r_2 P(1 - \dfrac{jP}{N})
\end{cases}
$$

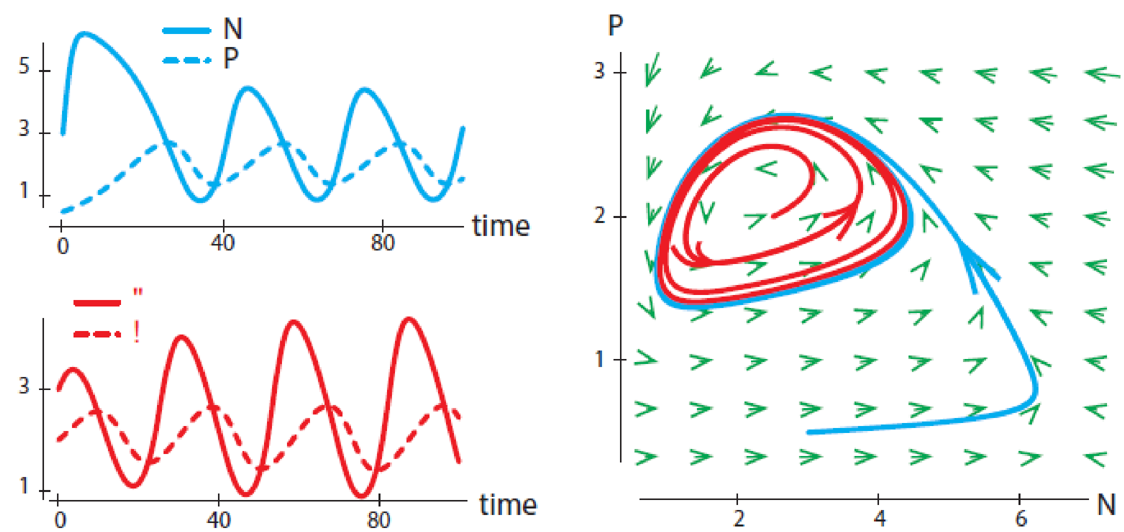


Figure 7.24: Two simulations of the Holling–Tanner model with w = 1 starting from different initial conditions

7.6 The Central Dogma of Molecular Biology

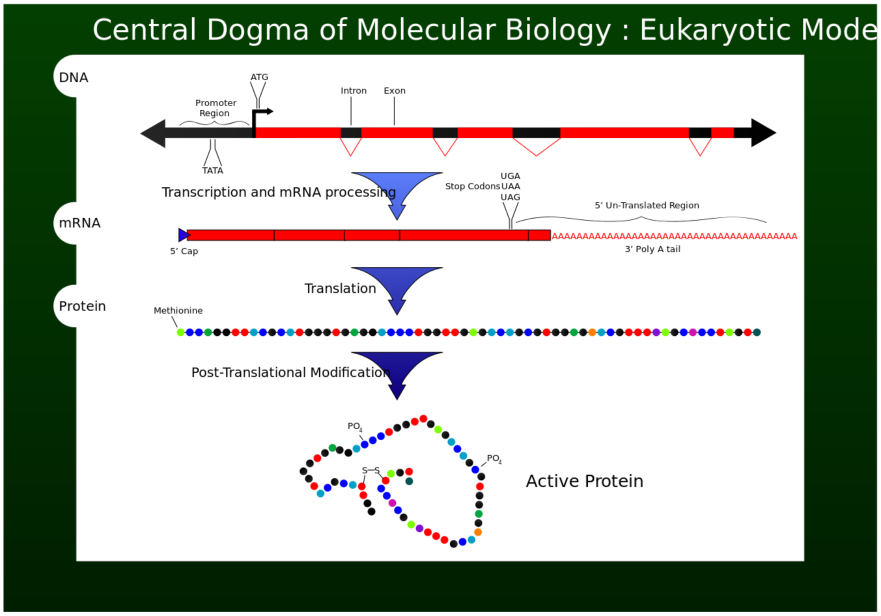


Figure 7.25: The Central Dogma of Molecular Biology.

This dogma and its social implications are proved to be wrong. The first blow is brought by Barbara McClintock, my intellectual hero. The point is that the central dogma ignored the feedback loop, which is called reverse transcript.
Hes1 is critical in neural development. It is under regulation that causes them to express in an oscillatory pattern.

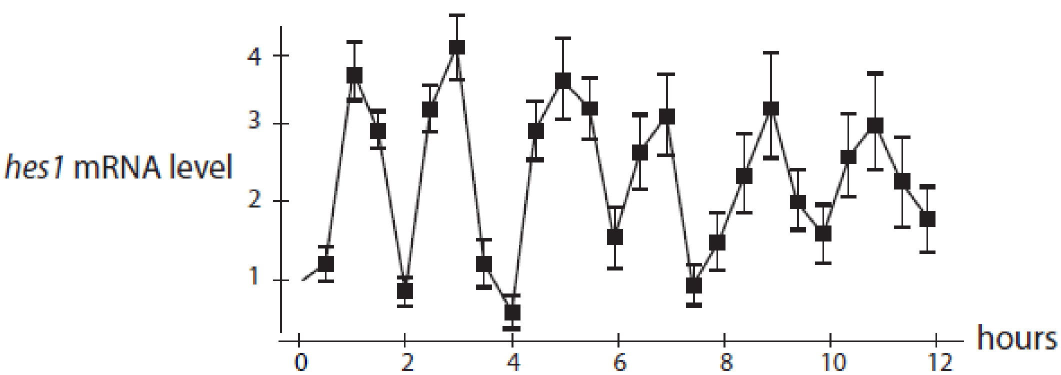


Figure 7.26: Two-hour oscillations in the expression of the gene Hes1. "Oscillatory expression of the bHLH factor Hes1 regulated by a negative feedback loop," by H. Hirata, S. Yoshiura, T. Ohtsuka, Y. Bessho, T. Harada, K. Yoshikawa, and R. Kageyama, (2002), Science 298(5594):840–843.

Let X be hes1 protein level, Y is hes1 RNA level, Z is a hypothesized interaction factor. The dynamic system is written as:

$$\begin{cases} \dfrac{dx}{dt} = by - cx - Axz \\ \\ \dfrac{dy}{dt} = -\dfrac{E}{1+x^2} - dy \\ \\ \dfrac{dz}{dt} = -Axz - \dfrac{F}{1+x^2} - Gz \end{cases}$$

This dynamic system produces the observed two-hour oscillation. One year later, Nicholas Monk proposed an alternative dynamic system that does not include the interactive factor, but hypothesized that the model need to take into account time delays.

$$\begin{cases} \dfrac{dR}{dt} = \dfrac{P_{\tau'}^2}{1+P_{\tau'}^2} \\ \\ \dfrac{dP}{dt} = R_\tau - kP \end{cases}$$

Monk showed that this model recovers the two-hour oscillation as well. The lesson is that time delay in the dynamic systems could explain oscillations.

7.7 Tacoma Narrows Bridge Collapse

An attractor of a dynamical system is a subset A of state space that has the following property:

For every X in some neighbourhood of A, $X \to A$ as $t \to \infty$

"Point attractor" is stable equilibrium point, while "limit cycle attractor" is stable oscillation. The bifurcation that comes from the stable equilibrium point to stable oscillation is called Hopf bifurcation.

Example: Tacoma Narrows Bridge

`https://www.youtube.com/watch?v=lXyG68_caV4`

Essentially, bifurcation is a qualitative change. So the issue of interest is qualitative dynamics: what is the cause of a qualitative change, or changes of forms of motion?

References

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