Computability Theory **Fall 2020**

Lecture 4: Turing Machines for Languages

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4.1 Turing Machines

Definition 4.1 *A Turing machine is a 7-tuple* $\{Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject}\}$*, where* Q, Σ, Γ *are all finite sets and:*

- Q *is the set of states,*
- Σ *is the imput alphabet not containing the blank symbol* $\sqcup$*,*
- Γ *is the tape alphabet, where* $\sqcup \in \Gamma$ *and* $\Sigma \subseteq \Gamma$*,*
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ *is the transition function,*
- $q_0 \in Q$ *is the start state,*
- $q_{accept} \in Q$ *is the accept state, and*
- $q_{jeject} \in Q$ *is the reject state, where* $q_{reject} \neq q_{accept}$.

Definition 4.2 *As a Turing machine computes, changes occur in the current state, the current tape contents, and the current head location. A setting of these three items is called a configuration of the Turing machine. For a state* q *and two strings* u *and* v *over the tape alphabet* Γ*, we write* uqv *for the configuration where the current state is* q*, the current tape contents is* uv*, and the current head location is the first symbol of* v*. The tape contains only blanks following the last symbol of* v*.*

Say that configuration C_1 *yields configuration* C_2 *if the Turing machine can legally go from* C_1 *to* C_2 *in a single step. Suppose that we have* a*,* b*,* c *in* Σ*, as well as* u *and* v *in* Σ^* *and states* q_i *and* q_j*. In that case,* uaq_ibv *and* uq_iacv *are two configurations. Say that* uaq_ibv *yields* uq_jacv *if in the transition function* $\delta(q_i, b) = (q_j, c, L)$*. For a rightward move, say that* uaq_ibv *yields* $uacq_jv$ *if in the transition function* $\delta(q_i, b) = (q_j, c, R)$*. For the left-hand end, the configuration* q_ibv *yields* q_jcv *if the transition is left-moving, and it yields* cq_jv *for the right-moving transition.*

The start configuration of M *on input* w *is the configuration* q_0w*, which indicates that the machine is in the start state* q_0 *with its head at the leftmost position on the tape. In an accepting configuration, the state of the configuration is* q_{accept}*. In a rejecting configuration, the state of the configuration is* q_{reject}*. Accepting and rejecting configurations are halting configurations and do not yield further configurations.*

A Turing machine M *accepts input* w *if a sequence of configurations* $C_1, C_2, \ldots, C_k$ *exists, where*

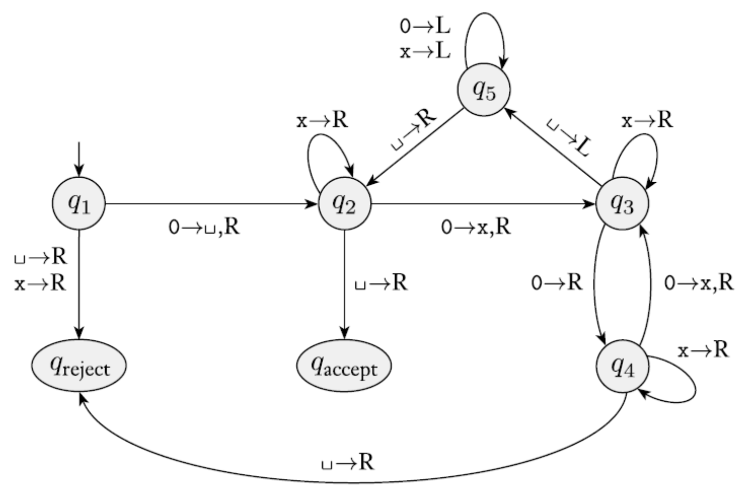
1. C_1 *is the start configuration of* M *on input* w*,*
2. *each* C_i *yields* C_{i+1}*, and*
3. C_k *is an accepting configuration.*

The collection of strings that M accepts is the language of M, or the language recognized by M, denoted $L(M)$. Given an input w, we say that M halts on w if M either accept or reject w. A machine that halts for all inputs is called a decider. A decider that recognizes some language also is said to decide that language.

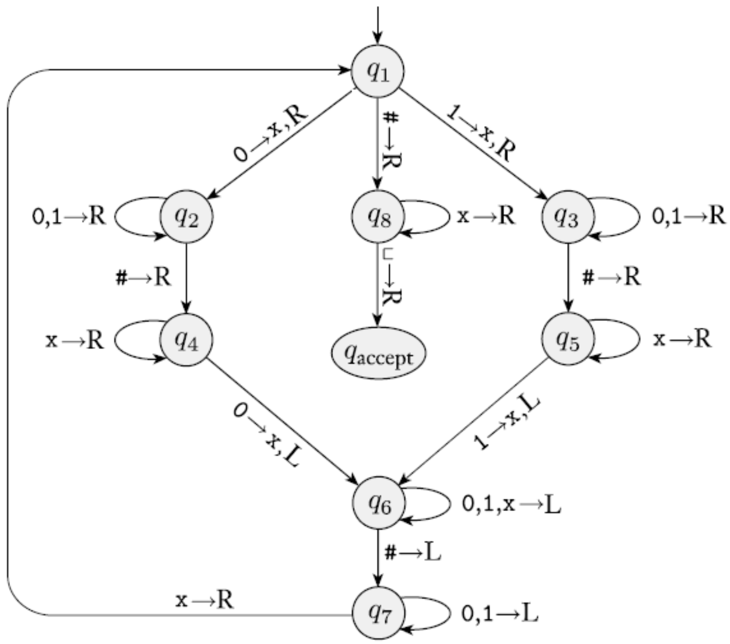
Definition 4.3 *Call a language Turing-recognizable if some Turing machine recognizes it.*

Definition 4.4 *Call a language Turing-decidable if some Turing machine decides it.*

Example A Turing machine M that decides $A = \{0^{2^n}\}$:
$Q = \{q_1, q_2, q_3, q_4, q_5, q_{accept}, q_{reject}\}$,
$\Sigma = \{0\}$,
$\Gamma = \{0, x, \sqcup\}$,
The start, accept, and reject states are q_1, q_{accept}, q_{reject}, respectively.
The transition function δ is described by the following state diagram:



Example A Turing machine M that decides $B = \{w\#w \mid w \in \{0,1\}^*\}$:
$Q = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_{accept}, q_{reject}\}$,
$\Sigma = \{0, 1, \#\}$,
$\Gamma = \{0, 1, \#, x, \sqcup\}$,
The start, accept, and reject states are q_1, q_{accept}, q_{reject}, respectively.
The transition function δ is described by the following state diagram:



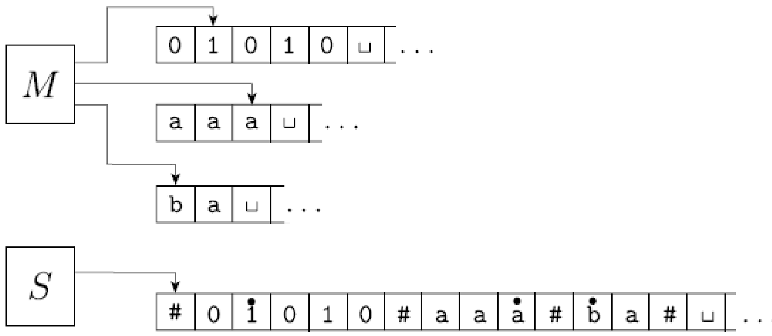
4.2 Turing Machines Variants

Definition 4.5 *A multitape Turing machine is a 7-tuple* $\{Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject}\}$, *where* Q, Σ, Γ *are all finite sets and:*

- Q *is the set of states,*
- Σ *is the imput alphabet not containing the blank symbol* $\sqcup$,
- Γ *is the tape alphabet, where* $\sqcup \in \Gamma$ *and* $\Sigma \subseteq \Gamma$,
- $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k$ *is the transition function,*
- $q_0 \in Q$ *is the start state,*
- $q_{accept} \in Q$ *is the accept state, and*
- $q_{jeject} \in Q$ *is the reject state, where* $q_{reject} \neq q_{accept}$.

Theorem 4.6 *Every multitape Turing machine has an equivalent single-tape Turing machine.*

Proof: We can convert a multitape Turing machine M to an equivalent single tape Turing machine S. The key idea is to show how to simulate M with S.



S = " On input $w = w_1 \cdots w_n$:

1. First S puts its tape into the format that represents all k tapes of M. The formatted tape contains $\#\dot{w}_1 w_2 \cdots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \cdots \#$

2. To simulate a single move, S scan its tape from the first $\#$, which marks the left-hand end, to the $(k+1)$th $\#$, which marks the right-hand end, in order to determine the symbols under the virtual heads. Then S makes a second pass to update the tapes according to the way that M's transition function dictates.

3. If at any point S moves one of the virtual heads to the right onto a $\#$, this action signifies that M has moved the corresponding head onto the previously unread blank portion of that tape. So S writes a blank symbol on this tape cell and shifts the tape contents, from this cell until the rightmost $\#$, one unit to the right. Then it continues the simulation as before. "

■

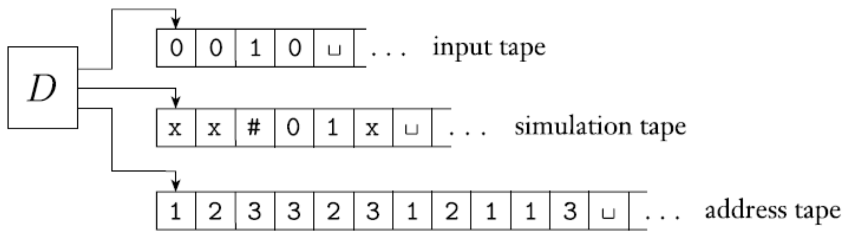
Corollary 4.7 *A language is Turing-recognizable if and only if some multitape Turing machine recognizes it.*

Definition 4.8 *A nondeterministic Turing machine is a 7-tuple* $\{Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject}\}$*, where* Q, Σ, Γ *are all finite sets and:*

- Q *is the set of states,*
- Σ *is the imput alphabet not containing the blank symbol* $\sqcup$*,*
- Γ *is the tape alphabet, where* $\sqcup \in \Gamma$ *and* $\Sigma \subseteq \Gamma$*,*
- $\delta : Q \times \Gamma \to \wp(Q \times \Gamma \times \{L, R\}^k)$ *is the transition function,*
- $q_0 \in Q$ *is the start state,*
- $q_{accept} \in Q$ *is the accept state, and*
- $q_{jeject} \in Q$ *is the reject state, where* $q_{reject} \neq q_{accept}$.

Theorem 4.9 *Every nondeterministic Turing machine has an equivalent deterministic Turing machine.*

Proof: The simulating deterministic Turing machine D has three tapes. By Theorem 4.6, this arrangement is equivalent to having a single tape. The machine D uses its three tapes in a particular way, as illustrated in the following figure. Tape 1 always contains the input string and is never altered. Tape 2 maintains a copy of N's tape on some branch of its nondeterministic computation. Tape 3 keeps track of D's location in N's nondeterministic computation tree.



Describe D as the following:

1. Initially, tape 1 contains the input w, and tapes 2 and 3 are empty.

2. Copy tape 1 to tape 2 and initialize the string on tape 3 to be ϵ

3. Use tape 2 to simulate N with input w on one branch of its nondeterministic computation. Before each step of N, consult the next symbol on tape 3 to determine which choice to make among those allowed by N's transition function. If no more symbols remain on tape 3 or if this nondeterministic choice is invalid, abort this branch by going to stage 4. Also go to stage 4 if a rejecting configuration is encountered. If an accepting configuration is encountered, accept the input.

4. Replace the string on tape 3 with the next string in the string ordering. Simulate the next branch of N's computation by going to stage 2.

$\blacksquare$

Corollary 4.10 *A language is Turing-recognizable if and only if some nondeterministic Turing machine recognizes it.*

Corollary 4.11 *A language is decidable if and only if some nondeterministic Turing machine decides it.*

Definition 4.12 *An enumerator is a 7-tuple* $\{Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject}\}$*, where* Q, Σ, Γ *are all finite sets and:*

- Q *is the set of states,*
- Σ *is the imput alphabet not containing the blank symbol* $\sqcup$*,*
- Γ *is the tape alphabet, where* $\sqcup \in \Gamma$ *and* $\Sigma \subseteq \Gamma$*,*
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\} \times \Sigma_\epsilon$ *is the transition function,*
- $q_0 \in Q$ *is the start state,*
- $q_{accept} \in Q$ *is the accept state, and*

- $q_{jeject} \in Q$ *is the reject state, where* $q_{reject} \neq q_{accept}$.

Definition 4.13 *The computation of an enumerator* E *is defined as in an ordinary Turing machine, except that it has two tapes, a working tape and a printing tape, both initially blank. At each step, the machine may write a symbol from* Σ *on the output tape, or nothing, as determined by* δ*. If* $\delta(q, a) = (r, b, L, c)$*, it means that in state* q*, reading* a*, enumerator* e *enters state* r*, write* b *on the work tape, moves the work tape head left, writes* c *on the output tape, and moves the output tape head to the right if* $c \neq \epsilon$*. Whenever state* q_{print} *is entered, the output tape is reset to blank and the head returns to the left-hand end. The machine halts when* q_{accept} *is entered.* $L(E) = \{w \in \Sigma^* |$ *appears on the work tape if* q_{print} *is entered.*$\}$

Theorem 4.14 *A language is Turing-recognizable if and only if some enumerator enumerates it.*

Proof: First we show that if we have an enumerator E that enumerates a language A, a Turing machine M recognizes A. The Turing machine M works in the following way.
$M =$ "On input w:

1. Run E every time that E outputs a string, compare it with w.

2. If w ever appears in the output of E, accept."

Clearly, M accepts those strings that appear on E's list. Now consider the other direction. If M recognizes a language A, we can construct the following enumerator E for A. Say that $s_1, s_2, s_3, \ldots$ is a list of all possible strings in Σ^*.
E = " Ignore the input.

1. Repeat the following for $i = 1, 2, 3, \ldots$.

2. Run M for i steps on each input, $s_1, s_2, \ldots, s_i$.

3. If any computations accept, print out the corresponding s_j."

If M accepts a particular string s, eventually it will appear on the list generated by E. In fact, it will appear on the list infinitely many times because M runs from the beginning on each string for each repetition of step 1. This procedure gives the effect of running M in parallel on all possible input string. ■

4.3 The Pumping Lemma

Theorem 4.15 *Pumping Lemma for regular languages: if* A *is a regular language, then there is a number* p*, which is called the pumping length, where if* s *is any string in* A *of length at least* p*, then* s *may be devided into three pieces,* $s = xyz$*, satisfying the following conditions:*

1. *for each* $i \geq 0$, $xy^iz \in A$,

2. $|y| > 0$, *and*

3. $|xy| \leq p$.

Proof: Let $M = \{Q, \Sigma, \delta, q_1, F\}$ be a DFA recognizing A and p be the number of the states of M. Let $s = s_1 s_2 \ldots s_n$ be a string in A of length n, where $n \geq p$. Let $r_1, \ldots, r_{n+1}$ be the sequence of states that M enters while processing s, so $r_{i+1} = \delta(r_i, s_i)$ for $1 \leq i \leq n$. This sequence has length $n+1$, which is at least $p+1$. Among the first $p+1$ elements in the sequence, two must be the same state, by the pigeonhole principle. We call the first of these r_j and the second r_l. Because r_l occors among the first $p+1$ places in a sequence starting at r_1, we have $l \leq p+1$. Now let $x = s_1 \cdots s_{j-1}$, $y = s_j \cdots s_{l-1}$, and $z = s_l \cdots s_n$.

As x takes M from r_1 to r_j, y takes M from r_j to r_j, and z takes M from r_j to r_{n+1}, which is an accept state, M must accept xy^iz for $i \geq 0$. We know that $j \neq l$, so $|y| > 0$; and $l \leq p+1$, so $|xy| \leq p$. Thus all three conditions of the pumping lemma are satisfied. ∎

Example Show that the language $C = \{0^n 1^n | \; n \geq 0\}$.

Proof: Assume to the contrary that C is regular. Let p be the pumping length given by the pumping lemma. Choose s to be the string 0^p1^p. Because s is a member of C and s has length more than p, the pumping lemma guarentees that s can be split into three pieces, $s = xyz$, where for any $o \geq 0$ the string xy^iz is in C. We consider three cases to show that this result is impossible.

1. The string y consists only of 0's. In this case, the string $xyyz$ has more 0's than 1's and so is not a member of C, violating condition 1 of the pumping lemma. Contradiction.
2. The string y consists only of 1's. This case also gives a contradiction.
3. The string y consists of both 0's and 1's. In this case, the string $xyyz$ may have the same number of 0's and 1's, but they will be out of order with some 1's before 0's. hence it is not a member of C. Another Contradiction.

Thus a contradiction is unavoidable if we make the assumption that C is regular, so C is not regular. ∎

Example Let $D = \{w | \; w \text{has an equal number of 0's and 1's}\}$. Show that D is not regular.

Proof: Assume to the contrary that D is regular. Let p be the pumping length given by the pumping lemma. Let s be the string 0^p1^p. Since s is a member of D and having length more than p, the pumping lemma guarantees that s can be split into three pieces $s = xyz$, where for any $i \geq 0$ the string xy^iz is in D. By condition 3 of the pumping lemma, $|xy| \leq p$, y must consist only of 0's, so $xyyz \notin C$, therefore s cannot be pumped. Contradiction. ∎

Theorem 4.16 *Pumping lemma for context-free languages: If A is a context-free language, then there is a number p, the pumping length, such that if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions*

1. *for each $i \geq 0$, $uv^ixy^iz \in A$*
2. *$|vy| > 0$*
3. *$|vxy| \leq p$*

See the theorem 2.34 for the proof details.

Example Show that the language $E = \{a^n b^n c^n\}$ is not context-free.

Proof: Assume that E is a CFL and construct a contradiction. Let p be the pumping length for E that is guaranteed to exist by the pumping lemma. Select the string $s = a^p b^p c^p$. Clearly s is a member of B and of length at least p. The pumping lemma states that s can be pumped, but we show that it cannot. In other words, we show that no matter how we divide s into $uvxyz$, one of the three conditions of the lemma is violated.

First, condition 2 stipulates that either v or y is nonempty. Then we consider one of two cases, depending on whether substrings v and y contain more than one type of alphabet symbol.

1. When both v and y contain only one type of alphabet symbol, v does not contain both a's and b's or both b's and c's, and the same holds for y. In this case, the string uv^2xy^2z cannot contain equal numbers of a's, b's, and c's. Therefore, it cannot be a member of E. That violates condition 1 of the lemma and is thus a contradiction.

2. When either v or y contains more than one type of symbol, uv^2xy^2z may contain equal numbers of the three alphabet symbols but not in the correct order. Hence it cannot be a member of E and a contradiction occurs.

One of these cases must occur, a contradiction is unavoidable. Therefore E is not a CFL. ■