

Lecture 7: The Equivalence of Computability and Recursiveness

*Lecturer: Renjie Yang***7.1 The Arithmetization of The Theory of Turing Machines**

Definition 7.1 Let M be an expression consisting of the symbols $\gamma_1, \gamma_2, \dots, \gamma_n$. Let a_1, a_2, \dots, a_n be the corresponding integers associated with these symbols. Then the **Gödel number** of M is the integer

$$gn(M) = r = \prod_{k=1}^n Pr(k)^{a_k}$$

If M is the empty expression, we let 1 be the Gödel number of M .

Note We adopt a convention to associate each symbol with an even number: $R \rightarrow 3, S \rightarrow 5, q_i \rightarrow 4i + 5, S_i \rightarrow 4i + 7$.

Example $gn(q_11Rq_2) = 2^9 \cdot 3^{11} \cdot 5^3 \cdot 7^{13}$

Corollary 7.2 If M and N are given expressions such that $gn(M) = gn(N)$, then $M = N$.

Proof: According to the Fundamental Theorem of Arithmetic, every natural number can be uniquely represented in the form $p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$, where p_1, \dots, p_k are distinct primes. Therefore if $M = N$ then their prime factorizations are the same. ■

Note A computation is a finite sequence of expressions; a Turing machine is a finite set of expressions.

Definition 7.3 If $n = gn(M)$, we also write $M = Exp(n)$

Definition 7.4 Let M_1, \dots, M_n be a finite sequence of expressions. Then the Gödel number of this sequence of expressions is the number

$$\prod_{k=1}^n Pr(k)^{gn(M_k)}$$

Example $gn(\{q_11Bq_1, q_1BRq_2\}) = 2^{2^9 \cdot 3^{11} \cdot 5^7 \cdot 7^9} \cdot 3^{2^9 \cdot 3^7 \cdot 5^3 \cdot 7^{13}}$

Corollary 7.5 No integer is the Gödel number both of an expression and of a sequence of expressions.

Proof: A Gödel number of an expression or a sequence of expressions is of the form $2^n \cdot m$. n is odd for expressions, and even for sequences of expressions. ■

Corollary 7.6 Two sequences of expressions that have the same Gödel number are identical.

Definition 7.7 Let Z be a Turing machine. Let M_1, \dots, M_n be any arrangement of the quadruples of Z without repetitions. Then, the Gödel number of the sequence M_1, \dots, M_n is called a **Gödel number of the Turing machine Z** .

Note A Turing machine consisting of n quadruples has $n!$ distinct Gödel numbers.

Definition 7.8 For each $n > 0$ and for each set of integers A , let $T_n(z, x_1, \dots, x_n, y)$ be the predicate that means, for given z, x_1, \dots, x_n, y that z is a Gödel number of a Turing machine Z , and that y is the Gödel number of a computation, with respect to Z , beginning with the instantaneous description $q_1(x_1, \dots, x_n)$.

Theorem 7.9 $T_n(z, x_1, \dots, x_n, y)$ is primitive recursive.

Proof: The proof proceeds by a detailed list of primitive recursive functions and predicates until we get $T_n(z, x_1, \dots, x_n, y)$.

Group 1 Functions and predicates which concern Gödel numbers of expressions and sequences of expressions:

$$1. n \text{ Gl } x = \bigwedge_{y=0}^x [(Pr(n)^y | x)] \wedge \neg (Pr(n)^{y+1} | x)]$$

The Gödel encoding of the n th symbol in the expression represented by x .

The Gödel encoding of the n th expression in the sequence of expression represented by x .

$$2. \mathcal{L}(x) = \bigwedge_{y=0}^x [(y \text{ Gl } x > 0) \wedge \forall i ((0 \leq i \leq x) \rightarrow (y + 1 + i) \text{ Gl } x = 0)]$$

The length of the expression represented by x .

The length of the sequence of expression represented by x .

$$3. GN(x) \leftrightarrow \neg \exists (1 \leq y \leq \mathcal{L}(x)) [(y \text{ Gl } x = 0) \wedge ((y + 1) \text{ Gl } x \neq 0)]$$

x is a Gödel Numbers of some expression or some sequence of expression with no empty expression.

$$4. Term(x, z) \leftrightarrow GN(z) \wedge \exists (1 \leq n \leq \mathcal{L}(x)) x = n \text{ Gl } z$$

z is a Gödel Numbers of some expression, and x is the Gödel Numbers of one of the symbols in the expression represented by z .

$$5. x * y = x \cdot \prod_{i=1}^{\mathcal{L}(x)} Pr(\mathcal{L}(x) + i)^{i \text{ GL } y}$$

If M and N are expressions, then $gn(MN) = gn(M) * gn(N)$.

If x and y are the Gödel numbers of the sequences of expressions M_1, \dots, M_n and N_1, \dots, N_p respectively, then $x * y$ is the Gödel numbers of the sequence $M_1, \dots, M_n, N_1, \dots, N_p$.

Group 2 Functions and Predicates which concern the basic structure of Turing machines:

$$6. IC(x) \leftrightarrow \exists (0 \leq y \leq x) (x = 4y + 9)$$

x is the number assigned to an internal configuration q_i

$$7. Al(x) \leftrightarrow \exists (0 \leq y \leq x) (x = 4y + 7)$$

x is the number assigned to an alphabet S_i

8. $Odd(x) \leftrightarrow \exists(0 \leq y \leq x)(x = 2y + 3)$
 x is an odd number ≥ 3 .
9. $Quad(x) \leftrightarrow GN(x) \wedge \mathcal{L}(x) \wedge IC(1 \text{ Gl } x) \wedge Al(2 \text{ Gl } x) \wedge Odd(3 \text{ Gl } x) \wedge IC(4 \text{ Gl } x)$
The expression represented by x is a quadruple.
10. $Inc(x, y) \leftrightarrow Quad(x) \wedge Quad(y) \wedge (1 \text{ Gl } x = 1 \text{ Gl } y) \wedge (2 \text{ Gl } x = 2 \text{ Gl } y) \wedge (x \neq y)$
 x and y are Gödel numbers of two incompatible quadruples beginning with the same two symbols.
11. $TM(x) \leftrightarrow GN(x) \wedge \forall(1 \leq n \leq \mathcal{L}(x))[Quad(n \text{ Gl } x) \wedge \forall(1 \leq m \leq \mathcal{L}(x)) \neg Inc(n \text{ Gl } x, m \text{ Gl } x)]$
 x is a Gödel number of a Turing machine.
12. $MR(0) = 2^{11}$,
 $MR(n+1) = 2^{11} * MR(n)$
 $MR(n)$ is the Gödel number of \bar{n} .
13. $CU(n, x) = 0$ if $n \text{ Gl } x \neq 11$,
 $CU(n, x) = 1$ if $n \text{ Gl } x = 11$.
 $CU(n, x)$ is the characteristic function of the predicate $n \text{ Gl } x \neq 11$, namely the n th symbol in the expression represented by x is not S_1 .
14. $Corn(x) = \sum_{n=1}^{\mathcal{L}(x)} CU(n, x)$
If x is the Gödel number of M , then $Corn(x) = \langle M \rangle$.
15. $U(y) = Corn(\mathcal{L}(y) \text{ Gl } y)$
If y is the Gödel number of a sequence of expression M_1, M_2, \dots, M_n , then $U(y) = \langle M_n \rangle$.
16. $ID(x) \leftrightarrow GN(x) \wedge \exists(1 \leq n \leq \mathcal{L}(x) \div 1)[IC(n \text{ Gl } x) \wedge \forall(1 \leq m \leq \mathcal{L}(x))(m = n \vee Al(m \text{ Gl } x))]$
 x is a Gödel number of an instantaneous description.
17. $Init_n(x_1, \dots, x_n) = 2^9 * MR(x_1) * 2^7 * MR(x_2) * 2^7 \dots * 2^7 * MR(x_n)$.
 $Init_n(x_1, \dots, x_n) = gn(q_1(\overline{x_1, \dots, x_n}))$.

Group 3 Functions and Predicates which concern the computations of Turing machines:

18. $Yield_1(x, y, z) \leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq F, G, r, s \leq x, 0 \leq t, u \leq y)$
 $[(x = F * 2^r * 2^s * G) \wedge (y = F * 2^t * 2^u * G)$
 $\wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Al(u) \wedge Term(2^r \cdot 3^s \cdot 5^u \cdot 7^t, z)]$.
 x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z , and $Exp(x) \rightarrow Exp(y)(Z)$, under the first rule of $\alpha \rightarrow \beta(Z)$:
(1) There exist expressions P and Q (possibly empty) such that
 α is Pq_iS_jQ ,
 β is Pq_lS_kQ ,
where
 Z contains $q_iS_jS_kq_l$.
19. $Yield_2(x, y, z) \leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq F, G, r, s \leq x, 0 \leq t, u \leq y)$
 $[(x = F * 2^r * 2^s * 2^t * G) \wedge (y = F * 2^s * 2^u * 2^t * G)$
 $\wedge IC(r) \wedge IC(u) \wedge Al(s) \wedge Al(t) \wedge Term(2^r \cdot 3^s \cdot 5^3 \cdot 7^u, z)]$.
 x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing

machine Z , and $Exp(x) \rightarrow Exp(y)(Z)$, under the second rule of $\alpha \rightarrow \beta(Z)$:

(2) There exist expressions P and Q (possibly empty) such that

$$\alpha \text{ is } Pq_iS_jS_kQ,$$

$$\beta \text{ is } PS_jq_lS_kQ,$$

where

$$Z \text{ contains } q_iS_jRq_l.$$

$$\begin{aligned} 20. \text{ Yield}_3(x, y, z) \leftrightarrow & ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq F, r, s \leq x, 0 \leq t \leq y) \\ & [(x = F * 2^r * 2^s) \wedge (y = F * 2^s * 2^t * 2^7) \\ & \wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Term(2^r \cdot 3^s \cdot 5^3 \cdot 7^t, z)]. \end{aligned}$$

x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z , and $Exp(x) \rightarrow Exp(y)(Z)$, under the third rule of $\alpha \rightarrow \beta(Z)$:

(3) There exist expressions P and Q (possibly empty) such that

$$\alpha \text{ is } Pq_iS_j,$$

$$\beta \text{ is } PS_jq_lS_0,$$

where

$$Z \text{ contains } q_iS_jRq_l.$$

$$\begin{aligned} 21. \text{ Yield}_4(x, y, z) \leftrightarrow & ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq F, G, r, s, t \leq x, 0 \leq u \leq y) \\ & [(x = F * 2^r * 2^s * 2^t * G) \wedge (y = F * 2^u * 2^r * 2^t * G) \\ & \wedge IC(s) \wedge IC(u) \wedge Al(r) \wedge Al(t) \wedge Term(2^s \cdot 3^t \cdot 5^5 \cdot 7^u, z)]. \end{aligned}$$

x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z , and $Exp(x) \rightarrow Exp(y)(Z)$, under the fourth rule of $\alpha \rightarrow \beta(Z)$:

(4) There exist expressions P and Q (possibly empty) such that

$$\alpha \text{ is } PS_kq_iS_jQ,$$

$$\beta \text{ is } Pq_lS_kS_jQ,$$

where

$$Z \text{ contains } q_iS_jLq_l.$$

$$\begin{aligned} 22. \text{ Yield}_5(x, y, z) \leftrightarrow & ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq G, r, s \leq x, 0 \leq t \leq y) \\ & [(x = 2^r * 2^s * G) \wedge (y = 2^t * 2^7 * 2^s * G) \\ & \wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Term(2^r \cdot 3^s \cdot 5^5 \cdot 7^t, z)]. \end{aligned}$$

x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z , and $Exp(x) \rightarrow Exp(y)(Z)$, under the fifth rule of $\alpha \rightarrow \beta(Z)$:

(5) There exist expressions P and Q (possibly empty) such that

$$\alpha \text{ is } q_iS_jQ,$$

$$\beta \text{ is } q_lS_0S_jQ,$$

where

$$Z \text{ contains } q_iS_jLq_l.$$

$$23. \text{ Yield}(x, y, z) \leftrightarrow \text{Yield}_1(x, y, z) \vee \text{Yield}_2(x, y, z) \vee \text{Yield}_3(x, y, z) \vee \text{Yield}_4(x, y, z) \vee \text{Yield}_5(x, y, z)$$

x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z , and $Exp(x) \rightarrow Exp(y)(Z)$.

$$\begin{aligned} 24. \text{ Fin}(x, z) \leftrightarrow & ID(x) \wedge TM(z) \wedge \exists(0 \leq F, G, r, s \leq x) \\ & [(x = F * 2^r * 2^s * G) \wedge IC(r) \wedge Al(s) \\ & \wedge \forall(1 \leq n \leq \mathcal{L}(z))((1 \text{ Gl } (n \text{ Gl } z) \neq r) \vee (2 \text{ Gl } (n \text{ Gl } z) \neq s))] \end{aligned}$$

z is a Gödel number of a Turing machine Z and x is the Gödel number of an instantaneous

description final with respect to Z , namely there is no yield rule in Z to further compute from x .

$$25. \text{Comp}(y, z) \leftrightarrow TM(z) \wedge GN(y) \wedge \forall(1 \leq n \leq \mathcal{L}(y))[Yield(n \text{ Gl } , (n+1) \text{ Gl } y, z)] \\ \wedge Fin(\mathcal{L}(y) \text{ Gl } y, z)$$

z is a Gödel number of a Turing machine Z and y is the Gödel number of a computation of Z .

By definition 7.8, $T_n(z, x_1, \dots, x_n, y) \leftrightarrow (1 \text{ Gl } y = Init_n(x_1, \dots, x_n)) \wedge \text{Comp}(y, z)$. Therefore it is primitive recursive according to the above construction. ■

7.2 Computability and Recursiveness

Theorem 7.10 *Let Z_0 be a Turing machine and let z_0 be a Gödel number of Z_0 . Then, the domain of the function $\Psi_{Z_0}^{(n)}(x^{(n)})$ is equal to the domain of $\min_y T_n(z_0, x^{(n)}, y)$. Moreover,*

$$\Psi_{Z_0}^{(n)}(x^{(n)}) = U(\min_y T_n(z_0, x^{(n)}, y)).$$

Also, if $T_n(z_0, x^{(n)}, y)$ is true for given $x^{(n)}$, then

$$y_0 = \min_y T_n(z_0, x^{(n)}, y).$$

Proof: For any given $x^{(n)}$, $\min_y T_n(z_0, x^{(n)}, y)$ is defined if and only if there exists an computation of Z_0 beginning with $q_1(\overline{x^{(n)}})$, that is, if and only if $\Psi_{Z_0}^{(n)}(x^{(n)})$ is defined. So the first statement is true.

When $y_0 = \min_y T_n(z_0, x^{(n)}, y)$ is defined, y_0 is the Gödel number of a computation of Z_0 beginning with $q_1(\overline{x^{(n)}})$, which means the second statement is true. Therefore, $\mathcal{L}(y_0) \text{ Gl } y_0$ is the Gödel number of the final instantaneous description α of this computation, and $U(y_0) = \text{Corn}(\mathcal{L}(y_0) \text{ Gl } y_0) = \langle \alpha \rangle = \Psi_{Z_0}^{(n)}(x^{(n)})$. So the final statement is true. ■

Corollary 7.11 *$f(x^{(n)})$ is partially computable if and only if there is a number z_0 such that*

$$f(x^{(n)}) = U(\min_y T_n(z_0, x^{(n)}, y))$$

Corollary 7.12 *Every (partially) computable function is (partially) recursive.*

Corollary 7.13 *A function is (partially) computable if and only if it is (partial) recursive.*

Corollary 7.14 *If a function is total and partial recursive, then it is recursive.*