Computability Theory **Fall 2020**

Lecture 6: Recursive Functions

Lecturer: Renjie Yang

6.1 Recursive Functions

Definition 6.1 *A function is* ***partial recursive*** *if it can be obtained by a finite number of applications of composition and minimalization beginning with the functions of the following list:*

1. $S(x) = x + 1$
2. $U_i^n(x_1, x_2, \ldots, x_n) = x_i, 1 \leqslant i \leqslant n$
3. $x + y$
4. $x \dot{-} y$
5. xy

Definition 6.2 *A function is* ***recursive*** *if it can be obtained by a finite number of applications of composition and minimalization of regular functions, beginning with the functions of the list of Definition 6.1.*

Corollary 6.3 *Every recursive function is total and is partial recursive.*

Theorem 6.4 *If a function is recursive or partial recursive, then it is computable or partially computable respectively.*

Examples

- $N(x) = 0$. $N(x) = U_1^1(x) \dot{-} U_1^1(x)$
- $\alpha(x) = 1 \dot{-} x$. $a(x) = S(N(x)) \dot{-} U_1^1(x)$
- $x^2 = U_1^1(x) \dot{-} U_1^1(x)$
- $\lfloor\sqrt{x}\rfloor$, the largest integer $\leqslant \sqrt{x}$

$$\begin{aligned}\lfloor\sqrt{x}\rfloor &= min_y[(y+1)^2 \dot{-} x > 0] \\ &= min_y[\alpha((S(U_2^2(x,y)))^2 \dot{-} U_1^2(x,y)) = 0]\end{aligned}$$

- $|x - y| = (x \dot{-} y) + (y \dot{-} x)$
- $\lfloor x/y \rfloor$. If $y \neq 0$, $\lfloor x/y \rfloor$ is the greatest integer $\leqslant \dfrac{x}{y}$. If $y = 0$, $\lfloor x/y \rfloor = 0$. We have

$$\begin{aligned}\lfloor x/y \rfloor &= min_z[y = 0 \vee y(z+1) > x] \\ &= min_z[y = 0 \vee y(z+1) \dot{-} x > 0] \\ &= min_z[y = 0 \vee \alpha(y(z+1) \dot{-} x) = 0] \\ &= min_z[y \cdot \alpha(y(z+1) \dot{-} x) = 0]\end{aligned}$$

- $R(x,y)$. If $y \neq 0$, $R(x,y)$ is the remainder on dividing x by y.

$$\frac{x}{y} = \lfloor x/y \rfloor + \frac{R(x,y)}{y}$$

We take $R(x,y) = x \dot{-} y\lfloor x/y \rfloor$. Note that $R(x,0) = x$ by the definition of $\lfloor x/y \rfloor$.

6.2 Primitive Recursive Functions

Theorem 6.5 *There exists recursive functions $J(x,y)$, $K(z)$, $L(z)$ such that*

$$\begin{gathered} J(K(z), L(z)) = z \\ K(J(x,y)) = x \\ K(J(x,y)) = y. \end{gathered}$$

Proof: Let z be any number. Let r be the largest number such that $1 + 2 + \cdots + r \leqslant z$. Let $x = z - (1 + 2 + \cdots + r)$, then $x \leqslant r$ because otherwise $1 + 2 + \cdots + (r+1) \leqslant z$. Let $y = r - x$. Then

$$\begin{aligned} z &= [1 + 2 + \cdots + (x+y)] + x \\ &= \frac{1}{2}(x+y)(x+y+1) + x \\ &= \frac{1}{2}[(x+y)^2 + 3x + y] \\ &= J(x,y) \end{aligned}$$

$J(x,y)$ is recursive: $J(x,y) = [(x+y)^2 + U_1^1(x) + U_1^1(x) + U_1^1(x) + U_1^1(y)]/S(S(N(x)))$.
Since

$$(2x + 2y + 1)^2 \leqslant 8z + 1 < (2x + 2y + 3)^2$$

therefore $\lfloor \frac{1}{2}(\lfloor\sqrt{8z+1}\rfloor + 1) \rfloor = x + y + 1$, thus

$$\begin{aligned} x + y &= \lfloor \frac{1}{2}(\lfloor\sqrt{8z+1}\rfloor + 1) \rfloor - 1 \\ 3x + y &= 2z - \lfloor \frac{1}{2}(\lfloor\sqrt{8z+1}\rfloor + 1) \rfloor^2 \end{aligned}$$

The determinant of the parameter matrix is not zero, so for each z, at most one x, y exists satisfying the set of equations. Let $x+y = Q_1(z) = [\frac{1}{2}(\lfloor \frac{1}{2}(\lfloor\sqrt{8z+1}\rfloor + 1) \rfloor \dot{-} 1$, $3x+y = Q_2(z) = 2z \dot{-} (Q_1(z))^2$. Then

$$\begin{aligned} x &= \frac{1}{2}[Q_2(z) \dot{-} Q_1(z)] = K(z) \\ y &= Q_1(z) \dot{-} \frac{1}{2}[Q_2(z) \dot{-} Q_1(z)] = L(z) \end{aligned}$$

are all recursive functions. ■

Theorem 6.6 *(Euclidean Algorithm) If $n > 0$ and if m is any natural number, then there exists one and only one pair of numbers q, r such that $m = nq + r$ and $0 \leqslant r < n$.*

Definition 6.7 *Let a, b, m be natural numbers. By the Euclidean algorithm we can write*

$$
\begin{aligned}
a &= q_1 m + r_1 \qquad 0 \leqslant r_1 < m \\
b &= q_2 m + r_2 \qquad 0 \leqslant r_2 < m
\end{aligned}
$$

if $r_1 = r_2$, we say that a is congruent to b modulo m, which is denoted by $a \equiv b$ (mod m).

Theorem 6.8 *(Chinese Remainder Theorem). Let $a_1, a_2, \ldots, a_k$ be any numbers, and let $m_1, m_2, \ldots m_k$ be relatively prime in pairs. Then there exists a number x such that*

$$x \equiv a_i (mod\ m_i) \qquad i = 1, 2, \ldots, k$$

Lemma 6.9 *Let v be divisible by the numbers $1, 2, \ldots, n$. Then the numbers $1 + v(i + 1)$, $i = 0, 1, 2, \ldots n$, are relatively prime in pairs.*

Proof: Let $m_i = 1+v(i+1)$. Since v is divisible by $1, 2, \ldots, n$, any divisor d of m_i other than 1 must be larger than n, since otherwise $d|1$. Suppose that $d|m_i$, $d|m_j$, $i > j$. Then $d|(i+1)m_j - (j+1)m_i$, which implies $d|(i - j)$. Since $i - j \leqslant n$, therefore $d = 1$. ■

Theorem 6.10 *Let $a_0, a_1, a_2, \ldots, a_n$ be a finite sequence of integers. Then there are integers u and v such that*

$$R(u, 1 + v(i + 1)) = a_i \qquad i = 0, 1, \ldots, n$$

Proof: Let A be the largest integer among $a_0, a_1, \ldots, a_n$, and let $v = 2A \cdot n!$. Let $m_i = 1 + v(i + 1)$. By lemma 6.9, the m_i are relatively prime in pairs. By the Chinese remainder theorem, there exists a number u such that

$$u \equiv a_i (\text{mod } m_i) \qquad i = 0, 1, \ldots, n$$

Since $a_i < v < m_i$,

$$R(u, m_i) = R(a_i, m_i) = a_i \qquad i = 0, 1, \ldots, n$$

Therefore $R(u, 1 + v(i + 1)) = R(u, m_i) = R(a_i, m_i) = a_i$ ■

Theorem 6.11 *There is a recursive function $T_i(w)$ such that, if $a_0, a_1, \ldots, a_n$ are any integers whatever, there exists a number w_0 such that $T_i(w_0) = a_i$, $i = 0, 1, \ldots, n$.*

Proof: For any given $a_0, a_1, \ldots, a_n$, by theorem 6.10 there exist numbers u and v such that

$$R(u, 1 + v(i + 1)) = a_i \qquad i = 0, 1, \ldots, n$$

Let $w_0 = J(u, v)$. Define $T_i(w) = R(K(w), 1 + [L(w)(i + 1)])$. Since R, K, L are recursive, $T_i(w)$ is also recursive. Then

$$
\begin{aligned}
T_i(w_0) &= R(K(J(u, v)), 1 + L(J(u, v)) \cdot (i + 1)) \\
&= R(u, 1 + v(i + 1)) \\
&= a_i \qquad i = 0, 1, \ldots, n
\end{aligned}
$$

■

Theorem 6.12 *Let $f(x^{(n)})$, $g(x^{(n+2)})$ be total functions. Then there exists at most one total function $h(x^{(n+1)})$ that satisfies the recursion equations*

$$
\begin{aligned}
h(0, x^{(0)}) &= f(x^{(n)}) \\
h(z + 1, x^{(n)}) &= g(z, h(z, x^{(n)}), x^{(n)})
\end{aligned}
$$

Proof: *Assume both $h_1(x^{(n)})$ and $h_2(x^{(n)})$ satisfy the above equations. The conclusion is apparent by simple induction.* ∎

Theorem 6.13 *Let $f(x^{(n)})$ and $g(x^{(n+2)})$ be total functions. Then there exists a total function $h(x^{(n+1)})$ that satisfies the recursion equations.*

Proof: Define a set of $(n+2)$-tuples $(y, x^{(n)}, u)$ that is called "satisfactory" by the following two conditions:

1. For each choice of $x^{(n)}$, $(0, x^{(n)}, f(x^{(n)})) \in S$
2. For each choice of $x^{(n)}$, if $(z, x^{(n)}, u) \in S$ then $(z+1, x^{(n)}, g(z, u, x^{(n)})) \in S$

Let Ω be the class of all satisfactory sets S. Since the set of all $(n+2)$-tuples of numbers is satisfactory, Ω is not empty. Let S_0 be the intersection of all satisfactory sets in Ω. S_0 is a satisfactory set by definition. By simple induction, it is direct to show that for each choice of y and $x^{(n)}$, there is one and only one value of u for which $(y, x^{(n)}, u) \in S_0$. In fact, if there is a value of u other than $g(z, u, x^{(n)})$ such that $(z+1, x^{(n)}, u) \in S_0$, then we can delete $(z+1, x^{(n)}, u)$ from S_0 and still be satisfactory, which contradicts the definition of S_0. ∎

Definition 6.14 *The operation of primitive recursion associates with the given total functions $f(x^{(n)})$ and $g(x^{(n+2)})$ the function $h(x^{(n+1)})$, where*

$$h(0, x^{(0)}) = f(x^{(n)})$$
$$h(z+1, x^{(n)}) = g(z, h(z, x^{(n)}), x^{(n)}).$$

Theorem 6.15 *Let $h(x^{(n+1)})$ be obtained from $f(x^{(n)})$ and $g(x^{(n+2)})$ by primitive recursion. If f and g are recursive, then so is h.*

Proof: By theorem 6.11, for each choice of $x^{(n)}$ and y, there exists at least one number w_0 such that

$$T_i(w_0) = h(i, x^{(n)}) \qquad i = 0, 1, \ldots, y.$$

Hence

$$\begin{aligned}
h(y, x^{(n)}) &= T_y(min_w[(T_0(w) = f(x^{(n)})) \wedge \forall z > 0((z < y) \rightarrow T_{z+1}(w) = g(z, T_z(w), x^{(n)}))]) \\
&= T_y(min_w[(T_0(w) = f(x^{(n)})) \wedge \forall z > 0((y \leqslant z) \vee T_{z+1}(w) = g(z, T_z(w), x^{(n)}))]) \\
&= T_y(min_w[(T_0(w) = f(x^{(n)})) \wedge y = min_z\{T_{z+1}(w) \neq g(z, T_z(w), x^{(n)})\}]) \\
&= T_y(min_w[(T_0(w) = f(x^{(n)})) \wedge y = min_z\{\alpha(|T_{z+1}(w) - g(z, T_z(w), x^{(n)})|) = 0\}]) \\
&= T_y(min_w[|T_0(w) - f(x^{(n)})| + |y - min_z\{\alpha(|T_{z+1}(w) - g(z, T_z(w), x^{(n)})|) = 0\}| = 0])
\end{aligned}$$

Therefore, $h(x^{(n+1)})$ is recursive. ∎

Definition 6.16 *A function is **primitive recursive** if it can be obtained by a finite number of applications of composition and primitive recursion beginning with the functions of the following list:*

1. $S(x) = x + 1$
2. $U_i^n(x_1, x_2, \ldots, x_n) = x_i, 1 \leqslant i \leqslant n$

3. $N(x) = 0$

Theorem 6.17 *If a function is primitive recursive, then it is recursive.*

Examples Some primitive recursive functions:

- $x + y$.
  $x + 0 = U_1^1(x)$,
  $x + (y + 1) = S(x + y)$
- $x \cdot y$.
  $x \cdot 0 = N(x)$,
  $x \cdot (y + 1) = (x \cdot y) + U_1^2(x, y)$
- $P(x)$, where $P(0) = 0$, $P(x) = x - 1$ if $x > 0$
  $P(0) = N(x)$
  $P(x + 1) = U_1^1(x)$
- $x \dot{-} y$
  $x \dot{-} 0 = U_1^1(x)$
  $x \dot{-} (y + 1) = P(x \dot{-} y)$
- $n!$
  $0! = S(N(0))$
  $(n + 1)! = n! \cdot S(n)$
- x^y
  $x^0 = S(N(x))$
  $x^{y+1} = x^y \cdot U_1^2(x, y)$
- $\alpha(x) = 1 \dot{-} x$
  $\alpha(x) = S(N(x)) \dot{-} U_1^1(x)$
- $|x - y|$
  $|x - y| = (x \dot{-} y) + (y \dot{-} x)$

Theorem 6.18 *A function is partial recursive if and only if it can be obtained by a finite number of applications of the operations of composition, primitive recursion, and minimalization to the functions in the list of Definition 6.16. The class of recursive functions will be obtained if minimalization is applied only to regular functions.*

Theorem 6.19 *If $f(k, x^{(p)})$ is (primitive) recursive, so are*

$$g(n, x^{(p)}) = \sum_{k=0}^{n} f(k, x^{(p)})$$

$$h(n, x^{(p)}) = \prod_{k=0}^{n} f(k, x^{(p)})$$

Proof:

$$g(0, x^{(p)}) = f(0, x^{(p)}),$$
$$g(n+1, x^{(p)}) = g(n, x^{(p)}) + f(n+1, x^{(p)}).$$

$$h(0, x^{(p)}) = f(0, x^{(p)}),$$
$$h(n+1, x^{(p)}) = h(n, x^{(p)}) \cdot f(n+1, x^{(p)}).$$

■

6.3 Recursive Sets and Predicates

Definition 6.20 *Let S be a set of n-tuples. Then, by the characteristic function of the set S, written $C_S(x_1, \ldots, x_n)$, is to be understood the total n-ary function whose value, for a given n-tuple $(a_1, \ldots, a_n)$, is 0 if $(a_1, \ldots, a_n) \in S$ and is 1 if $(a_1, \ldots, a_n) \notin S$.*

Example
$C_{R \cup S} = C_R \cdot C_S$
$C_{R \cap S} = C_R + C_S \dot{-} (C_R \cdot C_S)$
$C_{\overline{R}} = 1 \dot{-} C_R$

Definition 6.21 *A statement asserts a proposition that must be either true or false. An expression that contains variables and that becomes a statement when these variables are replaced by any numbers whatever is called a predicate. A predicate with n different variables is called an n-ary predicate. The extension of an n-ary predicate $P(x_1, \ldots, x_n)$ is the set of all n-tuples $(a_1, \ldots, a_n)$ for which $P(a_1, \ldots, a_n)$ is true, which is written as $\{x_1, \ldots, x_n | P(x_1, \ldots, x_n)\}$. We say that two predicates are equivalent if they have the same extension. The characteristic function of a predicate is the characteristic function of its extension.*

Definition 6.22 *Let S be a set of n-tuples. Then, we say that S is (primitive) recursive if its characteristic function $C_S(x^{(n)})$ is.*

Example Let n-tuples R and S be (primitive) recursive, then so are $R \cup S$, $R \cap S$, and $\overline{R}$.

Corollary 6.23 *$P(x^{(n)})$ is (primitive) recursive if and only if its characteristic function is.*

Example Let predicates P and Q be (primitive) recursive, then so are $P \vee Q$, $P \wedge Q$, and $\neg P$.
Proof: $C_{P \cup Q} = C_P \cdot C_Q$
$C_{P \cap Q} = C_P + C_Q \dot{-} (C_P \cdot C_Q)$
$C_{\neg P} = 1 \dot{-} C_P$ ■

Theorem 6.24 *If $P(y, x^{(n)})$ is an (primitive) recursive predicate, z is a natural number, then so are*

$$\exists y (y \leqslant z) \wedge P(y, x^{(n)})$$
$$\forall y (y \leqslant z) \to P(y, x^{(n)})$$

Proof: $C_{\exists y(y \leqslant z) \wedge P(y, x^{(n)})}(y, x^{(n)}) = \prod_{y=0}^{z} C_{P(y,x^{(n)})}(y, x^{(n)})$,
$\forall (y(y \leqslant z) \to P(y, x^{(n)})) \leftrightarrow \neg \exists y((y \leqslant z) \wedge \neg P(y, x^{(n)}))$. ■

Definition 6.25 *Let $P(y, x^{(n)})$ be an $(n+1)$-ary predicate, where $y \in \mathbb{N}$. Define the following $(n+1)$-ary total function*

$$\mathop{\mathfrak{M}}_{y=0}^{z} P(y, x^{(n)}) = \begin{cases} min_y[y \leqslant z \wedge P(y, x^{(n)})], & \textit{if such } y \textit{ exists} \\ \\ 0, & \textit{otherwise} \end{cases}$$

Theorem 6.26 *If $P(y, x^{(n)})$ is (primitive) recursive, then so also is $f(y, x^{(n)}) = \mathop{\mathfrak{M}}_{y=0}^{z} P(y, x^{(n)})$*

Proof:

$$\mathop{\mathfrak{M}}_{y=0}^{z} P(y, x^{(n)}) = \alpha(\prod_{y=0}^{z} C_p(y, x^{(n)})) \cdot \sum_{t=0}^{z} \prod_{y=0}^{t} C_p(y, x^{(n)})$$

■

Theorem 6.27 *The predicates $x < y$ and $x = y$ are primitive recursive.*

Proof: Their characteristic functions are $\alpha(y \dot{-} x)$ and $\neg(x < y) \wedge \neg(y < x)$ ■

Examples

- $y|x$, y is a devisor of x.
  $\exists z(z \geqslant 0) \wedge (z \leqslant x) \wedge (x = yz)$
- $Prime(x)$, x is a prime number.
  $(x > 1) \wedge \forall z[(z = 1) \vee (z = x) \vee \neg(z|x)]$
- $Pr(n)$, the n-th prime in order of magnitude, where the 0-th prime is 0.
  $Pr(0) = 0$
  $Pr(n+1) = \mathop{\mathfrak{M}}_{y=0}^{Pr(n)!+1} [Prime(y) \wedge y > Pr(n)]$
- $\lfloor x|2 \rfloor$, the largest integer $\leqslant \dfrac{x}{2}$.
  $\mathop{\mathfrak{M}}_{y=0}^{x} (2y + 2 > x)$
- $J(x, y)$
  $J(x, y) = \dfrac{(x+y)^2 + 3x + y}{2}$
- $K(z)$
  $\mathop{\mathfrak{M}}_{x=0}^{z} \exists y(y \geqslant 0 \wedge y \leqslant z \wedge z = J(x, y))$
- $L(z)$
  $\mathop{\mathfrak{M}}_{y=0}^{z} \exists x(x \geqslant 0 \wedge x \leqslant z \wedge z = J(x, y))$