Computability Theory **Fall 2020**

Lecture 7: The Equivalence of Computability and Recursiveness

Lecturer: Renjie Yang

7.1 The Arithmetization of The Theory of Turing Machines

Definition 7.1 *Let M be an expression consisting of the symbols $\gamma_1, \gamma_2, \ldots, \gamma_n$. Let $a_1, a_2, \ldots, a_n$ be the corresponding integers associated with these symbols. Then the* ***Gödel number*** *of M is the integer*

$$gn(M) = r = \prod_{k=1}^{n} Pr(k)^{a_k}$$

If M is the empty expression, we let 1 be the Gödel number of M.

Note We adopt a convention to associate each symbol with an even number: $R \to 3$, $S \to 5$, $q_i \to 4i + 5$, $S_i \to 4i + 7$.

Example $gn(q_1 1 R q_2) = 2^9 \cdot 3^{11} \cdot 5^3 \cdot 7^{13}$

Corollary 7.2 *If M and N are given expressions such that $gn(M) = gn(N)$, then $M = N$.*

Proof: According to the Fundamental Theorem of Arithmetic, every natural number can be uniquely represented in the form $p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$, where $p_1, \ldots, p_k$ are distinct primes. Therefore if $M = N$ then their prime factorizations are the same. ■

Note A computation is a finite sequence of expressions; a Turing machine is a finite set of expressions.

Definition 7.3 *If $n = gn(M)$, we also write $M = Exp(n)$*

Definition 7.4 *Let $M_1, \ldots, M_n$ be a finite sequence of expressions. Then the Gödel number of this sequence of expressions is the number*

$$\prod_{k=1}^{n} Pr(k)^{g_n(M_k)}$$

Example $gn(\{q_1 1 B q_1, q_1 B R q_2\}) = 2^{2^9 \cdot 3^{11} \cdot 5^7 \cdot 7^9} \cdot 3^{2^9 \cdot 3^7 \cdot 5^3 \cdot 7^{13}}$

Corollary 7.5 *No integer is the Gödel number both of an expression and of a sequence of expressions.*

Proof: A Gödel number of an expression or a sequence of expressions is of the form $2^n \cdot m$. n is odd for expressions, and even for sequences of expressions. ■

Corollary 7.6 *Two sequences of expressions that have the same Gödel number are identical.*

Definition 7.7 *Let Z be a Turing machine. Let $M_1, \ldots, M_n$ be any arrangement of the quadruples of Z without repetitions. Then, the Gödel number of the sequence $M_1, \ldots, M_n$ is called a* ***Gödel number of the Turing machine*** *Z.*

Note A Turing machine consisting of n quadruples has $n!$ distinct Gödel numbers.

Definition 7.8 *For each $n > 0$ and for each set of integers A, let $T_n(z, x_1, \ldots, x_n, y)$ be the predicate that means, for given $z, x_1, \ldots, x_n, y$ that z is a Gödel number of a Turing machine Z, and that y is the Gödel number of a computation, with respect to Z, beginning with the instantaneous description $q_1(\overline{x_1, \ldots, x_n})$.*

Theorem 7.9 *$T_n(z, x_1, \ldots, x_n, y)$ is primitive recursive.*

Proof: The proof proceeds by a detailed list of primitive recursive functions and predicates until we get $T_n(z, x_1, \ldots, x_n, y)$.

Group 1 Functions and predicates which concern Gödel numbers of expressions and sequences of expressions:

1. $n\ Gl\ x = \underset{y=0}{\overset{x}{\mathfrak{M}}}[(Pr(n)^y | x)] \wedge \neg(Pr(n)^{y+1} | x)]$
   The Gödel encoding of the nth symbol in the expression represented by x.
   The Gödel encoding of the nth expression in the sequence of expression represented by x.

2. $\mathcal{L}(x) = \underset{y=0}{\overset{x}{\mathfrak{M}}}[(y\ Gl\ x > 0) \wedge \forall i((0 \leqslant i \leqslant x) \rightarrow (y + 1 + i)\ Gl\ x = 0)]$
   The length of the expression represented by x.
   The length of the sequence of expression represented by x.

3. $GN(x) \leftrightarrow \neg\exists(1 \leqslant y \leqslant \mathcal{L}(x))[(y\ Gl\ x = 0) \wedge ((y + 1)\ Gl\ x \neq 0)]$
   x is a Gödel Numbers of some expression or some sequence of expression with no empty expression.

4. $Term(x, z) \leftrightarrow GN(z) \wedge \exists(1 \leqslant n \leqslant \mathcal{L}(x))x = n\ Gl\ z$
   z is a Gödel Numbers of some expression, and x is the Gödel Numbers of one of the symbols in the expression represented by z.

5. $x * y = x \cdot \prod_{i=1}^{\mathcal{L}(x)} Pr(\mathcal{L}(x) + i)^{i\ GL\ y}$
   If M and N are expressions, then $gn(MN) = gn(M) * gn(N)$.
   If x and y are the Gödel numbers of the sequences of expressions $M_1, \ldots, M_n$ and $N_1, \ldots, N_p$ respectively, then $x * y$ is the Gödel numbers of the sequence $M_1, \ldots, M_n, N_1, \ldots, N_p$.

Group 2 Functions and Predicates which concern the basic structure of Turing machines:

6. $IC(x) \leftrightarrow \exists(0 \leqslant y \leqslant x)(x = 4y + 9)$
   x is the number assigned to an internal configuration q_i

7. $Al(x) \leftrightarrow \exists(0 \leqslant y \leqslant x)(x = 4y + 7)$
   x is the number assigned to an alphabet S_i

8. $Odd(x) \leftrightarrow \exists(0 \leqslant y \leqslant x)(x = 2y + 3)$
   x is an odd number $\geqslant 3$.

9. $Quad(x) \leftrightarrow GN(x) \wedge \mathcal{L}(x) \wedge IC(1\ Gl\ x) \wedge Al(2\ Gl\ x) \wedge Odd(3\ Gl\ x) \wedge IC(4\ Gl\ x)$
   The expression represented by x is a quadruple.

10. $Inc(x, y) \leftrightarrow Quad(x) \wedge Quad(y) \wedge (1\ Gl\ x = 1\ Gl\ y) \wedge (2\ Gl\ x = 2\ Gl\ y) \wedge (x \neq y)$
    x and y are Gödel numbers of two incompatible quadruples beginning with the same two symbols.

11. $TM(x) \leftrightarrow GN(x) \wedge \forall(1 \leqslant n \leqslant \mathcal{L}(x))[Quad(n\ Gl\ x) \wedge \forall(1 \leqslant m \leqslant \mathcal{L}(x))\neg Inc(n\ Gl\ x, m\ Gl\ x)]$
    x is a Gödel number of a Turing machine.

12. $MR(0) = 2^{11}$,
    $MR(n + 1) = 2^{11} * MR(n)$
    $MR(n)$ is the Gödel number of $\overline{n}$.

13. $CU(n, x) = 0$ if $n\ Glx \neq 11$,
    $CU(n, x) = 1$ if $n\ Glx = 11$.
    $CU(n, x)$ is the characteristic function of the predicate $n\ Gl\ x \neq 11$, namely the nth symbol in the expression represented by x is not S_1.

14. $Corn(x) = \sum_{n=1}^{\mathcal{L}(x)} CU(n, x)$
    If x is the Gödel number of M, then $Corn(x) = \langle M \rangle$.

15. $U(y) = Corn(\mathcal{L}(y)\ Gl\ y)$
    If y is the Gödel number of a sequence of expression $M_1, M_2, \ldots, M_n$, then $U(y) = \langle M_n \rangle$.

16. $ID(x) \leftrightarrow GN(x) \wedge \exists(1 \leqslant n \leqslant \mathcal{L}(x) \dot{-} 1)[IC(n\ Gl\ x) \wedge \forall(1 \leqslant m \leqslant \mathcal{L}(x))(m = n \vee Al(m\ Gl\ x))])$
    x is a Gödel number of an instantaneous description.

17. $Init_n(x_1, \ldots, x_n) = 2^9 * MR(x_1) * 2^7 * MR(x_2) * 2^7 \ldots * 2^7 * MR(x_n)$.
    $Init_n(x_1, \ldots, x_n) = gn(q_1(\overline{x_1, \ldots, x_n}))$.

Group 3 Functions and Predicates which concern the computations of Turing machines:

18. $$\begin{aligned} Yield_1(x, y, z) \leftrightarrow\ & ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leqslant F, G, r, s \leqslant x, 0 \leqslant t, u \leqslant y) \\ & [(x = F * 2^r * 2^s * G) \wedge (y = F * 2^t * 2^u * G) \\ & \wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Al(u) \wedge Term(2^r \cdot 3^s \cdot 5^u \cdot 7^t, z)]. \end{aligned}$$
    x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z, and $Exp(x) \to Exp(y)(Z)$, under the first rule of $\alpha \to \beta(Z)$:
    (1) There exist expressions P and Q (possibly empty) such that
    $$\alpha \text{ is } Pq_iS_jQ, \qquad \beta \text{ is } Pq_lS_kQ,$$
    where
    $$Z \text{ contains } q_iS_jS_kq_l.$$

19. $$\begin{aligned} Yield_2(x, y, z) \leftrightarrow\ & ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leqslant F, G, r, s \leqslant x, 0 \leqslant t, u \leqslant y) \\ & [(x = F * 2^r * 2^s * 2^t * G) \wedge (y = F * 2^s * 2^u * 2^t * G) \\ & \wedge IC(r) \wedge IC(u) \wedge Al(s) \wedge Al(t) \wedge Term(2^r \cdot 3^s \cdot 5^3 \cdot 7^u, z)]. \end{aligned}$$
    x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing

machine Z, and $Exp(x) \to Exp(y)(Z)$, under the second rule of $\alpha \to \beta(Z)$:
(2) There exist expressions P and Q (possibly empty) such that

$$\alpha \text{ is } Pq_iS_jS_kQ,$$
$$\beta \text{ is } PS_jq_lS_kQ,$$

where

$$Z \text{ contains } q_iS_jRq_l.$$

20. $Yield_3(x,y,z) \leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leqslant F,r,s \leqslant x, 0 \leqslant t \leqslant y)$
$[(x = F * 2^r * 2^s) \wedge (y = F * 2^s * 2^t * 2^7)$
$\wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Term(2^r \cdot 3^s \cdot 5^3 \cdot 7^t, z)]$.
x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z, and $Exp(x) \to Exp(y)(Z)$, under the third rule of $\alpha \to \beta(Z)$:
(3) There exist expressions P and Q (possibly empty) such that

$$\alpha \text{ is } Pq_iS_j,$$
$$\beta \text{ is } PS_jq_lS_0,$$

where

$$Z \text{ contains } q_iS_jRq_l.$$

21. $Yield_4(x,y,z) \leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leqslant F,G,r,s,t \leqslant x, 0 \leqslant u \leqslant y)$
$[(x = F * 2^r * 2^s * 2^t * G) \wedge (y = F * 2^u * 2^r * 2^t * G)$
$\wedge IC(s) \wedge IC(u) \wedge Al(r) \wedge Al(t) \wedge Term(2^s \cdot 3^t \cdot 5^5 \cdot 7^u, z)]$.
x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z, and $Exp(x) \to Exp(y)(Z)$, under the fourth rule of $\alpha \to \beta(Z)$:
(4) There exist expressions P and Q (possibly empty) such that

$$\alpha \text{ is } PS_kq_iS_jQ,$$
$$\beta \text{ is } Pq_lS_kS_jQ,$$

where

$$Z \text{ contains } q_iS_jLq_l.$$

22. $Yield_5(x,y,z) \leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leqslant G,r,s \leqslant x, 0 \leqslant t \leqslant y)$
$[(x = 2^r * 2^s * G) \wedge (y = 2^t * 2^7 * 2^s * G)$
$\wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Term(2^r \cdot 3^s \cdot 5^5 \cdot 7^t, z)]$.
x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z, and $Exp(x) \to Exp(y)(Z)$, under the fifth rule of $\alpha \to \beta(Z)$:
(5) There exist expressions P and Q (possibly empty) such that

$$\alpha \text{ is } q_iS_jQ,$$
$$\beta \text{ is } q_lS_0S_jQ,$$

where

$$Z \text{ contains } q_iS_jLq_l.$$

23. $Yield(x,y,z) \leftrightarrow Yield_1(x,y,z) \vee Yield_2(x,y,z) \vee Yield_3(x,y,z) \vee Yield_4(x,y,z) \vee Yield_5(x,y,z)$
x and y are the Gödel numbers of instantaneous descriptions, z is a Gödel number of a Turing machine Z, and $Exp(x) \to Exp(y)(Z)$.

24. $Fin(x,z) \leftrightarrow ID(x) \wedge TM(z) \wedge \exists(0 \leqslant F,G,r,s \leqslant x)$
$[(x = F * 2^r * 2^s * G) \wedge IC(r) \wedge Al(s)$
$\wedge \forall(1 \leqslant n \leqslant \mathcal{L}(z))((1\ Gl\ (n\ Gl\ z) \neq r) \vee (2\ Gl\ (n\ Gl\ z) \neq s)]$
z is a Gödel number of a Turing machine Z and x is the Gödel number of an instantaneous

description final with respect to Z, namely there is no yield rule in Z to further compute from x.

25. $Comp(y,z) \leftrightarrow TM(z) \wedge GN(y) \wedge \forall(1 \leqslant n \leqslant \mathcal{L}(y))[Yield(n\ Gl\ ,(n+1)\ Gl\ y,z)]$
$\wedge Fin(\mathcal{L}(y)\ Gl\ y,z)$
z is a Gödel number of a Turing machine Z and y is the Gödel number of a computation of Z.

By definition 7.8, $T_n(z,x_1,\ldots,x_n,y) \leftrightarrow (1\ Gl\ y = Init_n(x_1,\ldots,x_n)) \wedge Comp(y,z)$. Therefore it is primitive recursive according to the above construction. ∎

7.2 Computability and Recursiveness

Theorem 7.10 *Let Z_0 be a Turing machine and let z_0 be a Gödel number of Z_0. Then, the domain of the function $\Psi^{(n)}_{Z_0}(x^{(n)})$ is equal to the domain of $min_y T_n(z_0,x^{(n)},y)$. Moreover,*

$$\Psi^{(n)}_{Z_0}(x^{(n)}) = U(min_y T_n(z_0,x^{(n)},y)).$$

Also, if $T_n(z_0,x^{(n)},y_0)$ is true for given $x^{(n)}$, then

$$y_0 = min_y T_n(z_0,x^{(n)},y).$$

Proof: For any given $x^{(n)}$, $min_y T_n(z_0,x^{(n)},y)$ is defined if and only if there exists an computation of Z_0 beginning with $q_1\overline{(x^{(n)})}$, that is, if and only if $\Psi^{(n)}_{Z_0}(x^{(n)})$ is defined. So the first statement is true.

When $y_0 = min_y T_n(z_0,x^{(n)},y)$ is defined, y_0 is the Gödel number of a computation of Z_0 beginning with $q_1\overline{(x^{(n)})}$. Therefore, $\mathcal{L}(y_0)\ Gl\ y_0$ is the Gödel number of the final instantaneous description α of this computation, and

$$U(y_0) = Corn(\mathcal{L}(y_0)\ Gl\ y_0) = \langle\alpha\rangle = \Psi^{(n)}_{Z_0}(x^{(n)}).$$

So the second statement is true.

If $T_n(z_0,x^{(n)},y_0)$ is true for given $x^{(n)}$, then y_0 is the Gödel number of a computation of Z_0 beginning with $q_1\overline{(x^{(n)})}$. Since our definition of Turing machine is deterministic, there is only one such computation procedure for the Turing machine Z_0. Therefore the final statement is true. ∎

Corollary 7.11 *(Kleene's Normal Form Theorem) $f(x^{(n)})$ is partially computable if and only if there is a number z_0 such that*

$$f(x^{(n)}) = U(min_y T_n(z_0,x^{(n)},y))$$

Proof: If $f(x^{(n)})$ is partially computable, then by definition 5.16, there is a Turing machine Z such that $f(x^{(n)}) = \Psi^{(n)}_Z(x^{(n)})$. According to theorem 7.10, $f(x^{(n)}) = \Psi^{(n)}_Z(x^{(n)}) = U(min_y T_n(z_0,x^{(n)},y))$.

For the other side, if there is a number z_0 such that $f(x^{(n)}) = U(min_y T_n(z_0,x^{(n)},y))$, since T and U are primitive recursive by theorem 7.9, $f(x^{(n)})$ is partial recursive. According to theorem 6.4, $f(x^{(n)})$ is partially computable. ∎

Corollary 7.12 *Every (partially) computable function is (partial) recursive.*

Proof: If a function is partial computable, then it is partial recursive according to corollary 7.11. For a computable function $f(x^{(n)})$, it is also a partial computable function, and its corresponding $U(min_y T_n(z_0, x^{(n)}, y))$ is defined for all $x^{(n)}$. So the minimization is of a regular function, by definition 6.2 $f(x^{(n)})$ is recursive. ■

Corollary 7.13 *(Equivalence) A function is (partially) computable if and only if it is (partial) recursive.*

Proof: Corollary 7.12 and theorem 6.4. ■

Corollary 7.14 *If a function is total and partial recursive, then it is recursive.*

Proof: Such a function is partial recursive, therefore partially computable. It is also total, therefore it is computable by definition 5.16, which implies it is recursive by the above corollary. ■