Lecture 12: Incompleteness

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12.1 Logic and Computability

Propositional Logic

- Propositional variables $p, q, r, \ldots$
- Connectives $\wedge, \vee, \neg, \rightarrow, \ldots$
- Semantics: assignment of truth values
- Procedures for calculating truth values: truth tables

Definition 12.1 *A propositional formula φ is valid, written $\models \varphi$, if φ is true under any truth assignment. If Γ is a set of formulas and φ is a formula, φ is a semantic consequence of Γ, written $\Gamma \models \varphi$, if φ is true under any truth assignment that makes every formula in Γ true.*

Definition 12.2 *A propositional formula φ is provable, written $\vdash \varphi$, if there is a formal derivation of φ. If Γ is a set of formulas and φ is a formula, φ is a deductive consequence of Γ, written $\Gamma \vdash \varphi$, if there is a formal derivation of φ from hypotheses in Γ.*

Theorem 12.3 *For any formula φ and any set of formulas Γ, $\Gamma \vdash \varphi$ if and only if $\Gamma \models \varphi$.*

First-order Logic

- Functions symbols $f_0, f_1, \ldots$
- Relation symbols $R_0, R_1, \ldots$
- Constant symbols $c_0, c_1, \ldots$
- Variables $x_0, x_1, \ldots$
- Logical symbols $\wedge, \vee, \rightarrow, \neg, \forall, \exists, =$
- (,)
- Terms, $r, s, t, \ldots$
- Formulas, $\varphi, \psi, \theta, \ldots$
- Substitutions: $\varphi[t/x]$

Definition 12.4 *A first-order formula φ is valid, written $\models \varphi$, if φ is satisfied under any model. If Γ is a set of formulas and φ is a formula, φ is a semantic consequence of Γ, written $\Gamma \models \varphi$, if φ is satisfied under any model that satisfies every formula in Γ.*

Definition 12.5 *A first-order formula φ is provable, written $\vdash \varphi$, if there is a formal derivation of φ. If Γ is a set of formulas and φ is a formula, φ is a deductive consequence of Γ, written $\Gamma \vdash \varphi$, if there is a formal derivation of φ from hypotheses in Γ.*

Theorem 12.6 *For any formula φ and any set of formulas Γ, $\Gamma \vdash \varphi$ if and only if $\Gamma \models \varphi$.*

Deductive system
Axioms:

- Propositional axioms: any instance of a valid propositional formula;
- Axioms involving quantifiers:
  $\forall x \varphi(x) \to \varphi(t)$
  $\varphi(t) \to \exists x \varphi(x)$
  $\forall x(\varphi \to \psi(x)) \to (\varphi \to \forall x \psi(x))$, x is not free in φ.
- Axioms for equality
  $\forall x(x = x)$
  $\forall x(x = y \to y = x)$
  $\forall x(x = y \wedge y = z \to z = y)$
  $\forall x_0, \ldots, x_k, y_0, \ldots, y_k(x_0 = y_0 \wedge \ldots \wedge x_k = y_k \wedge \varphi(x_0, \ldots, x_k) \to \varphi(y_0, \ldots, y_k))$.

Rules of Inference:

- Modus ponens: from φ and $\varphi \vdash \psi$ conclude ψ
- Generalization: from φ conclude $\forall x \varphi$
- From $\varphi \to \psi$ conclude $\exists x \varphi \to \psi$, if x is not free in ψ.

Note Similar to what we did for the formal theory of Turing machines, we can assign numerical codings for terms, formulas, and proofs or derivations, in such a way that the identification problems of these objects are computable. Similar to the construction of the T predicate, we can show that the corresponding predicates are primitive recursive.

12.2 Representability in Q

Language of Arithmetic

- A constant symbol 0
- Function symbols $+$, $\times$, S
- A relation symbol $<$

Non-logical Axioms of Q

1. $Sx = Sy \to x = y$
2. $0 \neq Sx$
3. $x \neq 0 \to \exists y(x = Sy)$

4. $x + 0 = x$

5. $x + Sy = S(x + y)$

6. $x \times 0 = x$

7. $x \times Sy = x \times y + x$

8. $x < y \leftrightarrow \exists z(Sz + x = y)$

Definition 12.7 *For each natural number n, define the numeral $\overline{n}$ to be $\underbrace{S \dots S}_{n \text{ times}} x$.*

Definition 12.8 *A function $f(x_0, \dots, x_k)$ from the natural numbers to the natural numbers is said to be representable in Q if there is a formula $\varphi_f(x_0, \dots, x_k, y)$ such that whenever $f(n_0, \dots, n_k) = m$, Q proves*

- $\varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m})$
- $\forall y(\varphi_f(\overline{n_0}, \dots, \overline{n_k}) \to \overline{m} = y)$

Theorem 12.9 *A function is representable in Q if and only if it is computable.*

Proof: For the forword direction, we can code terms, formulas and proofs in such a way that the relation "d is a proof of φ in theory Q" and function "the result of substituting the numeral of n for the code of variable v in the code of formula φ" are computable. Suppose the function f is represented by $\varphi_f(x_0, \dots, x_k, y)$. Then the algorithm for computing f is:

$$f(n_0, \dots, n_k) = L(\min_s(\text{"}K(s) \text{ is a proof of } \varphi(\overline{n_0}, \dots, \overline{n_k}, \overline{L(s)}) \text{ in } Q''))).$$

For the other direction, define a set of total functions C as follows: let C be the smallest set of functions containing

- 0
- $S(x)$
- $x + y$
- $x \times y$
- $U_i^n(x_1, \dots, x_n) = x_i, 1 \leqslant i \leqslant n$
- $\chi_=$

and closed under composition and unbounded search applied to regular functions. According to the definition of recursive functions, C is the set of recursive or computable functions, or that every computable functions are in C. By the 16 lemmas in Chapter 22 of Epstein's book, every function in C can be represented in Q. Therefore, every computable function can be represented in Q. ■

Definition 12.10 *A relation $R(x_0, \dots, x_k)$ on the natural numbers is represented in Q if there is a formula $\varphi_R(x_0, \dots x_k)$ such that whenever $R(n_0, \dots, n_k)$ is true, Q proves $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$, and whenever $R(n_0, \dots, n_k)$ is false, Q proves $\neg\varphi_R(\overline{n_0}, \dots, \overline{n_k})$.*

Theorem 12.11 *A relation is representable in Q if and only if it is computable.*

12.3 The First Incompleteness Theorem

Theorem 12.12 *Q is computable enumerable but not decidable.*

Definition 12.13 *A theory T is ω- consistent if the following holds: if $\exists x \varphi(x)$ is any sentence and T proves $\neg\varphi(\overline{0}, \overline{1}, \ldots)$ then T does not prove $\exists x \varphi(x)$.*

Theorem 12.14 *Let T be any ω-consistent theory that includes Q. Then T is not decidable.*

Lemma 12.15 *There is no binary computable relation $R(x, y)$, such that whenever $S(y)$ is a unary computable relation, there is some k such that for every y, $S(y)$ is true if and only if $R(k, y)$ is true.*

Theorem 12.16 *Let T be any consistent theory that includes Q. Then T is not decidable.*

Corollary 12.17 *The theory of arithmetic $\{\varphi | \langle \mathbb{N}, 0, S, +, \times, < \rangle \models \varphi\}$ is not decidable.*

Definition 12.18 *A theory T is said to be computably axiomatizable if it has a computable set of axioms A: $T = \{\varphi | A \vdash \varphi\}$.*

Lemma 12.19 *Suppose T is computably axiomatizable. Then T is computably enumerable.*

Lemma 12.20 *Suppose a theory T is complete and computably axiomatizable. Then T is computable.*

Theorem 12.21 *There is no complete, consistent, computabley axiomatized extension of Q.*