Computability Theory **Fall 2020**

Lecture 3: Context-Free Languages and Pushdown Automata

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3.1 Grammar

Sample English grammar rules:
sentence $\rightarrow$ subject predicate
subject $\rightarrow$ article adjective noun
predicate *rightarrow* verb object

Example The big dog chased the cat.

Definition 3.1 *A grammar is a 4-tuple $\{N, T, R, S\}$ consists of:*

- *A finite set N of grammar symbols called non-terminals;*
- *A finite set T of symbols called terminals, such that $N \cap T = \phi$;*
- *A finite set R of grammar rules of the form $\gamma \rightarrow \delta$ where γ and δ are strings over the symbol set $N \cup T$ with the following restrictions:*
  - *γ is not the empty string;*
  - *There is at least one production with S alone on the left hand side;*
  - *Each non-terminal must appear on the left hand side of some grammar rule;*
- *A non-terminal symbol S called start symbol.*

Example Let $\Sigma = \{a, b, c\}$, S is the start symbol. The following rules is a grammar for Σ^*:
$S \rightarrow \epsilon$
$S \rightarrow aS$
$S \rightarrow bS$
$S \rightarrow Sc$

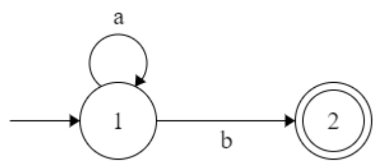
Example Languages and their corresponding grammar:
$\{a^n |\, n \in \mathbb{N}\}$ $S \rightarrow aS|\epsilon$
$\{a^n b^n |\, n \in \mathbb{N}\}$ $S \rightarrow aSb|\epsilon$
$\{(ab)^n |\, n \in \mathbb{N}\}$ $S \rightarrow abS|\epsilon$

Note A grammar is called a regular grammar if each of its grammar rule takes one of the following forms where the uppercase letters are non-terminals and w is a non-empty string of terminals:

- $S \rightarrow \epsilon$
- $S \rightarrow w$
- $S \rightarrow T$

- $S \rightarrow wT$

Note A regular language can be written by FSM, regular expression, or regular grammar. For example,
a^*b
$S \rightarrow aS; S \rightarrow b$
and the following FSM



all express the same language $\{b, ab, aab, aaab, \cdots\}$

3.2 Context-Free Grammar

Definition 3.2 *A context-free grammar is a 4-tuple* (V, Σ, R, S)*, where*

1. V *is a finite set called the variables;*
2. Σ *is a finite set, disjoint from* V*, called the terminals;*
3. R *is a finite set of rules, with each rule being a variable and a string of variables and terminals;*
4. $S \in V$ *is the start variable.*

Definition 3.3 *If* u*,* v *and* w *are strings of variables and terminals, and* $A \rightarrow w$ *is a rule of the grammar, we say that* uAv *yields* uwv*, written* $uAv \Rightarrow uwv$*. Say that* u *derives* v*, written* $u \overset{*}{\Rightarrow} v$*, if* $u = v$ *or if a sequence* $u_1, u_2, \ldots, u_k)$ *exists for* $k \geq 0$ *and*

$$u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_k \Rightarrow u.$$

The language of the grammar is $\{w \in \Sigma^* | \ S \overset{*}{\Rightarrow} w\}$.

Definition 3.4 *A context-free language is a language generated by a context-free grammar.*

Example $S \rightarrow (S)|SS|\epsilon$

Example $\{0^n1^n | \ n \geq 0\}$: $S \rightarrow \epsilon|0S1$

Example $L = \{w|w \in \{0,1\}^*\}$ and the number of 0's equals the number of 1's.

$S \rightarrow 0A|1B|\epsilon$
$A \rightarrow 1S|0AA$
$B \rightarrow 0S|1BB$

$S \rightarrow SAB|\epsilon$
$A \rightarrow 0S1|\epsilon$
$B \rightarrow 1S0|\epsilon$

Definition 3.5 *If a grammar generates the same string in several different ways, we say that the string is derived ambiguously in that grammar. If a grammar generates some string ambiguously, we say that the grammar is ambiguous.*

Example $S \to S + S | S \times S | a$

Theorem 3.6 *Every regular language is context-free.*

Proof: Given a DFA for the regular language, we can construct a context-free grammar that generates the same language through the following steps:

- make a variable for each state;
- make the variable for the starting state the starting variable;
- make a rule for each edge;
- Add an epsilon rule for each accept state.

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Definition 3.7 *A context-free grammar is in Chomsky normal form if every rule is of the form*

$$A \to BC$$
$$A \to a$$

where a is any terminal and A, B, and C are any variables, except that B and C may not be the start variable. In addition, we permit the rule $S \to \epsilon$, where S is the start variable.

Theorem 3.8 *Any context-free language is generated by a context-free grammar in Chomsky normal form.*

Proof: We prove this theorem by constructing an algorithm that make the transition in the following steps:

1. Add a new start variable S_0 and the rule $S_0 \to S$, where S was the original start variable.
2. For each of the ϵ rules $A \to \epsilon$, where A is not the start variable, we remove the rule. Then for each occurence of an A on the right-hand side of a rule, add a new rule with that occurrence deleted.
3. For every unit rule $A \to B$, we first remove it, and then whenever a rule $B \to u$ appears, we add the rule $A \to$ unless this was a unit rule previously removed.
4. Finally, replace each rule $A \to u_1 u_2 \cdots u_k$ with the rules $A \to u_1 A_1$, $A_1 \to u_2 A_2$, until $A_{k-2} \to u_{k-1} u_k$. Then replace any terminal u_i in the above rules with the new variable U_i and add the rule $U_i \to u_i$

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Example
$S \to ASA | aB$
$A \to B | S$
$B \to b | \epsilon$

Theorem 3.9 *Context-free languages are closed under union operation.*

Proof: Let L_1 and L_2 be two context-free languages. We can construct a context-free grammar for the union of the two by adding a grammar rule: $S_0 \to S_1|S_2$, where S_1 and S_2 are the start symbol for L_1 and L_2 respectively. $\blacksquare$

Theorem 3.10 *Context-free languages are closed under concatenation operation.*

Proof: Let L_1 and L_2 be two context-free languages. We can construct a context-free grammar for the union of the two by adding a grammar rule: $S_0 \to S_1S_2$, where S_1 and S_2 are the start symbol for L_1 and L_2 respectively. $\blacksquare$

Example A context-sensitive grammar: $L = \{1^n2^n3^n|\ n \geq 1\}$
$S \to 1SBC$
$S \to \epsilon$
$CB \to HB$
$HB \to HC$
$HC \to BC$
$1B \to 12$
$2B \to 22$
$2C \to 23$
$3C \to 33$

Note There could be more than one different grammars for the same language.
$S \to aS|aaS|b$

Theorem 3.11 *The class of regular languages is closed under the star operation.*

Proof: Let $N_1 = \{Q_1, \Sigma, \delta_1, q_1, F_1\}$ recognize A_1, construct $N = \{Q, \Sigma, \delta, q_0, F\}$ to recognize A_1^* as follows:

1. $q = q_0$, a new start state
2. $Q = \{q_0\} \cup Q_1$
3. $F = \{q_0\} \cup F_1$
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\epsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a), & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a), & q \in F_1 \text{ and } a \neq \epsilon \\ \delta_1(q,a) \cup \{q_1\}, & q \in F_1 \text{ and } a = \epsilon \\ \{q_1\}, & q = q_0 \text{ and } a = \epsilon \\ \phi, & q = q_0 \text{ and } a \neq \epsilon \end{cases}$$

$\blacksquare$

3.3 Pushdown Automata

Definition 3.12 *A pushdown automaton is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, F are all finite sets:*

1. *Q is the set of states,*
2. *Σ is the input alphabet,*
3. *Γ is the stack alphabet,*
4. *$\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \wp(Q \times \Gamma_\epsilon)$is the transition function,*
5. *$q_0 \in Q$ is the start state,*
6. *$F \subset Q$ is the set of accept states.*

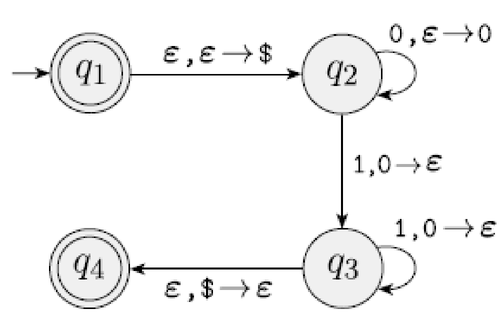
Definition 3.13 *A pushdown automaton $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts input w if w can be written as $w = w_1 w_2 \cdots w_m$, where each $w_i \in \Sigma_\epsilon$ and sequences of states $r_0, r_1, \ldots, r_m \in Q$ and strings $s_0, s_1, \ldots, s_m \in \Gamma^*$ exist that satisfy the following three conditions:*

1. *$r_0 = q_0$ and $s_0 = \epsilon$.*
2. *For $i = 0, \ldots, m-1$, we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Sigma_\epsilon$ and t in Γ^*.*
3. *$r_m \in F$.*

Example Let M be $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where
$Q = q_1, q_2, q_3, q_4$
$\Sigma = \{0, 1\}$
$\Gamma = \{0, \$\}$
$F = \{q_1, q_4\}$
δ is given by the following table:

Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$					
q_3				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4									

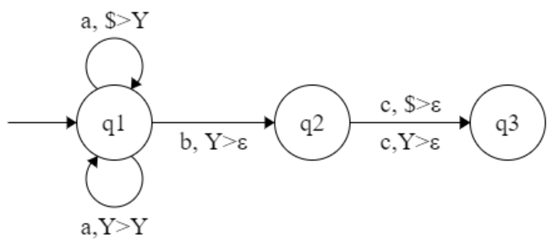
The machine M can also be described by the following state diagram:



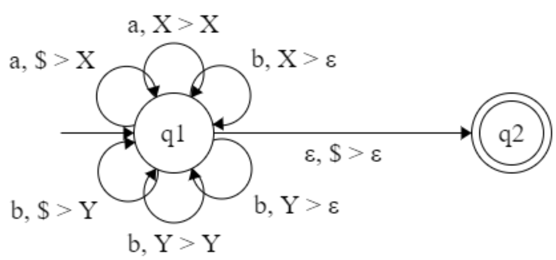
M is a pushdown automaton that recognizes language $\{0^n1^n | \ n \geq 0\}$

Note The stack of a pushdown automata can be non-empty when the computation finishes.

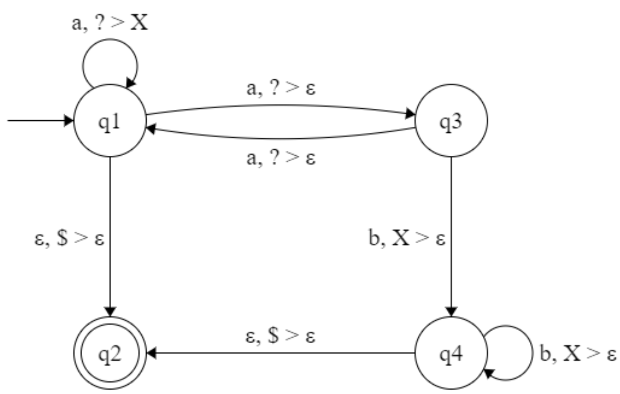
Example aa^*bc



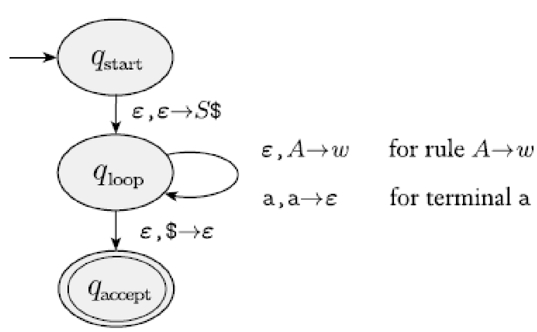
Example All strings over $\{a, b\}$ with the same number of a's and b's.



Example $S \rightarrow \epsilon | aSb | aaS$



Theorem 3.14 *If a language is context free, then some pushdown automaton recognizes it.*



Example Construct a pushdown automaton from the following grammar:

$$S \to aTb|b$$
$$T \to Ta|\epsilon$$