Computability Theory	**Fall 2020**

Lecture 9: Recursive Enumerable Sets(Unfinished)

Lecturer: Renjie Yang

9.1 Semicomputable Predicates

Definition 9.1 *A predicate $P(x^{(n)})$ is called* ***semicomputable*** *if there exists a partially computable function whose domain is the set $\{x^{(n)}|\ P(x^{(n)})\}$*

Theorem 9.2 *Every computable predicate is semicomputable.*

Proof: Let $R(x^{(n)})$ be computable. Then, $\{x^{(n)}|\ R(x^{(n)})\}$ is the domain of the partial computable function $min_y[C_R(x^{(n)}) + y = 0]$ ■

Theorem 9.3 *Let $R(x^{(n)}) \leftrightarrow \exists y P(y, x^{(n)})$, where $P(y, x^{(n)})$ is computable. Then $R(y, x^{(n)})$ is semicomputable.*

Proof: $\{x^{(n)}|\ R(x^{(n)})\}$ is the domain of the partial computable function $min_y[C_P(x^{(n)}) = 0]$ ■

Theorem 9.4 *Let $R(x^{(n)})$ be a semicomputable predicate. Then there exists a natural number z_0 such that $R(x^{(n)}) \leftrightarrow \exists y T_n(z_0, x^{(n)}, y)$.*

Proof: $R(x^{(n)})$ be a semicomputable predicate, then by definition $P(x^{(n)}|R(x^{(n)}))$ is the domain of a partially computable function $f(x^{(n)})$. By Kleene's normal form theorem, there is a number z_0 such that

$$f(x^{(n)}) = U(min_y Tn(z_0, x^n, y)).$$

Therefore

$$P(x^{(n)}|\ R(x^{(n)})) = \{x^{(n)}|\ \exists y Tn(z_0, x^{(n)}, y)\},$$

which means $R(x^{(n)}) \leftrightarrow Tn(z_0, x^{(n)}, y)$. ■

Corollary 9.5 *Let $R(x^{(n)})$ be a semicomputable predicate. Then there exists a computable predicate $P(y, x^{(n)})$ such that $R(x^{(n)}) \leftrightarrow \exists y P(y, x^{(n)})$.*

Theorem 9.6 *$R(x^{(n)})$ is computable if and only if both $R(x^{(n)})$ and $\neg R(x^{(n)})$ are semi-computable.*

Proof: For the forward direction, if $R(x^{(n)})$ is computable, then by the example following Corollary 6.23, $\neg R(x^{(n)})$ is also computable. Therefore both of them are semi-computable by theorem 9.2. ■

Theorem 9.7 *The predicate $\exists y T(x, x, y)$ is semicomputable, but is not computable.*

Proof: It suffices to show that the predicate $\neg\exists y T(x, x, y)$ is not semi-computable. Assume it was, then by theorem 9.4, there is a natural number z_0 such that

$$\exists y T(x, x, y) \leftrightarrow \exists T(z_0, x, y)$$

Set $x = z_0$ yields a contradiction. Therefore $\neg\exists y T(x, x, y)$ is not semi-computable. ■

Theorem 9.8 *Let $P(y, x^{(n)})$ be semicomputable, and let*

$$Q(x^{(n)}) \leftrightarrow \exists y P(y, x^{(n)}).$$

Then $Q(x^{(n)})$ is also semicomputable.

Proof: $P(y, x^{(n)})$ is semicomputable, by theorem 9.4, it can be written as $\exists z R(z, y, x^{(n)})$, where R is computable. Then

$$Q(x^{(n)}) \leftrightarrow \exists y \exists z R(z, y, x^{(n)}) \leftrightarrow \exists t R(K(t), L(t), x^{(n)}).$$

Since R, K, L are all computable, $Q(x^{(n)})$ is semicomputable by theorem 9.3. ■

Theorem 9.9 *Let $P(y, x^{(n)})$ be semicomputable, and let*

$$Q(z, x^{(n)}) \leftrightarrow \forall y((0 \leqslant y \leqslant z) \rightarrow P(y, x^{(n)})).$$

Then $Q(z, x^{(n)})$ is also semicomputable.

Proof: Since $P(y, x^{(n)})$ is semicomputable, there is a computable R such that

$$P(y, x^{(n)}) \leftrightarrow \exists u R(u, y, x^n),$$

u might be different for distinct y. Thus we can write

$$Q(z, x^{(n)}) \leftrightarrow \forall y((0 \leqslant y \leqslant z) \rightarrow \exists u_y R(u_y, y, x^n)).$$

Let w be the Gödel coding of the sequence $(u_0, u_1, \ldots, u_{y-1})$, then $u_i = (i+1)Glw$. Hence

$$Q(z, x^{(n)}) \leftrightarrow \exists w \forall y((0 \leqslant y \leqslant z) \rightarrow R((y+1)Glw, y, x^n)).$$

By theorem 6.24, $\forall y((0 \leqslant y \leqslant z) \rightarrow R((y+1)Glw, y, x^n))$ is computable, therefore $Q(z, x^{(n)})$ is semicomputable. ■

Theorem 9.10 *If $P(x^{(n)})$, $Q(x^{(n)})$ are semicomputable, then $P(x^{(n)}) \wedge Q(x^{(n)})$, $P(x^{(n)}) \vee Q(x^{(n)})$ are also semicomputable.*

Proof: Let $P(x^{(n)}) \leftrightarrow \exists y R(y, x^{(n)})$, $Q(x^{(n)}) \leftrightarrow \exists z S(z, x^{(n)})$, where R and S are computable. Then

$$P(x^{(n)}) \wedge Q(x^{(n)}) \leftrightarrow \exists y \exists z (R(y, x^{(n)}) \wedge S(z, x^{(n)})),$$

$$P(x^{(n)}) \vee Q(x^{(n)}) \leftrightarrow \exists y \exists z (R(y, x^{(n)}) \vee S(z, x^{(n)})),$$

which are semicomputable by theorem 9.8. ■

9.2 Recursive Enumerable Sets

Lemma 9.11 *Let $\{t | P(t)\}$ be the range of a partially computable function $f(x)$. Then $P(t)$ is semicomputable.*

Lemma 9.12 *Let $P(x)$ be semicomputable, and let $\{x | P(x) \neq \phi\}$. Then there exists an primitive recursive function whose range is $\{x | P(x)\}$.*

Theorem 9.13 *Let $S = \{x | P(x)\}$, and let $S \neq \phi$. Then the following statements are all equivalent:*

1. *$P(x)$ is semicomputable.*
2. *S is the range of a primitive recursive function.*
3. *S is the range of a recursive function.*
4. *S is the range of a partial recursive function.*

See Davis 5.4 for the proof.

Definition 9.14 *A set S is called recursively enumerable either if $S = \phi$ or if the equivalent conditions 1 - 4 of theorem 9.13 hold.*