Computability Theory **Fall 2020**

Lecture 8: Theorems on Computability

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Definition 8.1 *Write $f(x) \simeq g(x)$ if and only if either $f(x)$ and $g(x)$ are both undefined, or they are both defined and equal.*

Theorem 8.2 *(Kleene Normal Form) There is a primitive recursive relation $T(n, x, s)$ and a primitive recursive function U such that for each recursive function f there is a number n, such that*

$$f(x) \simeq U(min_y T_n(n, x, s))$$

Theorem 8.3 *(Enumeration) There is a universal partial computable function $\varphi(k, x)$ such that*

1. *$\varphi(k, x)$ is partial computable.*
2. *If $f(x)$ is any partial computable function, then there is a natural number k such that $f(x) \simeq \varphi(k, x)$ for every x.*

Definition 8.4 *Let φ_n denotes the nth unary partial computable function $\varphi(n, x)$. Use φ_n^k to denote the nth k-ary partial recursive function.*

Theorem 8.5 *(Universal Computable Function) There is a partial computable function $f(x, y)$ such that for each n and k and a sequence of numbers $a_0, a_1, \dots, a_{k-1}$, we have*

$$f(n, \langle a_0, a_1, \dots, a_{k-1} \rangle) \simeq \varphi_n^k(a_0, a_1, \dots, a_{k-1})$$

Theorem 8.6 *For each pair of natural numbers n and m, there is a primitive recursive function s_n^m such that for every sequence $x, a_0, \dots, a_{m-1}, y_0, \dots, y_{n-1}$, we have*

$$\varphi^n_{s_n^m(x, a_0, \dots, a_{m-1})}(y_0, \dots, y_{n-1}) \simeq \varphi_x^{m+n}(a_0, \dots, a_{m-1}, y_0, \dots, y_{n-1})$$

The Church-Turing Thesis

The Church-Turing thesis concerns the concept of an effective or systematic or mechanical method in logic, mathematics and computer science.

"Effective" and its synonyms "systematic" and "mechanical" are terms of art in these disciplines: they do not carry their everyday meaning. A method, or procedure, M, for achieving some desired result is called "effective" (or "systematic" or "mechanical") just in case:

1. M is set out in terms of a finite number of exact instructions (each instruction being expressed by means of a finite number of symbols);
2. M will, if carried out without error, produce the desired result in a finite number of steps;
3. M can (in practice or in principle) be carried out by a human being unaided by any machinery except paper and pencil;

4. M demands no insight, intuition, or ingenuity, on the part of the human being carrying out the method.

Turing's Thesis: L.C.M.s [logical computing machines: Turing's expression for Turing machines] can do anything that could be described as "rule of thumb" or "purely mechanical".

Church's Thesis: A function of positive integers is effectively calculable only if lambda-definable (or, equivalently, recursive).

Further reading: The SEP Entry: "The Church-Turing Thesis".