Lecture 8: Theorems on Computability

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8.1 Theorems on Computability

Definition 8.1 *We write $f(x) \simeq g(x)$ to mean that either $f(x)$ and $g(x)$ are both undefined, or they are both defined and equal.*

Theorem 8.2 *(Kleene's Normal Form Theorem) For any natural number $n > 0$, there are a primitive computable relation $T(n, x^{(n)}, y)$ and a primitive recursive function U such that for each partial computable (partial recursive) function $f(x^{(n)})$ there is a number z_0, such that*

$$f(x^{(n)}) \simeq U(min_y T_n(z_0, x^{(n)}, y))$$

Proof: This is Corollary 7.11 written in a slightly more general form. ■

Theorem 8.3 *(Effective Enumeration) For any natural number $n > 0$, there is a universal partial computable function $Un(k, x^{(n)})$ such that*

1. *$Un(k, x^{(n)})$ is partial computable.*
2. *If $f(x^{(n)})$ is any partial computable function, then there is a natural number k such that $f(x^{(n)}) \simeq Un(k, x^{(n)})$ for every sequence $x^{(n)} = (x_0, x_1, \dots, x_{n-1})$.*

Proof: Let $Un(k, x^{(n)}) \simeq U(min_y T_n(k, x^{(n)}, y))$ in Theorem 8.2. ■

Definition 8.4 *Let φ_i denotes the ith unary partial computable function $Un(i, x)$. Use φ_i^n to denote the ith n-ary partial recursive function $Un(i, x^{(n)})$.*

Theorem 8.5 *For any natural number $n > 0$, there is a partial computable function $f(i, x^{(n)})$ such that for each i and n and a sequence of numbers $a_0, a_1, \dots, a_{n-1}$, we have*

$$f(i, a_0, a_1, \dots, a_{n-1}) \simeq \varphi_i^n(a_0, a_1, \dots, a_{n-1})$$

Proof: Let $f(i, x^{(n)}) = Un(i, x^{(n)})$, the conclusion holds by the definition of φ_i^n. ■

Note While Theorem 8.5 defines a universal computable function for each k, it is also possible to define a binary function $g(i, x)$ which treats its second argument as the Gödel code for a finite sequence $(x_0, \dots, x_{n-1})$, and then computes in the same manner as the n-ary universal function so that we have $g(i, gn(x_0, \dots, x_{n-1})) \simeq f(i, x_0, \dots, x_{n-1})$. This provides a means of replacing the prior enumerations of any arbitrary n-ary partial computable functions with a single enumeration $\phi_i(x)$ of unary functions, such that $\phi_i(gn(x_0, \dots, x_{n-1})) \simeq f(i, gn(x_0, \dots, x_{n-1})) \simeq \varphi_i^n(x_0, \dots, x_{n-1})$. Thus the following theorem:

Theorem 8.6 *(Universal Computable Function) There is a partial computable function $g(x, y)$ such that for each i and n and sequence of numbers $a_0, \dots, a_{n-1}$ we have*

$$g(i, gn(a_0, \dots, a_{n-1})) \simeq \varphi_i^n(a_0, \dots, a_{n-1})$$

Proof: Let $g(i,x) \simeq Un(i,x)$, and define $Un(i, gn(a_0,\ldots,a_{n-1})) \simeq \varphi_i^n(a_0,\ldots,a_{n-1})$. ■

Theorem 8.7 *There is a primitive recursive function $\gamma(r,a)$ such that, for $n \leqslant 1$,*

$$\varphi_r^{1+n}(a, x^{(n)}) \simeq \varphi_{\gamma(r,a)}^n(x^{(n)})$$

Proof: For each value of a, let W_a be the Turing machine consisting of the following quadruples:

$$q_1 1 L q_1,$$
$$q_1 B L q_2,$$

$$\left.\begin{array}{l} q_{i+1}B1q_{i+1} \\ q_{i+1}1Lq_{i+2} \end{array}\right\} 1 \leqslant i \leqslant a$$

$$q_{a+2}B1q_{a+3},$$

W_a transforms $q_1(\overline{x^{(n)}})$ into $q_{y+3}(\overline{y, x^{(n)}})$.

Let r be a Gödel number of a Turing machine Z, and let $Z_a = W_a \cup Z^{(a+2)}$. Since the quadruples of $Z^{(a+2)}$ have precisely the same effect on $q_{(a+3)}(\overline{a, x^{(n)}})$ that those of Z have on $q_1(\overline{a, x^{(n)}})$, we have

$$\Psi_{Z_a}^{(n)}(x^{(n)}) = \Psi_Z^{(n)}(a, x^{(n)}) = \varphi_r^{1+n}(a, x^{(n)})$$

We now proceed to evaluate one of the Gödel number of Z_a as a function of r and y. The Gödel numbers of the quadruples that make up W_a are as follows:

$$b = gn(q_1 1 L q_1) = 2^9 \times 3^{11} \times 5^5 \times 7^9$$
$$c = gn(q_1 B L q_1) = 2^9 \times 3^7 \times 5^5 \times 7^{13}$$
$$d(i) = gn(q_{i+1}B1q_{i+1}) = 2^{4i+9} \times 3^7 \times 5^1 1 \times 7^{4i+9}, 1 \leqslant i \leqslant a$$
$$e(i) = gn(q_{i+1}1Lq_{i+1}) = 2^{4i+9} \times 3^{11} \times 5^5 \times 7^{4i+13}, 1 \leqslant i \leqslant a$$
$$f(a) = gn(q_{a+2}B1q_{a+3}) = 2^{4y+13} \times 3^7 \times 5^{11} \times 7^{4y+17}$$

Let

$$\varphi(a) = 2^b \times 3^c \times 5^{f(a)} \times \prod_{i=1}^{a}[Pr(i+3)^{d(i)}Pr(i+a+3)^{e(i)}],$$

then $\varphi(a)$ is a primitive recursive function, and for each a, $\phi(a)$ is a Gödel of W_a.

By theorem 7.9(6), if h is the Gödel number of a quadruple, then the Gödel number of the quadruple obtained from this one by replacing each q_i by q_{i+a+2} is

$$g(h,a) = 2^{1\,Gl\,h+4y+8} \times 3^{2\,Gl\,h} \times 5^{3\,Gl\,h} \times 7^{4\,Gl\,h+4y+8},$$

which is primitive recursive. Let

$$\theta(r,a) = \prod_{i=1}^{\mathcal{L}(r)} Pr(i)^{g(i\,Gl\,r,\,a)},$$

then $\theta(r,a)$ is a primitive recursive function and, for each a, $\theta(r,a)$ is a Gödel number of $Z^{(a+2)}$.

Let $\tau(x) = 1$ if x is a Gödel number of a Turing machine; 0, otherwise. Then by theorem 7.9(11), $\tau(x)$ is primitive recursive. Let

$$\gamma(r,a) = (\varphi(y) * \theta(r,a))\tau(r),$$

then $\gamma(r,a)$ is a primitive recursive function and, for each a, $\gamma(r,a)$ is a Gödel number of Z_a. Therefore,

$$\varphi_{\gamma(r,a)}^n(x^{(n)}) = \Psi_{Z_a}^{(n)}(x^{(n)}) = \Psi_Z^{(n)}(a, x^{(n)}) = \varphi_r^{1+n}(a, x^{(n)}).$$

It remains only to consider the case where r is not a Gödel number of a Turing machine. In that case, $\gamma(r,a)$ is 0, and thus not the Gödel number of a Turing machine. Therefore,

$$\varphi^n_{\gamma(r,a)}(x^{(n)}) \simeq \Psi^{(n)}_{Z_a}(x^{(n)}) \simeq \Psi^{(n)}_Z(a, x^{(n)}) \simeq \varphi^{1+n}_r(a, x^{(n)}).$$

■

Theorem 8.8 *(s-m-n) For each pair of natural numbers* n *and* m*, there is a primitive recursive function* s^m_n *such that for every sequence* $r, a_0, \ldots, a_{m-1}, x_0, \ldots, x_{n-1}$*, we have*

$$\varphi^n_{s^m_n(r,a_0,\ldots,a_{m-1})}(x_0, \ldots, x_{n-1}) \simeq \varphi^{m+n}_r(a_0, \ldots, a_{m-1}, x_0, \ldots, x_{n-1})$$

Proof: For $m = 0$, the result holds with $s^0_n(r) = r$. Suppose it is known for $m = k$. Then there is a primitive recursive function $s^k_n(r, a^{(k)})$ with the required property. Then, by theorem 8.7,

$$\varphi^{1+k+n}_r(a, a^{(k)}, x^{(n)}) \simeq \varphi^{k+n}_{\gamma(r,y)}(a^{(k)}, x^{(n)}) \simeq \varphi^n_{s^k_n(\gamma(r,a),a^{(k)})}(x^{(n)}).$$

Set $s^{k+1}_n(r, a, a^{(k)}) = s^k_n(\gamma(r, y), a^{(k)})$,

$$\varphi^{1+k+n}_r(a, a^{(k)}, x^{(n)}) \simeq \varphi^n_{s^{k+1}_n(r,a,a^{(k)})}(x^{(n)}),$$

therefore

$$\varphi^{m+n}_r(a^{(m)}, x^{(n)}) \simeq \varphi^n_{s^m_n(r,a^{(m)})}(x^{(n)})$$

for any natural number m. ■

8.2 The Church-Turing Thesis

The Church-Turing thesis concerns the concept of an effective or systematic or mechanical method in logic, mathematics and computer science.

"Effective" and its synonyms "systematic" and "mechanical" are terms of art in these disciplines: they do not carry their everyday meaning. A method, or procedure, M, for achieving some desired result is called "effective" (or "systematic" or "mechanical") just in case:

1. M is set out in terms of a finite number of exact instructions (each instruction being expressed by means of a finite number of symbols);
2. M will, if carried out without error, produce the desired result in a finite number of steps;
3. M can (in practice or in principle) be carried out by a human being unaided by any machinery except paper and pencil;
4. M demands no insight, intuition, or ingenuity, on the part of the human being carrying out the method.

Turing's Thesis: L.C.M.s [logical computing machines: Turing's expression for Turing machines] can do anything that could be described as "rule of thumb" or "purely mechanical".

Church's Thesis: A function of positive integers is effectively calculable only if lambda-definable (or, equivalently, recursive).

Further reading: The SEP Entry: "The Church-Turing Thesis".