

## Lecture 7: The Equivalence of Computability and Recursiveness

*Lecturer: Renjie Yang***7.1 The Arithmetization of The Theory of Turing Machines**

**Definition 7.1** Let  $M$  be an expression consisting of the symbols  $\gamma_1, \gamma_2, \dots, \gamma_n$ . Let  $a_1, a_2, \dots, a_n$  be the corresponding integers associated with these symbols. Then the **Gödel number** of  $M$  is the integer

$$gn(M) = r = \prod_{k=1}^n Pr(k)^{a_k}$$

If  $M$  is the empty expression, we let 1 be the Gödel number of  $M$ .

**Note** We adopt a convention to associate each symbol with an even number:  $R \rightarrow 3, S \rightarrow 5, q_i \rightarrow 4i + 5, S_i \rightarrow 4i + 7$ .

**Example**  $gn(q_11Rq_2) = 2^9 \cdot 3^{11} \cdot 5^3 \cdot 7^{13}$

**Corollary 7.2** If  $M$  and  $N$  are given expressions such that  $gn(M) = gn(N)$ , then  $M = N$ .

**Proof:** According to the Fundamental Theorem of Arithmetic, every natural number can be uniquely represented in the form  $p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ , where  $p_1, \dots, p_k$  are distinct primes. Therefore if  $M = N$  then their prime factorizations are the same. ■

**Note** A computation is a finite sequence of expressions; a Turing machine is a finite set of expressions.

**Definition 7.3** If  $n = gn(M)$ , we also write  $M = Exp(n)$

**Definition 7.4** Let  $M_1, \dots, M_n$  be a finite sequence of expressions. Then the Gödel number of this sequence of expressions is the number

$$\prod_{k=1}^n Pr(k)^{gn(M_k)}$$

**Example**  $gn(\{q_11Bq_1, q_1BRq_2\}) = 2^{2^9 \cdot 3^{11} \cdot 5^7 \cdot 7^9} \cdot 3^{2^9 \cdot 3^7 \cdot 5^3 \cdot 7^{13}}$

**Corollary 7.5** No integer is the Gödel number both of an expression and of a sequence of expressions.

**Proof:** A Gödel number of an expression or a sequence of expressions is of the form  $2^n \cdot m$ .  $n$  is odd for expressions, and even for sequences of expressions. ■

**Corollary 7.6** Two sequences of expressions that have the same Gödel number are identical.

**Definition 7.7** Let  $Z$  be a Turing machine. Let  $M_1, \dots, M_n$  be any arrangement of the quadruples of  $Z$  without repetitions. Then, the Gödel number of the sequence  $M_1, \dots, M_n$  is called a **Gödel number of the Turing machine  $Z$** .

**Note** A Turing machine consisting of  $n$  quadruples has  $n!$  distinct Gödel numbers.

**Definition 7.8** For each  $n > 0$  and for each set of integers  $A$ , let  $T_n(z, x_1, \dots, x_n, y)$  be the predicate that means, for given  $z, x_1, \dots, x_n, y$  that  $z$  is a Gödel number of a Turing machine  $Z$ , and that  $y$  is the Gödel number of a computation, with respect to  $Z$ , beginning with the instantaneous description  $q_1(x_1, \dots, x_n)$ .

**Theorem 7.9**  $T_n(z, x_1, \dots, x_n, y)$  is primitive recursive.

**Proof:** The proof proceeds by a detailed list of primitive recursive functions and predicates until we get  $T_n(z, x_1, \dots, x_n, y)$ .

**Group 1** Functions and predicates which concern Gödel numbers of expressions and sequences of expressions:

$$1. n \text{ Gl } x = \mathfrak{M}_{y=0}^x [(Pr(n)^y | x)] \wedge \neg (Pr(n)^{y+1} | x)]$$

The Gödel encoding of the  $n$ th symbol in the expression represented by  $x$ .

The Gödel encoding of the  $n$ th expression in the sequence of expression represented by  $x$ .

$$2. \mathcal{L}(x) = \mathfrak{M}_{y=0}^x [(y \text{ Gl } x > 0) \wedge \forall i ((0 \leq i \leq x) \rightarrow (y + 1 + i) \text{ Gl } x = 0)]$$

The length of the expression represented by  $x$ .

The length of the sequence of expression represented by  $x$ .

$$3. GN(x) \leftrightarrow \exists (1 \leq y \leq \mathcal{L}(x)) [(y \text{ Gl } x = 0) \wedge ((y + 1) \text{ Gl } x \neq 0)]$$

$x$  is a Gödel Numbers of some expression.

$$4. Term(x, z) \leftrightarrow GN(z) \wedge \exists (1 \leq n \leq \mathcal{L}(x)) x = n \text{ Gl } z$$

$z$  is a Gödel Numbers of some expression, and  $x$  is the Gödel Numbers of one of the symbols in the expression represented by  $z$ .

$$5. x * y = x \cdot \prod_{i=1}^{\mathcal{L}(x)} Pr(\mathcal{L}(x) + i)^{i \text{ GL } y}$$

If  $M$  and  $N$  are expressions, then  $gn(MN) = gn(M) * gn(N)$ .

If  $x$  and  $y$  are the Gödel numbers of the sequences of expressions  $M_1, \dots, M_n$ , and the  $N_1, \dots, M_n, N_1, \dots, N_p$ .

**Group 2** Functions and Predicates which concern the basic structure of Turing machines:

$$6. IC(x) \leftrightarrow \exists (0 \leq y \leq x) (x = 4y + 9)$$

$x$  is the number assigned to an internal configuration  $q_i$

$$7. Al(x) \leftrightarrow \exists (0 \leq y \leq x) (x = 4y + 7)$$

$x$  is the number assigned to an alphabet  $S_i$

$$8. Odd(x) \leftrightarrow \exists (0 \leq y \leq x) (x = 2y + 3)$$

$x$  is an odd number  $\geq 3$ .

9.  $Quad(x) \leftrightarrow GN(x) \wedge \mathcal{L}(x) \wedge IC(1 \text{ Gl } x) \wedge Al(2 \text{ Gl } x) \wedge Odd(3 \text{ Gl } x) \wedge IC(4 \text{ Gl } x)$   
The expression represented by  $x$  is a quadruple.
10.  $Inc(x, y) \leftrightarrow Quad(x) \wedge Quad(y) \wedge (1 \text{ Gl } x = 1 \text{ Gl } y) \wedge (2 \text{ Gl } x = 2 \text{ Gl } y) \wedge (x \neq y)$   
 $x$  and  $y$  are Gödel numbers of two incompatible quadruples beginning with the same two symbols.
11.  $TM(x) \leftrightarrow GN(x) \wedge \forall(1 \leq n \leq \mathcal{L}(x))[Quad(n \text{ Gl } x) \wedge \forall(1 \leq m \leq \mathcal{L}(x)) \neg Inc(n \text{ Gl } x, m \text{ Gl } x)]$   
 $x$  is a Gödel number of a Turing machine.
12.  $MR(0) = 2^{11}$ ,  
 $MR(n+1) = 2^{11} * MR(n)$   
 $MR(n)$  is the Gödel number of  $\bar{n}$ .
13.  $CU(n, x) = 0$  if  $n \text{ Gl } x \neq 11$ ,  
 $CU(n, x) = 1$  if  $n \text{ Gl } x = 11$ .  
 $CU(n, x)$  is the characteristic function of the predicate  $n \text{ Gl } x \neq 11$ , namely the  $n$ th symbol in the expression represented by  $x$  is not  $S_1$ .
14.  $Corn(x) = \sum_{n=1}^{\mathcal{L}(x)} CU(n, x)$   
If  $x$  is the Gödel number of  $M$ , then  $Corn(x) = \langle M \rangle$ .
15.  $U(y) = Corn(\mathcal{L}(y) \text{ Gl } y)$   
If  $y$  is the Gödel number of a sequence of expression  $M_1, M_2, \dots, M_n$ , then  $U(y) = \langle M_n \rangle$ .
16.  $ID(x) \leftrightarrow GN(x) \wedge \exists(1 \leq n \leq \mathcal{L}(x) \div 1)[IC(n \text{ Gl } x) \wedge \forall(1 \leq m \leq \mathcal{L}(x))(m = n \vee Al(m \text{ Gl } x))]$   
 $x$  is a Gödel number of an instantaneous description.
17.  $Init_n(x_1, \dots, x_n) = 2^9 * MR(x_1) * 2^7 * MR(x_2) * 2^7 \dots * 2^7 * MR(x_n)$ .  
 $Init_n(x_1, \dots, x_n) = gn(q_1(\overline{x_1, \dots, x_n}))$ .

**Group 3** Functions and Predicates which concern the computations of Turing machines:

18.  $Yield_1(x, y, z) \leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq F, G, r, s \leq x, 0 \leq t, u \leq y)$   

$$[(x = F * 2^r * 2^s * G) \wedge (y = F * 2^t * 2^u * G) \wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Al(u) \wedge Term(2^r \cdot 3^s \cdot 5^u \cdot 7^t, z)].$$
 $x$  and  $y$  are the Gödel numbers of instantaneous descriptions,  $z$  is a Gödel number of a Turing machine  $Z$ , and  $Exp(x) \rightarrow Exp(y)(Z)$ , under the first rule of  $\alpha \rightarrow \beta(Z)$ :  
(1) There exist expressions  $P$  and  $Q$  (possibly empty) such that
$$\begin{aligned} \alpha &\text{ is } Pq_i S_j Q, \\ \beta &\text{ is } Pq_l S_k Q, \end{aligned}$$
where
$$Z \text{ contains } q_i S_j S_k q_l.$$
19.  $Yield_2(x, y, z) \leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq F, G, r, s \leq x, 0 \leq t, u \leq y)$   

$$[(x = F * 2^r * 2^s * 2^t * G) \wedge (y = F * 2^s * 2^u * 2^t * G) \wedge IC(r) \wedge IC(u) \wedge Al(s) \wedge Al(t) \wedge Term(2^r \cdot 3^s \cdot 5^3 \cdot 7^u, z)].$$
 $x$  and  $y$  are the Gödel numbers of instantaneous descriptions,  $z$  is a Gödel number of a Turing machine  $Z$ , and  $Exp(x) \rightarrow Exp(y)(Z)$ , under the second rule of  $\alpha \rightarrow \beta(Z)$ :  
(2) There exist expressions  $P$  and  $Q$  (possibly empty) such that

$$\begin{aligned}\alpha &\text{ is } Pq_iS_jS_kQ, \\ \beta &\text{ is } PS_jq_lS_kQ,\end{aligned}$$

where

$$Z \text{ contains } q_iS_jRq_l.$$

$$\begin{aligned}20. \text{ Yield}_3(x, y, z) &\leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq F, r, s \leq x, 0 \leq t \leq y) \\ &\quad [(x = F * 2^r * 2^s) \wedge (y = F * 2^s * 2^t * 2^7) \\ &\quad \wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Term(2^r \cdot 3^s \cdot 5^3 \cdot 7^t, z)].\end{aligned}$$

$x$  and  $y$  are the Gödel numbers of instantaneous descriptions,  $z$  is a Gödel number of a Turing machine  $Z$ , and  $Exp(x) \rightarrow Exp(y)(Z)$ , under the third rule of  $\alpha \rightarrow \beta(Z)$ :

(3) There exist expressions  $P$  and  $Q$  (possibly empty) such that

$$\begin{aligned}\alpha &\text{ is } Pq_iS_j, \\ \beta &\text{ is } PS_jq_lS_0,\end{aligned}$$

where

$$Z \text{ contains } q_iS_jRq_l.$$

$$\begin{aligned}21. \text{ Yield}_4(x, y, z) &\leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq F, G, r, s, t \leq x, 0 \leq u \leq y) \\ &\quad [(x = F * 2^r * 2^s * 2^t * G) \wedge (y = F * 2^u * 2^r * 2^t * G) \\ &\quad \wedge IC(s) \wedge IC(u) \wedge Al(r) \wedge Al(t) \wedge Term(2^s \cdot 3^t \cdot 5^5 \cdot 7^u, z)].\end{aligned}$$

$x$  and  $y$  are the Gödel numbers of instantaneous descriptions,  $z$  is a Gödel number of a Turing machine  $Z$ , and  $Exp(x) \rightarrow Exp(y)(Z)$ , under the fourth rule of  $\alpha \rightarrow \beta(Z)$ :

(4) There exist expressions  $P$  and  $Q$  (possibly empty) such that

$$\begin{aligned}\alpha &\text{ is } PS_kq_iS_jQ, \\ \beta &\text{ is } Pq_lS_kS_jQ,\end{aligned}$$

where

$$Z \text{ contains } q_iS_jLq_l.$$

$$\begin{aligned}22. \text{ Yield}_5(x, y, z) &\leftrightarrow ID(x) \wedge ID(y) \wedge TM(z) \wedge \exists(0 \leq G, r, s \leq x, 0 \leq t \leq y) \\ &\quad [(x = 2^r * 2^s * G) \wedge (y = 2^t * 2^7 * 2^s * G) \\ &\quad \wedge IC(r) \wedge IC(t) \wedge Al(s) \wedge Term(2^r \cdot 3^s \cdot 5^5 \cdot 7^t, z)].\end{aligned}$$

$x$  and  $y$  are the Gödel numbers of instantaneous descriptions,  $z$  is a Gödel number of a Turing machine  $Z$ , and  $Exp(x) \rightarrow Exp(y)(Z)$ , under the fifth rule of  $\alpha \rightarrow \beta(Z)$ :

(5) There exist expressions  $P$  and  $Q$  (possibly empty) such that

$$\begin{aligned}\alpha &\text{ is } q_iS_jQ, \\ \beta &\text{ is } q_lS_0S_jQ,\end{aligned}$$

where

$$Z \text{ contains } q_iS_jLq_l.$$

$$23. \text{ Yield}(x, y, z) \leftrightarrow \text{Yield}_1(x, y, z) \vee \text{Yield}_2(x, y, z) \vee \text{Yield}_3(x, y, z) \vee \text{Yield}_4(x, y, z) \vee \text{Yield}_5(x, y, z)$$

$x$  and  $y$  are the Gödel numbers of instantaneous descriptions,  $z$  is a Gödel number of a Turing machine  $Z$ , and  $Exp(x) \rightarrow Exp(y)(Z)$ .

$$\begin{aligned}24. \text{ Fin}(x, z) &\leftrightarrow ID(x) \wedge TM(z) \wedge \exists(0 \leq F, G, r, s \leq x) \\ &\quad [(x = F * 2^r * 2^s * G) \wedge IC(r) \wedge Al(s) \\ &\quad \wedge \forall(1 \leq n \leq \mathcal{L}(z))((1 \text{ Gl } (n \text{ Gl } z) \neq r) \vee (2 \text{ Gl } (n \text{ Gl } z) \neq s))]\end{aligned}$$

$z$  is a Gödel number of a Turing machine  $Z$  and  $x$  is the Gödel number of an instantaneous description final with respect to  $Z$ , namely there is no yield rule in  $Z$  to further compute from  $x$ .

$$25. \text{Comp}(y, z) \leftrightarrow TM(z) \wedge GN(y) \wedge \forall(1 \leq n \leq \mathcal{L}(y))[\text{Yield}(n \text{ Gl } , (n+1) \text{ Gl } y, z)] \\ \wedge \text{Fin}(\mathcal{L}(y) \text{ Gl } y, z)$$

$z$  is a Gödel number of a Turing machine  $Z$  and  $y$  is the Gödel number of a computation of  $Z$ .

By definition 7.8,  $T_n(z, x_1, \dots, x_n, y) \leftrightarrow (1 \text{ Gl } y = \text{Init}_n(x_1, \dots, x_n)) \wedge \text{Comp}(y, z)$ . Therefore it is primitive recursive according to the above construction. ■

## 7.2 Computability and Recursiveness

**Theorem 7.10** *Let  $Z_0$  be a Turing machine and let  $z_0$  be a Gödel number of  $Z_0$ . Then, the domain of the function  $\Psi_{Z_0}^{(n)}(x^{(n)})$  is equal to the domain of  $\min_y T_n(z_0, x^{(n)}, y)$ . Moreover,*

$$\Psi_{Z_0}^{(n)}(x^{(n)}) = U(\min_y T_n(z_0, x^{(n)}, y)).$$

Also, if  $T_n(z_0, x^{(n)}, y)$  is true for given  $x^{(n)}$ , then

$$y_0 = \min_y T_n(z_0, x^{(n)}, y).$$

**Proof:** For any given  $x^{(n)}$ ,  $\min_y T_n(z_0, x^{(n)}, y)$  is defined if and only if there exists an computation of  $Z_0$  beginning with  $q_1(\overline{x^{(n)}})$ , that is, if and only if  $\Psi_{Z_0}^{(n)}(x^{(n)})$  is defined. So the first statement is true.

When  $y_0 = \min_y T_n(z_0, x^{(n)}, y)$  is defined,  $y_0$  is the Gödel number of a computation of  $Z_0$  beginning with  $q_1(\overline{x^{(n)}})$ , which means the second statement is true. Therefore,  $\mathcal{L}(y_0) \text{ Gl } y_0$  is the Gödel number of the final instantaneous description  $\alpha$  of this computation, and  $U(y_0) = \text{Corn}(\mathcal{L}(y_0) \text{ Gl } y_0) = \langle \alpha \rangle = \Psi_{Z_0}^{(n)}(x^{(n)})$ . So the final statement is true. ■

**Corollary 7.11**  *$f(x^{(n)})$  is partially computable if and only if there is a number  $z_0$  such that*

$$f(x^{(n)}) = U(\min_y T_n(z_0, x^{(n)}, y))$$

**Corollary 7.12** *Every (partially) computable function is (partially) recursive.*

**Corollary 7.13** *A function is (partially) computable if and only if it is (partial) recursive.*

**Corollary 7.14** *If a function is total and partial recursive, then it is recursive.*