The Idea of Proof: Part II

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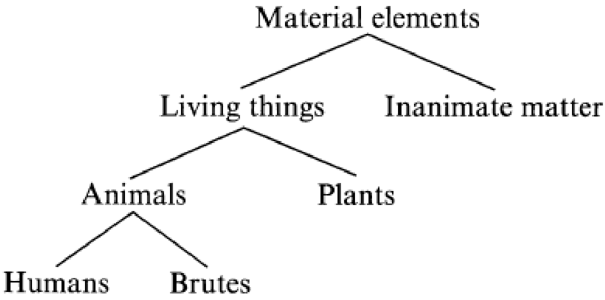
Outline

Aristotle's Conception of Scientific Explanation



The Method of Analysis and Synthesis

- According to both Plato and Aristotle, the objects of knowledge is that one thing or kind or property is a finite combination of other things or kinds or properties.
- Human is a combination of rational and animal; Triangle is a combination of closed, rectilinear, figure, and threesided.
- Any kind or property that can be the object of scientific knowledge can be analyzed into a combination of simple properties that cannot be further analyzed.
- Middle ages method for acquiring knowledge: "analyze" a thing into its simple properties (analysis), and then put it back together by combining those properties (synthesis).

The *Idea* Idea

- In Descartes' view, we do not analyze and synthesize things in themselves. We take apart our conceptions of things, our ideas of them.
- What we do in thought, then, is to try to find the simple ideas of which complex thoughts are compounded.
- An inquiry into the basis for knowledge must therefore be an inquiry into our psychology, into the operations of the mind.
- Such an inquiry should produce a method that could be shown to be perfectly reliable, and he claimed that he himself had found such a method.

Three Principles of Descartes' method

- What is clearly and distinctly conceived to be true cannot be false.
- The separation of thoughts of properties is a perfect indicator of the possible separation of the properties: properties that cannot be conceived of separately are necessarily coextensive, and properties that can be conceived of separately are not necessarily coextensive.
- A genuine recollection of a sequence of clear and distinct ideas cannot be false.

Descartes' Justification of the Three Principle

Descartes' argument is not (and should not be) taken seriously today:

1. Some thoughts, some clear and distinct ideas, are indubitable: "Cogito, ergo sum".
2. We can know with complete certainty that a benevolent God exists: 1. One cannot think of God without thinking that God exists, same as Saint Anselm's argument; 2. the cause of our idea of God must be at least as great as our idea itself.
3. Since God is perfect, he cannot be a deceiver. God created us, and God never lie, therefore whatever we clearly and distinctly perceive must be true.

Outline

1. The Problem of Deductive Inference
2. 2 Examples: Euclid's Geometry and Saint Anselm
3. Aristotle's Theory of Demonstration and Proof
4. Descartes: The *Idea* Idea
5. **Leibniz and the Mathematics of Reason**
6. Frege's New Logical World
7. Modern Logic

Leibniz and the Mathematics of Reason

Leibniz's viewpoint can be thought of as what you get if you do the following:

- Take the Platonic and Aristotelian view of the formal structure of what is known.
- Combine it with the method of analysis and synthesis.
- Abolish the distinction between properties that a thing has accidentally and properties that a thing has essentially and assume instead that every property something has, it has necessarily.

Leibniz and the Mathematics of Reason

- Leibniz assumed that every proposition consists of a predicate applied to a subject, and that in this regard the structure of language reflects the structure of the world.
- Substances don't have attributes. A substance just is a combination of attributes.
- The propositions that we assert and believe and perhaps know are about concepts: "All perfect people are happy".
- Every concept is just a list or combination of primitive concepts. All true propositions are true because the list of primitive concepts of the subject term is appropriately related to the list of primitive concepts of the predicate term.

Leibniz's Conception of Proof

Every true proposition can be given a proof:

1. Producing the combinations of simple concepts denoted by the predicate of the proposition and the subject of the proposition.
2. Showing that the concept of the predicate is included in the concept of the subject.

Leibniz's Conception of Proof

- Once a universal dictionary has been assembled that expresses each concept in terms of the simplest concepts, Leibniz thought that the production of scientific knowledge would become mechanical.
- He thought an algorithm or mechanical procedure could be found to carry out the second part of the procedure for giving proofs.
- The way to formulate such a procedure is to treat the problem as a part of algebra.
- Each simple term and each complex term should be given a letter or other symbol (Leibniz sometimes suggested using numbers as symbols for concepts), and then one would use algebraic methods to search for algebraic identities.

Leibniz's "Logical Calculi"

1. "$A = B$" is the same as "'$A = B$' is a true proposition."
2. "$A \neq B$ " is the same as "'$A = B$' is a false proposition."
3. $A = AA$; i.e., the multiplication of a letter by itself is here without effect.
4. $AB = BA$, i.e., transposition makes no difference.
5. "$A = B$" means that one can be substituted for the other, B for A or A for B, i.e., that they are equivalent.
6. "Not" immediately repeated destroys itself.
7. Therefore $A =$ not-not-A.
8. Further, "$A = B$" and "$Anot \neq B$" are equivalent.

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Relations and Leibniz's Metaphysics

Again, do **not** take any of these failed ideas seriously.

- Leibniz could not give any account of reasoning with relations.
- The absence of a theory about how to reason with relations would be less bothersome if relations could not be the subject of knowledge.
- One way to avoid real relations, therefore, is to suppose that there is only one substance. That was Spinoza's solution.
- Another way to solve the problem is to suppose that there are lots and lots of substances, but none of them stand in any relations to one another, or at least not in any relations that are the subject of scientific knowledge. That was Leibniz's solution.

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Intellectual Background

- How do we know what is true?
- How do we know the truths of geometry and arithmetic and physics, for example?
- Aristotle and Descartes: we know the truths of these subjects because we know certain fundamental principles through intuition and then validly deduce other truths from these fundamental principles.

Intellectual Background

- The seventeenth century succeeded in throwing off most of Aristotle's conception of knowledge and inquiry by establishing new and impressive sciences through other methods.
- Newton's Principia; William Harvey (1578 – 1657); William Gilbert (1544 – 1603)
- If science depends on inferences made from observation and experiment, what is the basis for our knowledge of mathematics and for the special certainty that propositions in geometry and arithmetic seem to have?

Frege, Logicism, and Logic

- Gottlob Frege (1848 – 1925), Professor of Mathematics at the University of Jena in Germany
- Frege's efforts opened the way to much of modern mathematics, modern logic, modern approaches to topics in economics, and the modern theory of computation.
- Every time you use a computer you take advantage of a device that emerged in the twentieth century as a result of work prompted by Frege's efforts.
- Frege stands to logic roughly as Newton stands to physics.

Frege's Idea: Logicism

- Arithmetic and geometry are certain and can be known by reason alone because arithmetic and geometry are nothing but logic;
- logic is certain and can be known by reason alone.
- The notions of number, numerical order, sum, and product can all be defined in purely logical terms.
- With those definitions, the basic laws of arithmetic should turn out to be purely logical propositions that are necessarily true.

Frege's Idea: Logicism

How could one reduce mathematics to logic when the available logical theories did not even permit one to derive the consequences of mathematical axioms?

Invent a better logic!

Frege's Logic for Sentences

- Let A, B, C, stand for any sentences.
- $\neg A$, $\neg A$, $\neg A$ stand for the sentences "A is not the case," "B is not the case," "C is not the case."
- $A \rightarrow B$ stands for the sentence "If A then B."
- The sentence "A and B" can be represented by the formula $\neg(A \rightarrow \neg B)$
- "Either A or B or both" can be represented by the formula $\neg A \rightarrow B$

Frege's Logic for Internal Sentence Structures

- Instead of assuming that sentences have a subject-predicate structure, Frege introduced names, variables, function symbols, and predicates
- Frege used these logical categories to represent the structure of sentences.
- Frege's Book: *Conceptual Notation: A Formula Language of Pure Thought Modeled on the Formula Language of Arithmetic.*

Frege's Logic for Internal Sentence Structures

- A name is any term that denotes a particular object: "the Moon," "Bertrand Russell," "Plato," "the number 3."
- A variable is a letter, x, y, z used in place of a name. It does not name an object but ranges over objects.
- A function symbol is a term that denotes some object when applied to a name or sequence of names. "Father of," "sum of," "product of."
- A predicate symbol denotes a property of individual things or a relation among things. "Loves," " is red ," " is between. "
- A quantifier is a phrase such as "every," "all," "some," "there exists,""there is," "there are."

Frege's Logic: Representation

- For every straight line segment, there exists an equilateral triangle having that line segment as one side: $\forall x(L(x) \to \exists y(T(y) \wedge S(x,y)))$
- Someone loves everyone: $\exists x \forall y L(x,y)$
- Everyone loves someone: $\forall x \exists y L(x,y)$
- People only criticize people that are not their friends: $\forall x(\forall y(P(x) \wedge P(y) \wedge C(x,y) \to \neg F(y,x)))$

Frege's Logic: Logical Truths

- $A \to (B \to A)$
- $(D \to (B \to A)) \to (B \to (D \to A))$
- $A \to \neg\neg A$
- $(c = d \to K) \to Kc/d$
  (where Kc/d is the result of substituting d for c wherever c occurs in formula K)
- $c = c$

Frege's Logic: Rules of Inference

- Substitution: From any set of premises at all, infer the result of substituting any formulas for A, B, C, and D, and of substituting any names for c and d in any of the forms of logical truths given above.
- Modus ponens: From $A \to B$ and A, infer B, where A and B are any formulas.
- Quantifier deletion From $\forall x F(x)$, infer $F(x)$ for any formula F.
- Conjunction introduction From any formulas A, B, infer $A \& B$.

Frege's Logic: Proof

- A proof in Frege's system is just a finite sequence of formulas such that every formula in the sequence either follows from preceding formulas by one of the rules of inference or else is an assumption of the proof.
- The important thing is that the notion of a proof in a formal language becomes perfectly definite, so that we can informally prove things about formal proofs, just as we give informal proofs about other definite mathematical objects, such as numbers or sets.

Frege's Logic: Proof

Example: Prove $A \to A$

1. $(D \to (B \to A)) \to (B \to (D \to A))$, logical truth
2. $(A \to (B \to A)) \to (B \to (A \to A))$, replacement rule D/A, 1
3. $A \to (B \to A)$, logical truth
4. $B \to (A \to A)$, Modus ponens, 2, 3
5. $(A \to (B \to A)) \to (A \to A)$, replacement rule B/$(A \to (B \to A))$, 4
6. $A \to (B \to A)$, logical truth

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7. $A \to A$, Modus ponents, 5, 6

The Virtues of Frege's System

- The rules of proof are sufficient to reconstruct valid deductive arguments in mathematics and science.
- The rules of formal proof are entirely explicit.
- They are so explicit, in fact, that it is a completely mechanical matter to determine whether or not a sequence of formulas is a proof. All that is required to test whether a purported proof is indeed a proof is to check whether the rules of inference have been properly applied.

The Theory of Meaning: Language and the World

- Frege's logical theory was founded on an analysis of the logical structure of language.
- His theory of proof uses nothing but grammatical features of formulas in a logical language.
- But what are formulas in such a language about?
- What is truth, and why are "logical truths" necessarily true?
- Do the proofs that can be constructed in Frege's system necessarily preserve truth?
- Does a proof exist for every valid deductive inference?

Theory of Models: the Mathematical Theory of Meaning

- Frege distinguished the *reference* of a phrase or a sentence from its sense.
- The phrases “morning star” and “evening star” refer to the same object, the planet Venus, but they have different senses.
- Frege held that each declarative sentence refers to its truth-value.
- Two expressions that have the same sense must have the same reference.

Theory of Models: the Mathematical Theory of Meaning

- Every conversation tacitly presupposes a universe of discourse.
- Suppose there is a set, called the domain, that contains all of the objects we wish to talk or reason about in a particular context.
- A discourse is about a domain and a specific collection of properties of members of that domain, and that each property determines the set of all objects of the domain having that property.
- Some formulas will come out true no matter what domain we choose, as long as it is not empty, and no matter what subsets of the domain individual predicates denote, even if they denote the empty subset.

Theory of Models: the Mathematical Theory of Meaning

Proposition: $\forall x(E(x) \vee \neg E(x))$ is a logically true formula.

Proof.

$\forall x(E(x) \vee \neg E(x))$ is true in a domain under any particular specification of what E denotes if and only if $E(x) \vee \neg E(x)$ is true of every element of the domain. $E(x) \vee \neg E(x)$ is true of a member of the domain if that individual is a member of the set denoted by E or a member of the complement (in the domain) of the set denoted by E. But every member of every nonempty domain is a member of E or of the complement of E, no matter what set E is. So $E(x) \vee \neg E(x)$ is true of every value of x. So, finally, $\forall x(E(x) \vee \neg E(x))$ is always true. □

Three Principal Questions about Deductive Reasoning

1. How can we determine whether or not a piece of reasoning from premises to a conclusion is a valid deductive argument?
2. How can we determine whether or not a conclusion is necessitated by a set of premises? If a conclusion is necessitated by a set of premises, how can we find a valid deductive argument that demonstrates that necessary connection?
3. What features of the structure of the world, the structure of language, and the relations among words, thoughts, and things make deductive reasoning possible?

Three Principal Questions about Deductive Reasoning

Problem 1: How can we determine whether or not a piece of reasoning from premises to a conclusion is a valid deductive argument?

- Given a deductive argument, we put the premises, the conclusion, and the intermediate steps of the argument into formal notation and then determine whether or not each step of the formal argument follows from preceding steps or from the premises by rules of proof or by rules that can be derived from those rules.
- Using Frege's theory, we can represent proofs in number theory, algebra, geometry, set theory, and those empirical sciences that use mathematical reasoning.
- David Hilbert (1862 – 1943), Euclid's geometry

Three Principal Questions about Deductive Reasoning

Problem 3: How can we determine whether or not a conclusion is necessitated by a set of premises?

- Leibniz hoped that by formulating all of science in a formal language, an "alphabet of thought," we would obtain an algorithm, a mechanical means, to derive all of the consequences of any proposition.
- No algorithm exists that will determine for every first-order formula whether or not that formula is logically true. Hence no algorithm exists that will determine for every set of premises and every possible conclusion whether or not the premises entail the conclusion. Leibniz dreamed an impossible dream.
- Theory of computation → theoretical computer science.

Three Principal Questions about Deductive Reasoning

Problem 2: What features of the structure of the world, the structure of language, and the relations among words, thoughts, and things make deductive reasoning possible?

- The names and predicates of a language denote objects, properties, and relations, and in the actual world some objects may happen to exemplify any particular property or relation, and some may not.
- The denotations and the facts of the world determine the truth-values of sentences.
- What constitutes the relation of denotation between words on the one hand and things, properties, and relations on the other remains mysterious.

Mysteries

- What is the relation of denoting? What makes it the case that a particular word or phrase on a particular occasion of utterance denotes a particular object or property?
- Solution 1: people have a mysterious and irreducible capacity to make words mean things.
- Solution 2: denoting is some sort of causal relation between objects or properties on the one hand and utterances of words or phrases on the other.
- Solution 3: there is no such relation as denoting.

Mysteries

We cannot give a perfectly adequate formal logical reconstruction of **quantum theory**.

- Superposition: $(L(e) \vee R(e)) \wedge \neg L(e) \wedge \neg R(e)$
- Solution 1: reject the physical theory and look for a better theory that does not involve a contradiction such as superposition.
- Solution 2: the physical theory is an excellent calculating device for predicting the outcomes of experiment but that the theory really says nothing about the constitution of matter.
- Solution 3: we should look for a logical theory in which the $(L(e) \vee R(e)) \wedge \neg L(e) \wedge \neg R(e)$ is consistent.

Medern Logic

- Frege's logical theory included what we now call the theory of sets. Bertrand Russell (1872 – 1970) shows that his theory is inconsistent:
- Is *the set of all sets that are not members of themselves* a member of itself?
- *First-order logic* permits quantifiers to bind variables that range over individuals but it does not contain variables that range over properties nor does it have quantifiers that bind such variables.

First Order Logic

1. Relational structures, the abstract structure of the possible worlds or circumstances that are of concern in mathematics and the sciences
2. Formalized languages, which differ from natural languages in their simplicity and in having explicit rules that make it possible to determine in a mechanical way whether or not a string of symbols is well formed.

First Order Logic

3. A semantics for formalized languages: specifies precisely the conditions under which a set of values for variables in the language satisfies a formula in the language.
4. A theory of proof: it does not use the notion of a relational structure or the notion of a set of values satisfying a formula.

First Order Logic

It is possible (informally) to prove that the theory of proof and the semantic account of entailment coincide perfectly

- **Soundness theorem**: If Δ is any finite set of formulas and there is a proof of a formula S from Δ, then Δ entails S.
- **Completeness theorem**: If Δ is any collection of formulas and $\underline{\Delta \text{ entails } S}$, then there is a proof of S from formulas in Δ.

Reference

- Clark Glymour. (2015) *Thinking Things Through: An Introduction to Philosophical Issues and Achievements*, 2nd edition. A Bradford Book.