Computational Physics, PHYS-UA 210, Problem Set #7

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1. Let $u(x,t) = X(x)T(t)$, we have:

$$XT' - \alpha X''T = 0 \implies \frac{T'}{T} = \alpha\frac{X''}{X} = k$$

for some constant k.

(a) If $k = 0$, then $\frac{T'}{T} = 0 \implies T' = 0 \implies T = A$. $\alpha\frac{X''}{X} = 0 \implies X'' = 0 \implies X = Bx + C$ applying the boundary condition (I am assuming the length of the rod is l) $\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(l,t)}{\partial x} = 0$ we see that $A\frac{\partial Bx + C}{\partial x} = AB = 0 \implies A = 0$ or $B = 0$ both of which result in constant solution which is not very interesting.

(b) If $k > 0$, then $\alpha\frac{X''}{X} = k \implies X'' - \frac{k}{\alpha}X = 0 \implies X(x) = Ae^{x\sqrt{\frac{k}{\alpha}}} + Be^{-x\sqrt{\frac{k}{\alpha}}}$, and $\frac{T'}{T} = k \implies T' - Tk = 0 \implies T = Ce^{tk}$. Applying the boundary conditions we have:

$$TX'(0) = TX'(l) = 0 \text{ (T can't be 0 else the function is 0)} \implies X'(0) = 0 \implies \sqrt{\frac{k}{\alpha}}(A - B) = 0$$

$$\implies A = B. \quad \text{and} \quad X'(l) = 0 \implies \sqrt{\frac{k}{\alpha}}(Ae^{l\sqrt{\frac{k}{\alpha}}} - Ae^{-l\sqrt{\frac{k}{\alpha}}}) = A\sqrt{\frac{k}{\alpha}}(e^{l\sqrt{\frac{k}{\alpha}}} - e^{-l\sqrt{\frac{k}{\alpha}}}) = 0$$

$\implies A = 0$ Since $A = B = 0$, we get a trivial solution.

(c) Lastly, if we let $k = -\lambda^2 < 0$ we have the following:

$$\frac{T'}{T} = -\lambda^2 \implies T' + \lambda^2 T = 0 \implies T = Ce^{-\lambda^2 t}$$

$$\alpha\frac{X''}{X} = -\lambda^2 \implies X'' + \frac{\lambda^2}{\alpha}X = 0 \implies X = A\cos\sqrt{\frac{\lambda^2}{\alpha}}x + B\sin\sqrt{\frac{\lambda^2}{\alpha}}x$$

Applying the boundary condition:

$$X'(0) = -A\sqrt{\frac{\lambda^2}{\alpha}}\sin\left(\sqrt{\frac{\lambda^2}{\alpha}}0\right) + B\sqrt{\frac{\lambda^2}{\alpha}}\cos\left(\sqrt{\frac{\lambda^2}{\alpha}}0\right) = B\sqrt{\frac{\lambda^2}{\alpha}} = 0 \implies B = 0$$

$$X'(l) = -A\sqrt{\frac{\lambda^2}{\alpha}}\sin\left(\sqrt{\frac{\lambda^2}{\alpha}}l\right) = 0$$

$$\implies \text{(Since we don't want A=0)} \quad \sqrt{\frac{\lambda_n^2}{\alpha}} = \frac{n\pi}{l} \implies \lambda_n^2 = \frac{n^2\pi^2\alpha}{l^2}$$

For each n we have a corresponding constant A_n, so the final solution will be a sum:

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)X_n(x) = \sum_{n=0}^{\infty} A_n e^{-\frac{n^2\pi^2\alpha}{l^2}t}\cos\left(\frac{n\pi x}{l}\right)$$

2. Converting the spatial equation to an eigenvalue problem, we get the following:

$$-\alpha X'' = \lambda^2 X \implies -\alpha(x)\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}\bigg|_{x=x_i} = \lambda^2 X(x)\bigg|_{x=x_i}$$

$$\implies -\alpha(x_i)\frac{X(x_{i-1}) - 2X(x_i) + X(x_{i+1})}{h^2} = \lambda^2 X(x_i)$$

Given the condition $x_0 = 0, x_N = l = 1$ and the boundary condition we have $X(x_0) = X(x_1)$ and $X(x_{N-1}) = X(x_N)$:

$$\begin{bmatrix} \frac{a(x_1)}{h^2} & -\frac{a(x_1)}{h^2} & 0 & 0 & \cdots & 0 \\ -\frac{a(x_2)}{h^2} & 2\frac{a(x_2)}{h^2} & -\frac{a(x_2)}{h^2} & 0 & \cdots & 0 \\ 0 & -\frac{a(x_3)}{h^2} & 2\frac{a(x_3)}{h^2} & -\frac{a(x_3)}{h^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & -\frac{a(x_{N-2})}{h^2} \\ 0 & 0 & 0 & \cdots & -\frac{a(x_{N-1})}{h^2} & -\frac{a(x_{N-1})}{h^2} \end{bmatrix} \times \begin{bmatrix} X(x_1) \\ X(x_2) \\ X(x_3) \\ \vdots \\ X(x_{N-1}) \end{bmatrix} = \lambda^2 \times \begin{bmatrix} X(x_1) \\ X(x_2) \\ X(x_3) \\ \vdots \\ X(x_{N-1}) \end{bmatrix}$$