10/26/17 Computational Physics Problem Set 7 Jeffrey Wang

1) Separation of variables assumes $u(x,t) = v(x) \cdot w(t)$

$$\Rightarrow \frac{du(x,t)}{dt} - \alpha \frac{\partial^2 u(x,t)}{\partial x^2} = v(x)\dot{w}(t) - \alpha w(t)\frac{\partial^2 v(x)}{\partial x^2} = 0$$

$$\Rightarrow v(x)\dot{w}(t) = \alpha w(t)\frac{\partial^2 v(x)}{\partial x^2} \Rightarrow \frac{\dot{w}(t)}{w(t)} = \alpha \frac{\frac{\partial^2 v(x)}{dx^2}}{v(x)} = C$$

$$\therefore w(t) = W_0 e^{Ct}$$

Now, for $v(x) = V_1 e^{\sqrt{\frac{C}{\alpha}}x} + V_2 e^{-\sqrt{\frac{C}{\alpha}}x}$

Boundary conditions require $X(0) = X(1) = 0$

So now, we must consider separate cases for α.

(i) $\alpha > 0$ $v(x) = V_1 e^{\sqrt{\frac{C}{\alpha}}x} + V_2 e^{-\sqrt{\frac{C}{\alpha}}x}$

Boundary conditions $\Rightarrow$ $V_1 + V_2 = 0$ $[v(0)]$

$V_1 e^{k} + V_2 e^{-k} = 0$ $[v(1)]$

$\Rightarrow V_1 = -V_2,\ V_1(e^{2\sqrt{\frac{C}{\alpha}}} - 1) = 0 \Rightarrow V_1 = V_2 = 0$, but trivial

(ii) $\alpha < 0$. $v(x) = V_1 \cos\left(\sqrt{\frac{C}{\alpha}}x\right) + V_2 \sin\left(\sqrt{\frac{C}{\alpha}}x\right)$

Boundary conditions require $\Rightarrow 0 = v(0) = V_1$

$$v(1) = 0 = B\sin\left(\sqrt{\frac{C}{\alpha}}\right)$$

$$\Rightarrow \sin\left(\sqrt{\frac{C}{\alpha}}\right) = 0$$

$$\Rightarrow \sqrt{\frac{C}{\alpha}} = n\pi \Rightarrow \frac{C}{\alpha} = n^2\pi^2$$

$\Rightarrow V_n(x) = V_{0_n}\sin(n\pi x)$ $\Big\}\ u(x,t) = B_n \sin(n\pi x) e^{-n^2\pi^2 t}$

and $w_n(t) = W_{0_n} e^{-n^2\pi^2 t}$

2) Finite difference method: Assuming N points, length $L \Rightarrow x_i = \frac{i}{N} L$

Equations are $\frac{\dot{w}(t)}{w(t)} = -C = \alpha \frac{\frac{\partial^2 v(x)}{\partial x^2}}{v(x)}$

We approximate derivatives recursively

$$\left.\frac{d v(x_i)}{dx}\right|_{x_i} \approx \frac{v(x_{i+1/2}) - v(x_{i-1/2})}{L/N}$$

$$\Rightarrow \frac{\partial v^2(x_i)}{dx^2} \approx \frac{\frac{d v(x_{i+\frac{1}{2}})}{dx} - \frac{d v(x_{i-1/2})}{dx}}{L/N}$$

$$\Rightarrow \frac{\partial v^2}{dx^2} \approx \frac{\frac{v(x_{i+1}) - v(x_i)}{\cancel{L/N}} - \frac{v(x_i) - v(x_{i-1})}{\cancel{L/N}}}{L/N}$$

$$\Rightarrow \frac{\partial v^2}{dx^2} \approx \frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{(L/N)^2}$$

If we define $\vec{v} = \{v(x_1), v(x_2), \ldots, v(x_N)\}$

$$\Rightarrow \frac{C}{\alpha}\vec{v} = M\vec{v}, \text{ where } M = \frac{1}{(\frac{L}{N})^2} \begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & -2 & 1 \\ 0 & & & 1 & -2 \end{bmatrix}$$