COMPUTATIONAL PHYSICS

Problem Set 7

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PROBLEM 1: *Analytically solve the heat equation*

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

for a rod of length L with Neumann boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0$$

using the method of separation of variables.

For the separation of variables, we assume that $u(x,t)$ is of the form

$$u(x,t) = f(x)g(t)$$

By plugging this substitution into the 1D heat equation (Eq. 1), we get this expression:

$$f(x)\frac{\partial g}{\partial t} = \alpha\ g(t)\frac{\partial^2 f}{\partial x^2}$$

If we divide this equation by $u(x,t)$, the equation becomes a useful form:

$$\begin{aligned} \frac{1}{u}\left(f(x)\frac{\partial g}{\partial t}\right) &= \frac{1}{u}\left(\alpha\ g(t)\frac{\partial^2 f}{\partial x^2}\right) \\ \frac{\dot{g}}{g} &= \alpha\ \frac{f''}{f} \end{aligned} \tag{2}$$

where the dot implies a time derivative and the double prime implies two spacial derivatives. For this to be a physical solution – that is, if an expression that involves dimensions of time were to equal an expression that involves dimensions of space – the two sides of Eq. 2 must be constant:

$$\frac{\dot{g}}{g} = \alpha\ \frac{f''}{f} = c$$

We now can work with each side individually. The time-dependent function $g(t)$ is the easiest to work with; simply use the method "separate-integrate:"

$$\begin{aligned} \frac{\mathrm{d}g/\mathrm{d}t}{g} &= c \\ \frac{\mathrm{d}g}{g} &= c\mathrm{d}t \\ \Rightarrow g(t) &= G_0 e^{ct} \end{aligned}$$

Since the Neumann boundary conditions only apply to the spacial dependence, this form of $g(t)$ is completely specified. c will be determined via the spacial dependence:

$$
\begin{aligned}
\alpha \, \frac{f''}{f} &= c \\
f'' &= \frac{c}{\alpha} f
\end{aligned}
$$

Solutions to this differential equation follow the form:

$$
f(x) = Ae^{x\sqrt{c/\alpha}} + Be^{-x\sqrt{c/\alpha}}
$$

From the boundary conditions, we can limit this further:

$$
\begin{aligned}
\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} &= 0 \\
g(t)f'(0) = g(t)f'(L) &= 0 \\
f'(0) = f'(L) = 0 & \\
\rightarrow A\sqrt{\frac{c}{\alpha}} - B\sqrt{\frac{c}{\alpha}} &= 0 \Rightarrow A = B
\end{aligned}
$$

Now let's consider the other boundary $f'(L) = 0$:

$$
A\sqrt{\frac{c}{\alpha}}e^{L\sqrt{c/\alpha}} - A\sqrt{\frac{c}{\alpha}}e^{-L\sqrt{c/\alpha}} = 0
$$

For this to work, either c is (trivially) 0 *or* c is negative; this would yield oscillatory behavior. Let $k^2 = \left|\frac{c}{\alpha}\right|$ and substitute in:

$$
\begin{aligned}
Ake^{ikL} - Ake^{-ikL} &= 0 \\
2iAk\sin(kL) &= 0 \\
\sin(kL) &= 0 \\
\rightarrow kL &= n\pi \\
k^2 &= \frac{n^2\pi^2}{L^2} \\
\Rightarrow c &= -\frac{\alpha n^2\pi^2}{L^2}
\end{aligned}
$$

This $\sin$ showed up for the *derivative* of $f(x)$. Furthermore, this is just for *one* eigenvector. The total spectrum of solutions make up the open domain where $n \in (-\infty, \infty) \,\&\, n \in \mathbb{Z}$. This means the full solution $u(x,t)$ is:

$$
u(x,t) = \sum_{n=-\infty}^{\infty} C_n \exp\left[-\frac{\alpha n^2\pi^2}{L^2}t\right]\cos\left(\frac{n\pi}{L}x\right) \tag{3}
$$

where C_n are just some constants that can be chosen at will.

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