Veritas
October 2017

Problem Set 7

1 Analytic Solution : Separation of Variables

The heat equation in one dimension is

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}$$

Separation of variables gives

$$u(x,t) = X(x)\,T(t) \rightarrow X(x)\frac{\partial T}{\partial t} = \alpha\, T(t) \frac{\partial^2 X}{\partial x^2}$$

Both sides must be equal to the same constant,

$$\frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} = \frac{1}{\alpha T(t)} \frac{\partial T}{\partial t} = \lambda$$

This gives a spatial equation and a temporal equation,

$$\frac{\partial^2 X(x)}{\partial x^2} = \lambda X(x) \qquad (\text{Spatial})$$

$$\frac{\partial T}{\partial t} = \alpha \lambda T(t) \qquad (\text{temporal})$$

First I'll solve the Spatial.

By inspection, we see that the solutions are,

$$X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$$

Next apply initial conditions:

$$\left.\frac{\partial X}{\partial x}\right|_{x=0} = -\overbrace{a\sqrt{\lambda}\sin(\sqrt{\lambda}\,x)}^{\to 0} + b\sqrt{\lambda}\cos(\sqrt{\lambda}\,x) = 0$$

$$\to b\sqrt{\lambda}\,(1) = 0$$

This means that either $b = 0$ or $\lambda = 0$.

$$\left.\frac{\partial X}{\partial x}\right|_{x=1} = -a\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + \overbrace{b\sqrt{\lambda}\cos(\sqrt{\lambda}\,x}^{\to 0} = 0) = 0$$

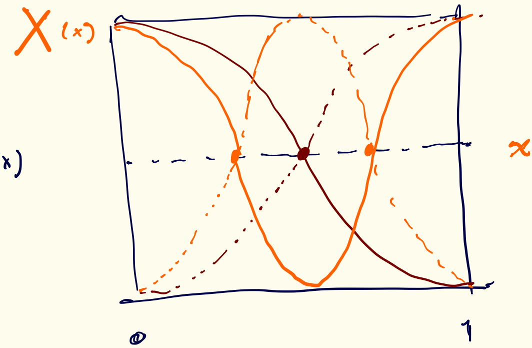
$$\to \sin(\sqrt{\lambda}) = 0$$

$$\to \boxed{\sqrt{\lambda} = n\pi} \qquad n = 1, 2, 3\ldots$$

So

$$\boxed{X(x) = \sum_{n=0}^{\infty} A_n \cos(n\pi x) \qquad (n = 1, 2, 3\cdots)}$$

where the coefficients are



$$A_m = \frac{2}{l}\int_0^l dx\, f(x)\cos(m\pi x)$$

$$= 2\int_0^1 dx\, f(x)\cos(m\pi x).$$

The temporal equation

$$\frac{\partial T}{\partial t} = \alpha \lambda T(t)$$

has the exponential solution

$$\boxed{T(t) = c e^{\alpha \lambda t}} .$$

Therefore, the complete solution is the product $X(x)\,T(t)$,

$$\text{i.e.} \quad \boxed{u(x,t) = \underbrace{\sum_{n=1}^{\infty} A_n \cos(n\pi x)\, e^{-\alpha n^2 \pi^2 t}}_{\beta = 0} + \underbrace{A_0}_{\lambda = 0}}$$

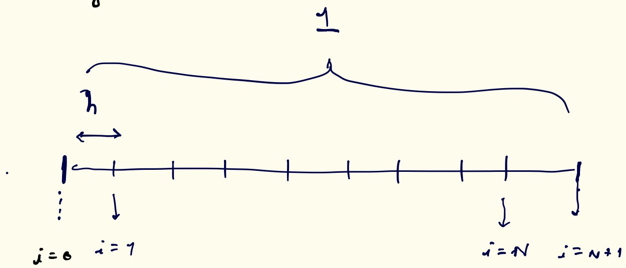
2 Numerical Solution : Finite Difference Method

From the previous section, the spatial equation is

$$\frac{\partial^2 X}{\partial x^2} = \lambda X(x) \longrightarrow \boxed{A X = \lambda X}.$$

which is an eigenvalue equation.

Numerically, we have to focus on a discretized interval,



Define the Derivative

$$\boxed{\frac{dX(x_i)}{dx} \approx \frac{X(x_{i+\frac{1}{2}}) - X(x_{i-\frac{1}{2}})}{h} = X'(x_i)}$$

where

$$\left\{ \begin{aligned} \frac{dX(x_{i+\frac{1}{2}})}{dx} &= \frac{X(x_{i+1}) - X(x_i)}{h} = X'(x_{i+\frac{1}{2}}) \\ \frac{dX(x_{i-\frac{1}{2}})}{dx} &= \frac{X(x_i) - X(x_{i-1})}{h} = X'(x_{i-\frac{1}{2}}) \end{aligned} \right\},$$

$$\rightarrow \boxed{\frac{\partial^2 X(x_i)}{\partial x^2} = \frac{X'(x_{i+\frac{1}{2}}) - X'(i-\frac{1}{2})}{h} = X''(x_i)}$$

Moving forward, we can write the spatial equation in terms of the derivative operator D^2, which is a matrix,

$$D^2 \vec{X} = \lambda \vec{X}$$

eigenvalue.

eigenvector

where

$$\vec{X}^T = (x_1, \ldots\, x_i, \ldots\, x_N).$$

If $i \neq 1, N$:

$$x_i'' = \frac{1}{h^2}(x$$

For $i = 1$,

$$x_1'' = \frac{X'(x_{1+\frac{1}{2}}) - X'(_{1-\frac{1}{2}})}{h}$$

$$= \frac{1}{h}\left[\frac{X(x_2) - X(x_1)] - [X(x_1) - X(x_0)}{h}\right.$$

$$\rightarrow \boxed{x_1'' = \frac{X(x_2) - X(x_0)}{h^2}.}$$

For $i = N$,

$$\boxed{x_N'' = \frac{X(x_{N+1}) - X(x_N)}{h^2}}. \quad \circledast$$

If $i \neq 1$ or N,

$$\boxed{x_i'' = \frac{1}{h^2}\left(X(x_{i+1}) + X(x_{i-1}) - 2X(x_i)\right)}.$$

This means that the spatial equation is

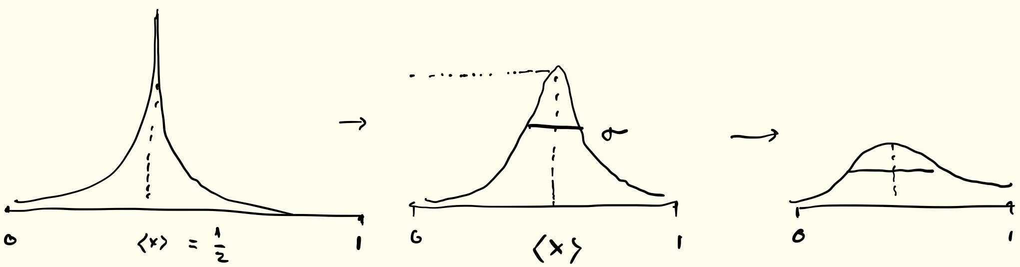
$$\underbrace{\begin{pmatrix} -1 & 1 & & & & & & & 0 \\ 1 & -2 & 1 & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & & \ddots & & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & \ddots & & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -2 \quad 1 \\ 0 & & & & & & & & 1 \quad -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ x_i \\ x_{i+1} \\ \vdots \\ x_N \end{pmatrix} = \lambda' \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ x_i \\ x_{i+1} \\ \vdots \\ x_N \end{pmatrix}$$

$$A \cdot \vec{x} = \lambda' \vec{x}$$

where $\lambda' = h^2 \lambda$.

The eigenvectors of A are the solutions to the spatial equation.

How a Gaussian evolves over time. What we expect to see is a sharply peaked function that flattens as time progresses:



③ If $\alpha(x)$

If $\alpha(x)$, then the Spatial equation becomes

$$\alpha(x) X''(x) = -\lambda X(x)$$

Defining $\alpha_i = \alpha(x_i)$,

$$\alpha_i X_i'' = \lambda X_i$$

$$\rightarrow \quad \alpha_i \left(\frac{1}{h^2} \left(X_{i+1} + X_{i-1} - 2X_i \right) \right) = \lambda X_i .$$

$$\underbrace{\begin{pmatrix}
-\alpha_1 & \alpha_1 & & & & & & & \\
\alpha_2 & -2\alpha_2 & \alpha_2 & & & & & O & \\
 & & \ddots & & & & & & \\
 & & \alpha_{i-1} & -2\alpha_{i-1} & \alpha_{i-1} & & & & \\
 & & & \alpha_i & -2\alpha_i & \alpha_i & & & \\
 & & & & \alpha_{i+1} & -2\alpha_{i+1} & \alpha_{i+1} & & \\
 & O & & & & & \ddots & & \\
 & & & & & & \alpha_{N-1} & -2\alpha_{N-1} & \alpha_{N-1} \\
 & & & & & & & \alpha_N & -2\alpha_N
\end{pmatrix}}_{\tilde{A}} \vec{x} = \lambda' \vec{x} .$$

Relating $\tilde{A}$ to A,

$$\tilde{A} = \Pi \cdot A$$

$$\Pi = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_N \end{pmatrix}$$