Appendix B: The Impulse Collision Method Algorithm

a.1) For a particle-particle interaction, calculate the relative position vector $\mathbf{x}_{(ab)}$

$$\mathbf{x}_{(ab)} = \mathbf{x}_{(a)} - \mathbf{x}_{(b)} \quad \text{and its norm} \quad x_{(ab)} = \left|\mathbf{x}_{(ab)}\right|$$

a.2) Compute the overlap (penetration depth)

a) $\Delta x_{(ab)} = r_{(a)} + r_{(b)} - x_{(ab)}$ – in case of particle-particle interaction

b) $\Delta x_{(as)} = r_{(a)} - \left\langle (\mathbf{x}_{(a)} - \mathbf{r}_{(s)}), \hat{\mathbf{n}}_{(s)} \right\rangle$ – in case of particle-surface interaction

The two objects interact (have a contact), if the overlap is *positive*, i.e. if

$$\Delta x_{(ab)} > 0 \quad \text{alt.} \quad \Delta x_{(as)} > 0$$

Proceed with steps (b) and (c) only for objects that overlaps.

b.1) For particle-particle interaction, calculate the unit collision direction from *b* to *a*, i.e.

$$\hat{\mathbf{n}}_{(ab)} = \mathbf{x}_{(ab)} / x_{(ab)}$$ (for particle-surface interaction we already have $\hat{\mathbf{n}}_{(as)} = \hat{\mathbf{n}}_{(s)}$)

b.2) Calculate the velocity projection $v^{\mathbf{n}}_{(ab)}$ on the unit collision direction $\hat{\mathbf{n}}_{(ab)}$ as scalar product

$$v^{\mathbf{n}}_{(ab)} = \left\langle \hat{\mathbf{n}}_{(ab)}, \mathbf{v}_{(ab)} \right\rangle$$

We can distinguish three cases:

$v^{\mathbf{n}}_{(ab)} < 0$ – impacting contact

$v^{\mathbf{n}}_{(ab)} = 0$ – resting contact

$v^{\mathbf{n}}_{(ab)} > 0$ – separating contact

Proceed with (c) ***only*** *in case of* ***impacting contact*** *when using the* ***impulse interaction*** *method.*

c.1) Calculate the reduced mass to particle mass ratio $m^{\text{red}}_{(a)} = m_{(ab)} m^{-1}_{(a)}$

a) $m^{\text{red}}_{(a)} = \dfrac{m_{(b)}}{m_{(a)} + m_{(b)}}$ – in case of ***particle-particle*** interaction, when $m_{(a)} \approx m_{(b)}$

b) $m^{\text{red}}_{(a)} = 1$ – in case of ***particle-surface*** interaction, when $m_{(a)} \ll m_{(s)}$

c.2) Calculate the scalar component of the velocity jolt $v^{\text{jolt}}_{(ab)}$

$$v^{\text{jolt}}_{(a)} = -(1+e)\, m^{\text{red}}_{(a)}\, v^{\mathbf{n}}_{(ab)}$$ (note that $v^{\mathbf{n}}_{(ab)} < 0$ for impacting contact, so $v^{\text{jolt}}_{(a)} > 0$)

c.3) Calculate the tangential unit vector of contact

$$\hat{\mathbf{t}}_{(ab)} = \mathbf{v}^{\mathbf{t}}_{(ab)} / \left|\mathbf{v}^{\mathbf{t}}_{(ab)}\right| \quad \text{where} \quad \mathbf{v}^{\mathbf{t}}_{(ab)} = \mathbf{v}_{(ab)} - \hat{\mathbf{n}}_{(ab)} \left\langle \hat{\mathbf{n}}_{(ab)}, \mathbf{v}_{(ab)} \right\rangle$$

c.4) Calculate the velocity jolt (where $\mu > 0$ in case of friction)

$$\mathbf{v}^{\text{jolt}}_{(a)} = v^{\text{jolt}}_{(a)}\, (\hat{\mathbf{n}}_{(ab)} - \mu\, \hat{\mathbf{t}}_{(ab)})$$ and **clip** $\mu v^{\text{jolt}}_{(a)}$ term not to exceed $v^{\mathbf{t}}_{(ab)}$ component **(!)**

c.5) Calculate the position projection

$$\mathbf{x}^{\text{jolt}}_{(a)} = k_{\text{p}}\, m^{\text{red}}_{(a)}\, \Delta x_{(ab)}\, \hat{\mathbf{n}}_{(ab)}$$ (note that $\Delta x_{(ab)} < 0$ since objects are in contact)

c.6) Apply the velocity jolt and the position projection when appropriate (e.g. in the odd half-step of the leapfrog integrator to keep external forces balanced)

$$\mathbf{v}'_{(a)} = \mathbf{v}_{(a)} + \mathbf{v}^{\text{jolt}}_{(a)}$$

$$\mathbf{x}'_{(a)} = \mathbf{x}_{(a)} + \mathbf{x}^{\text{jolt}}_{(a)}$$

Notes on Two-body Collision Impulses

Let us introduce a velocity jolt $\mathbf{v}_{(a)}^{\text{jolt}}$ as the change of the velocity of a particle *a* during the hardcore interaction, i.e.

$$\mathbf{v}_{(a)}^{\text{jolt}} = \mathbf{v}'_{(a)} - \mathbf{v}_{(a)} \tag{B.1}$$

Then, from Eq. (5.11) $\mathbf{v}'_{(a)} = \mathbf{v}_{(a)} + m_{(a)}^{-1}\, \mathbf{j}_{(ab)}$ we have

$$\mathbf{v}_{(a)}^{\text{jolt}} = m_{(a)}^{-1}\, \mathbf{j}_{(ab)} \tag{B.2}$$

and further, from Eq. (5.14) $\mathbf{j}_{(ab)} = j\, \hat{\mathbf{n}}_{(ab)}$, we get expression for the velocity jolt

$$\mathbf{v}_{(a)}^{\text{jolt}} = j\, m_{(a)}^{-1}\, \hat{\mathbf{n}}_{(ab)} \tag{B.3}$$

On the other hand, substituting impulse *j* from Eq. (5.16)

$$j = -(1+e)\frac{1}{m_{(a)}^{-1} + m_{(b)}^{-1}} \left\langle \hat{\mathbf{n}}_{(ab)}, \mathbf{v}_{(ab)} \right\rangle \tag{5.16}$$

into Eq. (B.3) yields

$$\mathbf{v}_{(a)}^{\text{jolt}} = -(1+e)\frac{1}{m_{(a)}^{-1} + m_{(b)}^{-1}} \left\langle \hat{\mathbf{n}}_{(ab)}, \mathbf{v}_{(ab)} \right\rangle m_{(a)}^{-1}\, \hat{\mathbf{n}}_{(ab)} \tag{B.4}$$

Denoting as $v_{(ab)}^{\mathbf{n}} = \left\langle \hat{\mathbf{n}}_{(ab)}, \mathbf{v}_{(ab)} \right\rangle$ the projection of the velocity $\mathbf{v}_{(ab)}$ on the unit collision direction $\hat{\mathbf{n}}_{(ab)} = \mathbf{x}_{(ab)} / x_{(ab)}$ (from particle *b* to particle *a*), and substituting into Eq. (B.4), we get

$$\mathbf{v}_{(a)}^{\text{jolt}} = -(1+e)\frac{m_{(b)}}{m_{(a)} + m_{(b)}} v_{(ab)}^{\mathbf{n}} \hat{\mathbf{n}}_{(ab)} \,. \tag{B.5}$$

At this point, it is convenient to introduce a *reduced mass* $m_{(ab)}$ of a two-body system as $m_{(ab)}^{-1} = m_{(a)}^{-1} + m_{(b)}^{-1}$ and the reduced mass to particle *a* ratio $m_{(a)}^{\text{red}} = m_{(ab)} m_{(a)}^{-1}$ i.e.

$$m_{(a)}^{\text{red}} = m_{(ab)} m_{(a)}^{-1} = \frac{m_{(b)}}{m_{(a)} + m_{(b)}} \tag{B.6}$$

Substituting the last reduced mass to particle ratio into (B.5) gives

$$\mathbf{v}_{(a)}^{\text{jolt}} = -(1+e)\, m_{(a)}^{\text{red}}\, v_{(ab)}^{\mathbf{n}}\, \hat{\mathbf{n}}_{(ab)} \,. \tag{B.7}$$

If we further denote as $v_{(a)}^{\text{jolt}}$ the scalar component of the velocity jolt $v_{(a)}^{\text{jolt}} = \mathbf{v}_{(a)}^{\text{jolt}} / \hat{\mathbf{n}}_{(ab)}$, then

$$\mathbf{v}_{(a)}^{\text{jolt}} = v_{(a)}^{\text{jolt}}\, \hat{\mathbf{n}}_{(ab)} \tag{B.8}$$

and from Eq. (B.5)

$$v_{(a)}^{\text{jolt}} = -(1+e)\, m_{(a)}^{\text{red}}\, v_{(ab)}^{\mathbf{n}} \tag{B.9}$$

Combining Eq. (B.8) with Eq. (B.3), we get also relationships

$$v_{(a)}^{\text{jolt}} = j\, m_{(a)}^{-1} \quad \text{i.e.} \quad j = m_{(a)} v_{(a)}^{\text{jolt}} \,. \tag{B.10}$$

Notes on Collision Impulses with the Static Surface

We can assume that the mass of the surface is $m_{(b)} \to \infty$. In this case the reduced mass to particle ratio $m_{(a)}^{\text{red}} \to 1$ and Eq. (B.9) becomes

$$v_{(a)}^{\text{jolt}} = (1+e)\, v_{(ab)}^{\mathbf{n}} \tag{B.11}$$

Notes on Simplified Frictional Collision Impulses

Applying the triple vector product rule $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\langle\mathbf{a},\mathbf{c}\rangle - \mathbf{c}\langle\mathbf{a},\mathbf{b}\rangle$ to modified Eq. (5.25)

$$\hat{\mathbf{t}}_{(ab)} = \mathbf{v}_{(ab)}^{\mathbf{t}} / \left|\mathbf{v}_{(ab)}^{\mathbf{t}}\right| \quad \text{where} \quad \mathbf{v}_{(ab)}^{\mathbf{t}} = \hat{\mathbf{n}}_{(ab)} \times (\mathbf{v}_{(ab)} \times \hat{\mathbf{n}}_{(ab)}) \tag{5.25}$$

which gives

$$\mathbf{v}_{(ab)}^{\mathbf{t}} = \mathbf{v}_{(ab)} \left\langle \hat{\mathbf{n}}_{(ab)}, \hat{\mathbf{n}}_{(ab)} \right\rangle - \hat{\mathbf{n}}_{(ab)} \left\langle \hat{\mathbf{n}}_{(ab)}, \mathbf{v}_{(ab)} \right\rangle \tag{B.12}$$

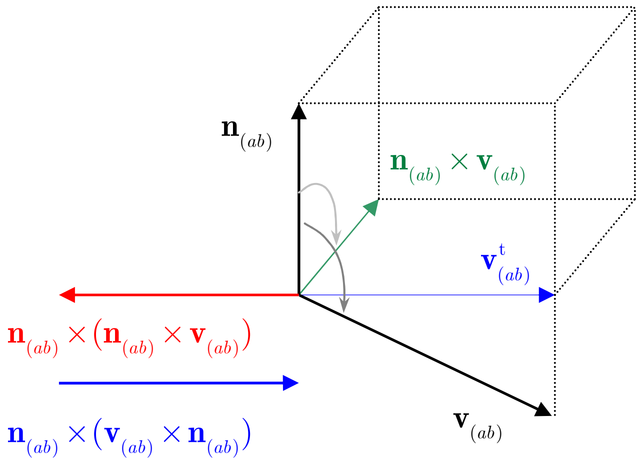
Recalling from earlier that $v_{(ab)}^{\mathbf{n}} = \left\langle \hat{\mathbf{n}}_{(ab)}, \mathbf{v}_{(ab)} \right\rangle$ and since $\left\langle \hat{\mathbf{n}}_{(ab)}, \hat{\mathbf{n}}_{(ab)} \right\rangle = 1$, the last expression becomes

$$\mathbf{v}_{(ab)}^{\mathbf{t}} = \mathbf{v}_{(ab)} - \hat{\mathbf{n}}_{(ab)} \left\langle \hat{\mathbf{n}}_{(ab)}, \mathbf{v}_{(ab)} \right\rangle \tag{B.13}$$

and

$$\hat{\mathbf{t}}_{(ab)} = \mathbf{v}_{(ab)}^{\mathbf{t}} / \left|\mathbf{v}_{(ab)}^{\mathbf{t}}\right|$$

> Note that Eq. (B.13) can be calculated for *any* number of spatial dimensions (1, 2 or 3), while Eq. (5.25) can be calculated only for three-dimensional vectors (cross product operation requires arrays of length 3).
>
> From performance view, Eq. (B.13) consists of one scalar multiplication (with velocity projection scalar product calculated earlier) and vector subtraction. This is much faster than two cross products from Eq. (5.25) and also easier to "vectorize" for many-particle system (e.g. in MATLAB).



Finally, since

$$\mathbf{j}_{(ab)} = j\, \hat{\mathbf{n}}_{(ab)} - \mu j\, \hat{\mathbf{t}}_{(ab)} \tag{B.14}$$

it follows

$$\mathbf{v}_{(a)}^{\text{jolt}} = v_{(a)}^{\text{jolt}}\, \hat{\mathbf{n}}_{(ab)} - \mu\, v_{(a)}^{\text{jolt}}\, \hat{\mathbf{t}}_{(ab)} \tag{B.15}$$

Notes on Position Projections

A position projections factor k_p is introduced into equations (5.27) and (5.28) found in *Projections* subsection in *Notes*. The modified expressions for the position projections (with the factor k_p) are then

$$\mathbf{x}'_{(a)} = \mathbf{x}_{(a)} + k_{\mathrm{p}} \cdot \delta_{(a)} \hat{\mathbf{n}}_{(ab)} \tag{5.27}$$

$$\left[\ \mathbf{x}'_{(b)} = \mathbf{x}_{(b)} + k_{\mathrm{p}} \cdot \delta_{(b)} \hat{\mathbf{n}}_{(ba)}, \quad \hat{\mathbf{n}}_{(ba)} = -\hat{\mathbf{n}}_{(ab)} \ \right] \tag{5.28}$$

The projection factor k_p values are in range from 0 to 1, where the factor $k_p = 0$ disables position projections and $k_p = 1$ enables projections in full extent.

Other values $0 < k_p < 1$ make position projections to act like a spring (softening displacements).

Let us denote with $\mathbf{x}^{\mathrm{jolt}}_{(a)}$ a position projection, i.e. as the change of the position of a particle a during the hardcore interaction

$$\boxed{\mathbf{x}^{\mathrm{jolt}}_{(a)} = \mathbf{x}'_{(a)} - \mathbf{x}_{(a)}} \tag{B.16}$$

Then, from Eq. (5.27) $\mathbf{x}'_{(a)} = \mathbf{x}_{(a)} + k_{\mathrm{p}} \cdot \delta_{(a)} \hat{\mathbf{n}}_{(ab)}$ we have

$$\mathbf{x}^{\mathrm{jolt}}_{(a)} = \delta_{(a)} \ \hat{\mathbf{n}}_{(ab)}$$

Since, from Eq. (5.29)

$$\delta_{(a)} = \frac{m_{(b)}}{m_{(a)} + m_{(b)}} \Delta x_{(ab)} = m^{\mathrm{red}}_{(a)} \ \Delta x_{(ab)} \tag{B.17}$$

the position projection is

$$\boxed{\mathbf{x}^{\mathrm{jolt}}_{(a)} = k_{\mathrm{p}} \ m^{\mathrm{red}}_{(a)} \ \Delta x_{(ab)} \ \hat{\mathbf{n}}_{(ab)}} \tag{B.18}$$

Position projection factor seen as spring stiffness

Substituting $\delta_{(a)}$ from Eq. (5.29) in *Notes* into earlier modified Eq. (5.27) (with k_p), we get

$$\mathbf{x}'_{(a)} - \mathbf{x}_{(a)} = k_{\mathrm{p}} \cdot \frac{m^{-1}_{(a)}}{m^{-1}_{(a)} + m^{-1}_{(b)}} \Delta x_{(ab)} \cdot \hat{\mathbf{n}}_{(ab)} \tag{B.19}$$

On the other side, from the integrator algorithm perspective, the position projection can be seen as there is an ad-hoc spring force $\mathbf{f}_{(a)} = k_{\mathrm{s}} \Delta x_{(ab)} \cdot \hat{\mathbf{n}}_{(ab)}$ giving the same position displacement during the integrator time-step (but not influencing the velocity, however)

$$\mathbf{x}'_{(a)} - \mathbf{x}_{(a)} = \mathbf{f}_{(a)} m^{-1}_{(a)} \cdot h^2 \quad \text{i.e.}$$

$$\mathbf{x}'_{(a)} - \mathbf{x}_{(a)} = \left(k_{\mathrm{s}} \Delta x_{(ab)} \cdot \hat{\mathbf{n}}_{(ab)} \right) \cdot m^{-1}_{(a)} \cdot h^2 \tag{B.20}$$

Combining equations (B.19) and (B.20) yields

$$k_{\mathrm{p}} \cdot \frac{m^{-1}_{(a)}}{m^{-1}_{(a)} + m^{-1}_{(b)}} \Delta x_{(ab)} \cdot \hat{\mathbf{n}}_{(ab)} = \left(k_{\mathrm{s}} \Delta x_{(ab)} \cdot \hat{\mathbf{n}}_{(ab)} \right) \cdot m^{-1}_{(a)} \cdot h^2 \tag{B.21}$$

Finally, canceling $\Delta x_{(ab)} \hat{\mathbf{n}}_{(ab)} \cdot m^{-1}_{(a)}$ and substituting $m^{-1}_{(ab)} = m^{-1}_{(a)} + m^{-1}_{(b)}$ yields the relation

$$k_{\mathrm{p}} = k_{\mathrm{s}} m^{-1}_{(ab)} \cdot h^2 \tag{B.22}$$

Since the semi-implicit Euler integrator method is assumed stable for $(k_{\mathrm{s}} / m) \cdot h^2 = (\omega h)^2 < 1$, then, for $m_{(a)} = m_{(b)} = m$, it follows from (B.22), that the same method is stable for the position projection factors $k_{\mathrm{p}} < 0.5$.

Orientation of a plane and plane impenetrability

Inequality (5.8) for contact and expression (5.9) for overlap in section 5.2.2 in *Notes* can produce bugs. Namely, since $\left|(\mathbf{x}_{(a)} - \mathbf{r})^{\mathrm{T}} \hat{\mathbf{n}}\right|$ is an *absolute* (always positive) distance between a particle and a plane, the inequality (5.9) does not distinguish sides of the plane.

As one of the sides of the plane is impenetrable (the side opposite to the normal unit vector $\hat{\mathbf{n}}$), the correct expression for the overlap (actually the *penetration*) is:

$$\Delta x_{(a)} = r_{(a)} - (\mathbf{x}_{(a)} - \mathbf{r})^{\mathrm{T}} \hat{\mathbf{n}} \qquad \text{modified (5.9)}$$

and the condition for contact is/when:

$$\Delta x_{(a)} \geq 0 \qquad \text{modified (5.8)}$$

Motivation 1

The equations (5.8) and (5.9) are mentioned to be used for detecting contacts in *collisions* with surfaces. However, the equations fail with a finite time-step integrator in case when:

- a particle has a high velocity, or
- a particle has a very small size, or
- time-step is too large.

In such cases the integrator can "tunnel" a particle through the surface without noticing that there was a contact (left case on Figure A.1)! In particular, tunneling might happen if $r < h\,v$, where r is particle radius, h is time-step size and v is particle velocity. However, **a box wall** or **ground surfaces** should not be 'transparent' for particles in classical mechanics (right case on Figure A.1).

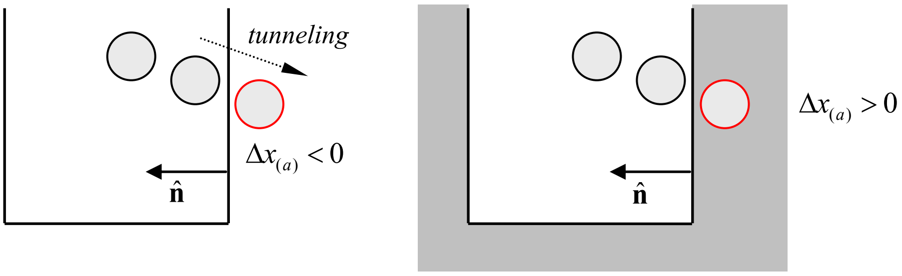


Figure A.1: Tunneling of a particle with unmodified (5.9) giving $\Delta x_{(a)} < 0$ (left case)

So, the collision detection <u>algorithm</u> based on Eqs. (5.8-9) has *a bug*, which makes usage of the equations erroneous. Otherwise, one has to *ensure* that particle velocities in a system *does not become larger* than r/h, which seems as an unreasonable requirement.

Motivation 2

Another way to look on this problem is to perceive an Euclidian plain as a limit of the sphere with center $\mathbf{R}_{(s)}$ at infinity and having infinite radius $R_{(s)} \to \infty$. In this case, $\Delta x_{(a)} = r_{(a)} - (\mathbf{x}_{(a)} - \mathbf{r})^{\mathrm{T}} \hat{\mathbf{n}}$ follows from Eq. (5.4), i.e. from $\Delta x_{(a)} = r_{(a)} + R_{(s)} - \left|\mathbf{x}_{(a)} - \mathbf{R}_{(s)}\right|$ (given here without proof).

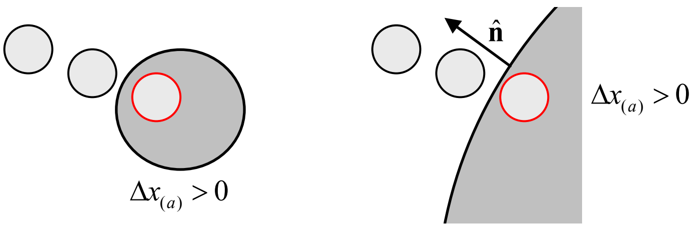


Figure A.2: Collision with a plain seen as a collision with sphere of an infinite radius