Lab3 Appendix C: Angular Velocity Decay

Initialization

General assumptions when solving equations

In[1]:=
```
$Assumptions = R > 0 ⋀ 0 ≤ ϕ ≤ 2 π ⋀ 0 ≤ θ ≤ π ⋀
   ρ > 0 ⋀ m > 0 ⋀ C_D ≥ 0 ⋀
   vx ∈ Reals ⋀ vy ∈ Reals ⋀ vz ∈ Reals ⋀ ωx ∈ Reals ⋀ ωy ∈ Reals ⋀ ωz ∈ Reals;
```

Definition of the vector norm symbol

In[2]:= $|v\_| := \sqrt{v_{[[1]]}^2 + v_{[[2]]}^2 + v_{[[3]]}^2}$

Equations

Velocity of the surface element δA at position **r** relative to the fluid is

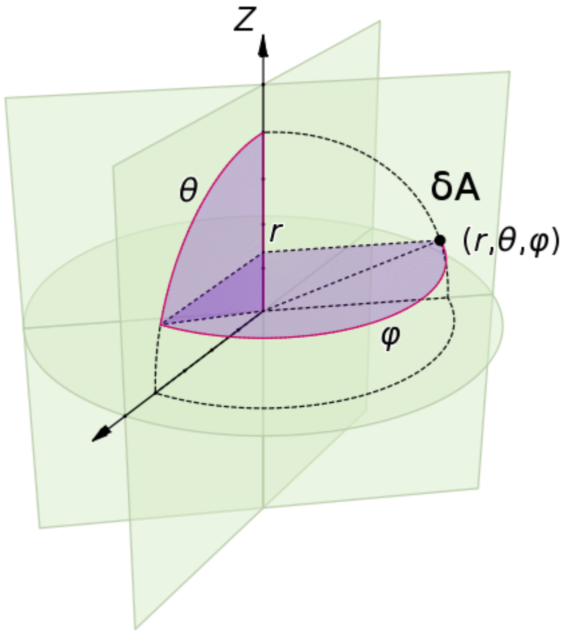
In[3]:= $\mathcal{U} := \mathbf{v} + \omega \times \mathbf{r}$

where **v** is velocity of the ball throught the fluid and $\boldsymbol{\omega}$ is angular velocity of the ball.

Let us assume that an infinitesimal drag force at the surface element δA is

In[4]:= $\delta f := -\frac{1}{2} \rho\, C_D\, \delta A\, |\mathcal{U}|^{\gamma-1}\, \mathcal{U}$

where ρ is mass density of the fluid, C_D is a constant that depends on the surface material, and γ characterizes type of the flow ($\gamma = 1$ for laminar flow and $\gamma = 2$ for turbulent flow).



An infinitesimal torque caused by the drag force is then

In[5]:= $\delta\tau := \mathbf{r} \times \delta f$

where the net torque is $\tau = \int \delta\tau$ integrated across the surface.

Angular acceleration of the ball can be expressed from the net torque as

In[6]:= $\dot{\omega} := \mathcal{I}^{-1}\, \tau$

where $\mathcal{I}$ is the moment of inertia of the ball (i.e. sphere)

In[7]:= $\mathcal{I} := \frac{2}{5} m R^2$

Parameters

Taking as a parameter ball cross section area A

In[8]:=
$$A := \pi R^2$$

together with a constant C_w that depends on type of surface and moment of inertia

In[9]:=
$$C_w := \frac{20}{3} C_D$$

and introducing constant of proportionality k_w

In[10]:=
$$k_w := \frac{1}{2} \frac{\rho A C_w}{m}$$

Coordinate System

Spherical Coordinates

Position of the surface element δA in in spherical coordinates is

In[11]:=
```
r := {
  R Cos[ϕ] Sin[θ],
  R Sin[ϕ] Sin[θ],
  R Cos[θ]
 }
```

The area of the surface element is

In[12]:=
$$\delta A := R^2 \, \delta\Omega$$

where $\delta\Omega$ is an infinitesimal solid angle subtended by the surface element, in spherical coordinates expressed as

In[13]:=
$$\delta\Omega := \text{Sin}[\theta] \, \delta\theta \, \delta\phi$$

Velocity of the ball throught the fluid **v** in cartesian coordinates

In[14]:=
```
v := {vx, vy, vz}
```

Angular velocity ω of the ball in cartesian coordinates

In[15]:=
```
ω := {ωx, ωy, ωz}
```

Velocity of the surface element relative to the fluid in cartesian coordinates

In[16]:=
```
{{"𝓊x =", "𝓊y =", "𝓊z ="}, 𝓊} // Transpose // TableForm
```

Out[16]//TableForm=

$$\mathcal{u}_x = vx + R\,\omega y\,\text{Cos}[\theta] - R\,\omega z\,\text{Sin}[\theta]\,\text{Sin}[\phi]$$
$$\mathcal{u}_y = vy - R\,\omega x\,\text{Cos}[\theta] + R\,\omega z\,\text{Cos}[\phi]\,\text{Sin}[\theta]$$
$$\mathcal{u}_z = vz - R\,\omega y\,\text{Cos}[\phi]\,\text{Sin}[\theta] + R\,\omega x\,\text{Sin}[\theta]\,\text{Sin}[\phi]$$

Sanity-Check

Total area of the ball should be $\int \delta A = 4\pi R^2$

In[17]:=
$$\int_0^{\pi}\int_0^{2\pi}\left(\frac{\delta A}{\delta\theta\,\delta\phi}\right) d\phi\, d\theta$$

Out[17]=
$$4\pi R^2$$

Solving Equations

Let's find angular acceleration $\alpha = \frac{\tau}{k_w I}$ by integrationg the net torque $\tau = \int \delta\tau$ across the whole surface.

Laminar Flow

In[18]:=
```
γ = 1; θ =.; ϕ =.;
v = {vx, vy, vz};
ω = {ωx, ωy, ωz};
```

In[21]:=
$$\delta\alpha[\gamma] = \frac{\delta\tau}{I\ \delta\theta\ \delta\phi}\ \text{// FullSimplify;}$$

In[22]:=
$$\alpha[\gamma] = \int_0^{\pi}\int_0^{2\pi} \delta\alpha[\gamma]\, d\phi\, d\theta;$$

In[23]:=
```
{{"ω̇x =", "ω̇y =", "ω̇z ="}, α[γ]} // Transpose // TableForm
```

Out[23]//TableForm=

$$\dot{\omega}_x = -\frac{10\,\pi\,R^2\,\rho\,\omega x\,C_D}{3\,m}$$

$$\dot{\omega}_y = -\frac{10\,\pi\,R^2\,\rho\,\omega y\,C_D}{3\,m}$$

$$\dot{\omega}_z = -\frac{10\,\pi\,R^2\,\rho\,\omega z\,C_D}{3\,m}$$

In[24]:=
$$\text{Solve}\left[\{\alpha x, \alpha y, \alpha z\} == \alpha[\gamma]\ \&\&\ k_w == \frac{1}{2}\frac{\rho\,A\,C_w}{m}, \{\alpha x, \alpha y, \alpha z\}\right][[1]]\ \text{// TableForm}$$

Out[24]//TableForm=

$$\alpha x \to -\omega x\ k_w$$

$$\alpha y \to -\omega y\ k_w$$

$$\alpha z \to -\omega z\ k_w$$

Conclusion:

$$\boldsymbol{\alpha} == -k_w\ \boldsymbol{\omega}$$

▼ Turbulent Flow, ω = 0

In[25]:=
```
γ = 2; θ =.; ϕ =.;
v = {vx, vy, vz};
ω = {0, 0, 0};
```

In[28]:=

$$\delta\alpha[\gamma] = \frac{\delta\tau}{I\,\delta\theta\,\delta\phi} \text{ // FullSimplify;}$$

In[29]:=

$$\alpha[\gamma] = \int_0^{\pi}\int_0^{2\pi} \delta\alpha[\gamma]\, d\phi\, d\theta;$$

In[30]:=

$$\{\{\text{"}\dot{\omega}_x \text{ =", "}\dot{\omega}_y \text{ =", "}\dot{\omega}_z \text{ ="}\}, \alpha[\gamma]\} \text{ // Transpose // TableForm}$$

Out[30]//TableForm=

$\dot{\omega}_x$	=	0
$\dot{\omega}_y$	=	0
$\dot{\omega}_z$	=	0

In[31]:=

$$\text{Solve}\left[\{\alpha x, \alpha y, \alpha z\} == \alpha[\gamma] \,\&\&\, k_w == \frac{1}{2}\frac{\rho\, A\, C_w}{m}, \{\alpha x, \alpha y, \alpha z\}\right][[1]] \text{ // TableForm}$$

Out[31]//TableForm=

$\alpha x \to 0$
$\alpha y \to 0$
$\alpha z \to 0$

Conclusion:

$\boldsymbol{\alpha} == 0$ for $\boldsymbol{\omega} == 0$

▼ Turbulent Flow, v = 0

In[32]:=
```
γ = 2; θ =.; ϕ =.;
v = {0, 0, 0};
ω = {0, 0, ωz};
```

In[35]:=
$$\delta\alpha[\gamma] = \frac{\delta\tau}{I\,\delta\theta\,\delta\phi} \text{ // FullSimplify;}$$

In[36]:=
$$\alpha[\gamma] = \int_0^{\pi}\int_0^{2\pi} \delta\alpha[\gamma]\, d\phi\, d\theta;$$

In[37]:=
$$\{\{"\dot\omega_x =", "\dot\omega_y =", "\dot\omega_z ="\}, \alpha[\gamma]\} \text{ // Transpose // TableForm}$$

Out[37]//TableForm=

$$\dot\omega_x = 0$$
$$\dot\omega_y = 0$$
$$\dot\omega_z = -\frac{15\,\pi^2\,R^3\,\rho\,\omega z\,\mathrm{Abs}[\omega z]\,C_D}{16\,m}$$

In[38]:=
$$\text{Solve}\left[\{\alpha x, \alpha y, \alpha z\} == \alpha[\gamma]\ \&\&\ k_w == \frac{1}{2}\,\frac{\rho\,A\,C_w}{m}, \{\alpha x, \alpha y, \alpha z\}\right][[1]] \text{ // TableForm}$$

Out[38]//TableForm=

$$\alpha x \to 0$$
$$\alpha y \to 0$$
$$\alpha z \to -\frac{9}{32}\,\pi\,R\,\omega z\,\mathrm{Abs}[\omega z]\,k_w$$

Conclusion:

$\alpha_z == -k_w\,\omega_z\,|R\,\omega_z|$ for $\mathbf{v} == 0$.

▼ Turbulent Flow, vx ≠ 0, $\omega z \neq 0$, $\theta == \pi/2$

```
γ = 2; θ = π / 2; ϕ =.; R =.;
v = {vx, 0, 0};
ω = {0, 0, ωz};
```

$$\delta\alpha[\gamma] = \frac{\delta\tau}{k_w\, \mathcal{I}\, \delta\theta\, \delta\phi} \text{ // FullSimplify}$$

$$\left\{0,\ 0,\ \frac{3\,(-R\,\omega z + vx\,\text{Sin}[\phi])\,\sqrt{vx^2 + R^2\,\omega z^2 - 2\,R\,vx\,\omega z\,\text{Sin}[\phi]}}{8\,\pi\,R}\right\}$$

$$\alpha 3 = \int \delta\alpha[\gamma]_{[\![3]\!]}\, d\phi \text{ // FullSimplify}$$

$$\Bigg(-2\,R\,vx\,\omega z\,\text{Cos}[\phi]\,\left(vx^2 + R^2\,\omega z^2 - 2\,R\,vx\,\omega z\,\text{Sin}[\phi]\right) + (vx - R\,\omega z)^2 \Bigg(\left(vx^2 + 7\,R^2\,\omega z^2\right)\,\text{EllipticE}\left[\frac{1}{4}\,(\pi - 2\,\phi),\ -\frac{4\,R\,vx\,\omega z}{(vx - R\,\omega z)^2}\right] - (vx + R\,\omega z)^2\,\text{EllipticF}\left[\frac{1}{4}\,(\pi - 2\,\phi),\ -\frac{4\,R\,vx\,\omega z}{(vx - R\,\omega z)^2}\right]\Bigg)\sqrt{\frac{vx^2 + R^2\,\omega z^2 - 2\,R\,vx\,\omega z\,\text{Sin}[\phi]}{(vx - R\,\omega z)^2}}\Bigg) \Big/ \left(8\,\pi\,R^2\,\omega z\,\sqrt{vx^2 + R^2\,\omega z^2 - 2\,R\,vx\,\omega z\,\text{Sin}[\phi]}\right)$$

```
(α3 /. ϕ → 2 π) - (α3 /. ϕ → 0) // FullSimplify
```

$$-\frac{1}{8\,\pi\,R^2\,\omega z}\,\text{Abs}[vx - R\,\omega z]\,\Bigg(\left(vx^2 + 7\,R^2\,\omega z^2\right)\,\text{EllipticE}\left[\frac{\pi}{4},\ -\frac{4\,R\,vx\,\omega z}{(vx - R\,\omega z)^2}\right] + \left(vx^2 + 7\,R^2\,\omega z^2\right)\,\text{EllipticE}\left[\frac{3\,\pi}{4},\ -\frac{4\,R\,vx\,\omega z}{(vx - R\,\omega z)^2}\right] - (vx + R\,\omega z)^2\,\left(\text{EllipticF}\left[\frac{\pi}{4},\ -\frac{4\,R\,vx\,\omega z}{(vx - R\,\omega z)^2}\right] + \text{EllipticF}\left[\frac{3\,\pi}{4},\ -\frac{4\,R\,vx\,\omega z}{(vx - R\,\omega z)^2}\right]\right)\Bigg)$$

Continued with manual simplification...

```
In[39]:= γ = 2; θ = π / 2; ϕ =.; R =.;
v = {vx, 0, 0};
ω = {0, 0, ωz};
vx =.; ωz =.;
```

In[43]:= $\text{E1E} := \text{EllipticE}\left[\frac{\pi}{4}, -\frac{4\,\text{R}\,\text{vx}\,\omega\text{z}}{(\text{vx} - \text{R}\,\omega\text{z})^2}\right] + \text{EllipticE}\left[\frac{3\pi}{4}, -\frac{4\,\text{R}\,\text{vx}\,\omega\text{z}}{(\text{vx} - \text{R}\,\omega\text{z})^2}\right];$

In[44]:= $\text{E1F} := \text{EllipticF}\left[\frac{\pi}{4}, -\frac{4\,\text{R}\,\text{vx}\,\omega\text{z}}{(\text{vx} - \text{R}\,\omega\text{z})^2}\right] + \text{EllipticF}\left[\frac{3\pi}{4}, -\frac{4\,\text{R}\,\text{vx}\,\omega\text{z}}{(\text{vx} - \text{R}\,\omega\text{z})^2}\right];$

In[45]:= $\text{E3} := -\frac{\text{Abs}[\text{vx} - \text{R}\,\omega\text{z}]}{8\,\pi\,\text{R}^2\,\omega\text{z}}\left(\left(\text{vx}^2 + 7\,\text{R}^2\,\omega\text{z}^2\right)\text{E1E} - (\text{vx} + \text{R}\,\omega\text{z})^2\,\text{E1F}\right)$

In[46]:= $\text{E3} := -\frac{\text{Abs}[\text{vx} - \text{R}\,\omega\text{z}]}{8\,\pi}\,\omega\text{z}\left(\left(\left(\frac{\text{vx}}{\text{R}\,\omega\text{z}}\right)^2 + 7\right)\text{E1E} - \left(\frac{\text{vx}}{\text{R}\,\omega\text{z}} + 1\right)^2\text{E1F}\right)$

```
In[47]:= Limit[E3, vx → 0, Assumptions → ωz ∈ Reals ∧ R > 0]
```

Out[47]= $-\frac{3}{4}\,\text{R}\,\omega\text{z}\,\text{Abs}[\omega\text{z}]$

```
In[48]:= Limit[E3, ωz → 0]
Out[48]= 0

In[49]:= ωz = 1; vx = -1.001; R = 1; E3 // N
Out[49]= -1.27419
```

In[50]:= $\omega\text{z} = 1;\ \text{vx} = -10^6;\ \text{R} = 1;\ \text{E3} // \text{N}$

Out[50]= -1.12498×10^6

In[51]:= $\omega\text{z} = 1;\ \text{vx} = -10^7;\ \text{R} = 1;\ \text{E3} // \text{N}$

Out[51]= -1.12652×10^7

In[52]:= $\omega\text{z} = 10^4;\ \text{vx} = -1;\ \text{R} = 1;\ \text{E3} // \text{N}$

Out[52]= -7.5×10^7

In[53]:= $\omega\text{z} = 10^5;\ \text{vx} = -1;\ \text{R} = 1;\ \text{E3} // \text{N}$

Out[53]= -7.5×10^9

In[54]:= $\omega\text{z} = 1;\ \text{vx} = -1.001;\ \text{R} = 10^8;\ \text{E3} // \text{N}$

Out[54]= -7.5×10^7

In[55]:= $\omega\text{z} = 1;\ \text{vx} = -1.001;\ \text{R} = 10^9;\ \text{E3} // \text{N}$

Out[55]= -7.5×10^8

There exists:

1) square dependence of angular velocity,
2) linear dependence of linear velocity, and
3) linear dependence on radius.

Conclusion:

$$\alpha_z \propto -k_w\,\omega_z\,\left(|v_x| + |R\,\omega_z|\right)$$

Turbulent Flow, golf ball, numerical analysis

In[56]:=
```
γ = 2; θ =.; ϕ =.;
ρ = 1.3; m = 0.045; R = 0.02; C_D = 1; C_w = 10;
ωz =.; vx =.;
v := {vx, 0, 0}
ω := {0, 0, ωz}
```

Numerical solution

In[61]:=
$$\delta\alpha[\gamma] = \frac{\delta\tau}{I\,\delta\theta\,\delta\phi} \;//\; \text{FullSimplify};$$

In[62]:=
```
foo[vx_, ωz_] = δα[γ][[3]];
```

In[63]:=
```
sol[vx_, ωz_] := NIntegrate[foo[vx, ωz], {ϕ, 0, 2 π}, {θ, 0, π}]
```

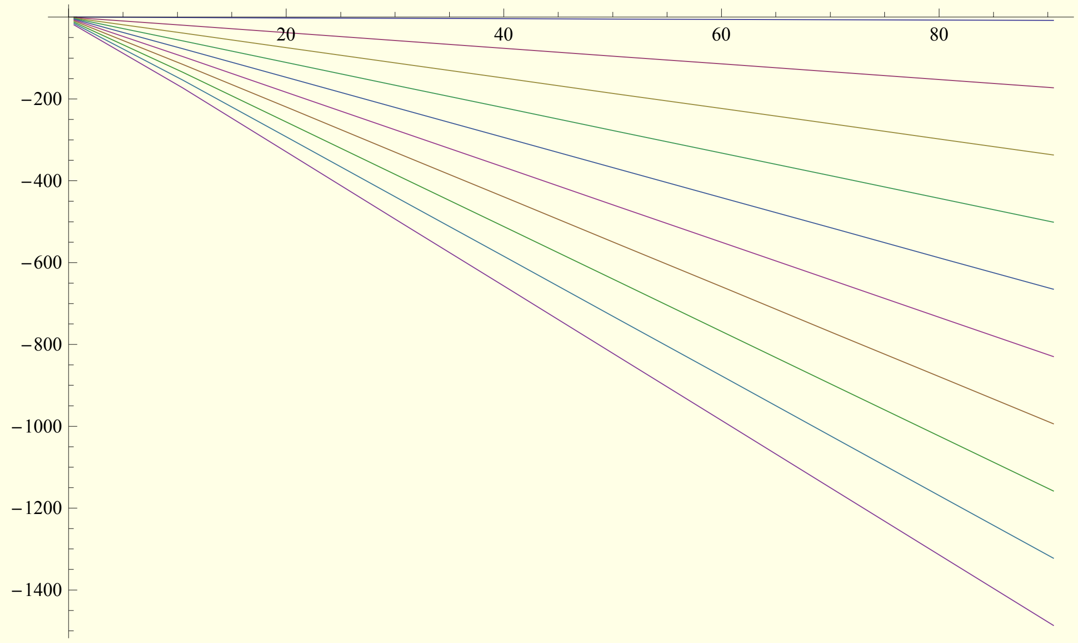
Guessed solution

In[64]:=
$$\text{guessed}[vx\_, \omega z\_] := -\frac{1}{2}\,\frac{\rho\,A\,C_w}{m}\,\omega z\,(R\,\omega z + vx)$$

Comparison of numerical solution (top diagram) and guessed solution (bottom diagram) where abscissa = v_x and ordinate = $\dot{\omega}$

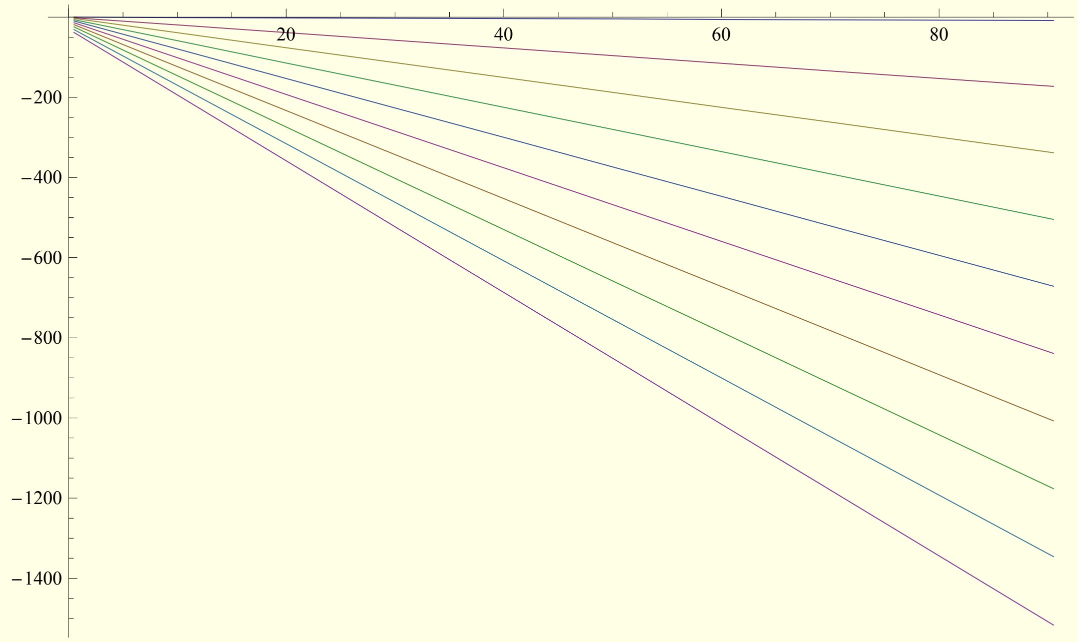
In[65]:=
```
ListLinePlot[Table[Table[{vx, sol[vx, ωz]}, {vx, 0.5, 100, 10}], {ωz, 0.5, 100, 10}]]
```

Out[65]=



In[66]:=
```
ListLinePlot[
 Table[Table[{vx, guessed[vx, ωz]}, {vx, 0.5, 100, 10}], {ωz, 0.5, 100, 10}]]
```

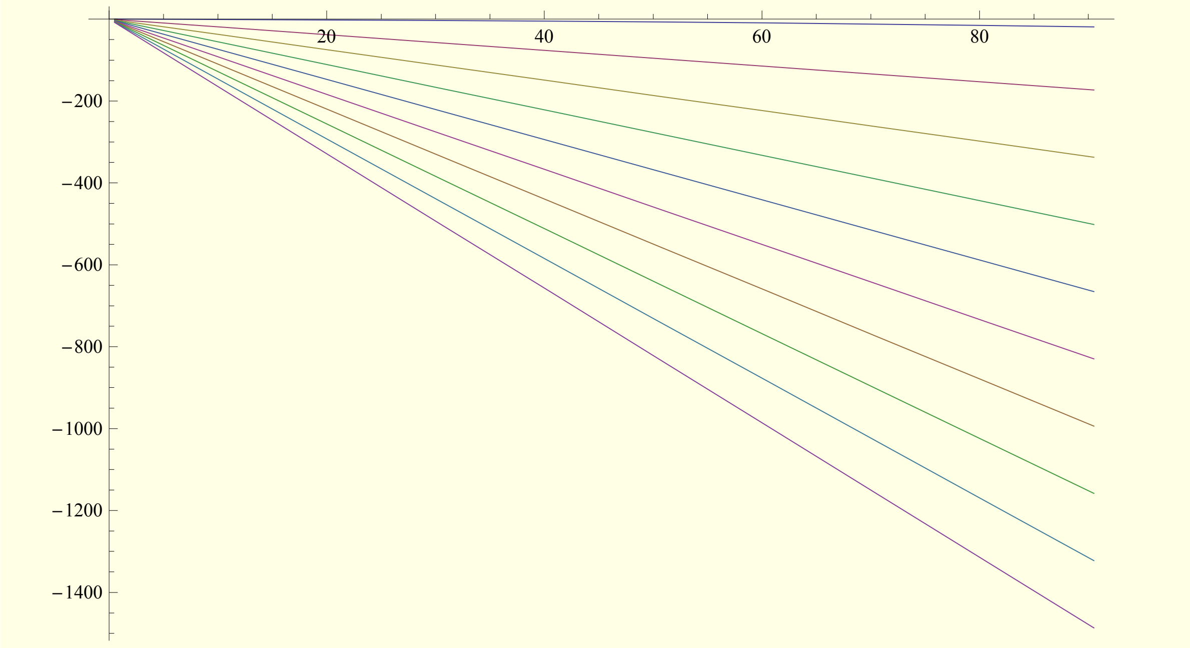
Out[66]=



Comparison of numerical solution (top diagram) and guessed solution (bottom diagram) where abscissa = ω_z and ordinate = $\dot{\omega}$

In[67]:=
```
ListLinePlot[Table[Table[{ωz, sol[vx, ωz]}, {ωz, 0.5, 100, 10}], {vx, 0.5, 100, 10}]]
```

Out[67]=



In[68]:=
```
ListLinePlot[
 Table[Table[{ωz, guessed[vx, ωz]}, {ωz, 0.5, 100, 10}], {vx, 0.5, 100, 10}]]
```

Out[68]=