Lab1: Shadowing a Particle

Background

Analyze the trajectory $\mathbf{x}(t)$ of a particle of mass m in a given force field $\mathbf{f}(\mathbf{x})$ and with given initial conditions $\mathbf{x}(t_0)$ and $\mathbf{v}(t_0)$. Study the dependency of the trajectory on disturbances in the initial conditions by simulating a system of particles with slightly different initial conditions. Study the dependency of the numerically computed trajectory on the time-step size h by comparing with a given reference trajectory representing the exact solution.

System of N_p particles in 2D of mass m is governed by Newton's equations of motion. The force $\mathbf{f}(\mathbf{x})$ and potential energy $E_\mathrm{p}(\mathbf{x})$ for each particle is:

$$\mathbf{f}(\mathbf{x}) = m\,\mathbf{g} - \sum_{k=1}^{5} \frac{f_k}{r_k^2} \exp\left(-\frac{\|\mathbf{x}-\mathbf{p}_k\|^2}{2\,r_k^2}\right)(\mathbf{x}-\mathbf{p}_k)$$

$$E_\mathrm{p}(\mathbf{x}) = -m\,\mathbf{g}\cdot\mathbf{x} - \sum_{k=1}^{5} f_k \exp\left(-\frac{\|\mathbf{x}-\mathbf{p}_k\|^2}{2\,r_k^2}\right)$$

This force and potential represents gravity $\mathbf{g}$ and $k = 1, 2, ..., 5$ spherically symmetric attractive force fields with center position p_k, strength f_k and radius r_k, respectively.

General Parameters

$N_\mathrm{p} = 500, \quad m = 1\,\mathrm{kg}, \quad L = 5\,\mathrm{m}$

Initial Conditions

$t_0 = 0\,\mathrm{s}, \quad t_\mathrm{f} = 3\,\mathrm{s}, \quad h = 0.01\,\mathrm{s},$

$\mathbf{x}(t_0) = (-L, L), \quad \mathbf{v}(t_0) = (5, 10)\,\mathrm{m/s}$

Physical Constants

$\mathbf{g} = (0, -9.81)\,\mathrm{m/s^2}$

Symmetric Attractive Force Fields

$f_1 = 32\,\mathrm{N}, \quad f_2 = 40\,\mathrm{N}, \quad f_3 = 28\,\mathrm{N}, \quad f_4 = 16\,\mathrm{N}, \quad f_5 = 20\,\mathrm{N}$

$r_1 = 0.3\,L, \quad r_2 = 0.2\,L, \quad r_3 = 0.4\,L, \quad r_4 = 0.5\,L, \quad r_5 = 0.3\,L$

$p_1 = (-0.2, 0.8)\,L,$

$p_2 = (-0.3, -0.8)\,L,$

$p_3 = (-0.6, 0.1)\,L,$

$p_4 = (0.4, 0.7)\,L,$

$p_5 = (0.8, -0.3)\,L$

Position and Velocity Disturbances

$\delta\mathbf{x} = (\delta\mathrm{x}, \delta\mathrm{y}); \quad \delta\mathrm{x}, \delta\mathrm{y} \in (-0.02, 0.02)\,L$

$\delta\mathbf{v} = (\delta\mathrm{v}_x, \delta\mathrm{v}_y); \quad \delta\mathrm{v}_x, \delta\mathrm{v}_y \in (-0.02\,\mathbf{v}(t_0), 0.02\,\mathbf{v}(t_0))$

Parameters

```
Remove["`*"]; (* Remove all global symbols *)
```

General Parameters

```
L := 5; (* Characteristic length of the system *)
```

```
m := 1; (* Particle mass *)
```

Initial Conditions

```
t0 = 0; (* Initial time *)
```

```
tf = 3; (* Final time *)
```

```
x0 := {-L, -L}; (* Initial position *)
```

```
v0 := {5, 10}; (* Initial velocity *)
```

Physical Constants

```
g := {0, -9.81}; (* Acceleration of gravity *)
```

Symmetric Attractive Force Fields

```
f := {32, 40, 28, 16, 20}; (* Force strength *)
```

```
r := {0.3, 0.2, 0.4, 0.5, 0.3} L; (* Force radius *)
```

$$p := \begin{pmatrix} -0.2 & 0.8 \\ -0.3 & -0.8 \\ -0.6 & 0.1 \\ 0.4 & 0.7 \\ 0.8 & -0.3 \end{pmatrix} L; \text{(* Center position of the force *)}$$

Forces and Energy

Total Force

$$F[x\_?VectorQ] := m\,g - \sum_{k=1}^{5} \left(\frac{f_{[[k]]}}{r_{[[k]]}{}^2} \, Exp\left[-\frac{Norm[x - p_{[[k]]}]^2}{2\, r_{[[k]]}{}^2} \right] (x - p_{[[k]]}) \right);$$

Potential and Kinetic Energy

$$Ep[x\_?VectorQ] := -m\,g.x - \sum_{k=1}^{5} \left(f_{[[k]]} \, Exp\left[-\frac{Norm[x - p_{[[k]]}]^2}{2\, r_{[[k]]}{}^2} \right] \right);$$

$$Ek[v\_?VectorQ] := \frac{m\,v.v}{2};$$

Equations of Motion

```
SolveODE[
   t0_, tf_,
   x0_?VectorQ, v0_?VectorQ,
   method_: Automatic, h_: Automatic
  ] := Module[
   {X, V, sol},
   sol = NDSolve[
     {
      (* Differential Equations: *)
          X'[t] == V[t],
          m V'[t] == F[X[t]],
      (* Initial conditions: *)
          X[t0] == x0,
          V[t0] == v0
     },
     {X, V}, (* Dependent variables *)
     {t, t0, tf} (* Range of the independent variable *)
     , Method → method
     , StartingStepSize → h
     , MaxStepSize → h
     , MaxSteps → Infinity
    ];
   ({X, V} /. sol)[[1]]
  ];
```

■ Reference Solution, RK4, h = 1e-4

```
X$ref = {-3.983 781 315 868 414, -5.118 978 158 404 967};
Etot$ref = 13.008 837 864 965 010;
```

■ Solution, Runge-Kutta, h = 0.01

```
{X, V} = SolveODE[t0, tf, x0, v0, "ExplicitRungeKutta", 0.01];
```

```
NumberForm[Norm[X[tf] - X$ref], 20]
```

$3.377978811938265\times10^{-11}$

```
NumberForm[Abs[Ep[X[tf]] + Ek[V[tf]] - Etot$ref], 20]
```

$2.967155410260602\times10^{-10}$

■ Solution, Forward Euler, h = 0.01

```
{X, V} = SolveODE[t0, tf, x0, v0, "ExplicitEuler", 0.01];
```

```
NumberForm[Norm[X[tf] - X$ref], 20]
```

0.2517073657616581

```
NumberForm[Abs[Ep[X[tf]] + Ek[V[tf]] - Etot$ref], 20]
```

2.341932983396703

■ Solution, Automatic

```
{X, V} = SolveODE[t0, tf, x0, v0];
```

```
NumberForm[Norm[X[tf] - X$ref], 20]
```

$5.85117449696673\times10^{-7}$

```
NumberForm[Abs[Ep[X[tf]] + Ek[V[tf]] - Etot$ref], 20]
```

$3.176574857377545\times10^{-6}$

Plots

```
<< VectorFieldPlots`;
```

```
pathPlot := ParametricPlot[
   {X[t][[1]], X[t][[2]]},
   {t, t0, tf},
   PlotStyle → {Thickness[0.005], Ticks → {1, 1}, Red}
  ];
```

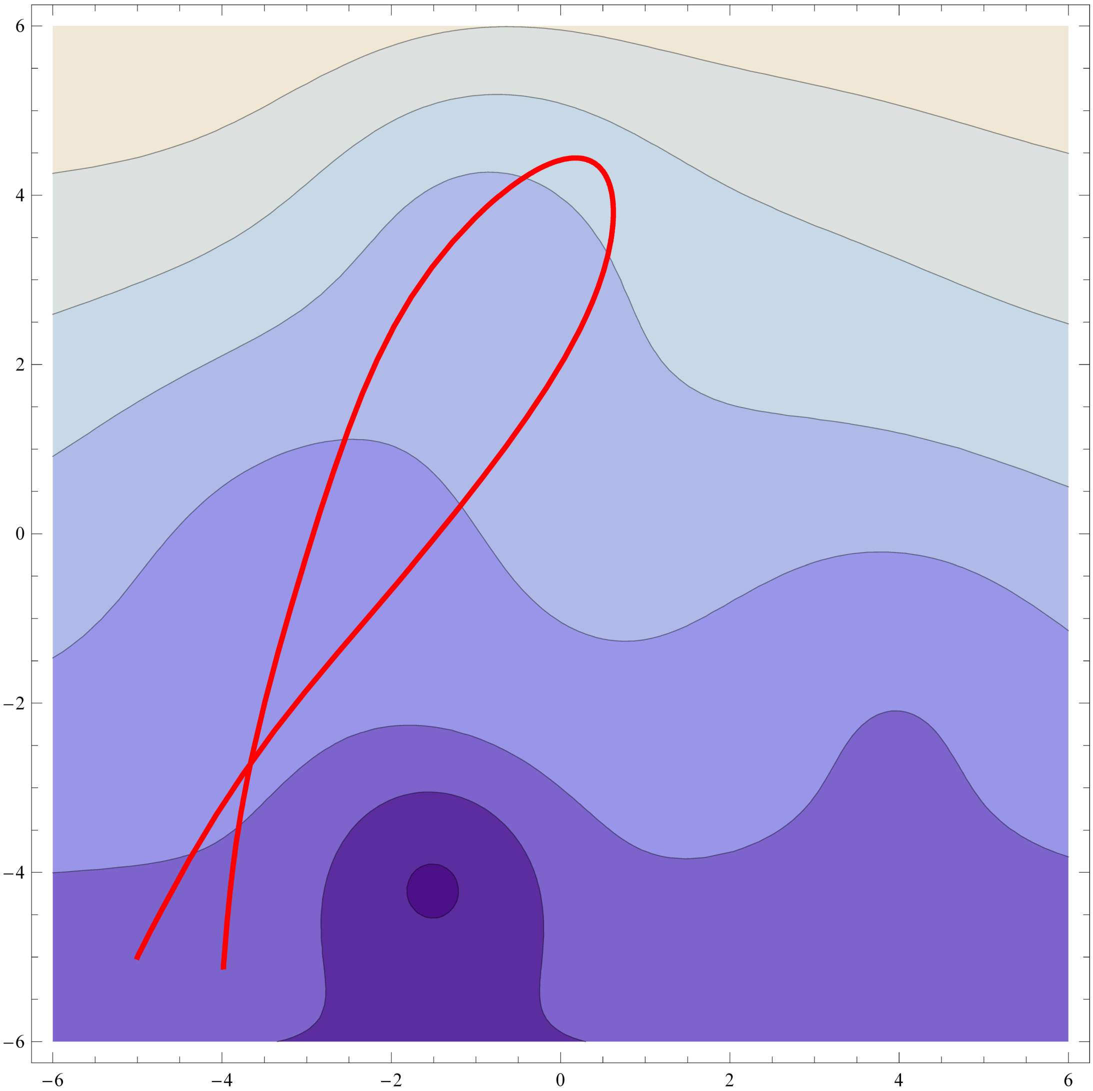
```
forceFieldPlot := VectorFieldPlot[
   F[{x1, x2}],
   {x1, -1.2 L, 1.2 L}, {x2, -1.2 L, 1.2 L},
   PlotPoints → 20
  ];
```

```
potentialPlot := ContourPlot[
   Ep[{x1, x2}],
   {x1, -1.2 L, 1.2 L}, {x2, -1.2 L, 1.2 L}
  ];
```

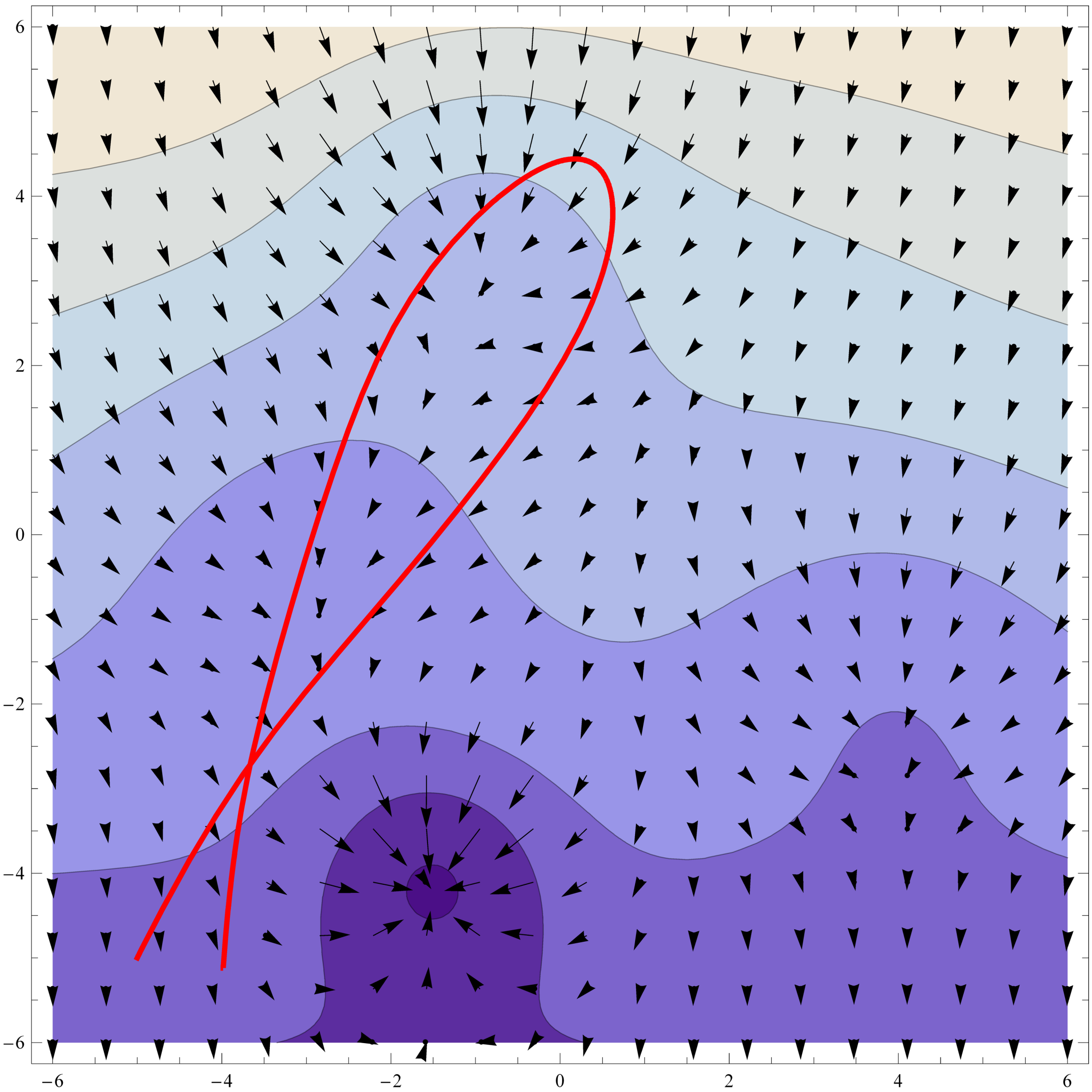
```
pathPlot3D := ParametricPlot3D[
   {X[t][[1]], X[t][[2]], Ep[X[t]]},
   {t, t0, tf},
   PlotStyle → {Thickness[0.005], Red}
  ];
```

```
potentialPlot3D := Plot3D[
   Ep[{x1, x2}],
   {x1, -1.2 L, 1.2 L}, {x2, -1.2 L, 1.2 L}
  ];
```

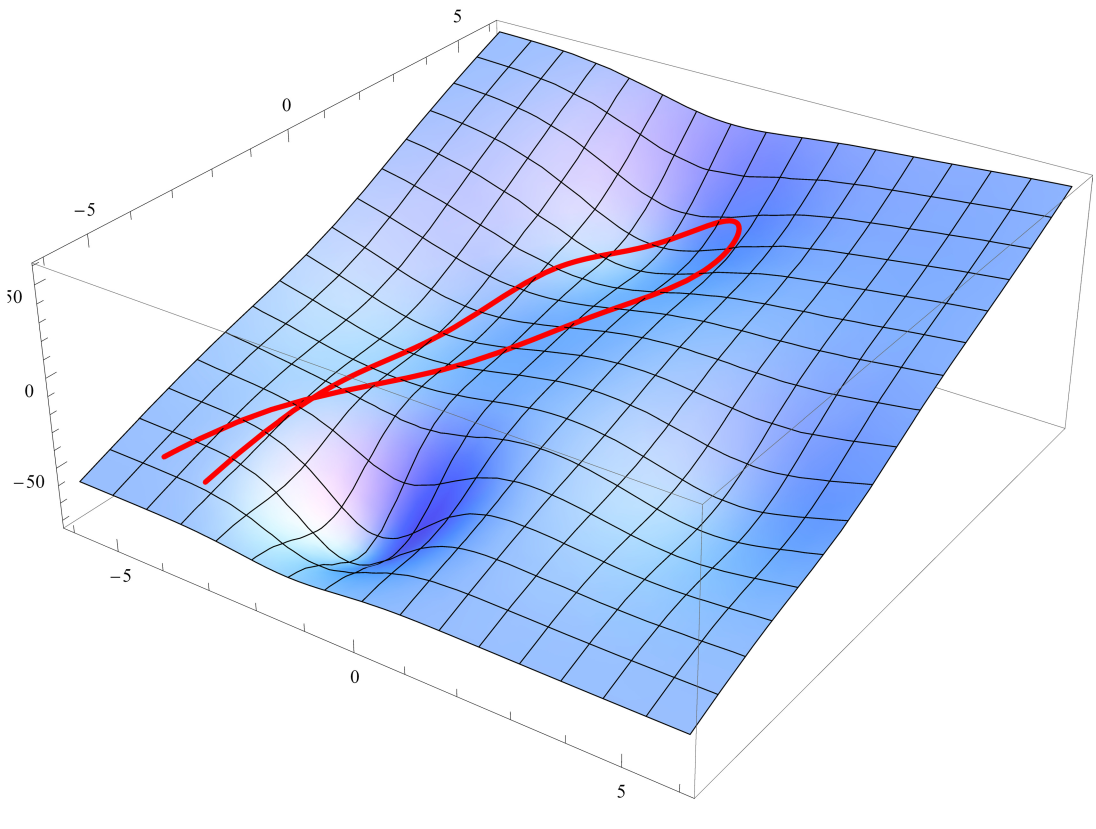
```
Show[ potentialPlot, pathPlot ]
```



```
Show[potentialPlot, forceFieldPlot, pathPlot]
```



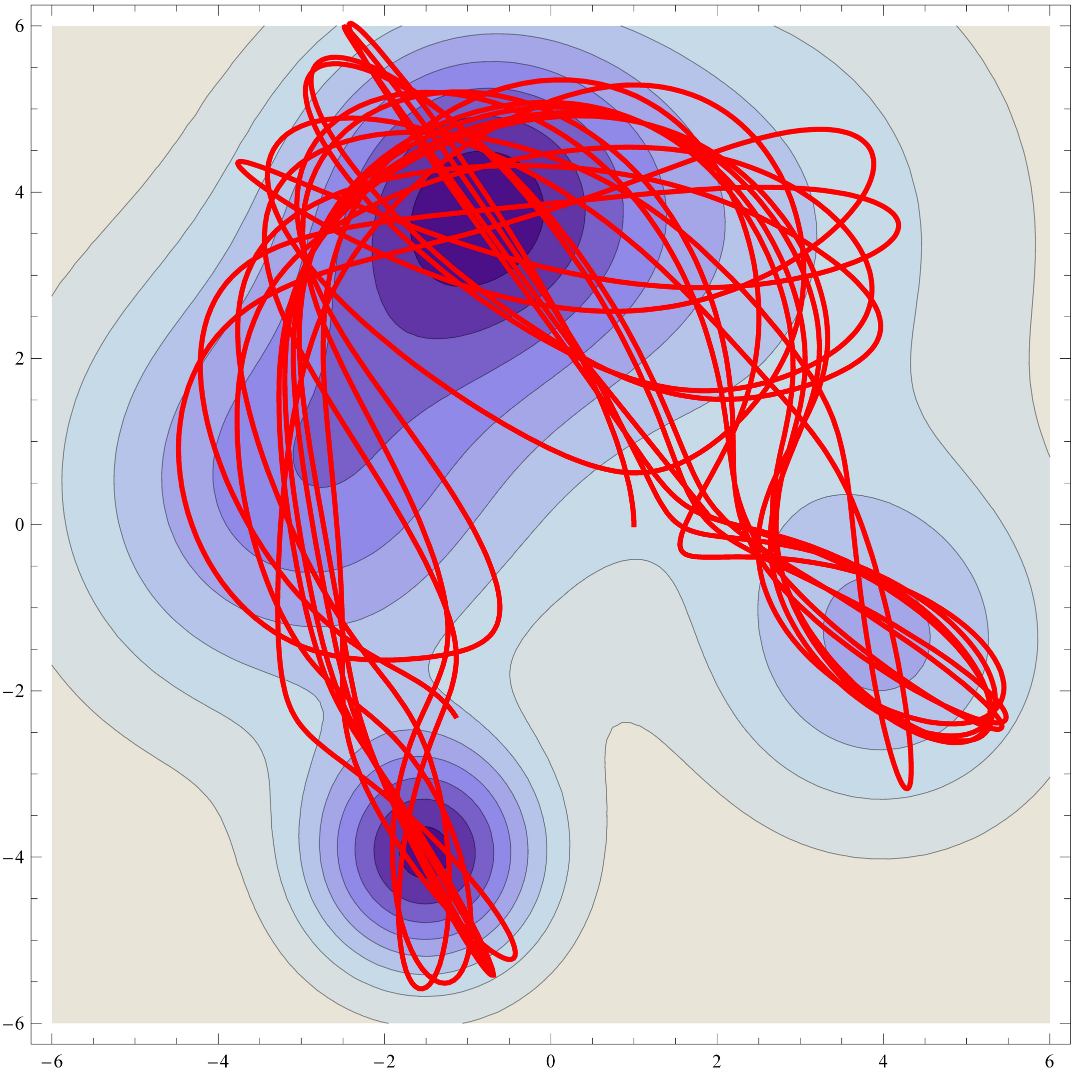
```
Show[potentialPlot3D, pathPlot3D]
```



Solution without Gravitation Field

```
g = {0, 0};
tf = 100;
{X, V} = SolveODE[ t0, tf, {1, 0}, {0, 1} ];
```

```
Show[ potentialPlot, pathPlot ]
```



```
Show[ potentialPlot3D, pathPlot3D ]
```