Supplementary: Solitary death in coupled limit-cycle oscillators with higher-order interactions

Subhasanket Dutta, Umesh Kumar Verma, and Sarika Jalan*
Complex Systems Lab, Department of Physics, Indian Institute of Technology Indore, Khandwa Road, Simrol, Indore-453552, India

The dynamical equation of uncoupled Stuart-Landau (SL) oscillator [1] can be written by,

$$\dot{z}(t) = (a^2 - |z(t)|^2)z + i\omega z \tag{S1}$$

Here z is a complex variable depicting the dynamical state of an oscillator with ω being its intrinsic frequency. The oscillator has one unstable fixed point acting as a center for a stable circular limit cycle of radius a.

The dynamical evolution of an uncoupled SL oscillator is governed by Eq. S1. Upon substituting $z = x + iy$, the resulting equation is,

$$\begin{aligned} \dot{x} &= (1 - x^2 - y^2)x - wy \\ \dot{y} &= (1 - x^2 - y^2)y + wx \end{aligned}$$

Now we are considering an ensemble of N SL oscillators, which are interacting with each other via higher-order interaction. The dynamics of coupled SL oscillators are given by,

$$\dot{z}_j(t) = (1 - |z_j(t)|^2)z_j + i\omega z_j + \frac{\varepsilon}{N^2}\sum_{k=1}^{N}\sum_{l=1}^{N} z_k z_l \tag{S2}$$

Substituting $z_j = r_j e^{\iota\theta_j}$ in above equation we get,

$$\begin{aligned} \dot{r}_j &= (1 - r_j^2)r_j + \frac{\varepsilon}{N^2}\sum_{k,l=1}^{N} r_k r_l cos(\theta_k + \theta_l - \theta_j) \\ \dot{\theta}_j &= \omega_j + \frac{\varepsilon}{N^2}\sum_{k,l=1}^{N} \frac{r_k r_l}{r_j} sin(\theta_k + \theta_l - \theta_j) \end{aligned}$$

The cartesian form of above equation are,

$$\begin{aligned} \dot{x}_j &= (1 - x_j^2 - y_j^2)x_j - w_j y_j - \frac{\varepsilon}{N^2}\sum_{k,l=1}^{N}(x_k x_l - y_l y_k), \\ \dot{y}_j &= (1 - x_j^2 - y_j^2)y_j + w_j x_j - \frac{\varepsilon}{N^2}\sum_{k,l=1}^{N}(x_k y_l + x_l y_k) \end{aligned}$$

Coupling between SL oscillators may be responsible for the birth of new fixed points and the change in the stability properties of existing ones. In the following, we define an order parameter that quantifies the variance of fluctuation of the dynamical variables over a time span, which tends to 0 for both amlitude death(AD) and oscillation death

* sarika@iiti.ac.in

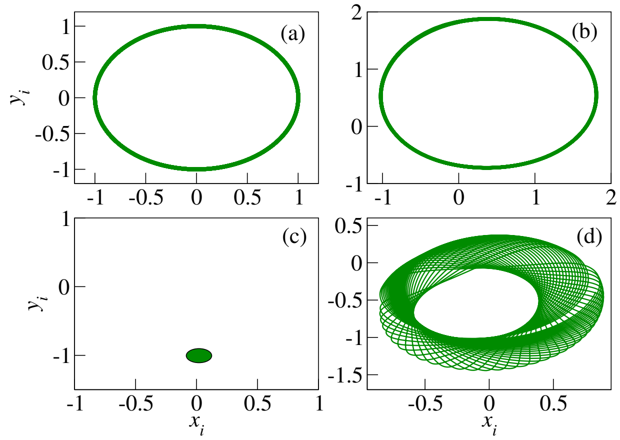

FIG. S1. Trajectory of the coupled SL oscillator governed by Eq. S2 in (x, y) phase space for (a) incoherent state ($\varepsilon = 1.0$), (b) synchronized state ($\varepsilon = 3.0$), (c) oscillation death ($\varepsilon = 4.0$) (d) torus ($\varepsilon = 8.0$).

(OD) cases:

$$A = \frac{1}{N}\sum_{i=1}^{N}(\langle x_i\rangle_{max,t} - \langle x_i\rangle_{min,t}),$$

a. ***Trajectory:*** Initially, in the incoherent state the trajectories are elliptical which is shown in Fig. S1(a). However, there is no presence of a limit cycle since different initial conditions assume different closed orbits. Upon increasing the coupling strength the system attains a synchronized state, where the shape of the orbit deforms which is shown in Fig. S1(b). Coupled system stabilized at a coupling-dependent steady state for a higher value of coupling, which is shown in Fig. S1(c). We also observed toroid orbit for a higher value of the coupling strength which is shown in Fig. S1(d).

b. ***Effects of noise:*** To check the robustness of the results, we analyze the effects of noise. We include Gaussian noise $\xi_j(t)$ in Eq. S2. The dynamics equation of coupled SL oscillators in the presence of noise can be written as

$$\dot{z}_j(t) = (1 - |z_j(t)|^2)z_j + i\omega z_j + \frac{\varepsilon}{N^2}\sum_{k=1}^{N}\sum_{l=1}^{N} z_k z_l + \gamma\xi_j(t)$$

where γ is the strength of Gaussian noise. We set noise strength $\gamma = 0.001$ and calculate amplitude order parameter A in both forward and backward directions (Fig. S2). In the forward direction, we observe a revival of oscillation in the presence of noise which is not observed in the absence of noise. Due to the presence of noise in the forward transition, the initial conditions in each value of coupling strength can be slightly different for all the synchronized oscillators. Then, at a critical coupling strength, the system oscillates.

c. ***Conjugate coupling in higher order:*** Here, we propose another form of higher-order coupling, where three oscillators interact via a multiplicative conjugate coupling. The dynamics of the coupled system can be written as,

$$\dot{z}_j(t) = (1 - |z_j(t)|^2)z_j + i\omega z_j + \frac{\varepsilon}{N^2}\sum_{k=1}^{N}\sum_{l=1}^{N} z_k z_l^*$$

An introduction of the higher-order conjugate coupling between a pair of connected nodes and upon substituting $z = x + iy$, the resulting equation is,

$$\begin{aligned}\dot{x}_j &= (1 - x_j^2 - y_j^2)x_j - w_j y_j + \frac{\varepsilon}{N^2}\sum_{k,l=1}^{N}(x_k x_l + y_l y_k),\\ \dot{y}_j &= (1 - x_j^2 - y_j^2)y_j + w_j x_j \end{aligned} \tag{S3}$$

In this model, the dynamics are a little different than in the previous model. We first calculate the synchronization order parameter S in both forward and backward directions which is shown in Fig. S3(a). Here we can see that the

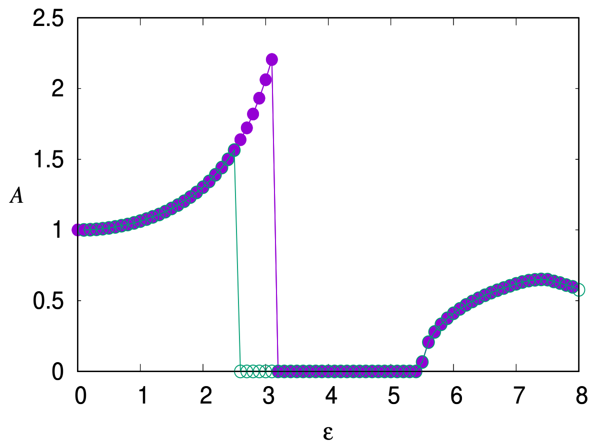

FIG. S2. Amplitude order parameter A plotted with coupling strength ε in both forward and backward direction in the presence of noise. The other parameters are $\omega = 4$, $N = 1000$ and $\gamma = 0.001$.

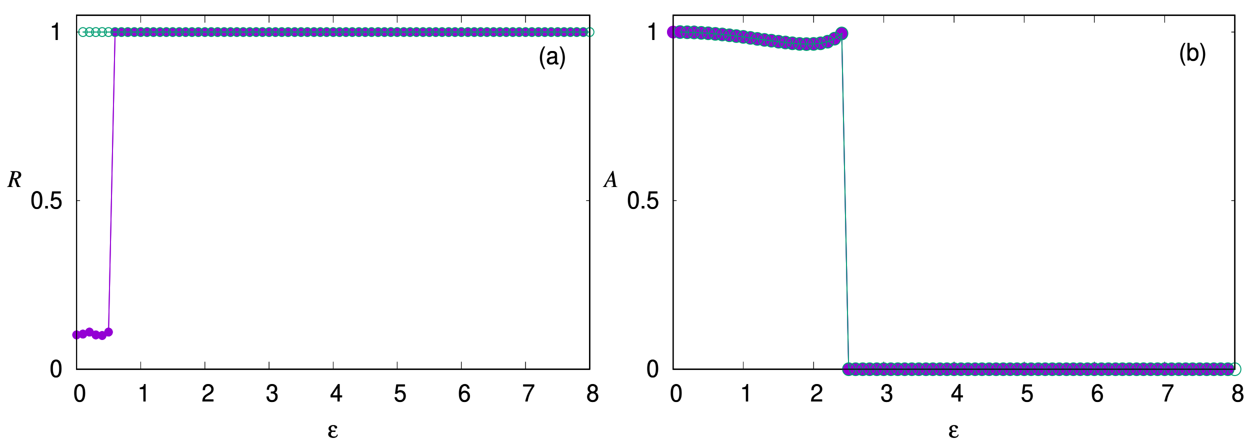

FIG. S3. (a) Synchronization order parameter R and (b) Amplitude order parameter A calculated in both forward and backward direction of coupled SL oscillators governed by Eq. S3. The other parameters are $N = 1000$ and $\omega = 4.0$.

order parameter shows a sudden transition in the forward direction, however, there is no transition in the backward direction. The amplitude order parameter A is also calculated in both forward and backward directions (Fig. S3(b)). Here, we can see that both forward and backward transition points occur at the same value, showing no hysteresis. In this case, we also observe that there is no increase in the amplitude of the oscillator after synchronization.

d. ***Stuart-Landau oscillators with pairwise interaction:*** Next, we consider an ensemble of N SL oscillators, which interact via pairwise interaction [2]. The dynamics of coupled SL oscillators are given by,

$$\dot{z}_j(t) = (1 - |z_j(t)|^2)z_j + i\omega z_j + \frac{\varepsilon}{N^2}\sum_{k=1}^{N} F_k \tag{S4}$$

where $F_k = z_k$ or z_k^*. In the case of pairwise interaction, when both variable x and y get positive feedback (i.e. we are considering $F_k = z_k$ in coupling term in the Eq. S4) we observe that only the amplitude of the coupled system is increasing, which is shown in Fig. S4(a). On the other hand when the x variable is getting positive feedback and y variable getting negative feedback (i.e. we consider $F_k = z_k^*$ in the coupling term of Eq. S4) we observe a sudden transition from oscillatory state to death state in both forward and backward continuation (Fig. S4(b)). Here both forward and backward transition points are the same.

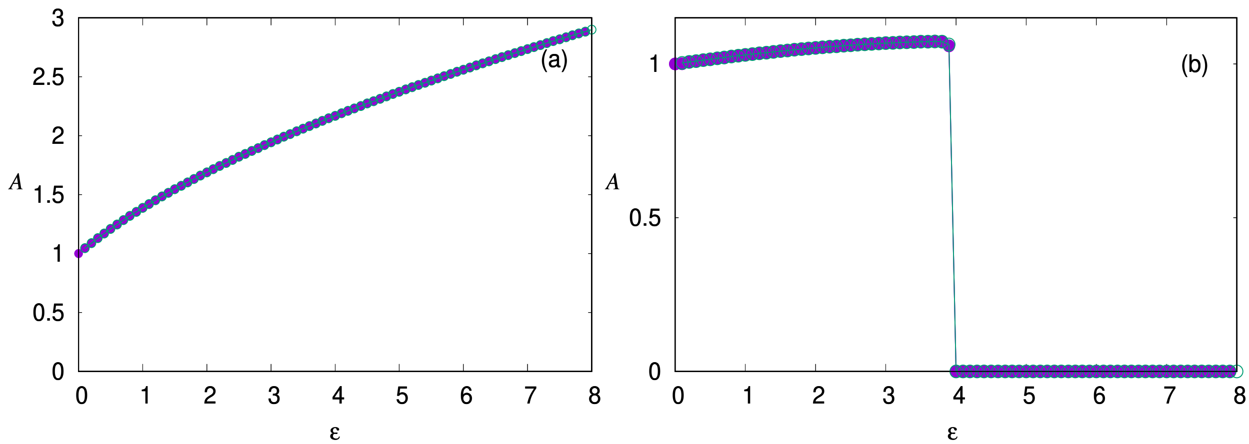

FIG. S4. Amplitude order parameter A calculated in both forward and backward direction of coupled SL oscillators of Eq. S4 (a) positive feedback coupling in both variable x and y (b) positive feedback in x variable and negative feedback coupling in variable y. The other parameters are $N = 1000$, and $\omega = 4.0$.

[1] Ramana Reddy D. V., Sen A., and Johnston G. L. Time Delay Induced Death in Coupled Limit Cycle Oscillators. *Phys. Rev.Lett.* **80**, 5109 (1998).

[2] Sathiyadevi K., Premraj D., Banerjee T., and Lakshmanan M. Additional complex conjugate feedback-induced explosive death and multistabilities. *Phys. Rev. E*, **106**, 024215 (2022).