

About the course....

- Course site: <https://complexity-methods.github.io>
- We refer to websites that have assignments and additional literature, one we call “the course book” ... https://darwin.pwo.ru.nl/skunkworks/courseware/1819_CMBS/book/
- Literature
- Most assignments about analyses were designed for **R**. You do not need to learn to **script/code**, you mainly need to be able to load data and run scripts. There will be instructions in lecture notes and assignment solutions that should give you enough information to work out what to do.
- There's also a module for JAMOVI
- This afternoon we'll help you get started with the software



Complexity Methods for Behavioural Science

Day 1: Intro Mathematics of Change

1D dynamical models

Notes on Deterministic Chaos

Extra (if time permits):

2D models (coupled dynamics)

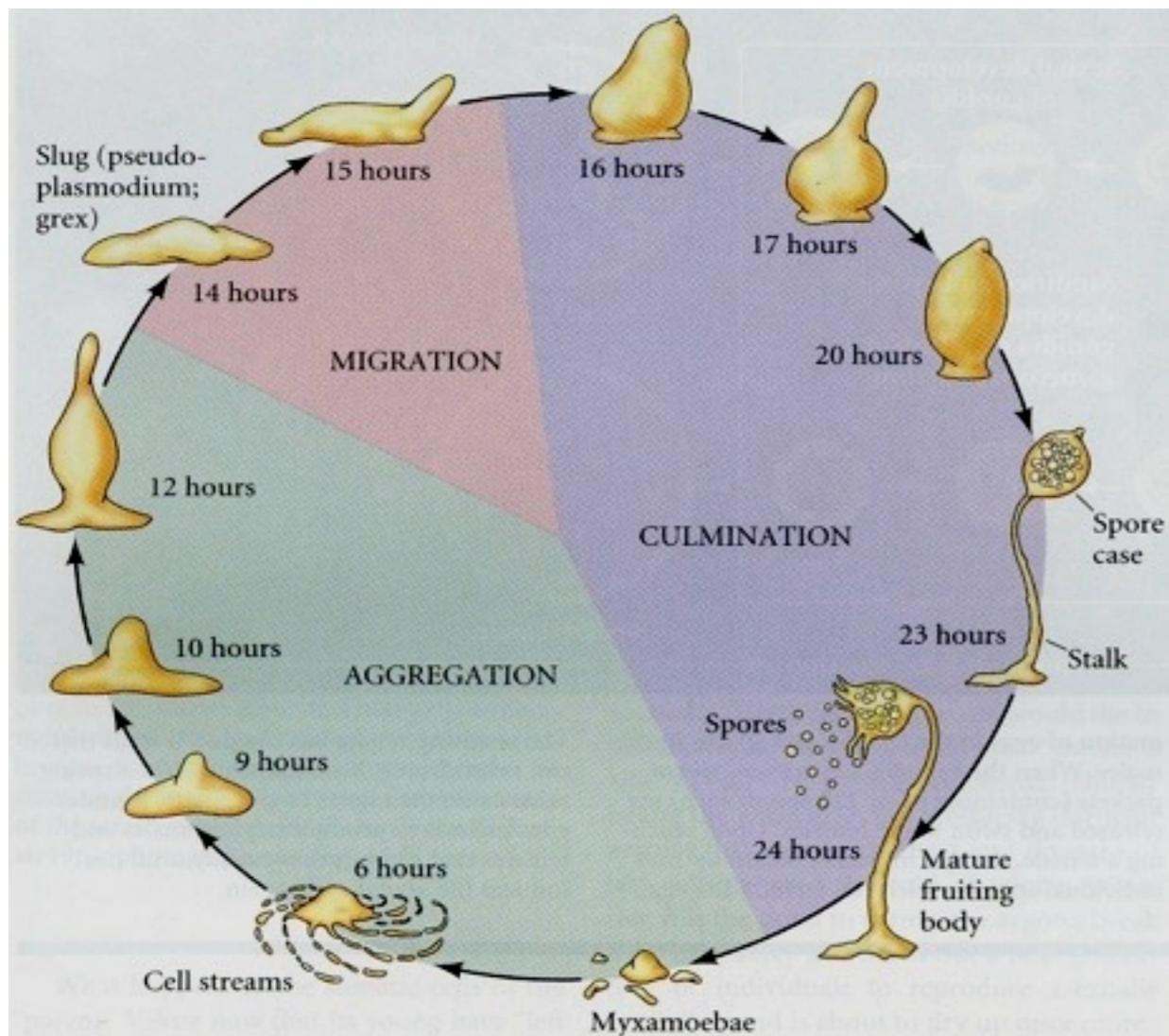
Fitting parameters of analytic solutions

Behavioural Science Institute

Radboud University Nijmegen



Emergence and Self-Organization: The life-cycle of *Dictyostelium*

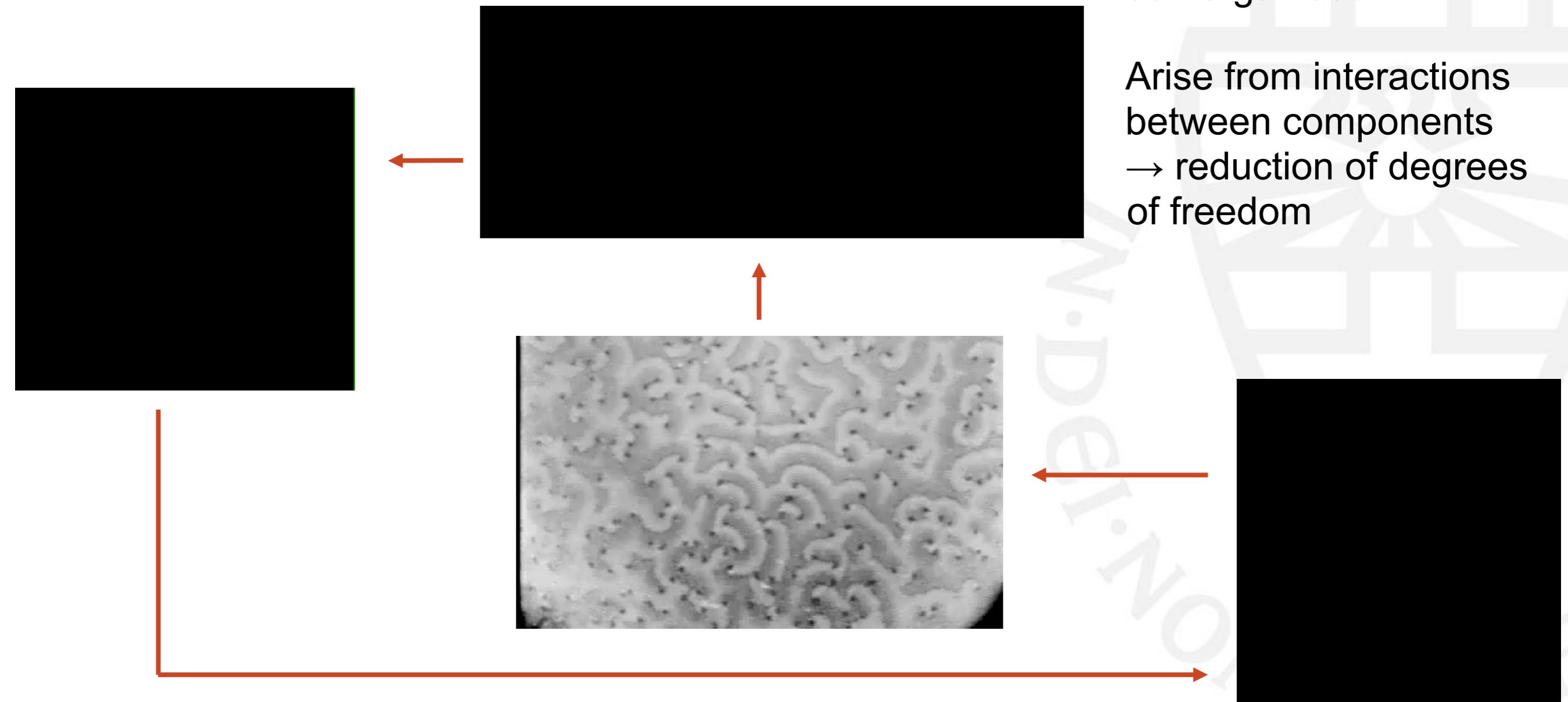


1. Free living myxamoebae feed on bacteria and divide by fission.
2. When food is exhausted they aggregate to form a mound, then a multicellular slug.
3. Slug migrates towards heat and light.
4. Differentiation then ensues forming a fruiting body, containing spores.
5. It all takes just 24 hrs.
6. Released spores form new amoebae.

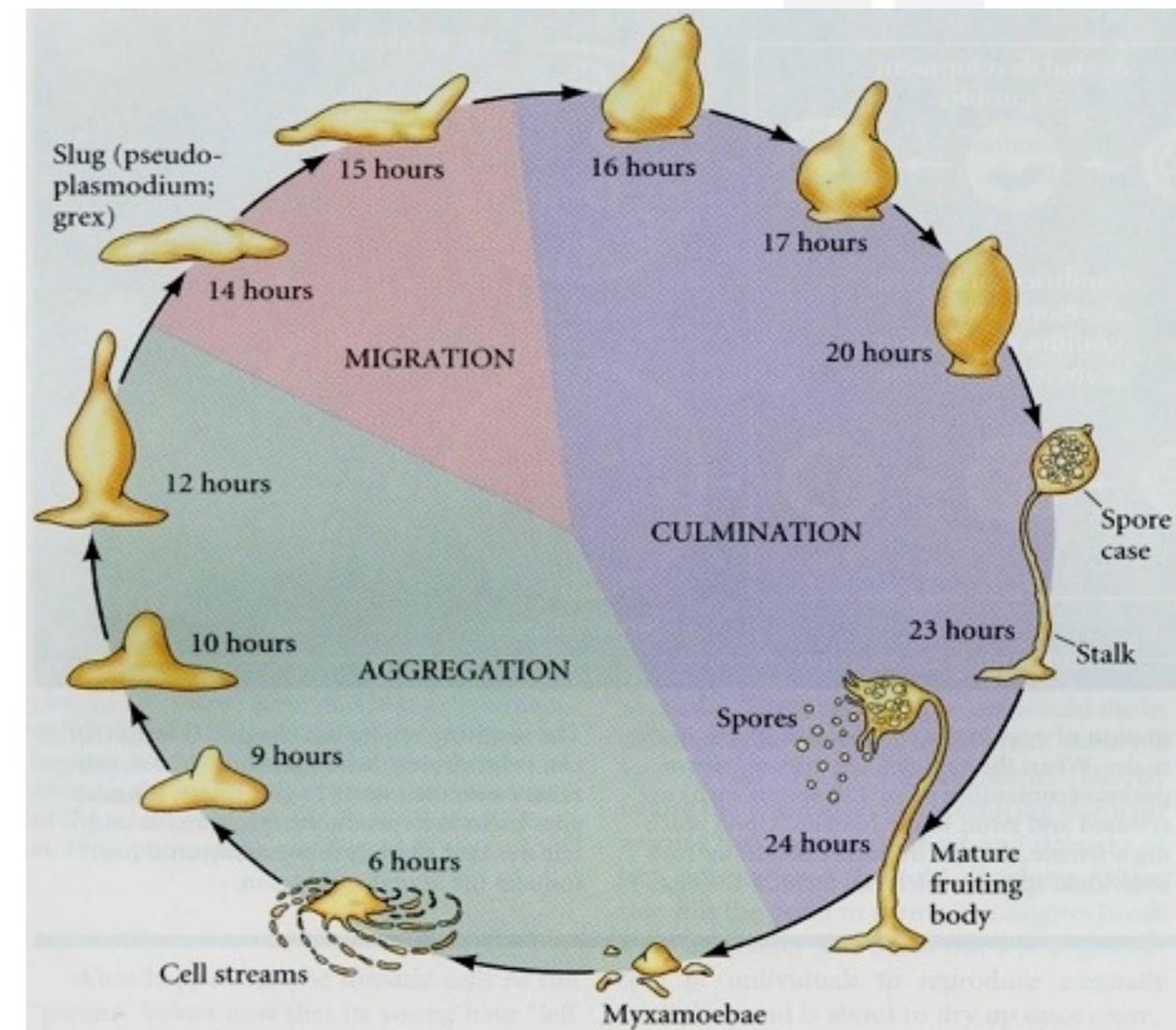
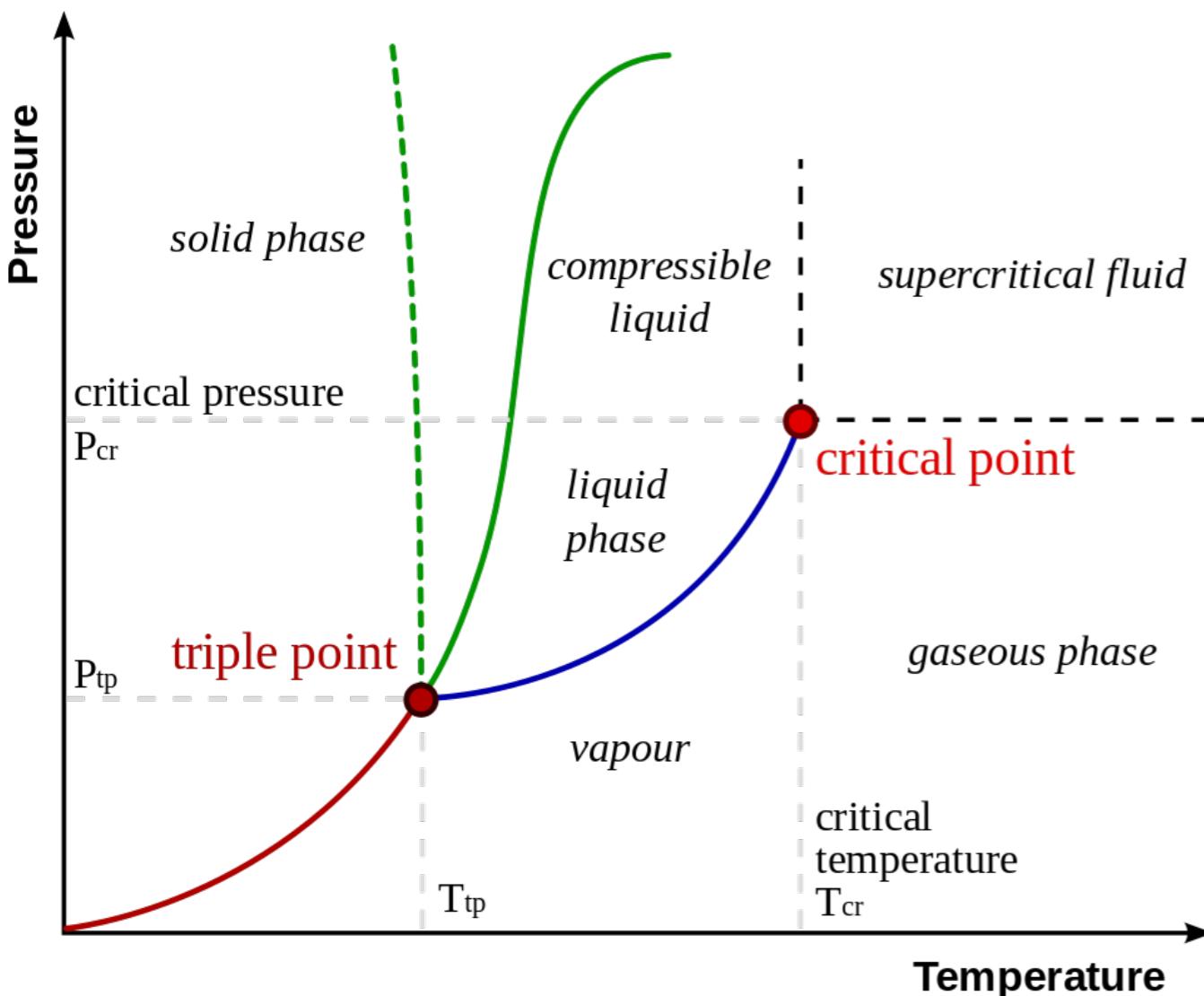
Order parameter: Labelling states of a complex system

Forms are emergent,
self-organised:

Arise from interactions
between components
→ reduction of degrees
of freedom



Phase Diagram & Order parameter



The order parameter is often a qualitative description of a macro state / global organisation of the system, conditional on the control parameters:

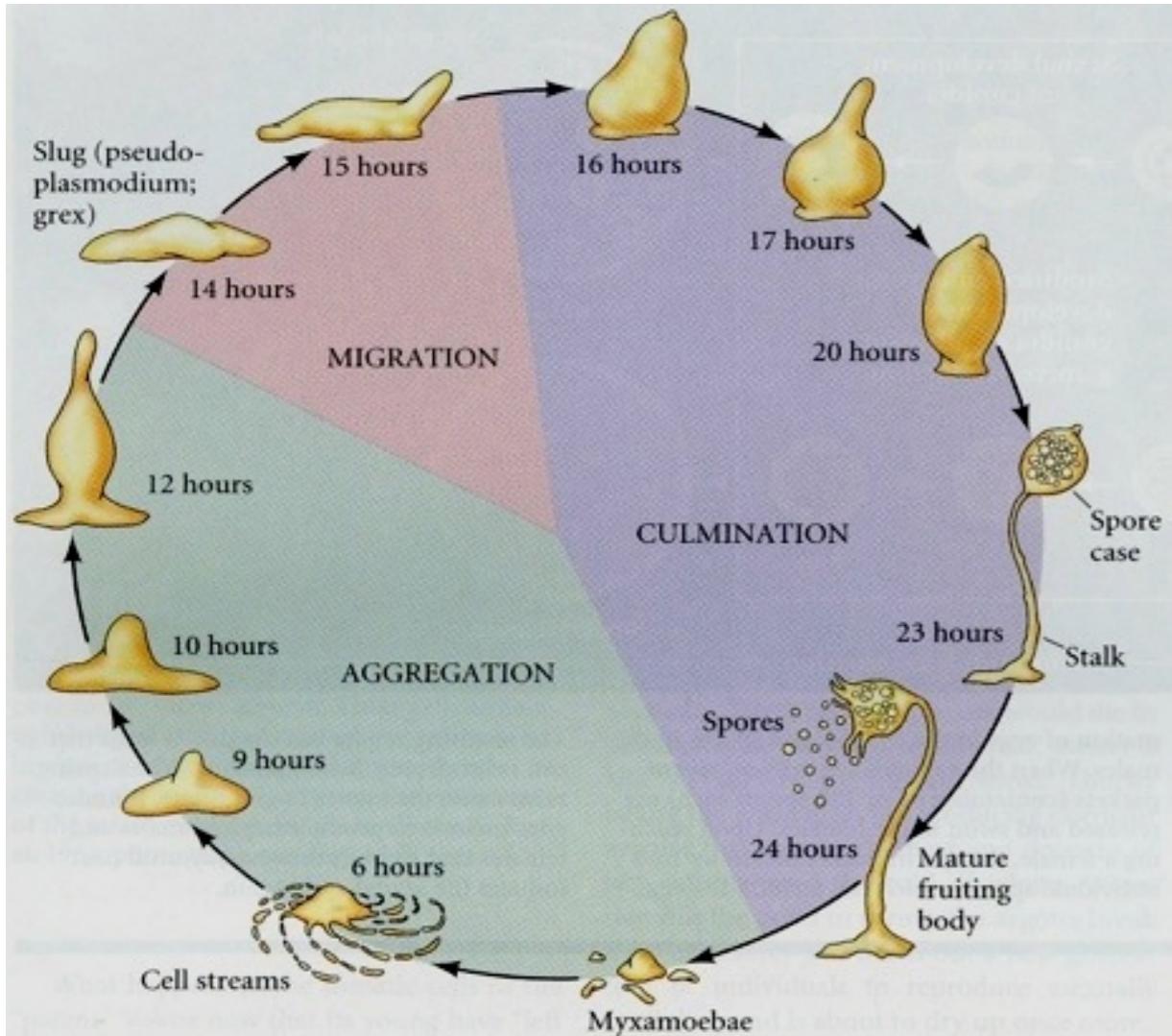
H_2O : Ice (Solid), Water (Liquid), Steam (Vapour)

Disctyostelium: Aggregation (Mound), Migration (Slug), Culmination (Fruiting Body)



Dynamic Metaphor vs. Dynamic Measure

Metaphor: State Space / Order Parameter
Measures: Attractor strength / Stability



Order parameter: the qualitatively different states

Control parameter: available food (actually concentration of a chemical that is released if they are starving)

Experiments:

Find out if the process is reversible... add food

perturb the system during the various phases...

the degrees of freedom of the individual components are increasingly constrained by the interaction:

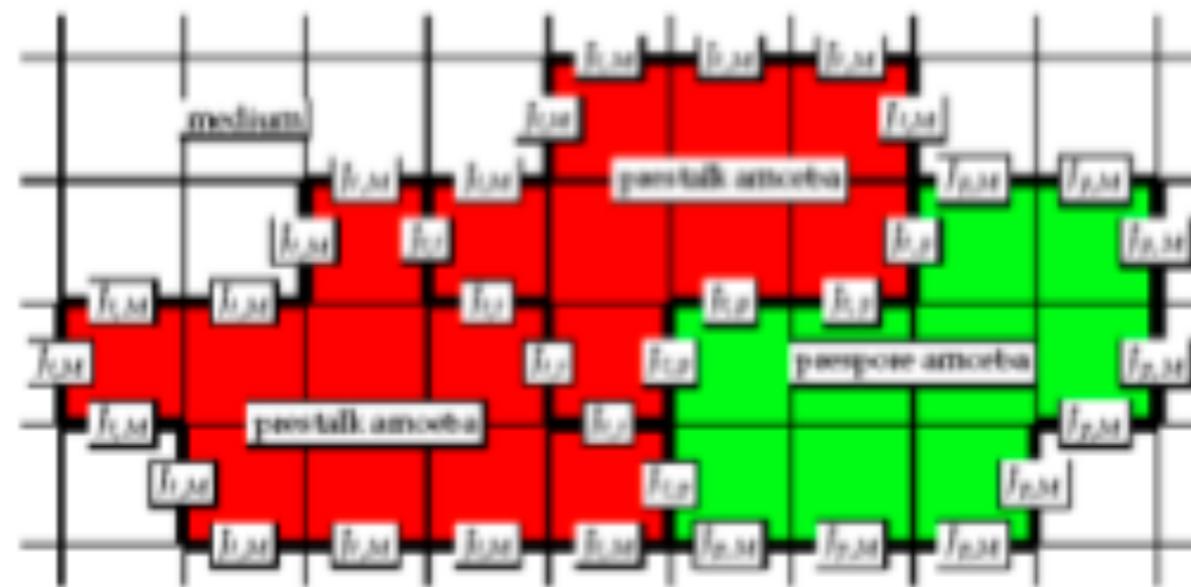
free living amoebae... slug... immovable sporing pod

nb State space and Phase Space (or: Diagram) are different concepts, but often used interchangeably to describe a State Space... see slide 18

From Pattern Formation to Morphogenesis

Multicellular Coordination in *Dictyostelium Discoideum*

A.F.M. Marée (2000). PhD Thesis, UU.



Two-Scale Cellular Automata with Differential Adhesion

$$H_\sigma = \sum_{\text{all } \sigma, \sigma' \text{ neighbours}} \frac{J_{\tau_\sigma, \tau_{\sigma'}}}{2} + \sum_{\text{all } \sigma, \text{medium neighbours}} J_{\tau_\sigma, \tau_{\text{medium}}} + \lambda(v_\sigma - V)^2, \quad (1.1)$$

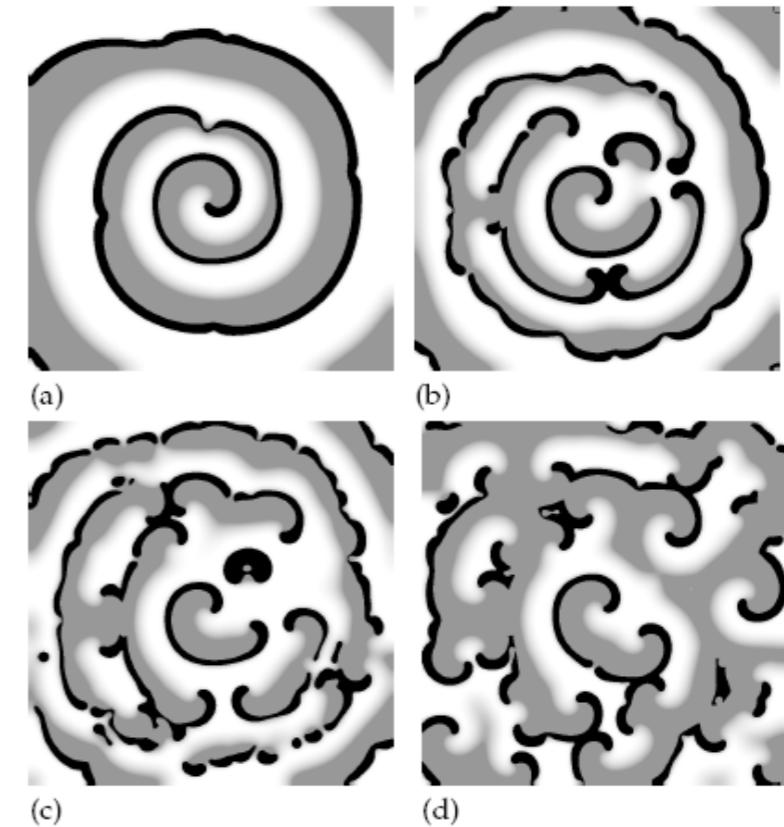
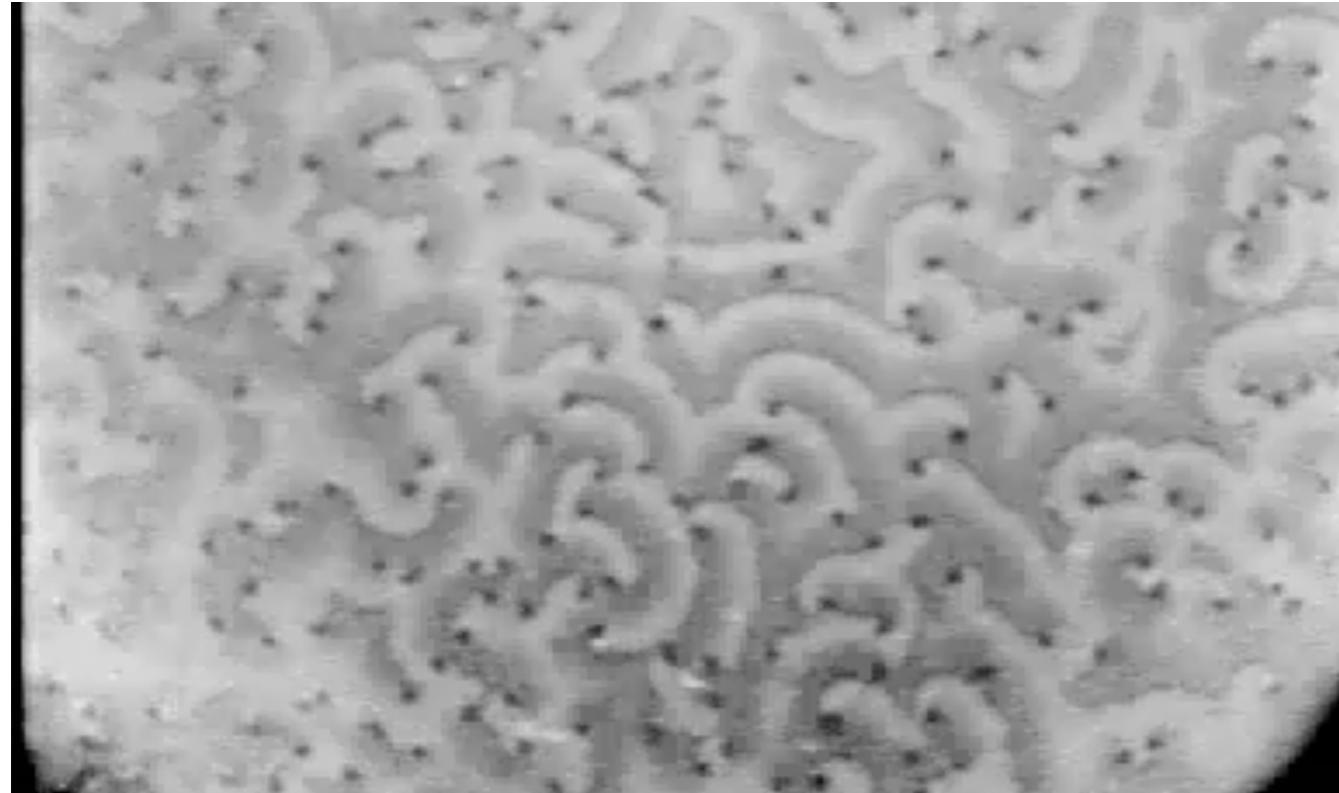
Mathematical model of *Dictyostelium*

Radboud University Nijmegen



Spiral Breakup in Excitable Tissue due to Lateral Instability

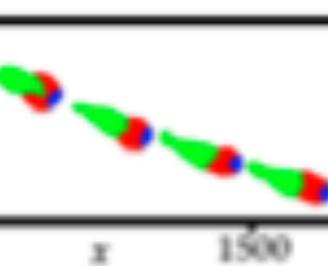
Marée, A. F. M., & Panlov, A.V. (1997). *Physical Review Letters*, 78, 1819-1822.



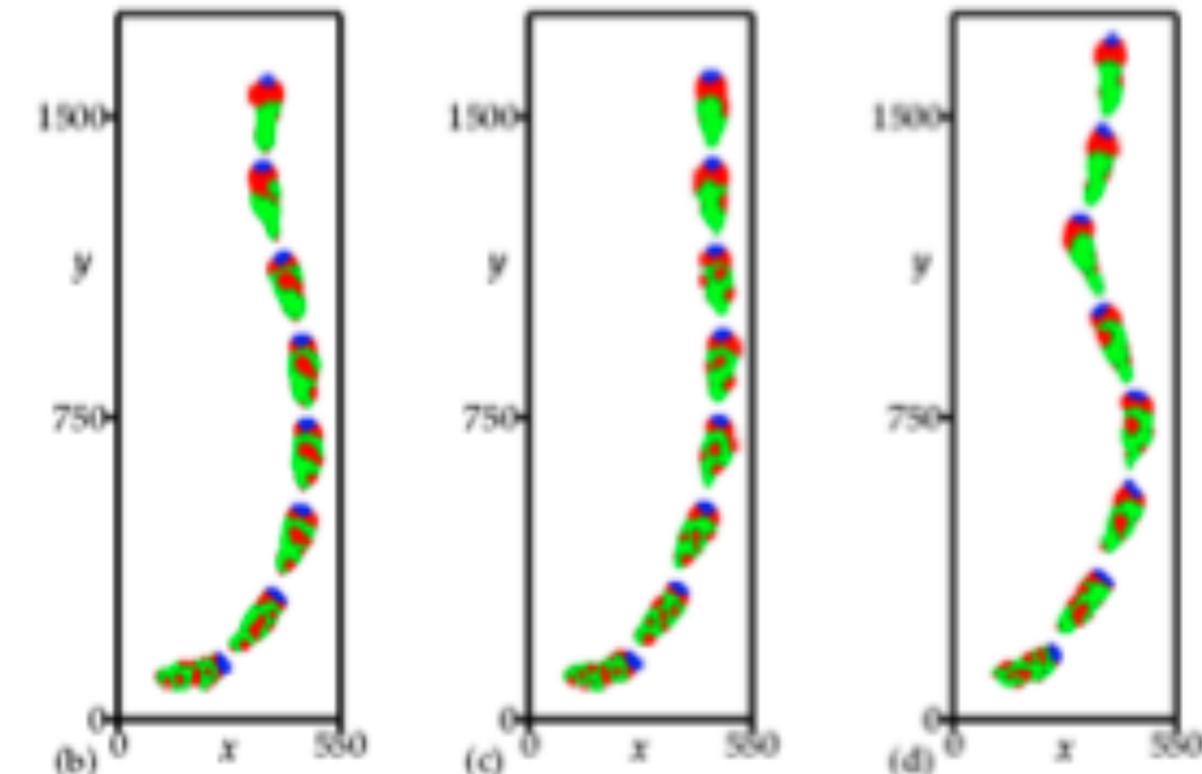
$$\frac{\partial e}{\partial t} = \Delta e - f(e) - g,$$

$$\frac{\partial g}{\partial t} = D_g \Delta g + \varepsilon(e, g)(ke - g),$$

Mathematical model of Dictyostelium



(a)



(b)

(c)

(d)

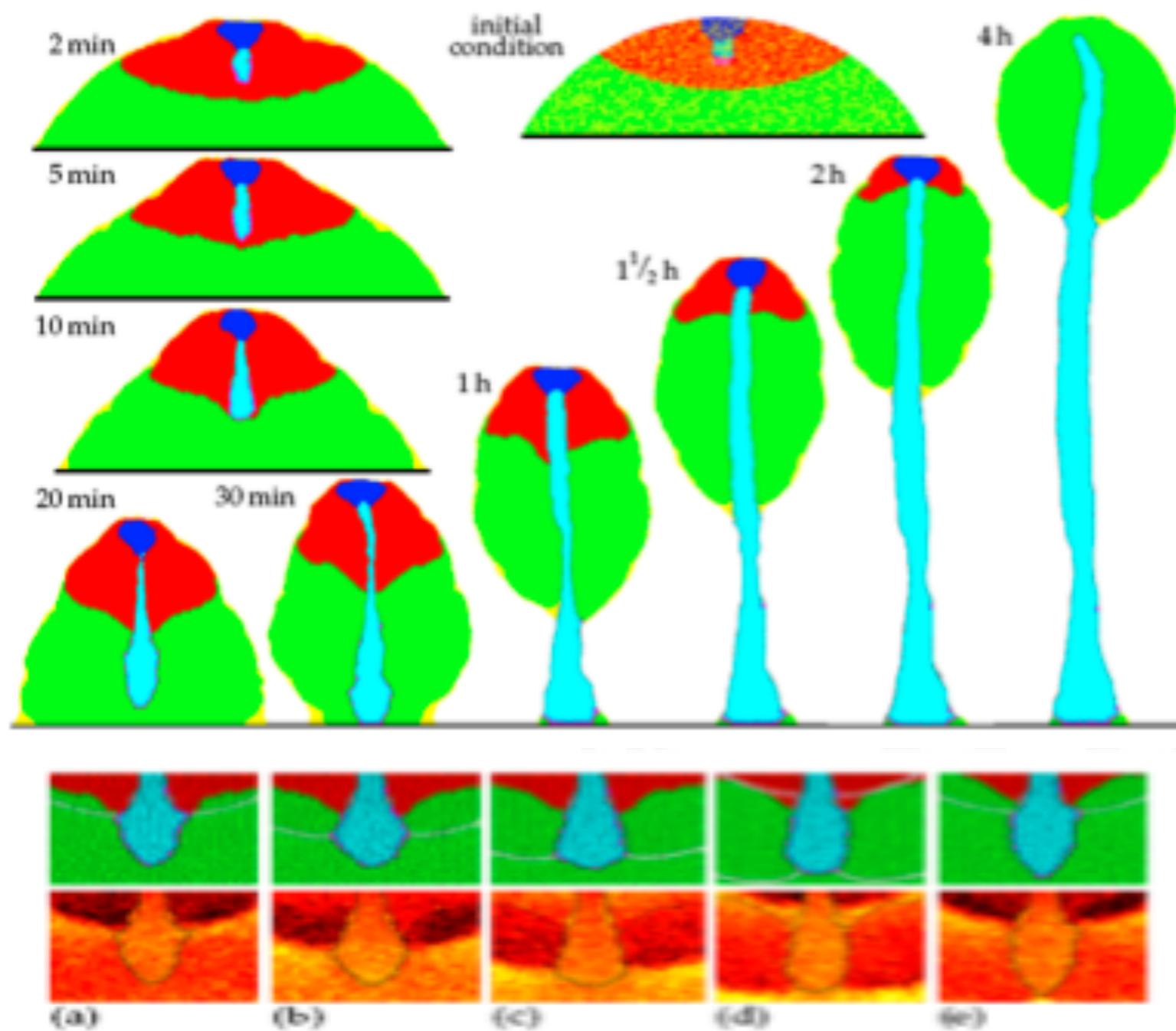
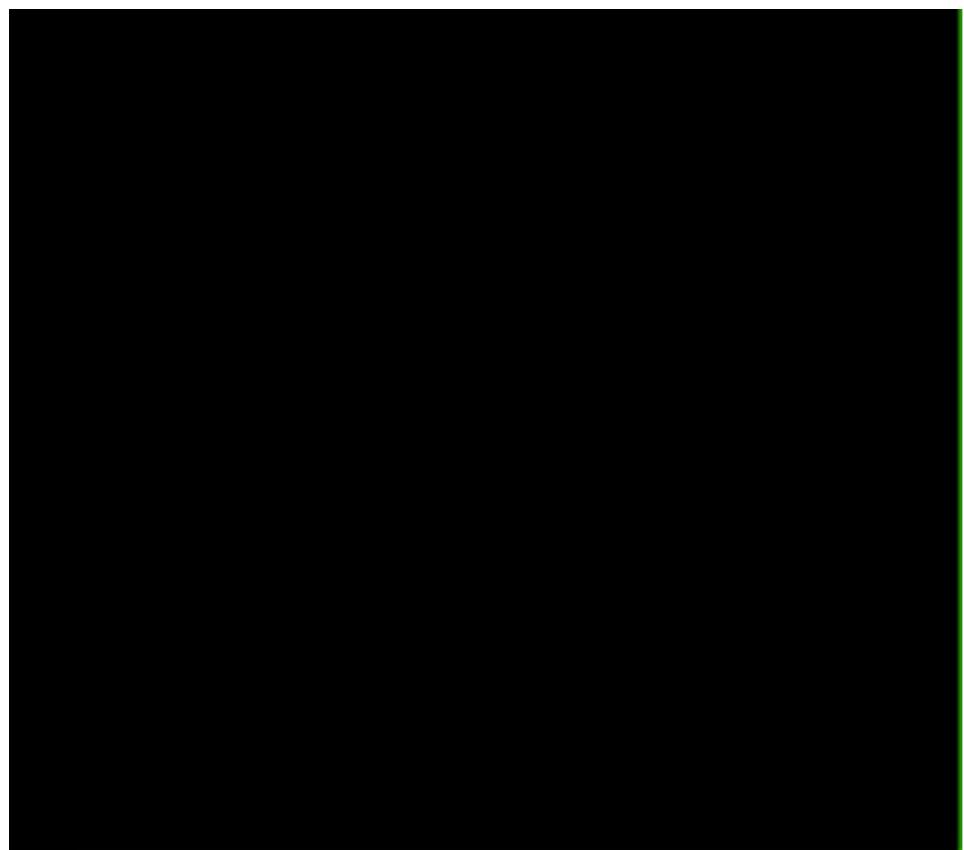
$$H_\sigma = \sum \frac{J_{\text{cell,cell}}}{2} + \sum J_{\text{cell,medium}} + \lambda(v - V)^2,$$

$$\left. \begin{array}{lcl} \frac{\partial c}{\partial t} & = & D_c \Delta c - f(c) - r, \\ \frac{\partial r}{\partial t} & = & \varepsilon(c)(kc - r), \\ \frac{\partial c}{\partial t} & = & D_i \Delta c - d_i(c - c_0), \end{array} \right\} \begin{array}{l} \text{inside the amoebae} \\ \text{outside the amoebae} \end{array}$$

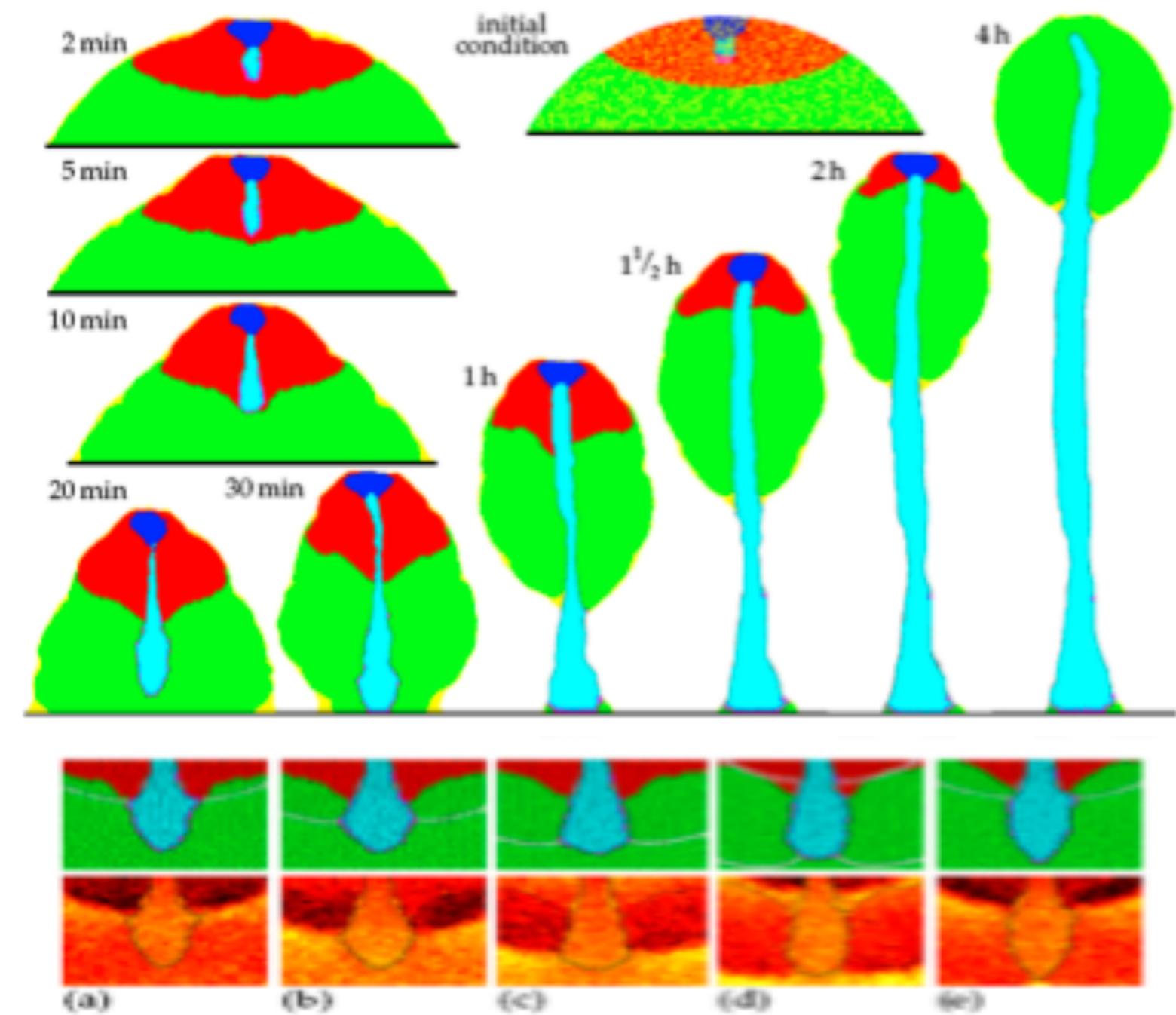
$$\Delta H' = \Delta H - \mu(c_{\text{automaton}} - c_{\text{neighbour}}),$$

Mathematical model of *Dictyostelium*





Mathematical model of *Dictyostelium*



Mathematical model of *Dictyostelium*

Two Metaphors to explain Human Behaviour

Machine Metaphor

- Parts exist for each other, but not by means of each other
- Parts act together to meet the things purpose, but their actions have nothing to do with the thing's construction
- **Open to efficient cause (predicative logic)**
- Human behaviour: **Computation; Information processing**

Organism Metaphor

- Parts are both causes and effects of the thing, both means and end
- Parts act together but also construct and maintain themselves as a whole
- **Closed to efficient cause (impredicative logic)**
- Human Behaviour: **Concinnity; Embodied and Embedded**

Concinnity: Harmony in the arrangement or interarrangement of parts with respect to a whole.



Two types of mathematical formalism:

Random events / processes
Linear
Efficient causes

Random events / processes
Deterministic events / processes
Linear / Nonlinear
Efficient causes / Circular causality

component dominant dynamics

The Law of Large Numbers (Bernouilli, 1713) +
The Central Limit Theorem (de Moivre, 1733) +
The Gauss-Markov Theorem (Gauss, 1809) +
Statistics by Intercomparison (Galton, 1875) =
Social Physics (Quetelet, 1840)

Collectively known as:

The Classical Ergodic Theorems

Molenaar, P.C.M. (2008). On the implications of the classical ergodic theorems:
Analysis of developmental processes has to focus on intra individual variation. *Developmental Psychobiology*, 50, 60-69

interaction dominant dynamics

Deterministic chaos (Lorenz, 1972)
(complexity, nonlinear dynamics, predictability)

Takens' Theorem (1981)
(phase space reconstruction)

Systems far from thermodynamic equilibrium
(Prigogine, & Stengers, 1984)

SOC / $\frac{1}{f^\alpha}$ noise (Bak, 1987)
(self-organized criticality, interdependent measurements)

Fractal geometry (Mandelbrot, 1988)
(self-similarity, scale free behaviour, infinite variance)

Aczel's Anti-Foundation Axiom (1988)
(hyperset theory, circular causality, complexity analysis)



Two types of mathematical formalism for two types of systems

component dominant dynamics

Jakob Bernoulli (1654-1704): [The application of the Law of large numbers in chance theory] to predict the weather next month or year, predicting the winner of a game which depends partly on psychological and or physical factors or to the investigation of matters which depend on hidden causes, which can interact in a multitude of ways is completely futile!" Vervaet (2004)

A system is ergodic iff:

The averaged behaviour of an observed variable in a substantial ensemble of individuals (space-average) is expected to be equivalent to the average behaviour of an individual observed over a substantial amount of time (time average)

f.i. Throw 100 dice at once, and then throw 1 die 100 times in a row... The expected value will be similar for both measurements

interaction dominant dynamics

Deterministic chaos (Lorenz, 1972)
(complexity, nonlinear dynamics, predictability)

Takens' Theorem (1981)
(phase space reconstruction)

Systems far from thermodynamic equilibrium
(Prigogine, & Stengers, 1984)

SOC / $\frac{1}{f^\alpha}$ noise (Bak, 1987)

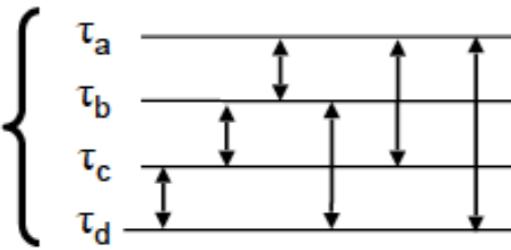
(self-organized criticality, interdependent measurements)

Fractal geometry (Mandelbrot, 1988)
(self-similarity, scale free behaviour, infinite variance)

Aczel's Anti-Foundation Axiom (1988)
(hyperset theory, circular causality, complexity analysis)

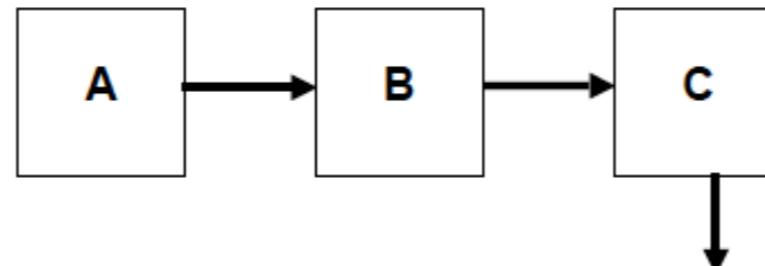


Interaction dominant dynamics



Behavior emerges from interaction between many processes on different timescales in body and environment

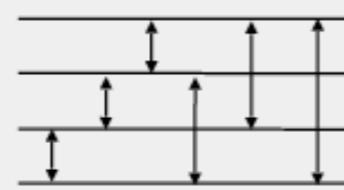
Component dominant dynamics



Behavior is the result of a linear arrangement of a virtual architecture of cognitive components and processes

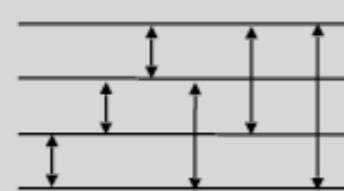
Place of measurement of efficient causes

Environment



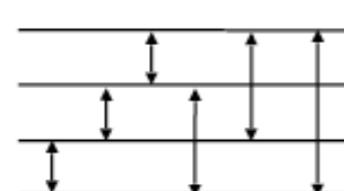
Environmental factors, Performance and perception measures, Social interactions, ...

Body



Genetic, immunological, endocrine systems. Biophysical composition, physiology, Organic chemistry, ...

CNS



Structure and function of the cortex, cerebellum, brainstem, neural pathways. Neurochemistry, ...

Cognitive components and processes

“Lack of Group to Individual Generalizability”

Lack of group-to-individual generalizability is a threat to human subjects research

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Edited by David L. Donoho, Stanford University, Stanford, CA, and approved May 25, 2018 (received for review July 4, 2017)

Only for ergodic processes will inferences based on group-level data generalize to individual experience or behavior. Because human social and psychological processes typically have an individually variable and time-varying nature, they are unlikely to be ergodic. In this paper, six studies with a repeated-measure design were used for symmetric comparisons of interindividual and intraindividual variation. Our results delineate the potential scope and impact of nonergodic data in human subjects research. Analyses across six samples (with 87–94 participants and an equal number of assessments per participant) showed some degree of agreement in central tendency estimates (mean) between groups and individuals across constructs and data collection paradigms. However, the variance around the expected value was two to four times larger within individuals than within groups. This suggests that literatures in social and medical sciences may overestimate the accuracy of aggregated statistical estimates. This observation could have serious consequences for how we understand the consistency between group and individual correlations, and the generalizability of conclusions between domains. Researchers should explicitly test for equivalence of processes at the individual and group level across the social and medical sciences.

research methodology | replicability | idiographic science | generalizability | ecological fallacy

Inferences made in social and medical research typically result from statistical tests conducted on aggregated data. The implicit assumption is that group-derived estimates can be applied

consistency between individual and group variability before generalizing results across levels of analysis. We will refer to this latter condition as the “group-to-individual generalizability” of a given statistical estimate. However, whether couched in prosaic terms, or within formal mathematical theorems, researchers have not systematically examined such generalizability in extant literatures, despite a number of calls to do so throughout the years (cf. refs. 6–11). Hitherto, the highest-impact publications in medical and social sciences have been largely based on data aggregated across large samples, with best-practice guidelines almost exclusively based on statistical inferences from group designs. The worst-case scenario—a global, uniform absence of group-to-individual generalizability due to nonergodicity in the social and medical sciences—would undermine the validity of our scientific canon in these domains. However, even moderate incongruities between group and individual estimates could result in imprecise or potentially invalid conclusions. We argue that this possibility should be formally tested, wherever possible, to be ruled out.

Ergodicity, the Ecological Fallacy, and Simpson’s Paradox

The ergodic theorem is a general and formal mathematical expression that deals with the generalizability of statistical phenomena across levels and units of analysis. [While a more thorough explication of the ergodic theorem is outside of the scope of the present paper, readers are referred to Molenaar (1) for a comprehensive mathematical treatment of ergodicity in human subjects research.] Ergodic theory postulates that the

There are two important takeaways from these findings:

(i) **Aggregated estimates did not consistently agree with individual estimates [...]** the present study provides an empirical demonstration of [*the implausibility of ergodicity in human subjects research*] across multiple settings and constructs.

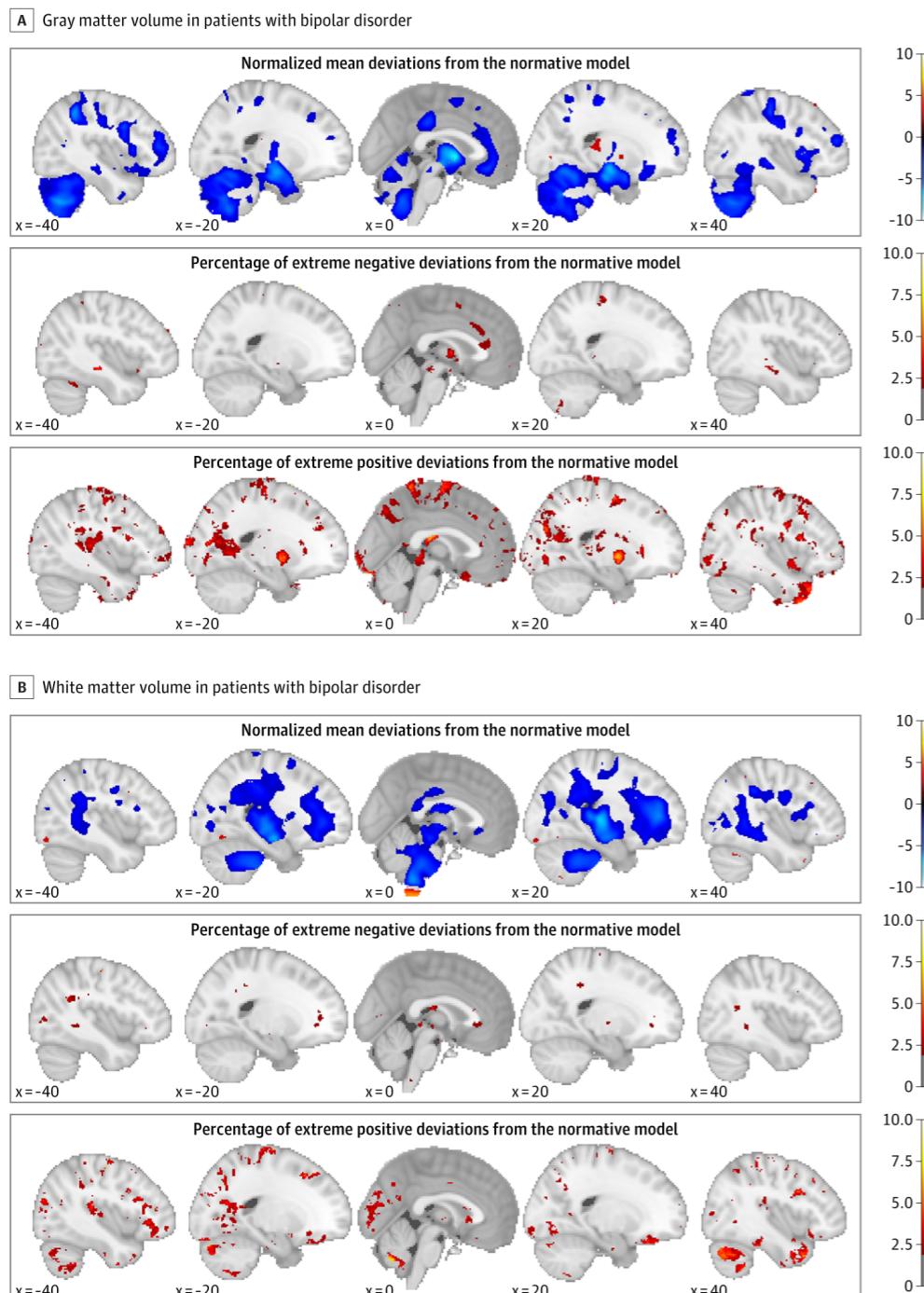
(ii) **even in the best-case scenario, we should not think of a correlation in group data as an estimate that generalizes to any given individual in the population.** Only 68% of all individual correlational values fall within a range that would be predicted by group data to cover 99.7% of all possible correlations—a discrepancy of nearly 32%.

The worst-case scenario is clearly dire: It is plausible that **inattention to nonergodicity and a lack of group-to-individual generalizability threaten the veracity of countless studies, conclusions, and best-practice recommendations.**

PSYCHOLOGICAL AND
COGNITIVE SCIENCES

MEDICAL SCIENCES

The idea of the average patient is a noninformative construct



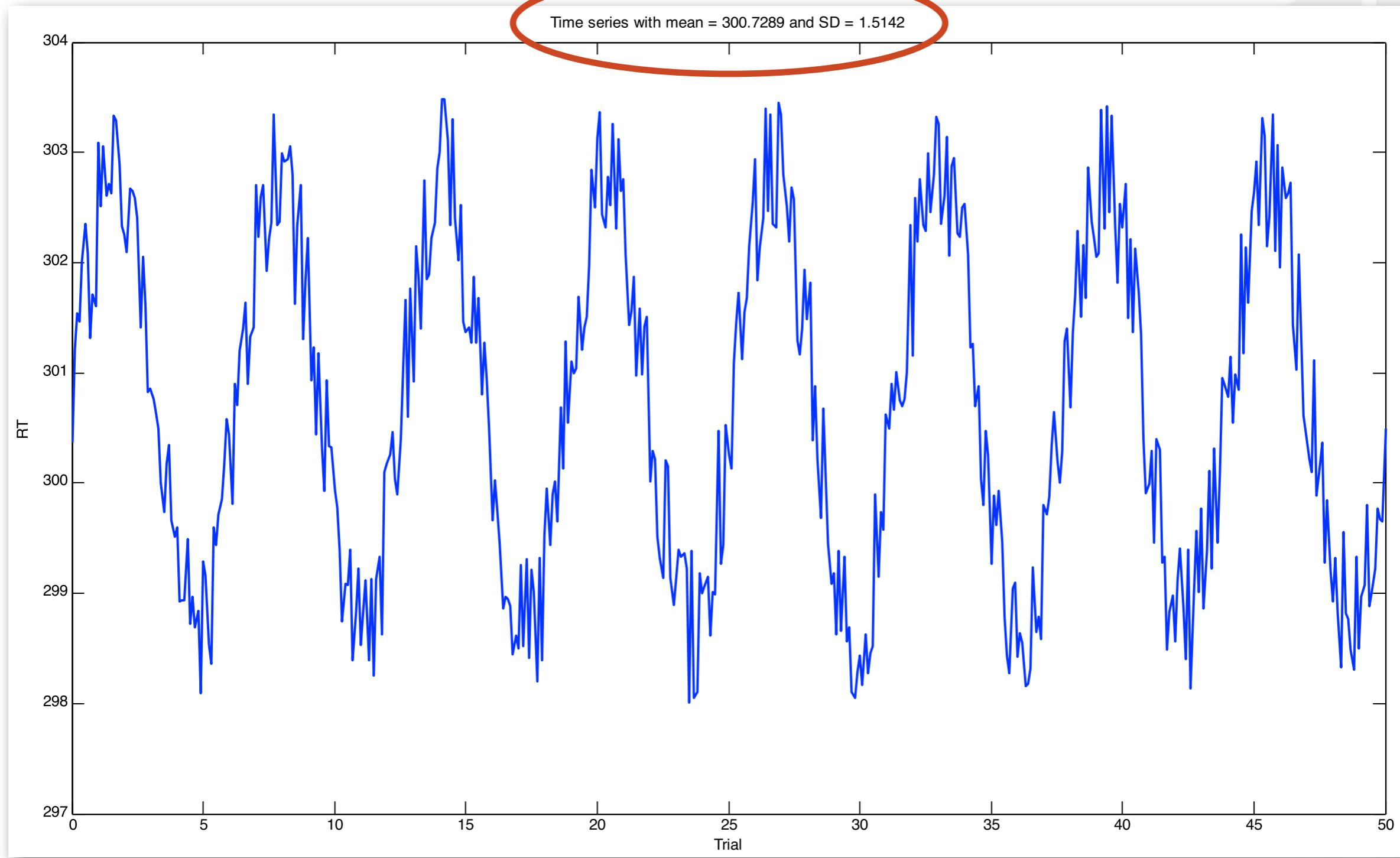
Question: Is the focus on the average patient disinguing interindividual differences among patients with mental disorders?

Findings: In this study of magnetic resonance imaging data from 218 patients with schizophrenia spectrum disorders and 256 healthy control individuals, mapping of interindividual differences in brain structure revealed that **only a few brain loci had the same abnormalities in more than 2% of patients with the same disorder despite robust group-level differences** in multiple brain regions between patients and control individuals.

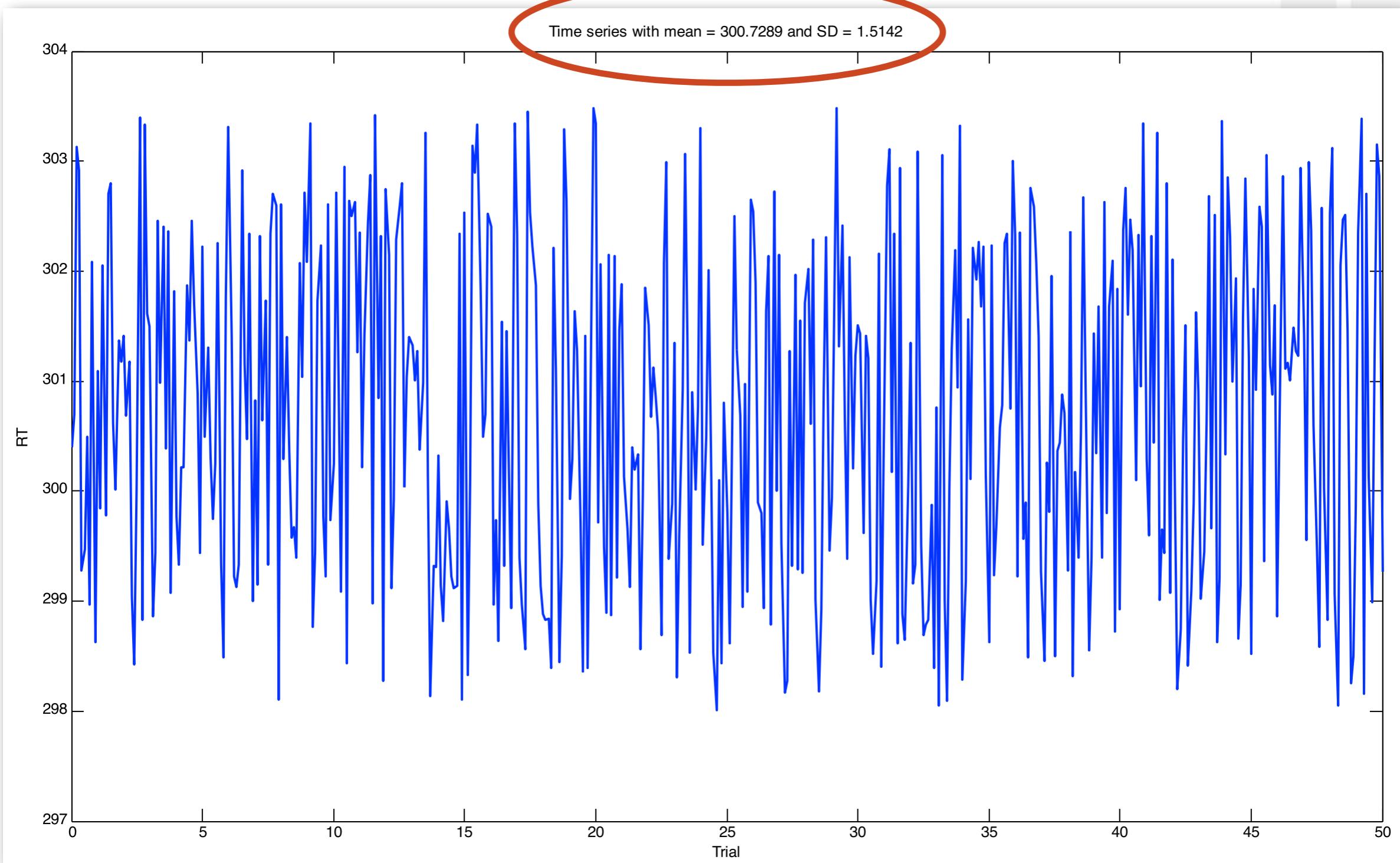
Meaning: These findings suggest that **the idea of the average patient is a noninformative construct that falls apart when mapping interindividual differences** and provide a framework toward precision medicine in psychiatry.

<https://www.ru.nl/donders/@1179539/very-few-similarities-between-brains-schizophrenia/>

Importance of TIME-evolutionary processes



Importance of TIME-evolutionary processes



Order vs. Disorder

"order is essentially the arrival of redundancy in a system, a reduction of possibilities"

Von Förster (2003)

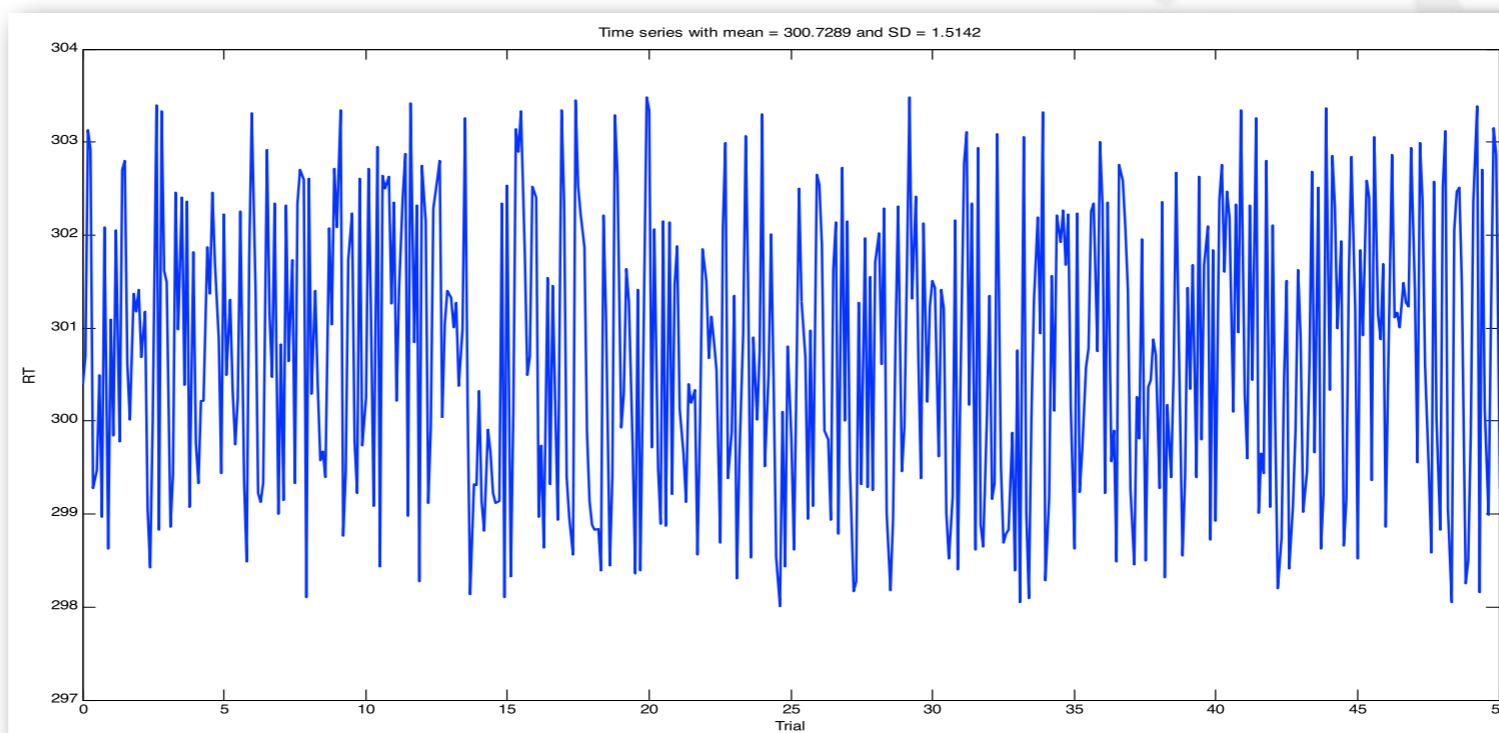
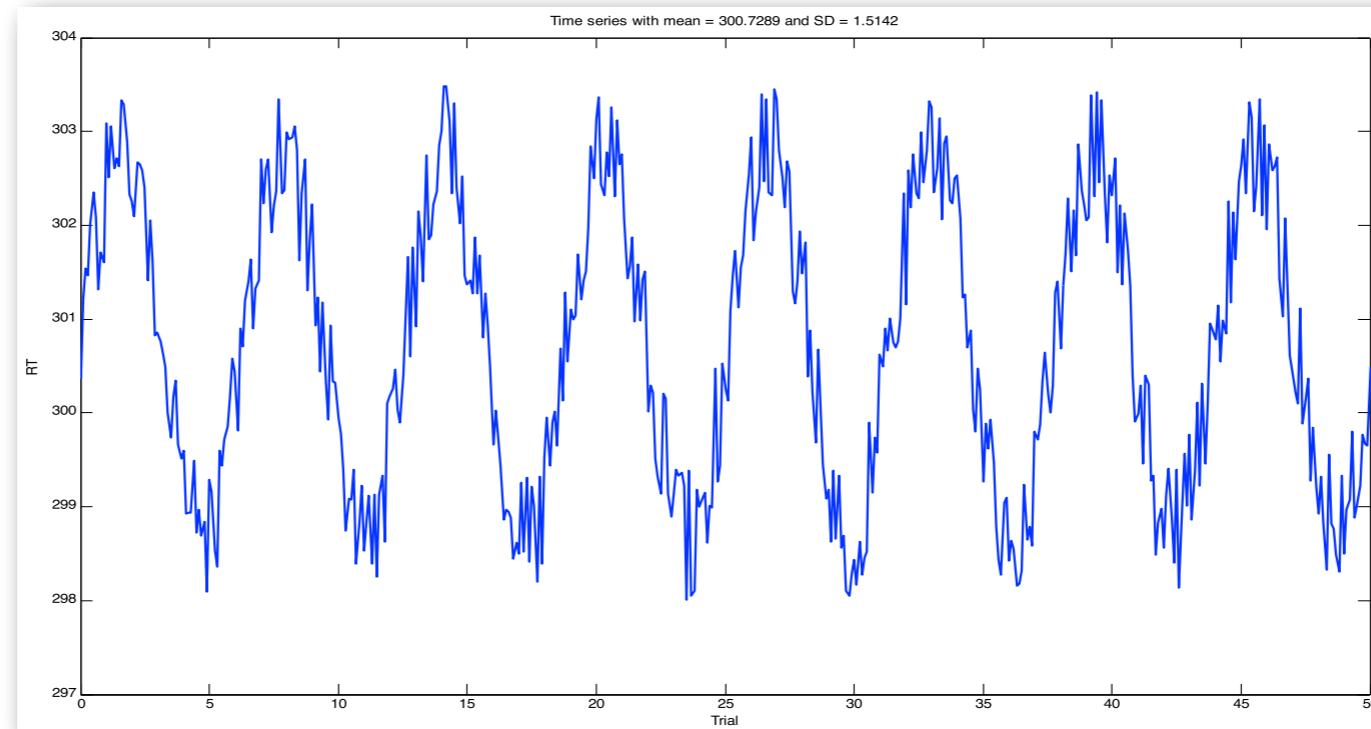
Entropy: Measure of ‘disorder’ in a system

However see Information Theory: “The amount of information we need to describe the different states (behaviour) of a system.”

General definition: Relative absence of order/redundancy in a system. The degrees of freedom a system has available for generating its behaviour: Possibility.



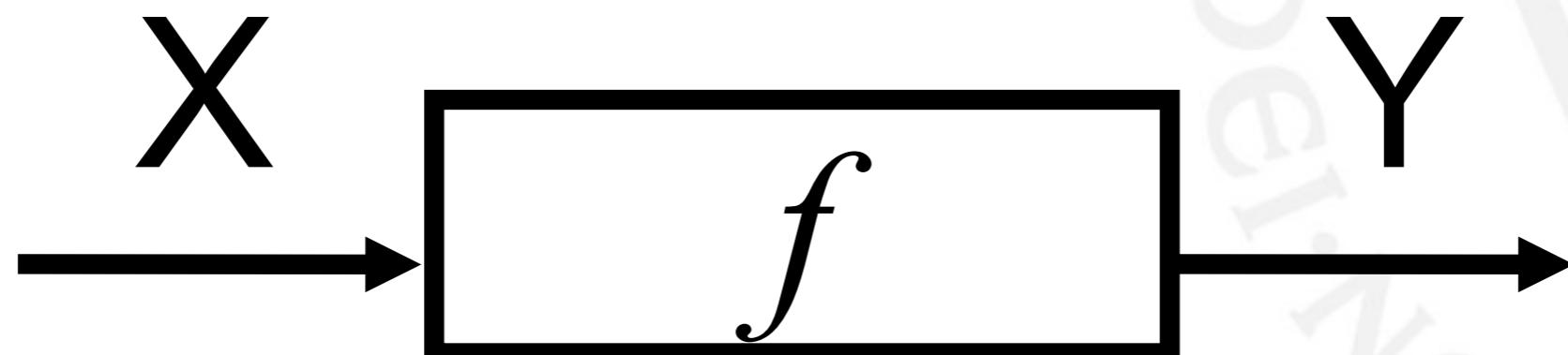
Which series has a higher entropy?



The mathematics of change

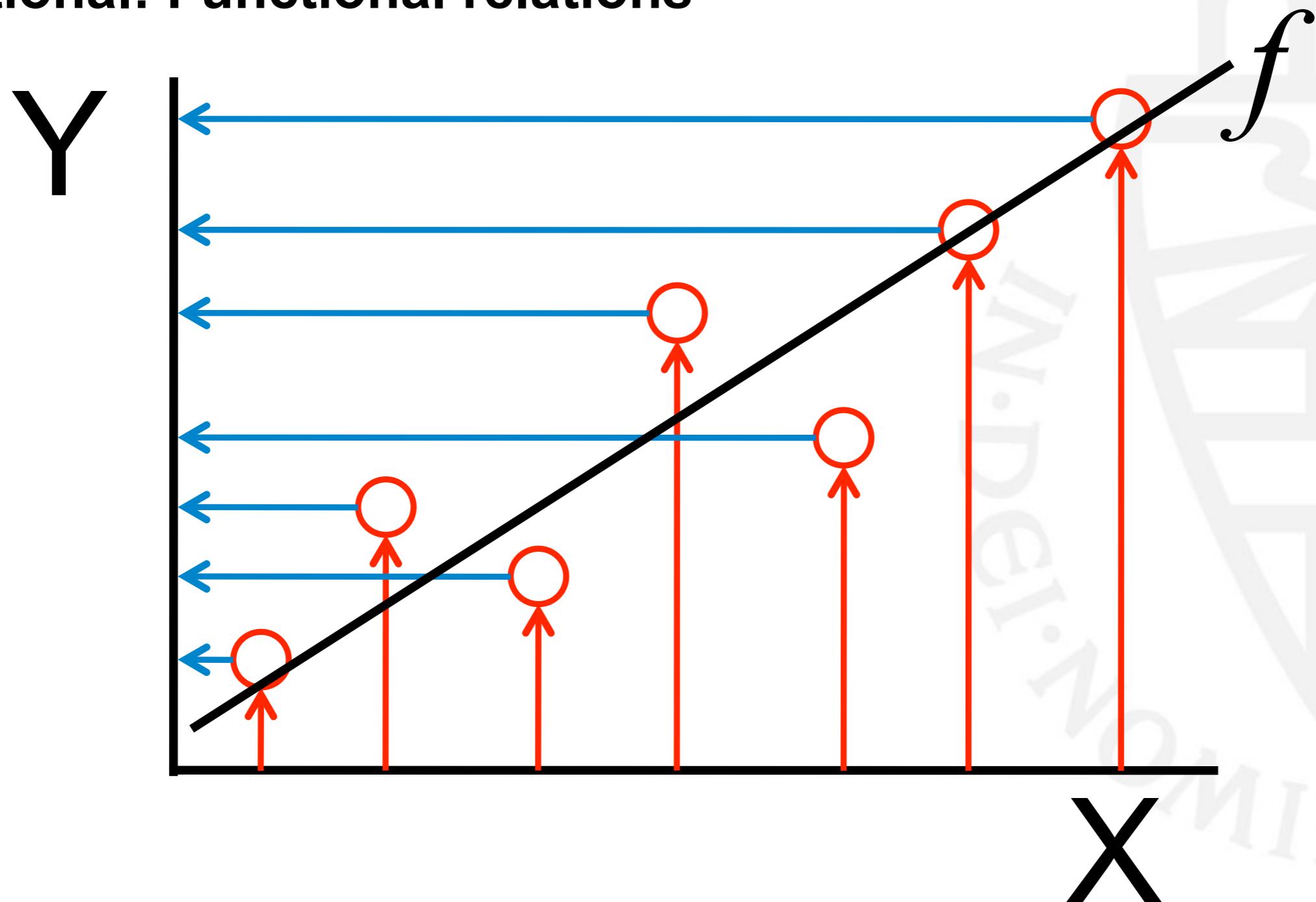
Traditional: Functional relations

$$Y = f(X)$$



The mathematics of change

Traditional: Functional relations



The mathematics of change

Complex systems however:

- Consist of feedback loops
- Are recurrent / recursive
- Have history
- Are characterised by multiplicative interactions between components

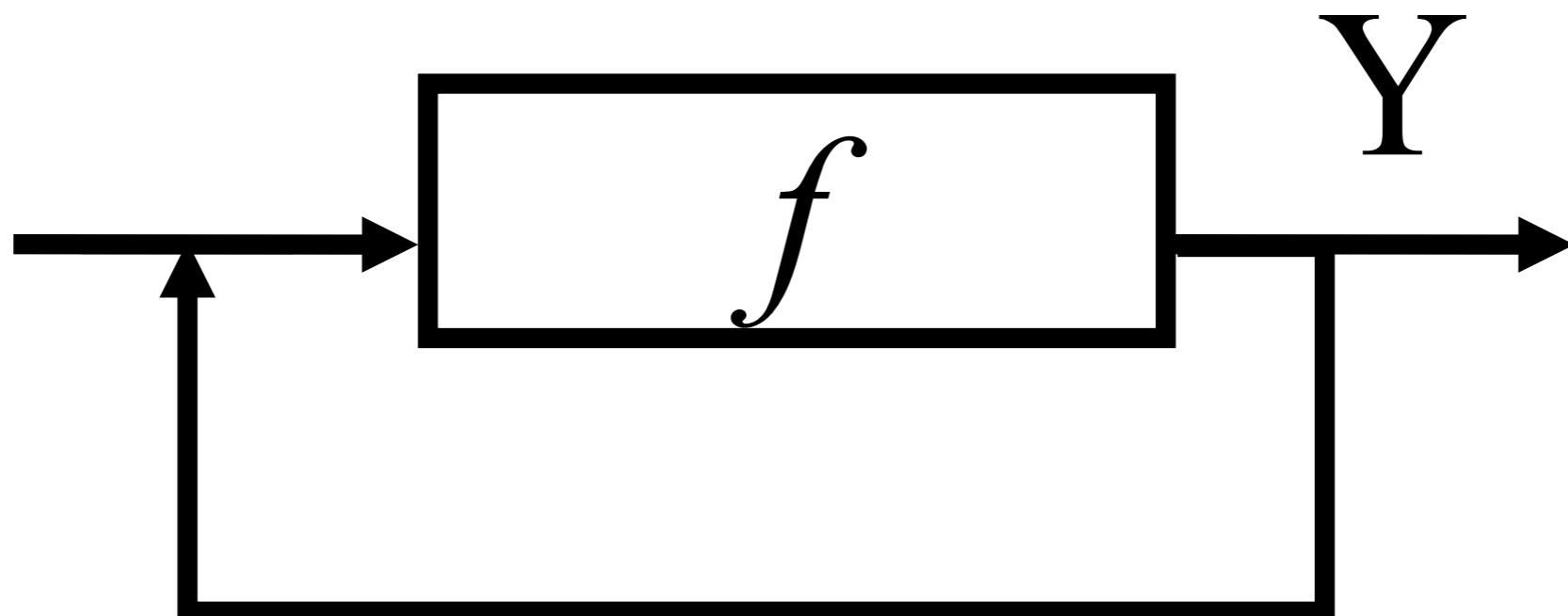
¹refs



The mathematics of change

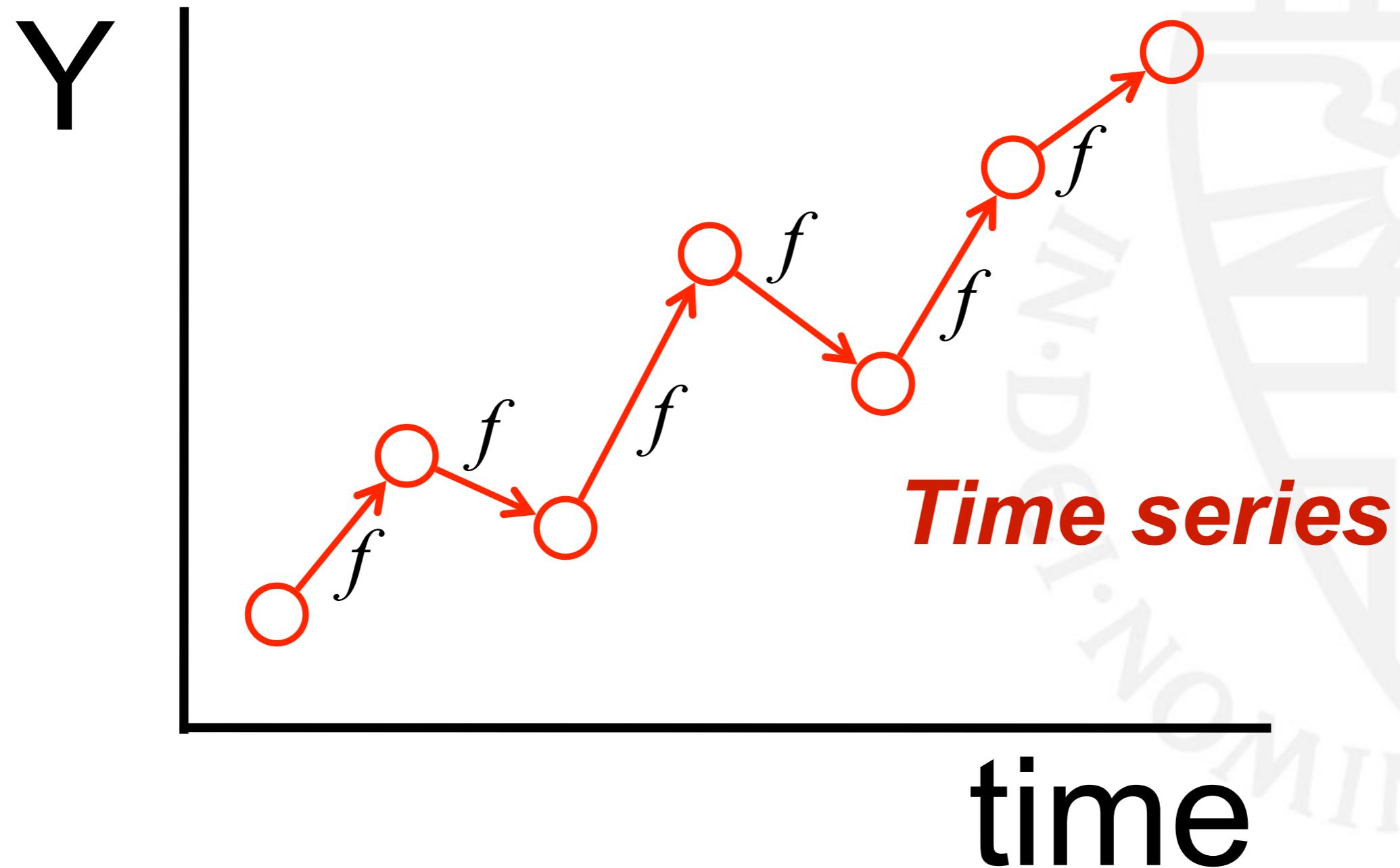
Complex systems: Recurrent processes / Feedback

$$\hat{Y} = f(Y)$$



The mathematics of change

Complex systems: Recurrent processes / Feedback



Two Flavors: Flows & Maps

Dynamical models of psychological processes can be formulated in:

‘Clock’ time

Continuous System

~ Flow ~

(Differential equation)

‘Metronome’ time

Discrete System

... Map ...

(Difference equation)



PARAMETERS & BIFURCATIONS

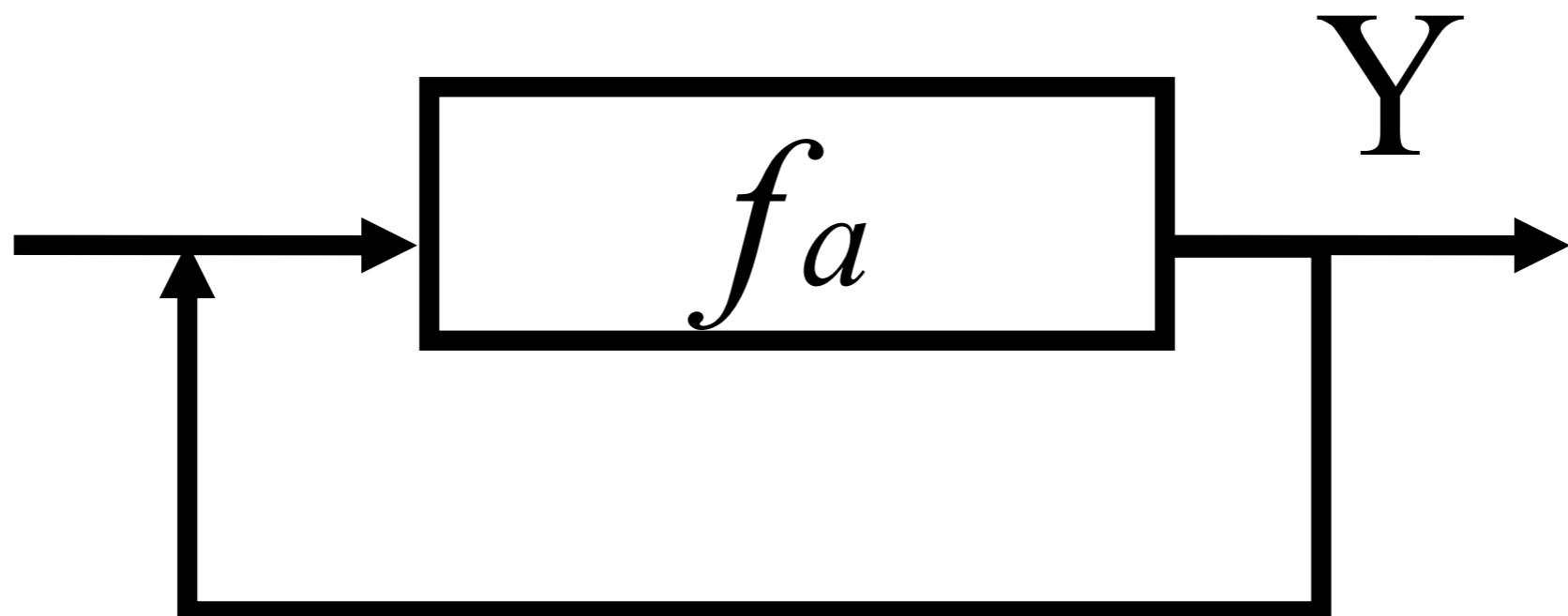
EXAMPLE 1:
The Linear Map
(Linear Growth)



The linear map

Dynamic Models: Parameter

$$\hat{Y} = f_a(Y)$$



The Linear Map ...

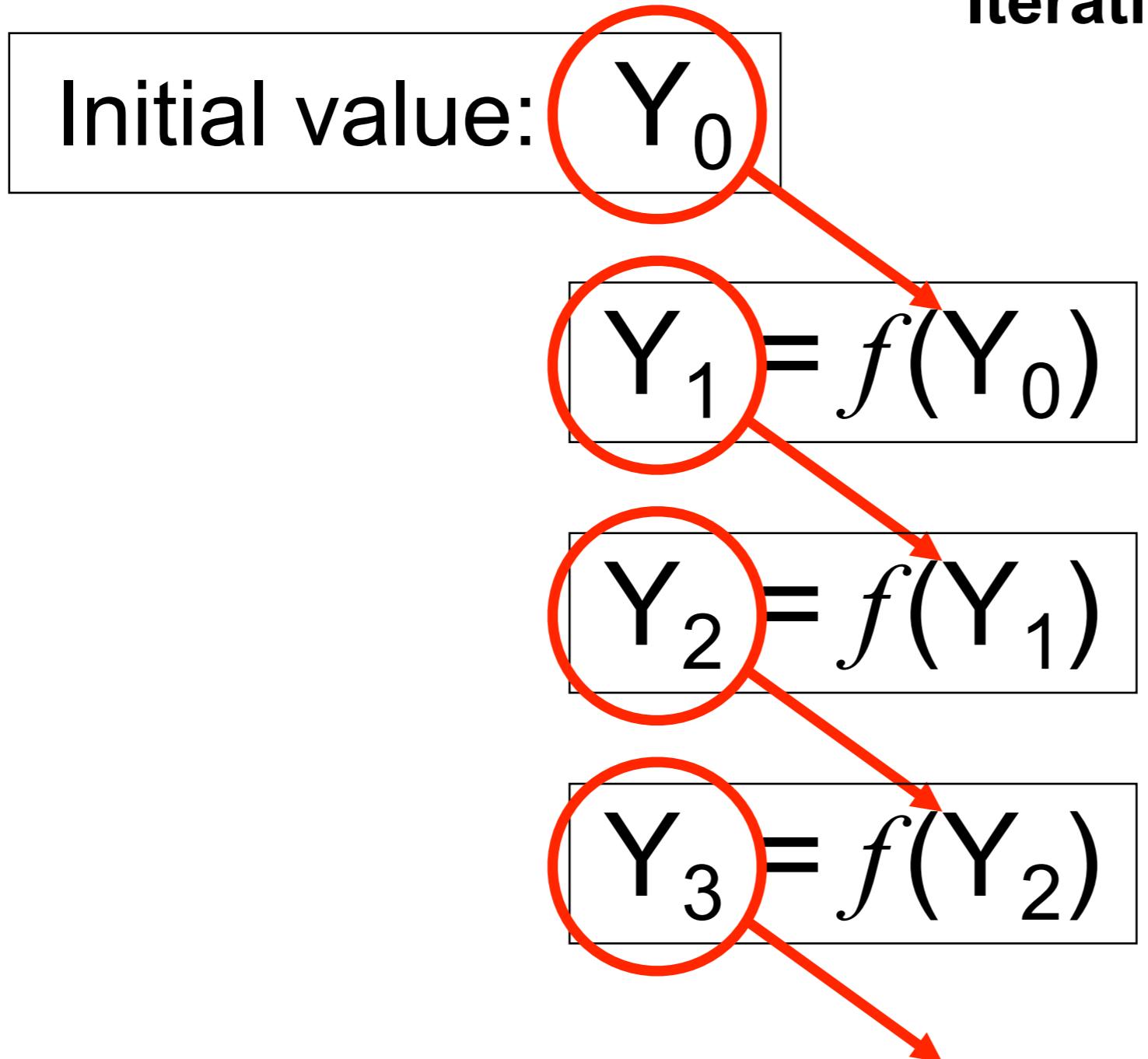
The (rate of) change of the state of a system is proportional to its current state:

$$Y_{i+1} = a \cdot Y_i$$

...Iteration...



The Linear Map



Iteration in general just means applying the function over and over again starting with an initial value

and subsequently to the result of the previous step

The Linear Map

$$Y_{i+1} = f(Y_i)$$

$$i = 0: \quad Y_0 \rightarrow Y_1 = f(Y_0)$$

$$i = 1: \quad Y_1 \rightarrow Y_2 = f(Y_1) = f(f(Y_0)) = f^2(Y_0)$$

$$i = 2: \quad Y_2 \rightarrow Y_3 = f(Y_2) = \dots = f^3(Y_0)$$

$$\vdots \qquad \vdots$$

$${}^{1\text{refs}} \quad i = n: \quad Y_n \rightarrow Y_{n+1} = f(Y_n) = \dots = f^n(Y_0)$$

Linear Map: Iteration with a parameter

$$Y_{i+1} = a \cdot Y_i$$

$$i = 0: \quad Y_0 \rightarrow Y_1 = a \cdot Y_0$$

$$i = 1: \quad Y_1 \rightarrow Y_2 = a \cdot Y_1 = a \cdot a \cdot Y_0 = a^2 \cdot Y_0$$

$$i = 2: \quad Y_2 \rightarrow Y_3 = a \cdot Y_2 = \dots = a^3 \cdot Y_0$$

⋮
⋮
⋮

$$i = n: \quad Y_n \rightarrow Y_{n+1} = a \cdot Y_n = \dots = a^{n+1} \cdot Y_0$$



Linear Map: Iteration with a Parameter

$$Y_{i+1} = a \cdot Y_i$$

$0 < a < 1$

$a > 1$

$a = 1$

$-1 < a < 0$

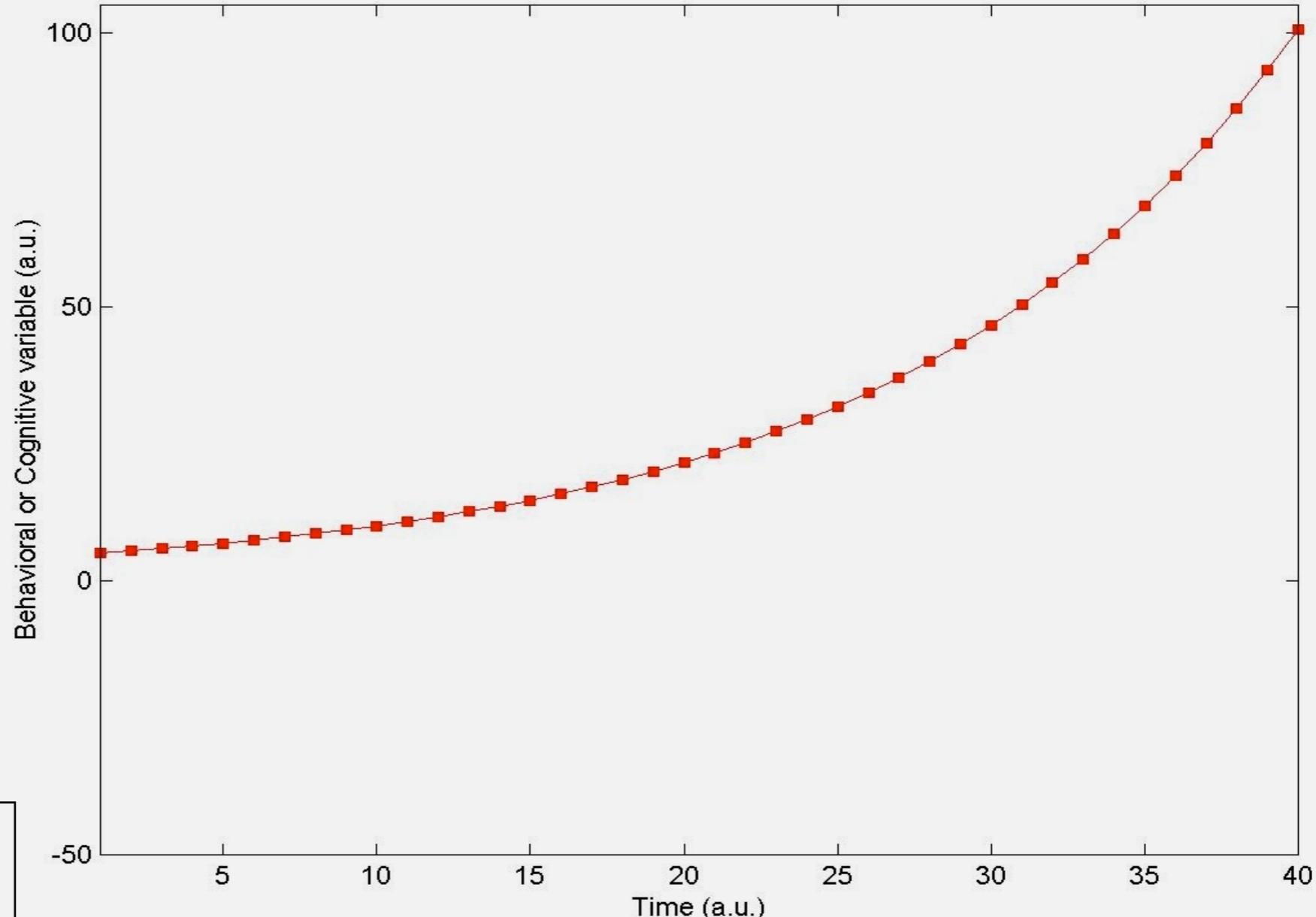
$a < -1$

$a = -1$

Y_0 nonspecific

Linear Map: Iteration with a Parameter

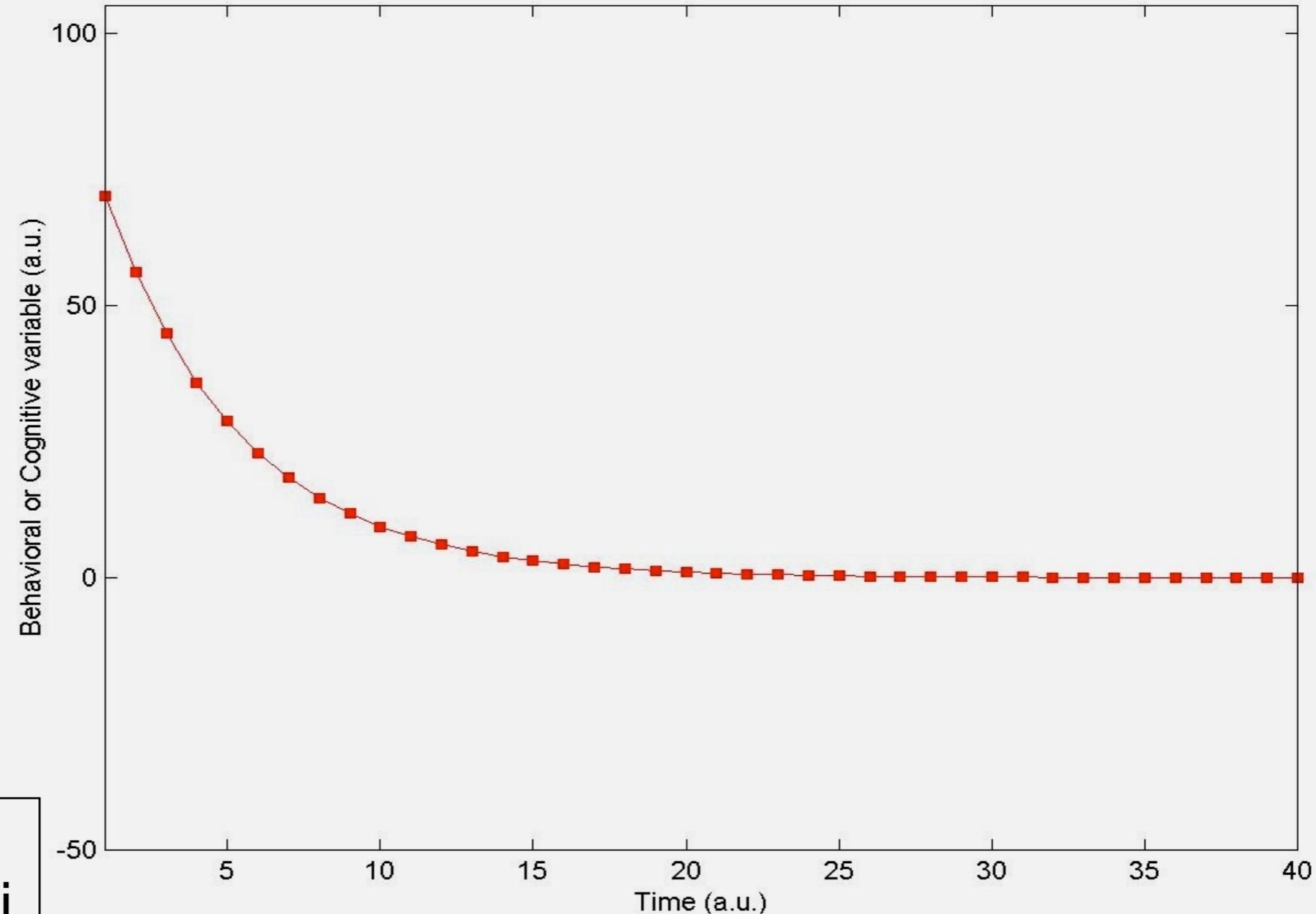
$$a = 1.08$$
$$Y_0 = 5$$



$$Y_{i+1} = a \cdot Y_i$$

Linear Map: Iteration with a Parameter

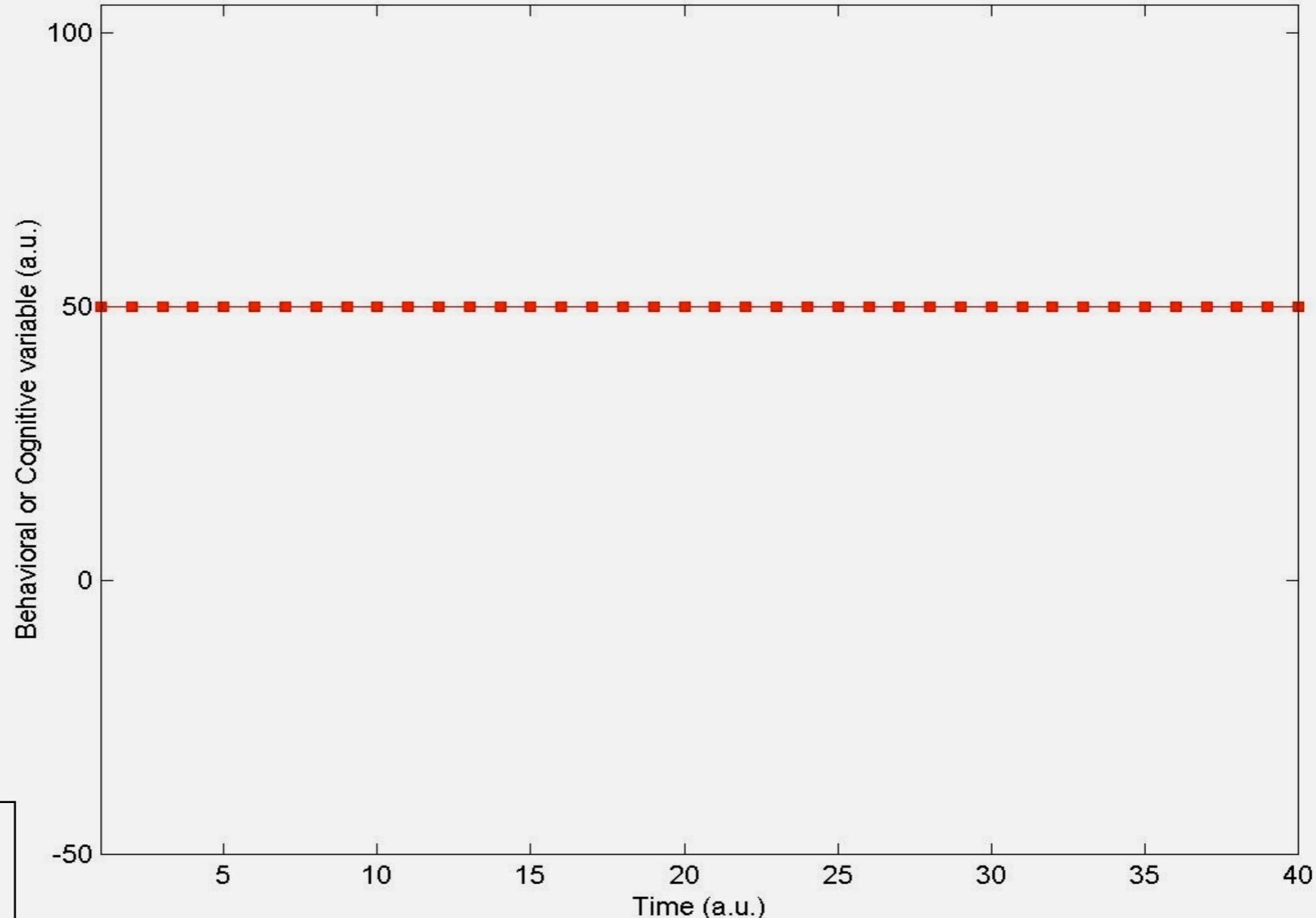
$a = 0.8$
 $Y_0 = 70$



$$Y_{i+1} = a \cdot Y_i$$

Linear Map: Iteration with a Parameter

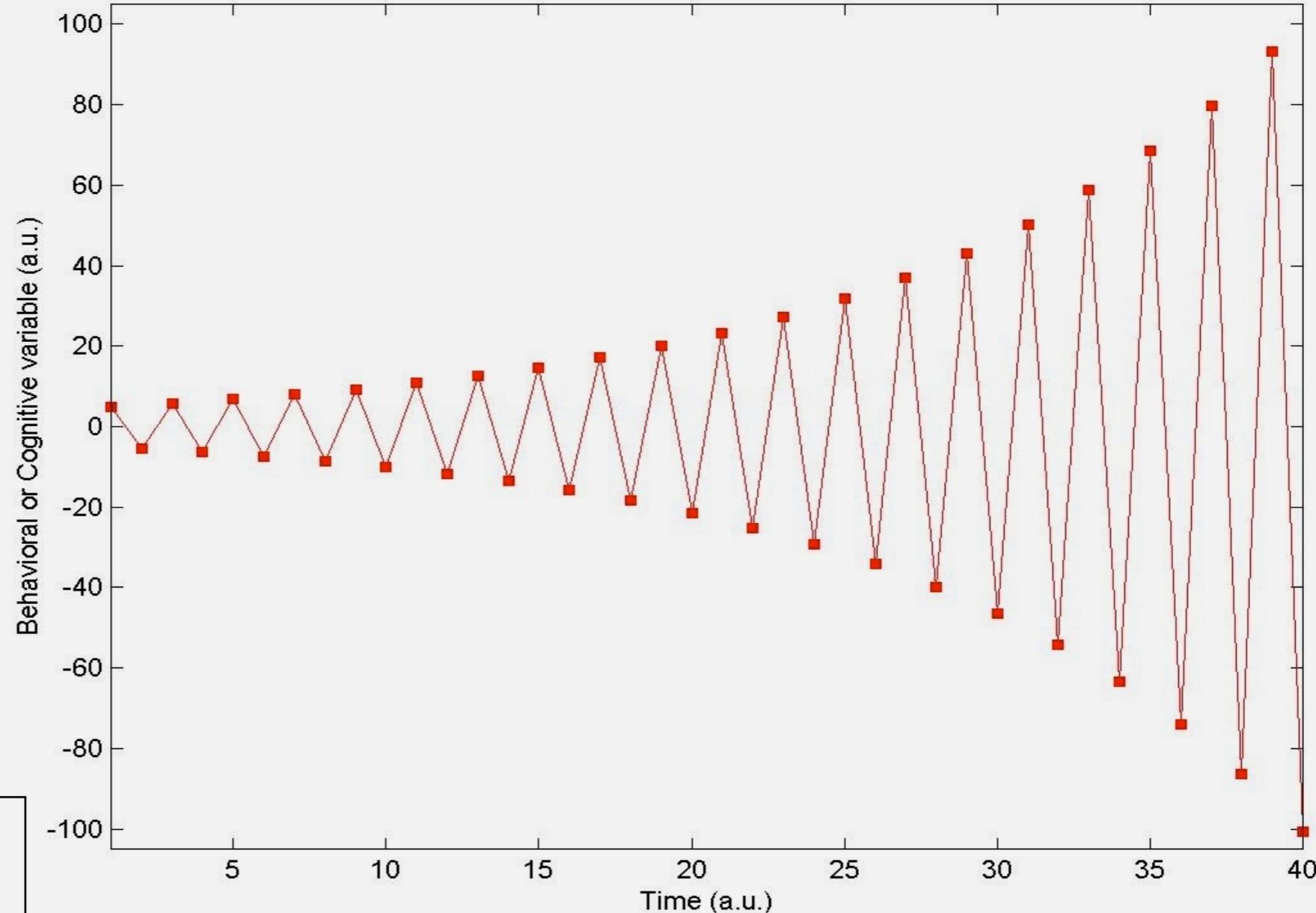
$a = 1.00$
 $Y_0 = 50$



$$Y_{i+1} = a \cdot Y_i$$

Linear Map: Iteration with a Parameter

$$a = -1.08$$
$$Y_0 = 5$$



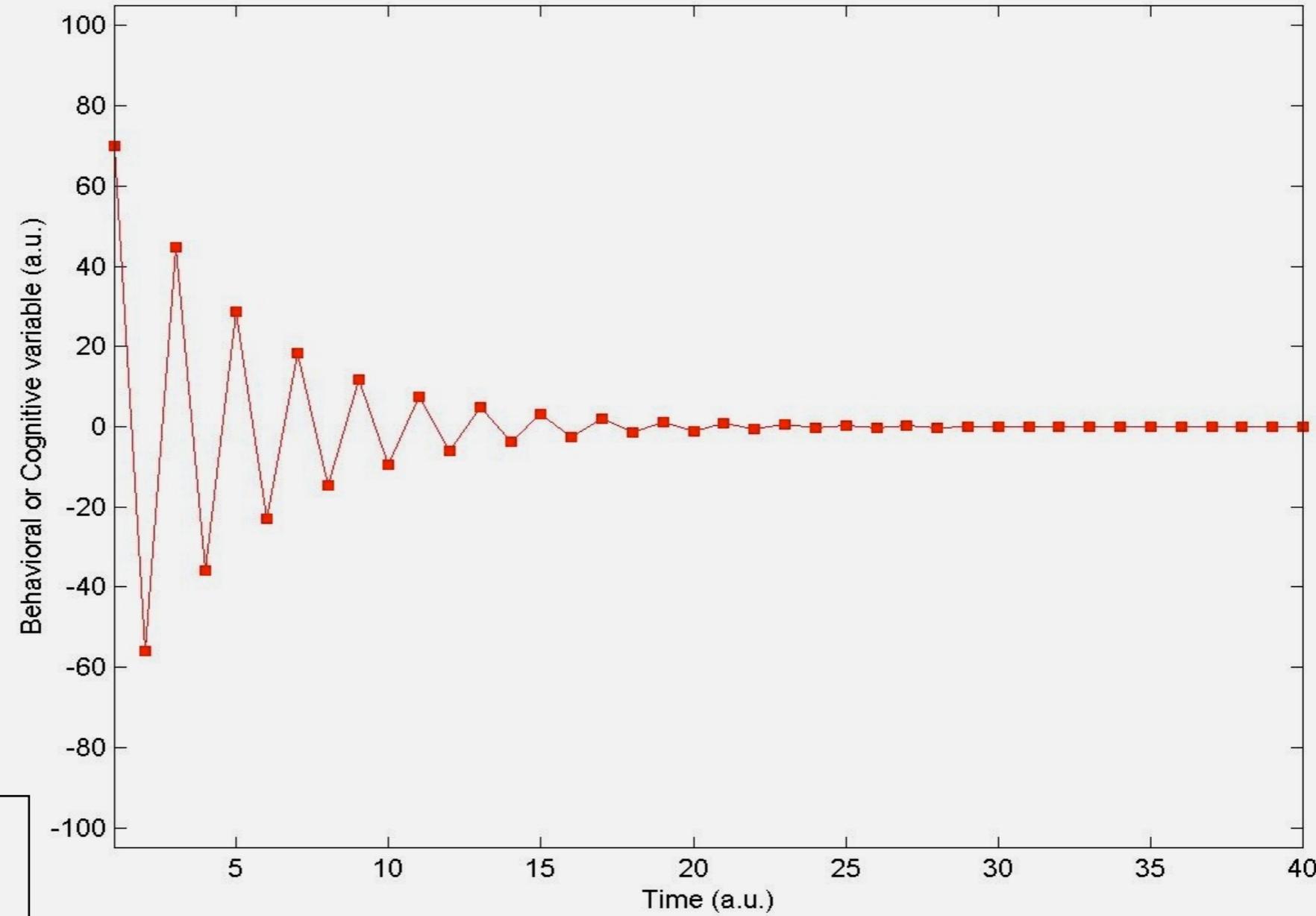
$$Y_{i+1} = a \cdot Y_i$$

¹refs

Linear Map: Iteration with a Parameter

$a = -0.8$
 $Y_0 = 70$

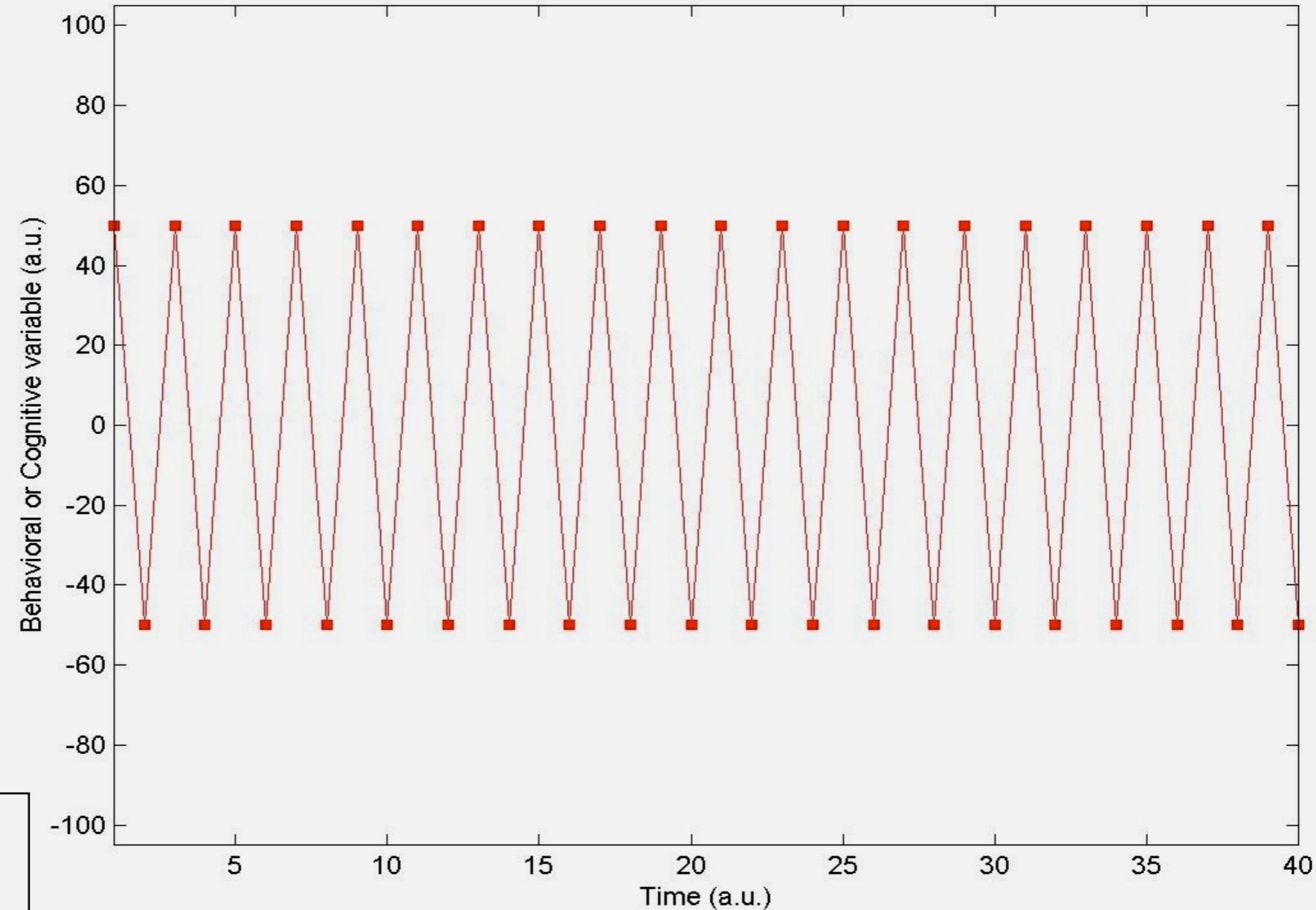
$$Y_{i+1} = a \cdot Y_i$$



Linear Map: Iteration with a Parameter

$a = -1.00$
 $Y_0 = 50$

$Y_{i+1} = a \cdot Y_i$



Linear Map: Iteration with a Parameter

Some interesting differences compared to a linear model:

- Change of behaviour over iterations
 - ▶ *Simple model vs. “time” or “occasion” as a predictor*
- Qualitatively different behaviour
 - ▶ *One model produces at least four different types of behaviour*
 - ▶ *Not by adding predictors (components), by changing one parameter*



PARAMETERS & BIFURCATIONS

EXAMPLE 2:

The Logistic Map
(restricted growth)



Logistic Map ...

$$L_{i+1} = r L_i (1 - L_i)$$

- Simplest nontrivial model often used as an introduction to DST and Chaos theory.
- Well-known model in ecology, physics, economics and social sciences.
- ‘Styled’ version of Van Geert’s model for language growth. (*Next meeting*)



Logistic Map: Iteration

$$L_{i+1} = r L_i (1 - L_i)$$

$$i = 0: \quad L_0 \rightarrow L_1 = r L_0 (1 - L_0)$$

$$\begin{aligned} i = 1: \quad L_1 &\rightarrow L_2 = r L_1 (1 - L_1) \\ &= r r L_0 (1 - L_0) (1 - r L_0 (1 - L_0)) \\ &= -r^3 L_0^4 + 2r^3 L_0^3 - r^2 (1+r) L_0^2 + r^2 L_0 \end{aligned}$$



Logistic Map: Parameter

$$L_{i+1} = r L_i (1 - L_i)$$

$r = 0.90$

$r = 1.90$

$r = 2.90$

$r = 3.30$

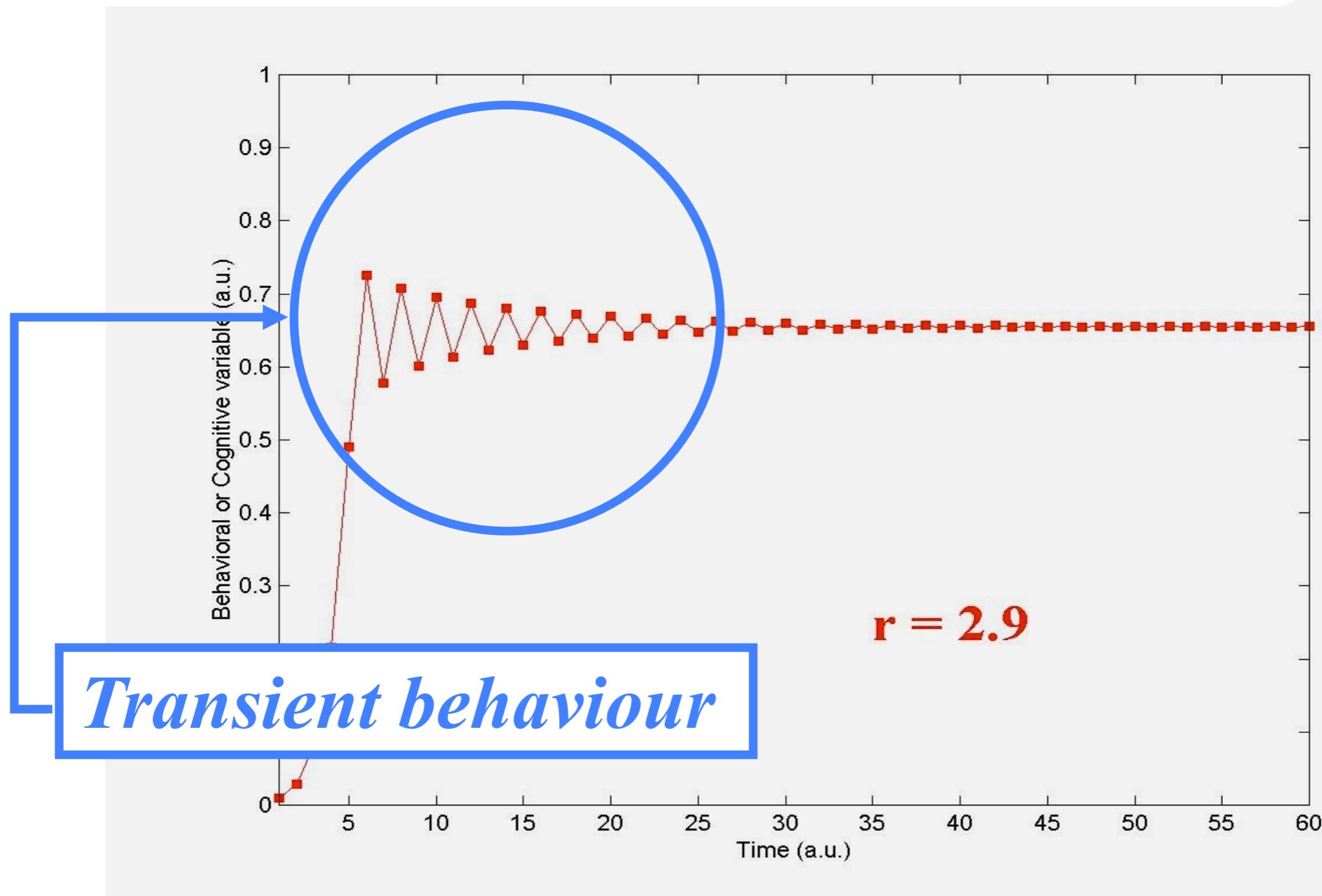
$r = 3.52$

$r = 3.90$

L_0 small

Logistic Map: Graphs

$$L_{i+1} = r L_i (1 - L_i)$$



An ecology of growth models?
Same principle!

Basic Growth Models: Exponential + Restricted Growth

$$Population = rN \times \left(\frac{K - N}{K} \right)$$

Additional Parameter: Carrying Capacity

$$CognitiveGrowth = L_i \left(1 + r \times \frac{K - L_i}{K} \right)$$

$$StylizedLogistic = r Y_i \times \left(\frac{1 - Y_i}{1} \right)$$

Bifurcation Diagram



Bifurcation Diagram - Phase Diagram

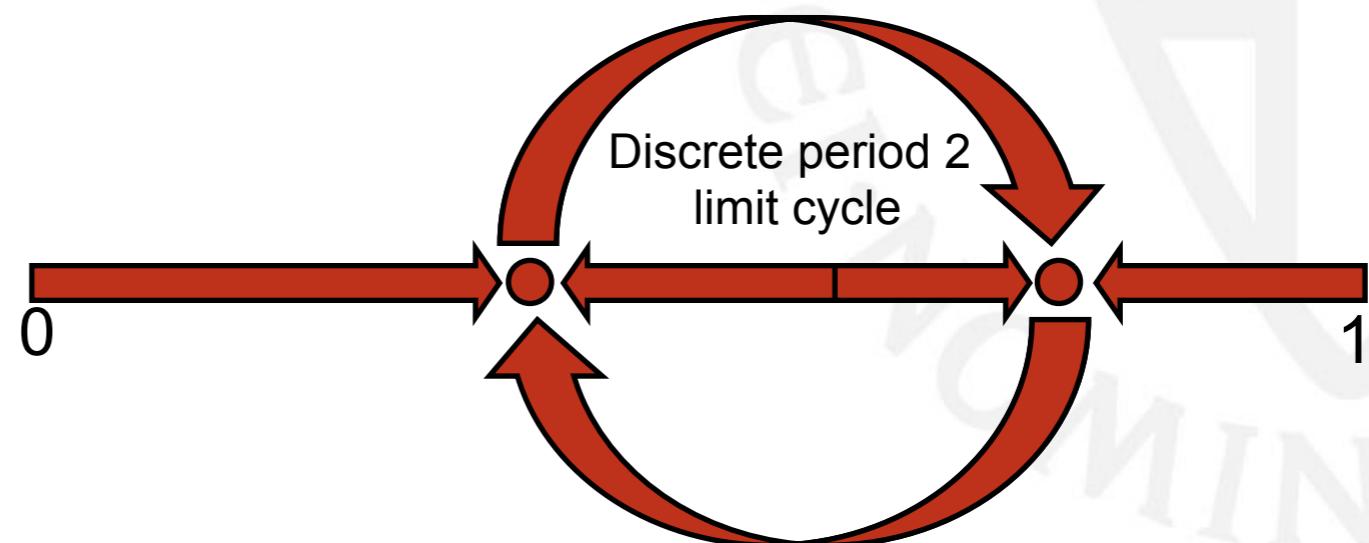
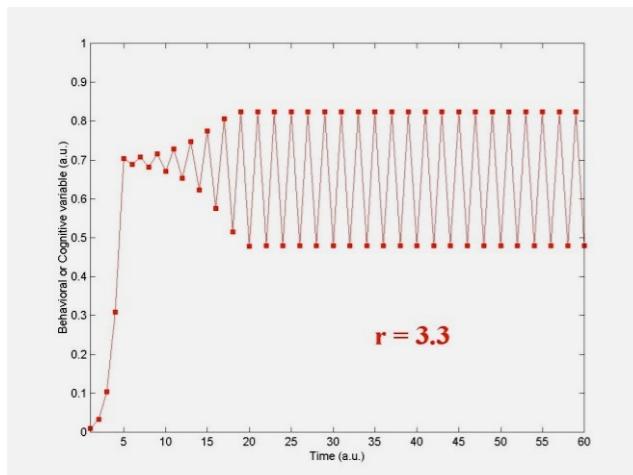
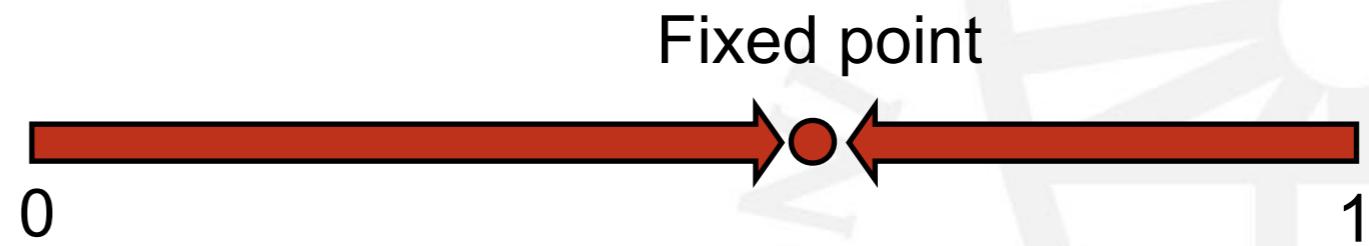
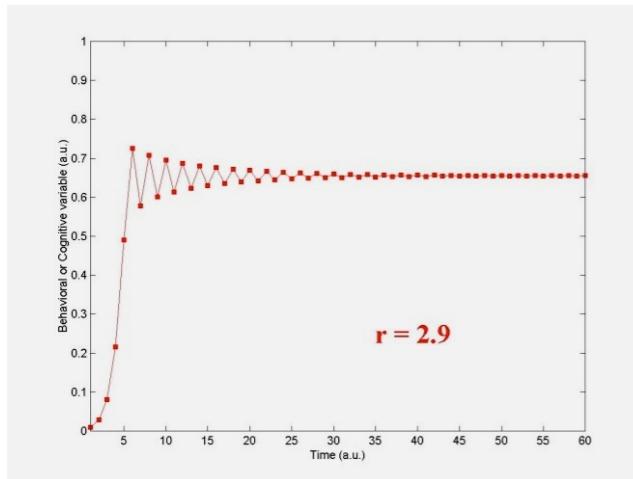
A graphical representation of the possible states a dynamical system can end up in for different values of one or more parameters.

- The parameter is called the **control parameter**.
- The end states are called **attractors**.
- The change from one attractor (or set) to another is called a **bifurcation**.



End states are attractors in state space: Attractor types

State Space is an abstract space used to represent the behaviour of a system. Its dimensions are the variables of the system. Thus a point in the phase space defines a potential state of the system. The points actually achieved by a system depend on its iterative function and initial condition (starting point).



State space, Attractor types

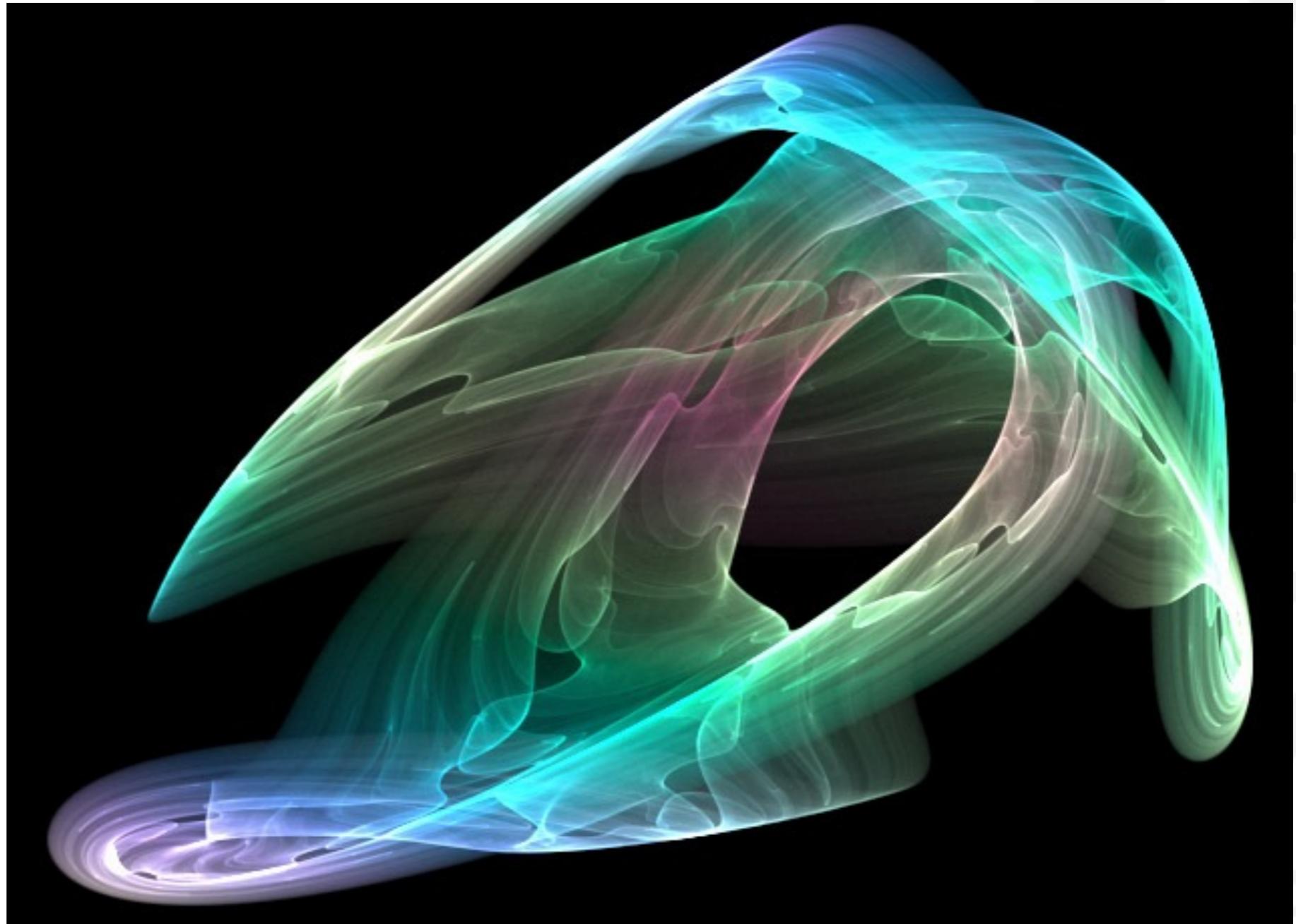
“Saturn”
attractor

Strange attractors
are quasi periodic
and bounded

Bottom line:

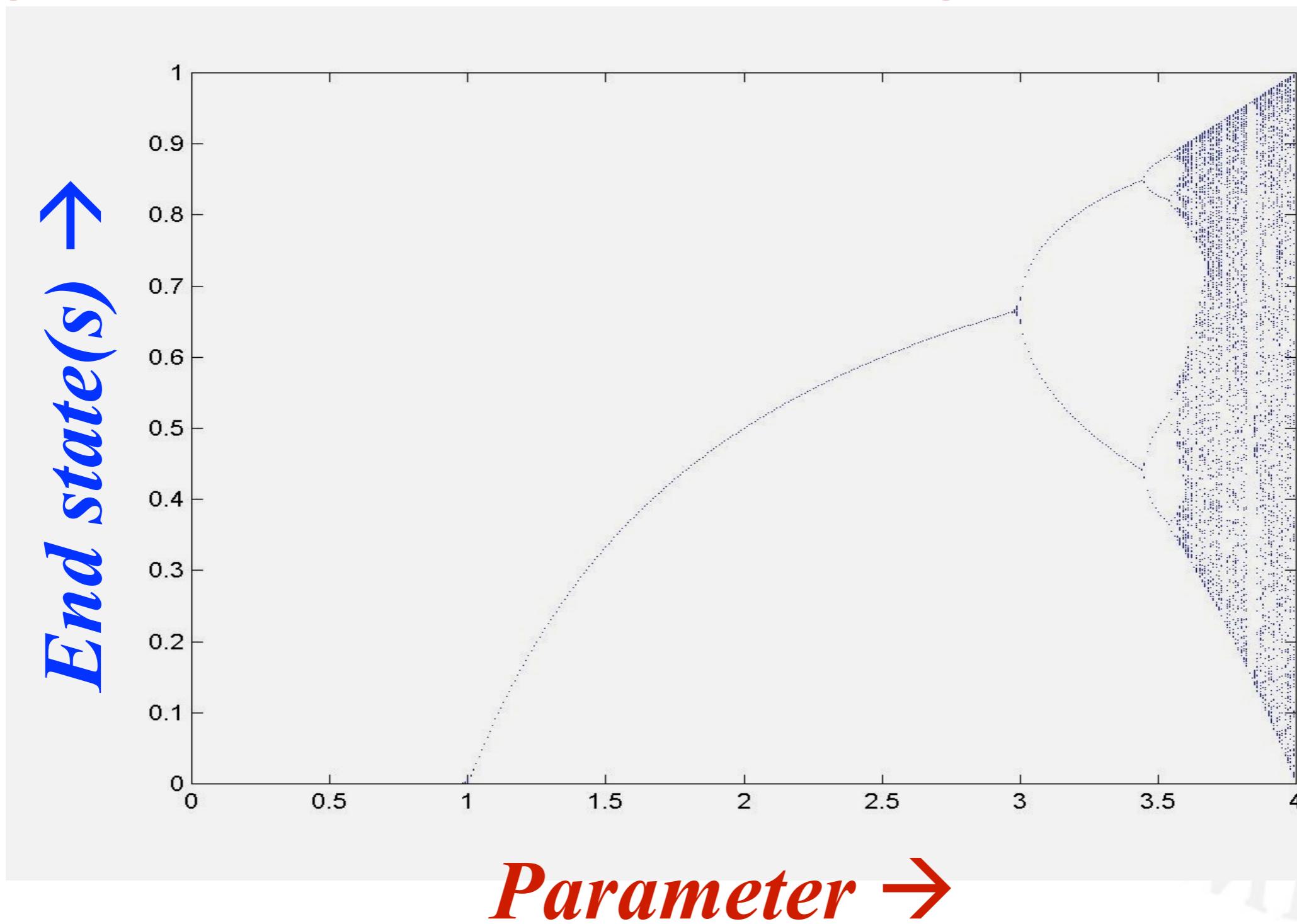
An attractor means
a limited region
of state space
is visited.

Not all DF actually
available
to the system
are used.

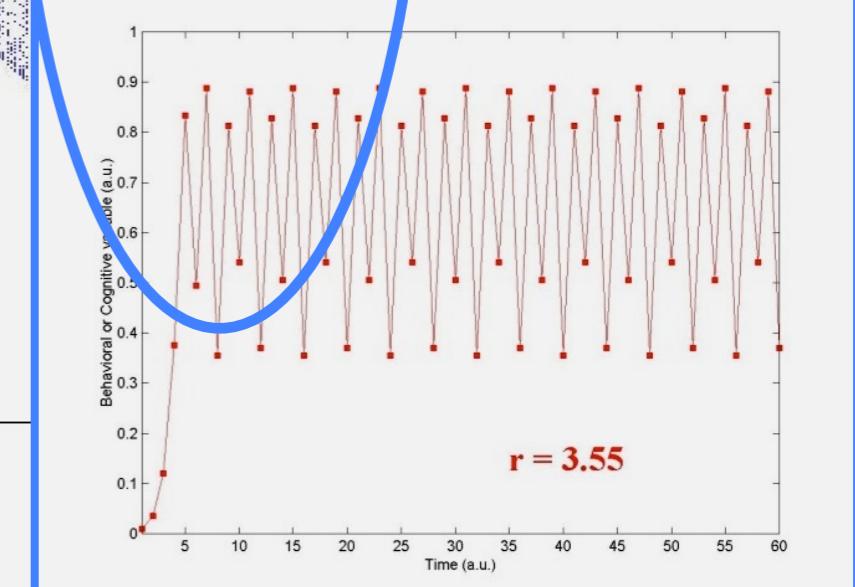
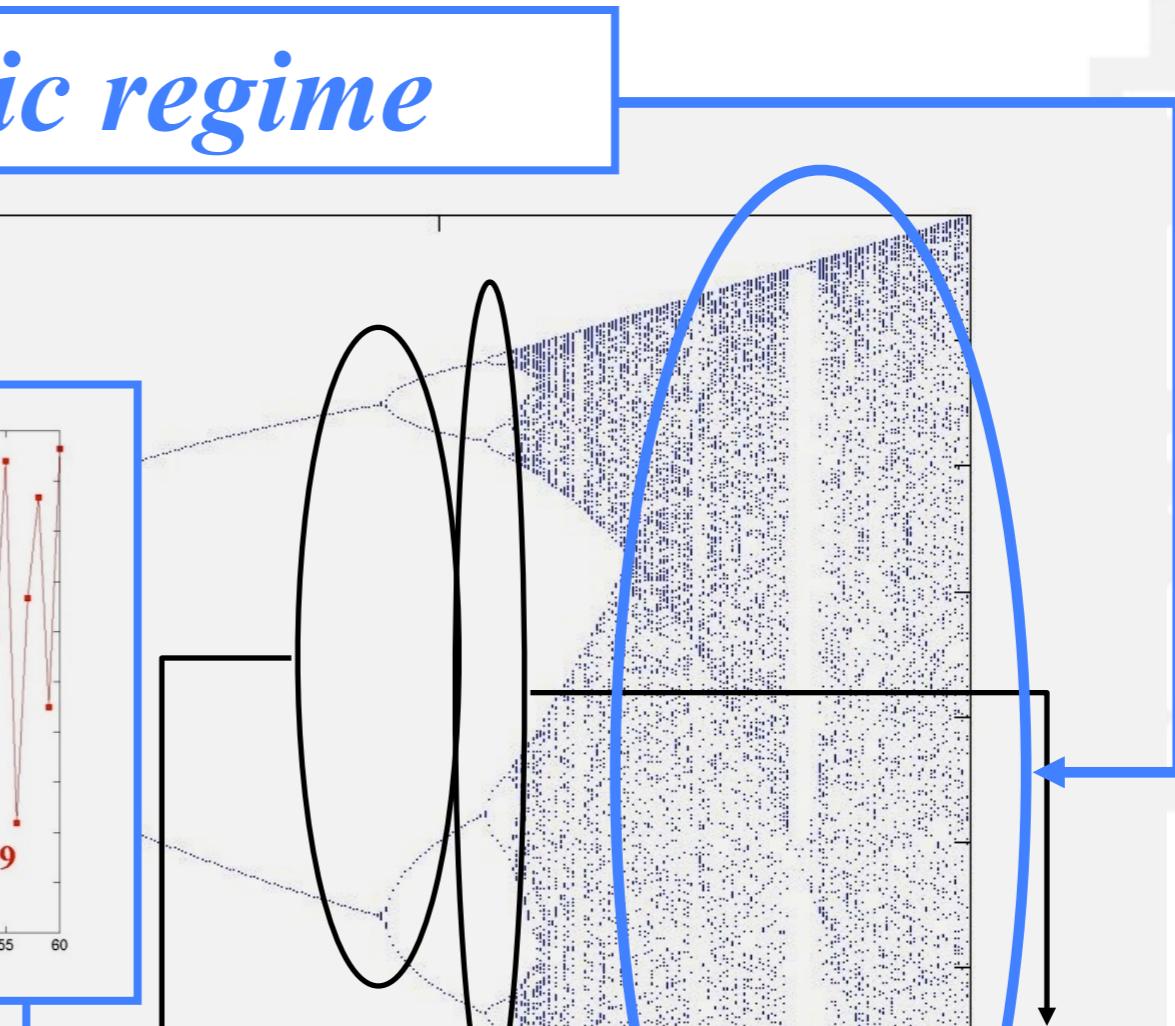
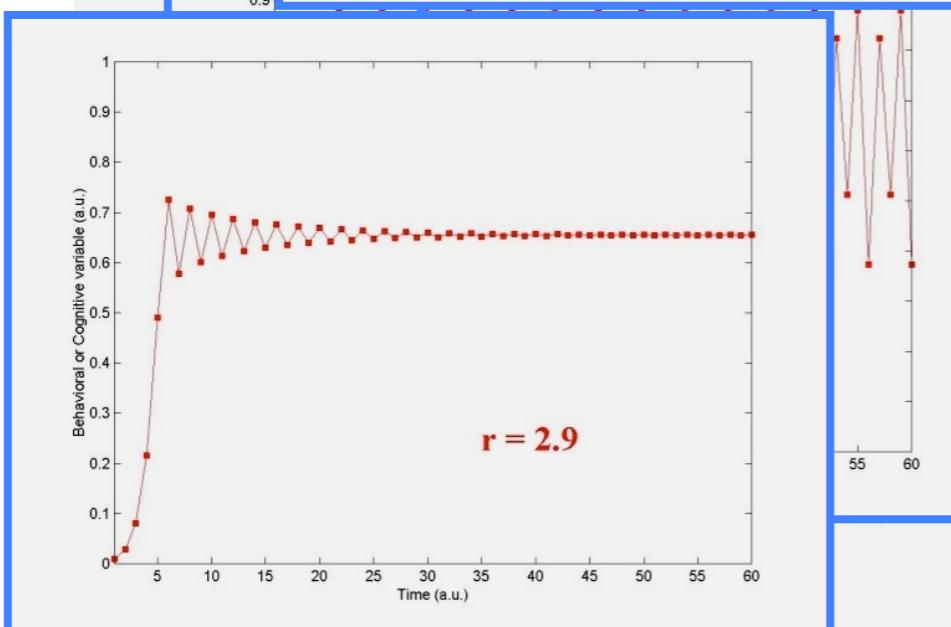
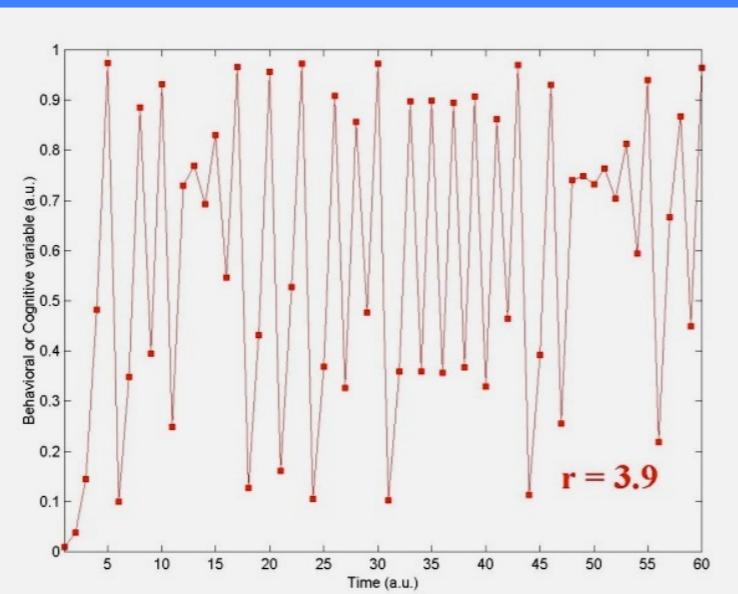
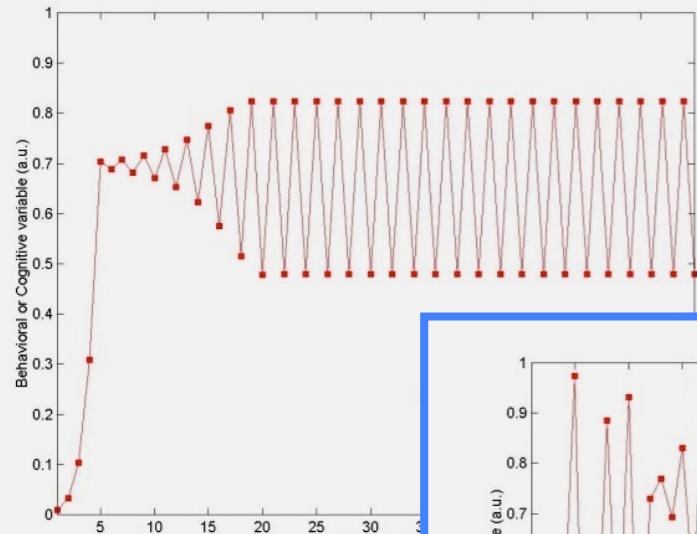


<http://www.da4ga.nl/wp-content/uploads/2012/03/PastedGraphic-2-1.jpg>

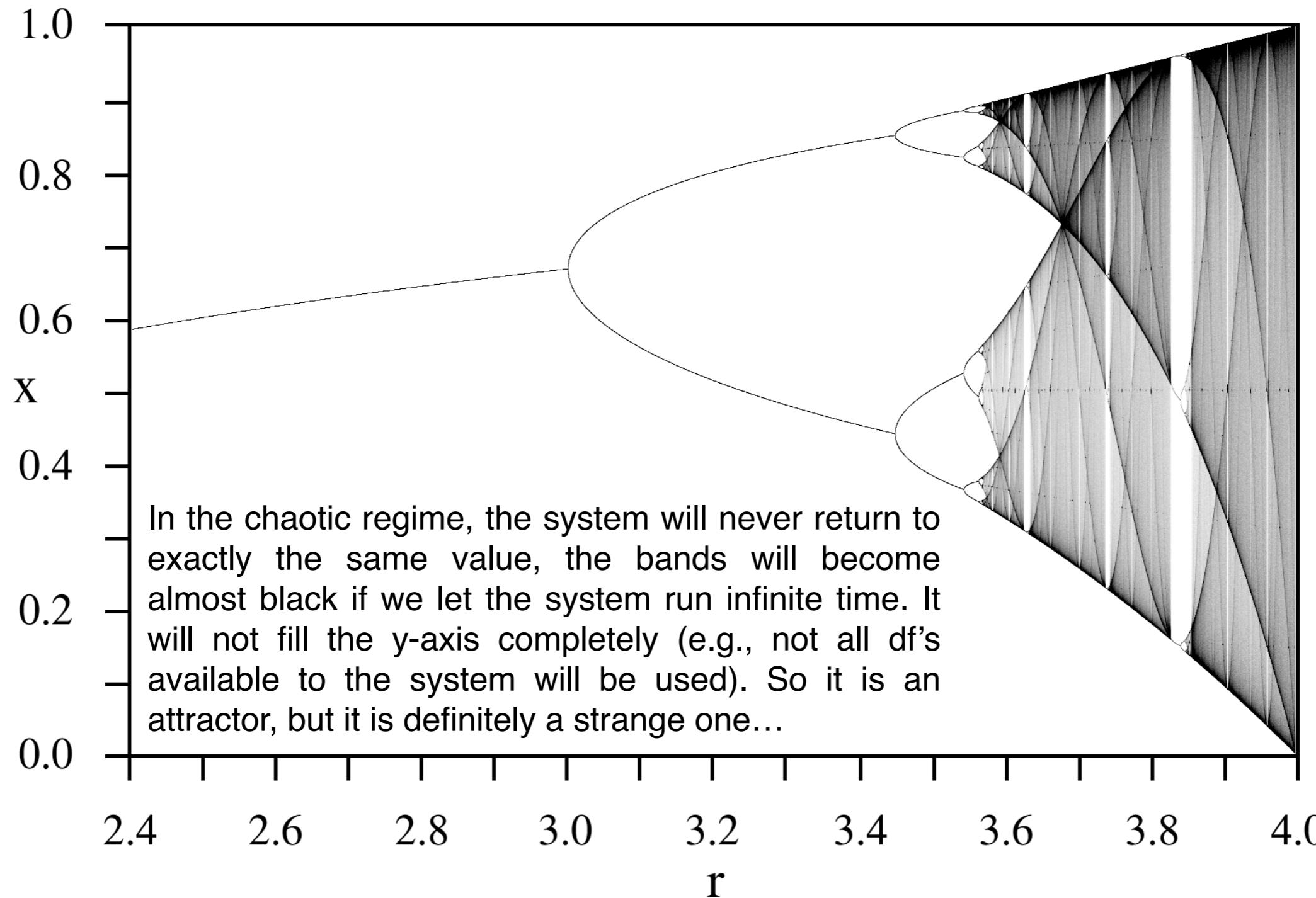
Logistic Map: Bifurcation Diagram



Chaotic regime



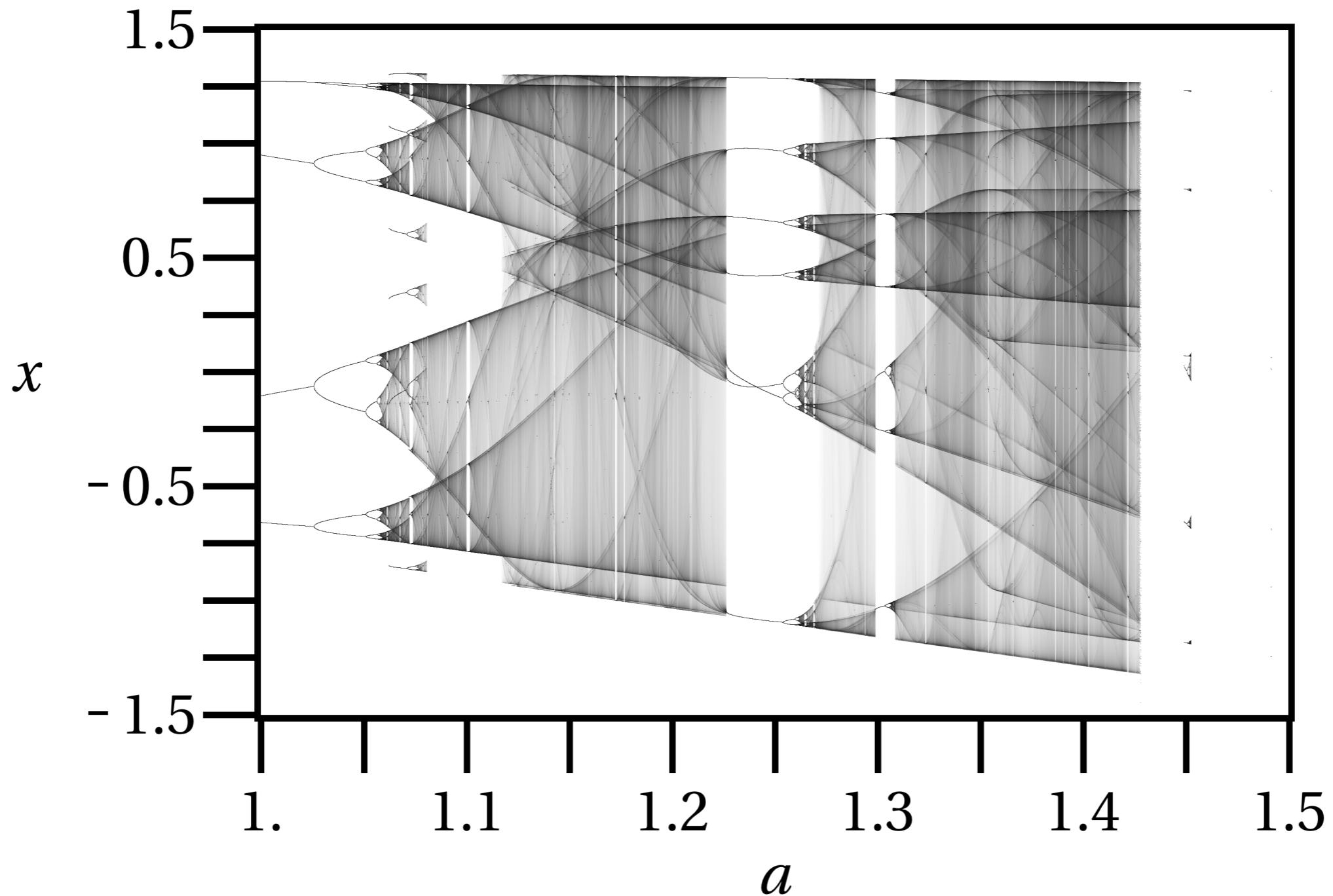
Logistic Map: Bifurcation Diagram



http://upload.wikimedia.org/wikipedia/commons/7/7d/LogisticMap_BifurcationDiagram.png



Henon Map: Bifurcation Diagram



http://upload.wikimedia.org/wikipedia/commons/c/cd/Henon_bifurcation_map_b%3D0.3.png

DETERMINISTIC CHAOS

1refs



CHAOS, TURBULENCE and other unsolved mysteries

“Turbulence is the most important unsolved problem of classical physics”

- Richard Feynman (1918 - 1988)

“I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment:

*One is quantum electrodynamics,
and the other is the turbulent motion of fluids.*

And about the former I am rather optimistic.”

- Horace Lamb (1849 - 1934)



Deterministic Chaos

Table 12-1. Summary of the Hierarchy of Dynamic Systems.

Type	Constraints	Description
Zero	Absolute	Constant state
I	Analytic integrals	Solvable dynamic system
II	Approximate analytic integrals	Amenable to perturbation theory
III	Quasi-deterministic; smooth but erratic trajectory	Chaotic dynamic system
IV	Rigorously defined only by averages over time or state space	Turbulent/stochastic

Table 12-2. A few examples of the types of dynamic systems.

Type	Examples
Zero	Images, gravity models, structures
I	Gear trains, 2-body problem, physical pendulum
II	Satellite orbits, lunar and planetary theories
III	Climatology, Lorenz equations, discrete logistic equation
IV	Quantum mechanics, turbulent flow, statistical mechanics

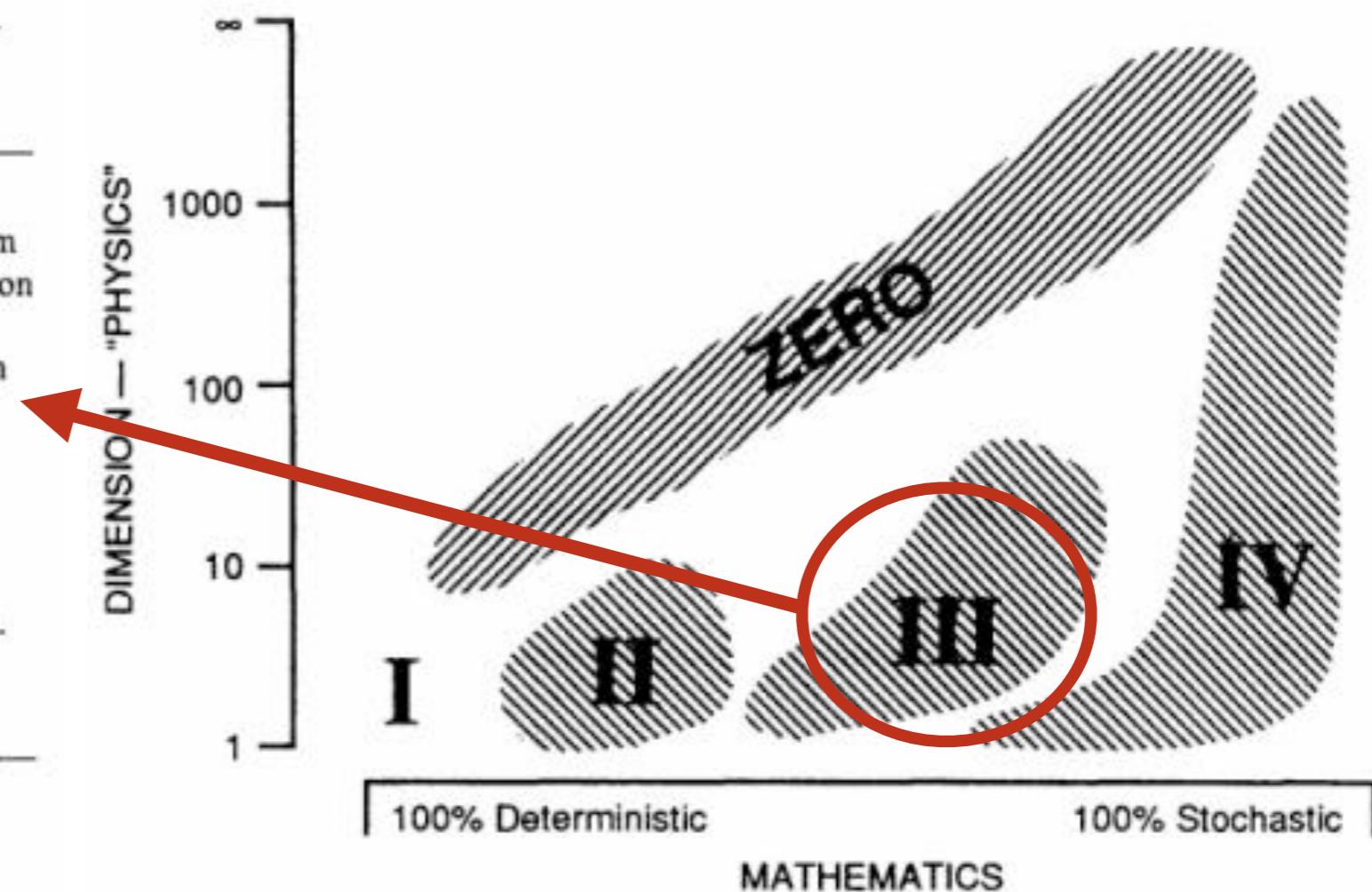


Figure 12-1. Schematic representation of the Hierarchy of Dynamic Systems.

Deterministic Chaos

There is no real definition of chaos, but there are at least four ingredients:

*The dynamics is **a-periodic** and **bounded**, and the system is **deterministic** and **sensitively depends on initial conditions**.*



Deterministic Chaos... Paradox?

Something that is ***deterministic***, is:

- *Mathematically exact;*
- *Predictable.*

Something that is '***chaotic***', shows:

- *Disorderly behaviour;*
- *Extreme sensitivity.*

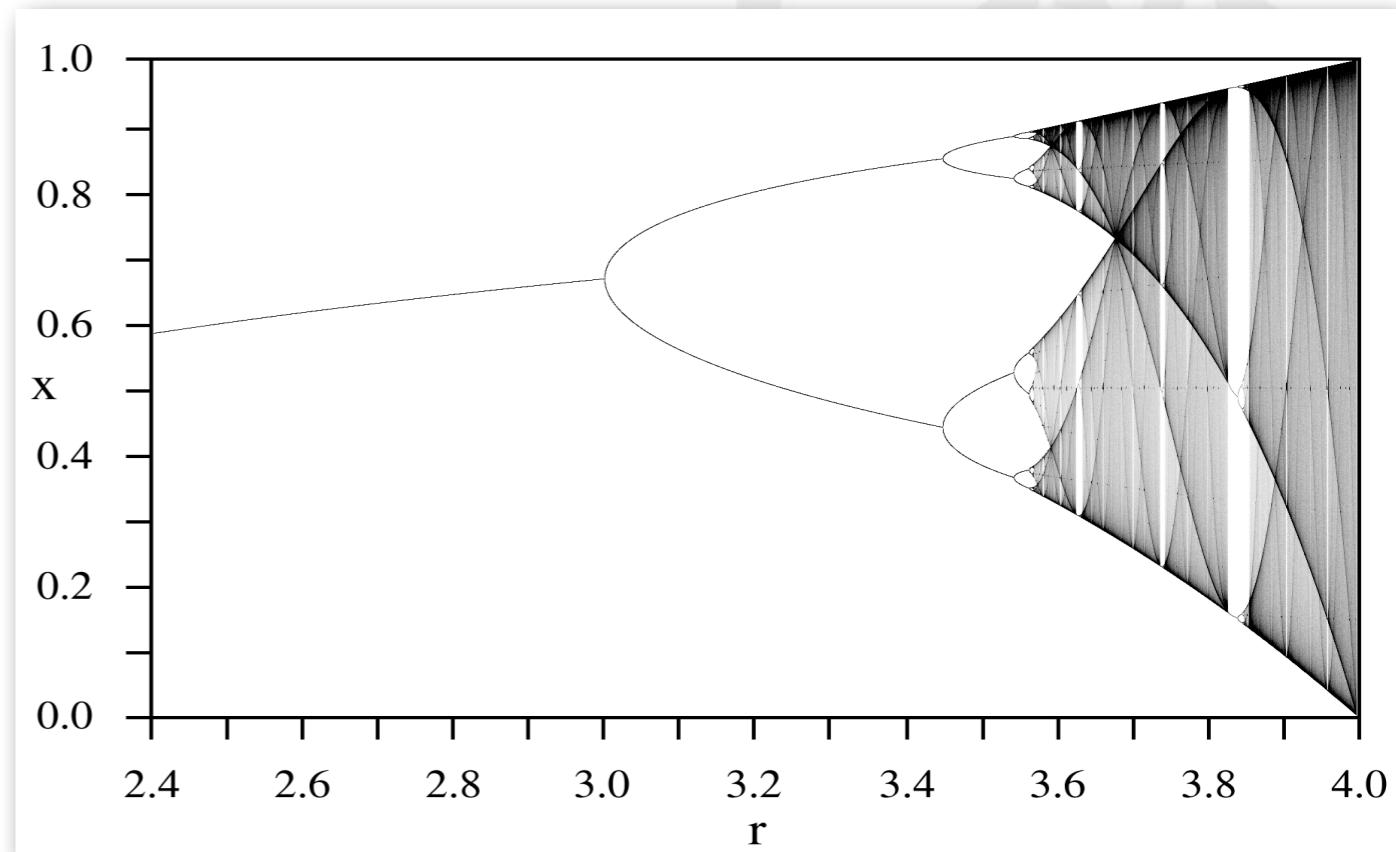


CHAOS, TURBULENCE and other unsolved mysteries

Chaotic regime of the logistic map represented by the bifurcation diagram

Transitions between regimes:

- Order to Order
- Order to Chaos
- Chaos to Order
- Chaos to Chaos



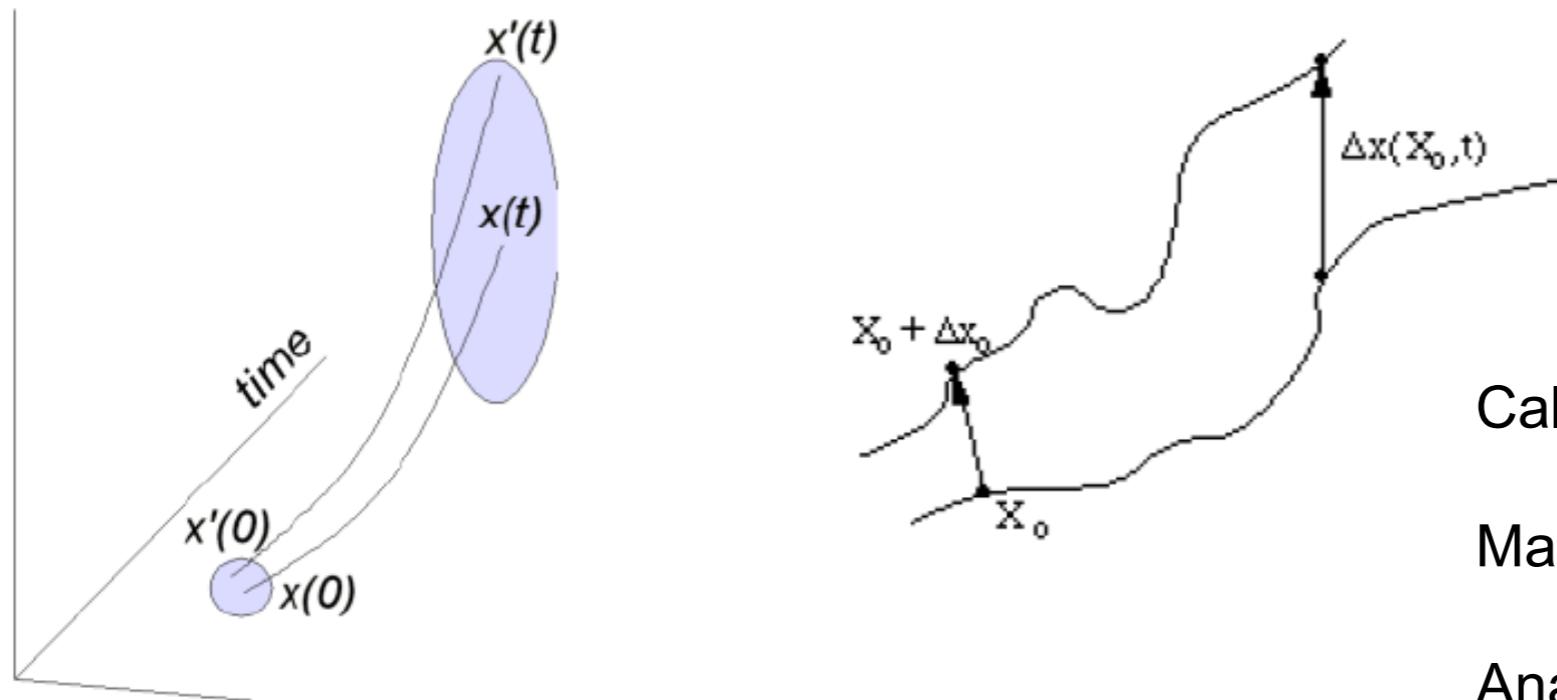
Why this happens at these parameter settings is.... unknown

CHAOS, TURBULENCE and other unsolved mysteries

What can we say about chaos?

4. Sensitive dependence on initial conditions

The *Lyapounov Exponent* characterises (quantifies) the rate of separation of two infinitesimally close trajectories in state space.



Calculate if you have a model

May be experimentally accessible

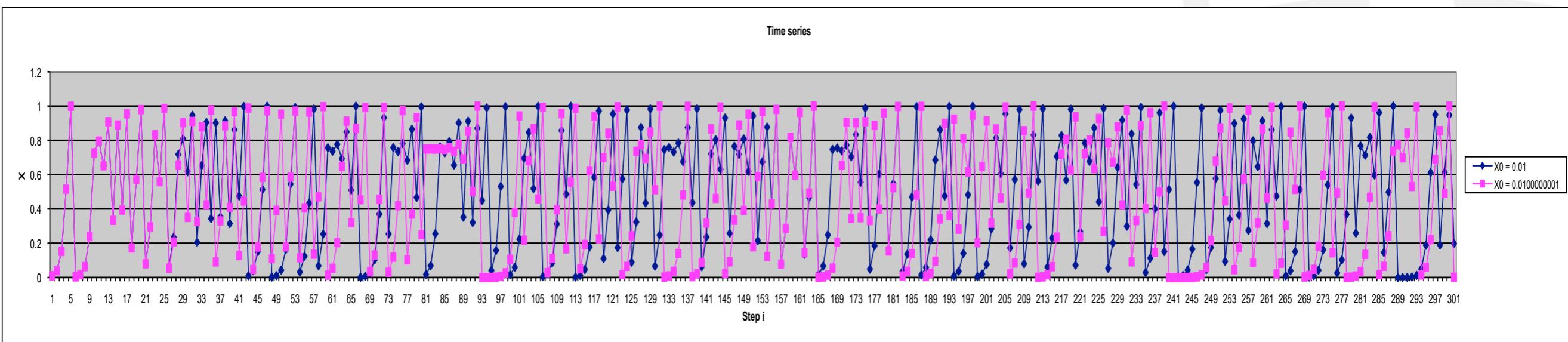
Analytic techniques (in R) are available

Sensitive Dependence on Initial Conditions

What can we say about deterministic chaos and complexity?

$X_0 = 0.01$

$X_0 = 0.0100000001$

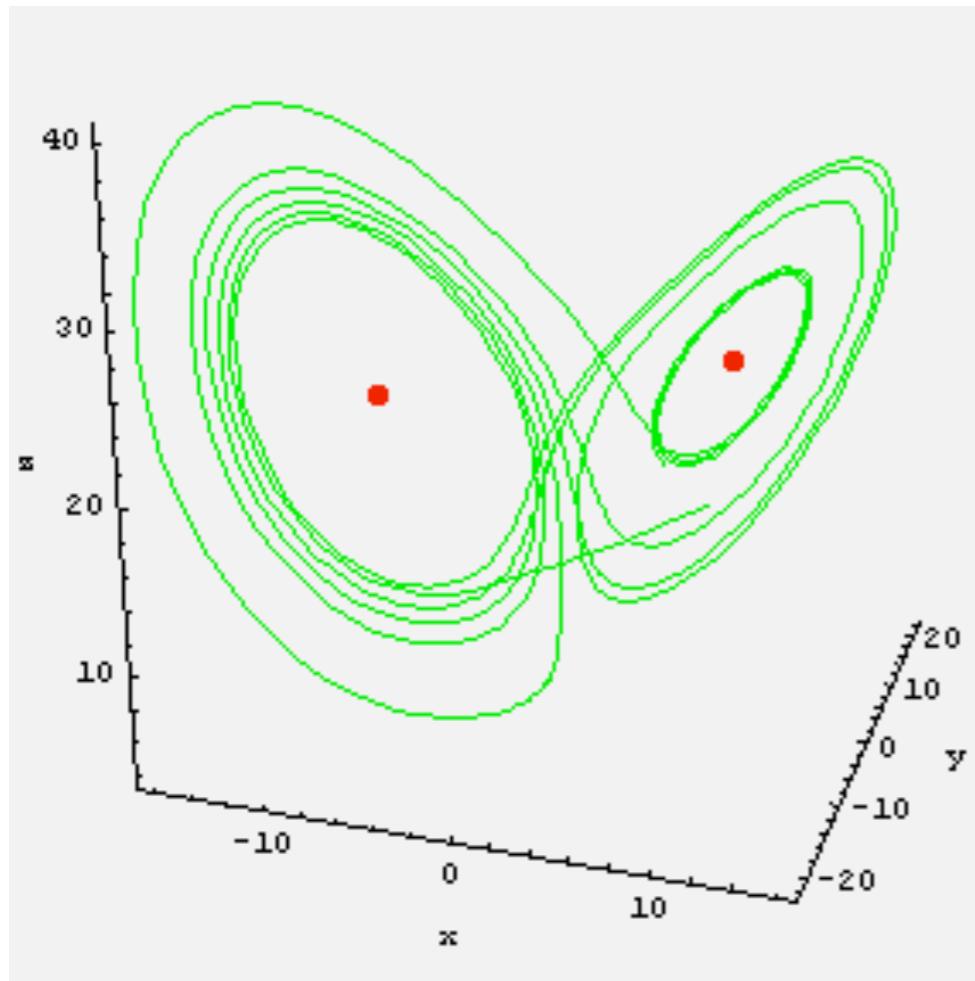


Tiny differences in initial conditions can yield diverging time-evolutions of system states

Lorenz observed this in his models of the upper atmosphere:

The divergence was so extreme it resembled a butterfly flapping its wings -or not- could be the difference between weather developing as a hurricane or a summer breeze

Lorenz Attractor

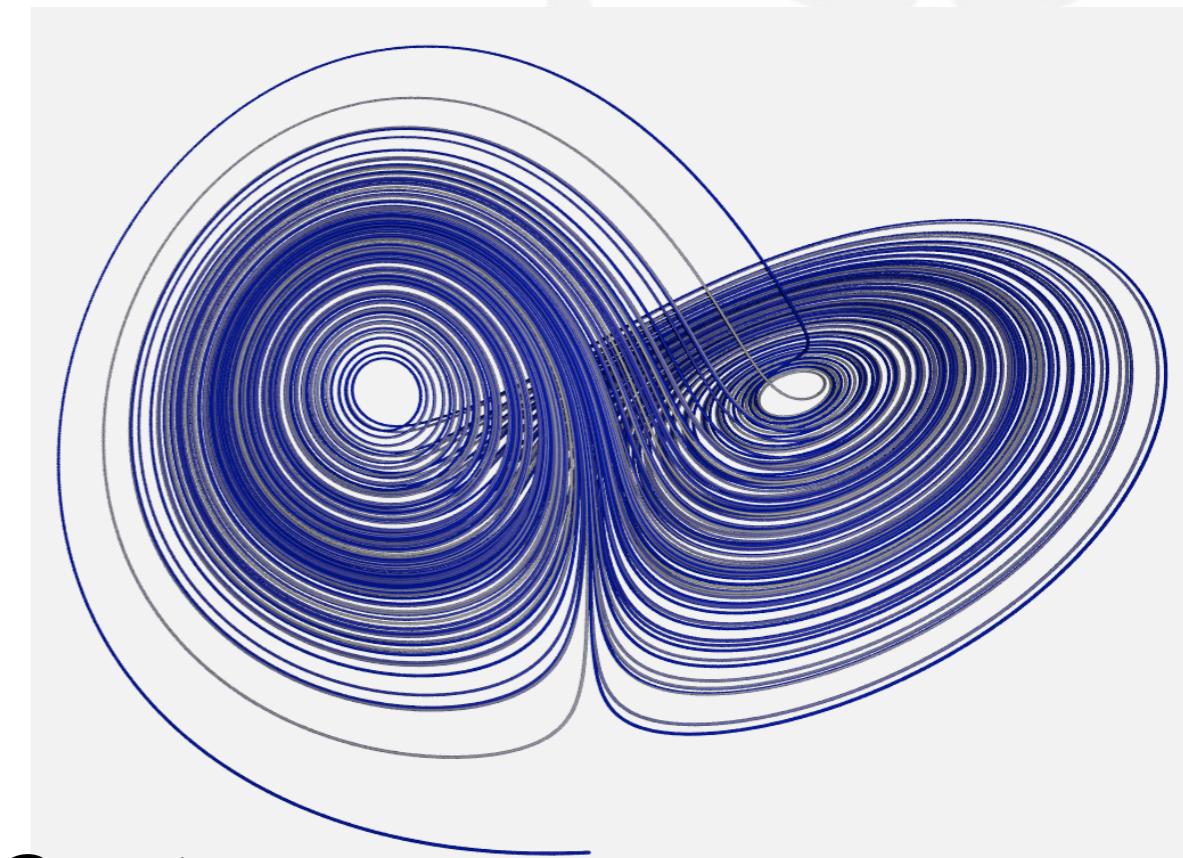


$$\begin{aligned}\frac{dx}{dt} &= a(y - x), \\ \frac{dy}{dt} &= x(b - z) - y, \\ \frac{dz}{dt} &= xy - cz.\end{aligned}$$

Deterministic Chaos

Maps: linear map, 1D state space

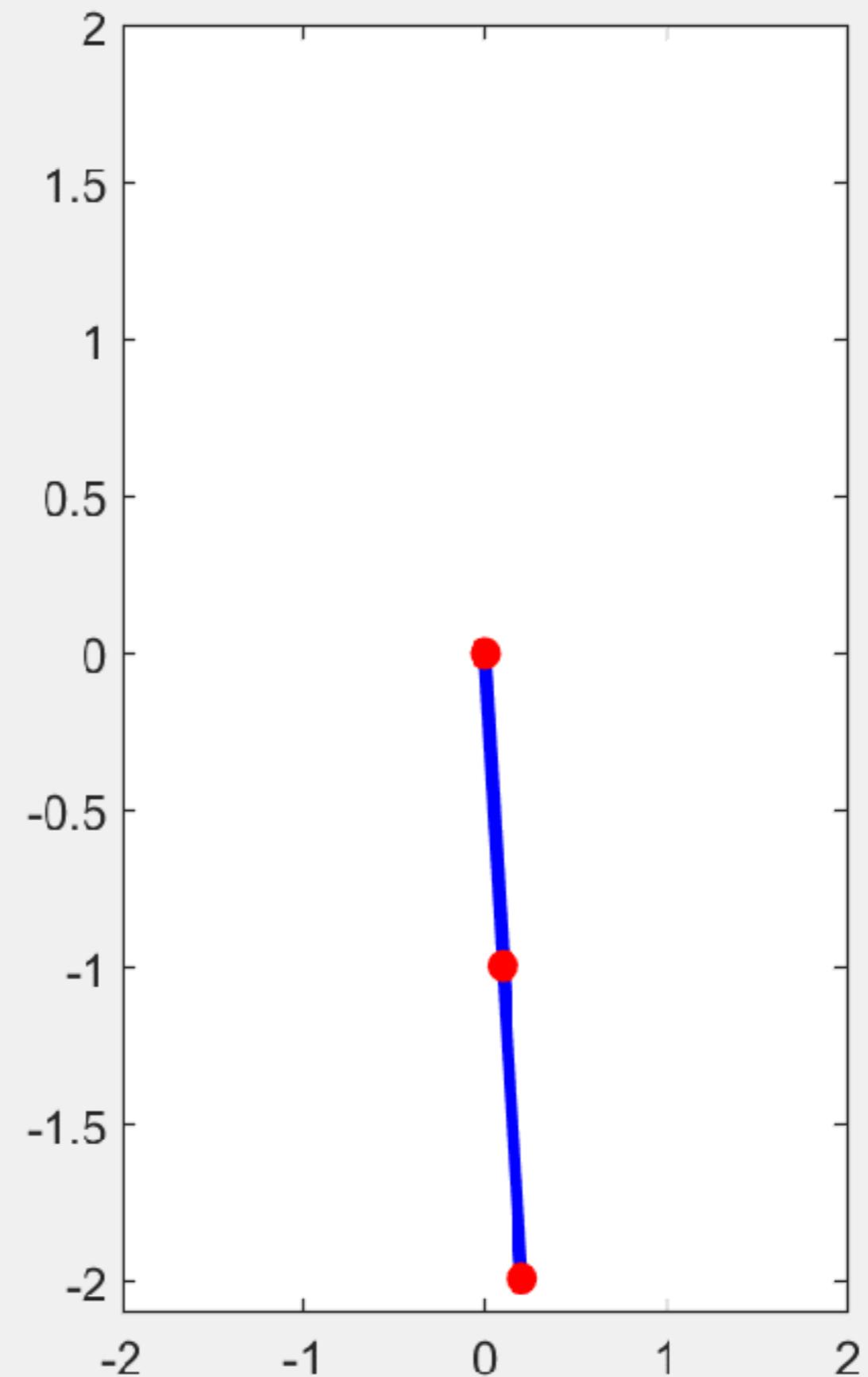
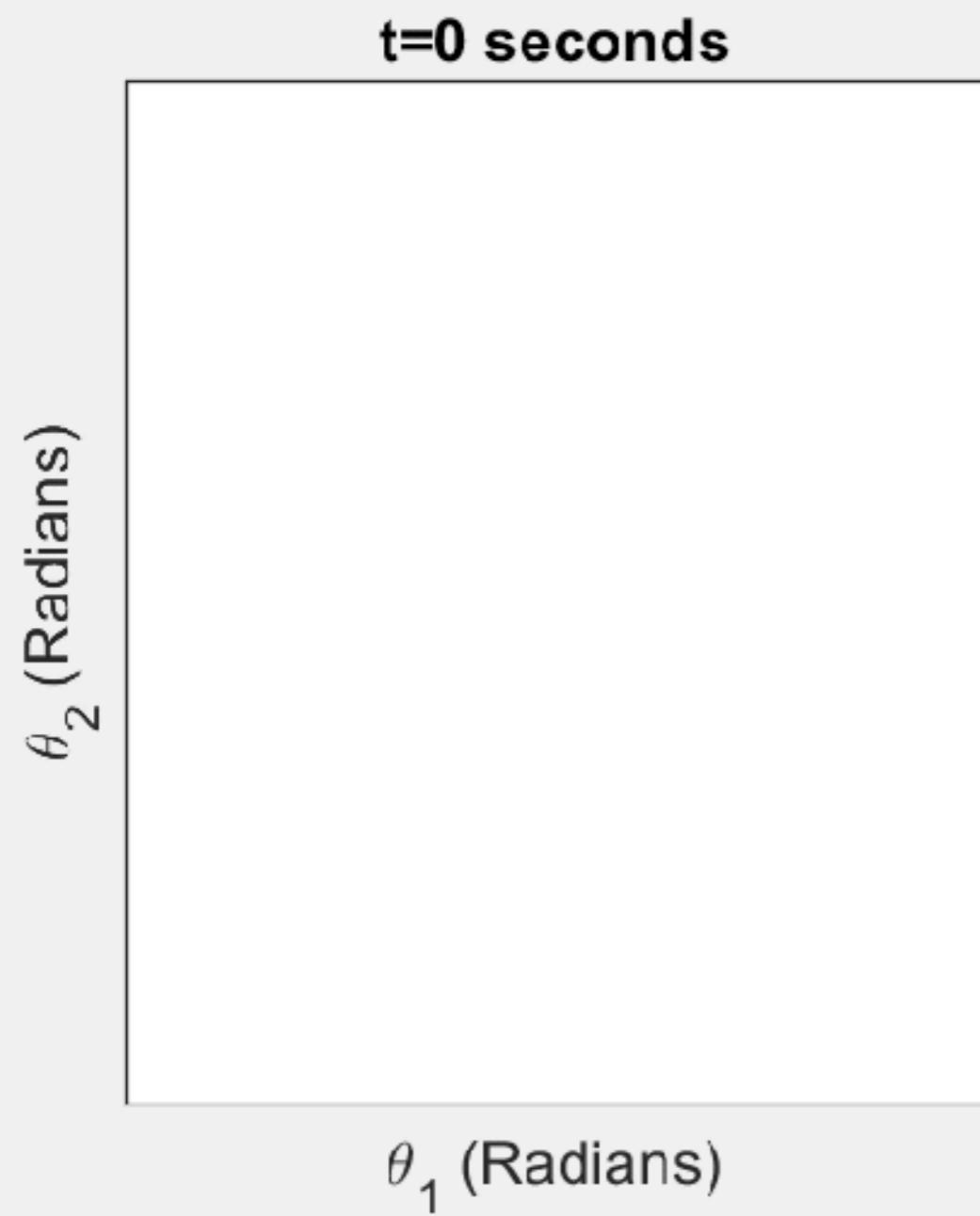
Flows: Need 3 coupled ODEs
(ordinary differential equations)
Minimum is 3D state space



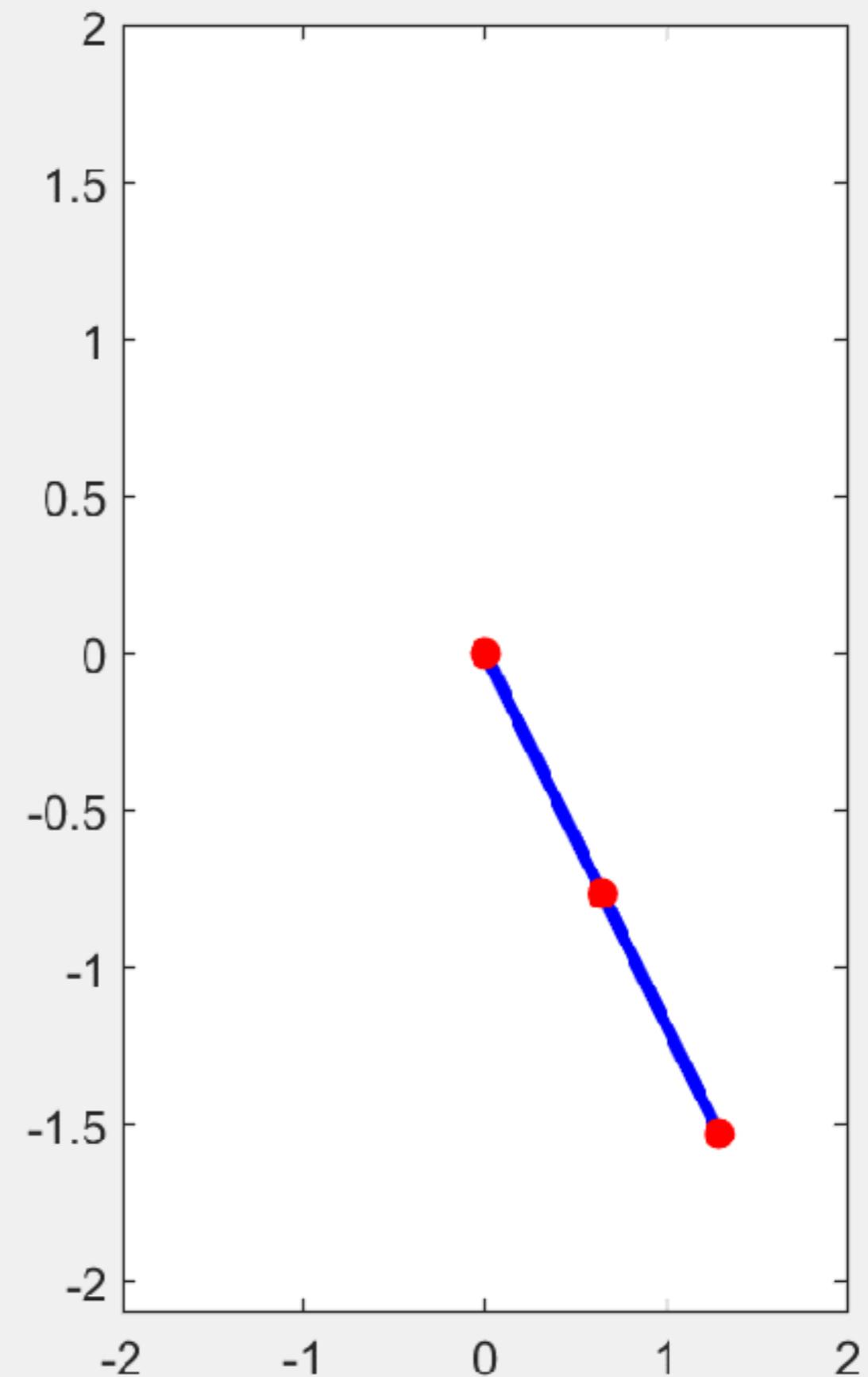
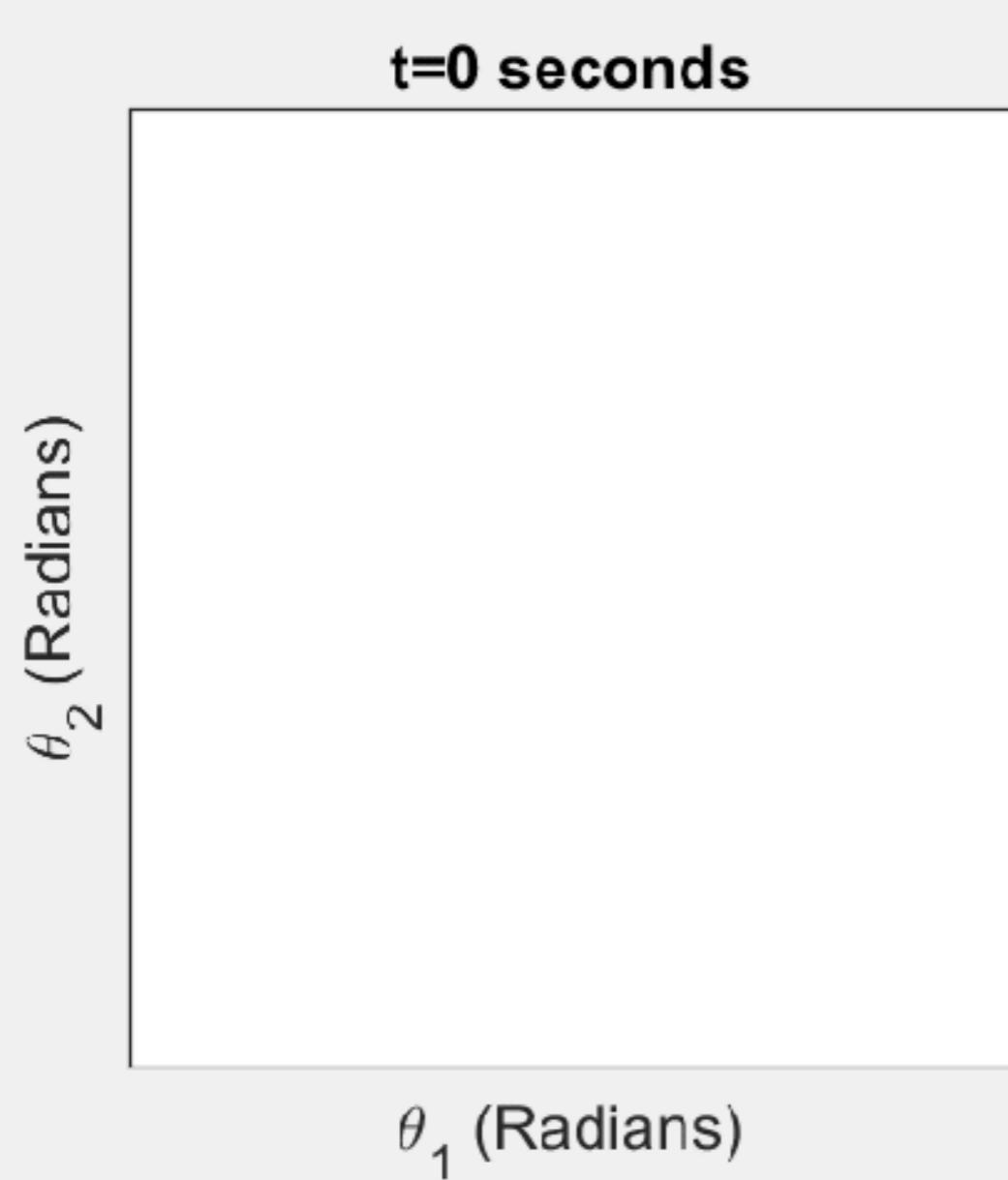
Lorenz about chaos, fractals, SOC, etc.:
“Study of things that look random -but are not”



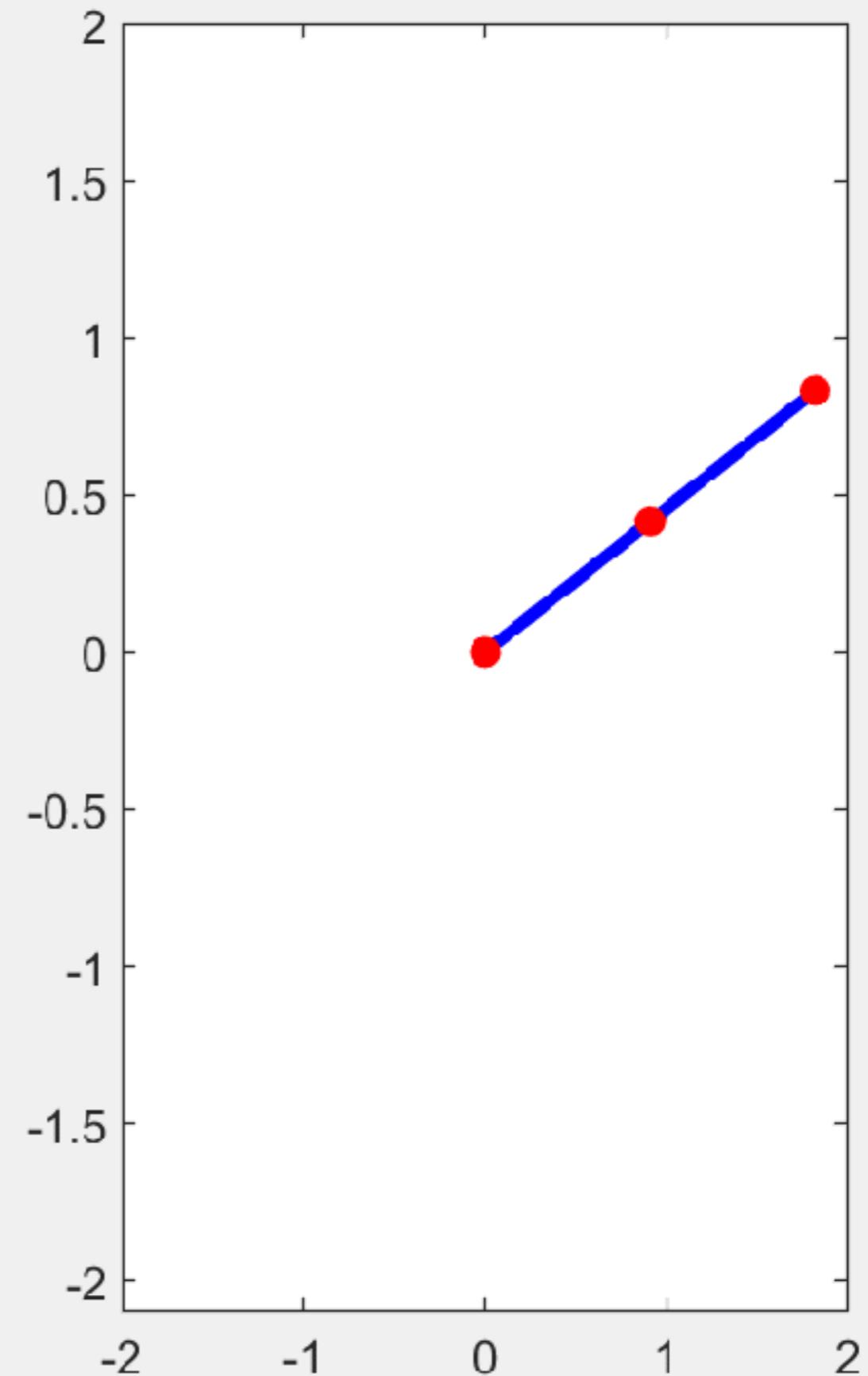
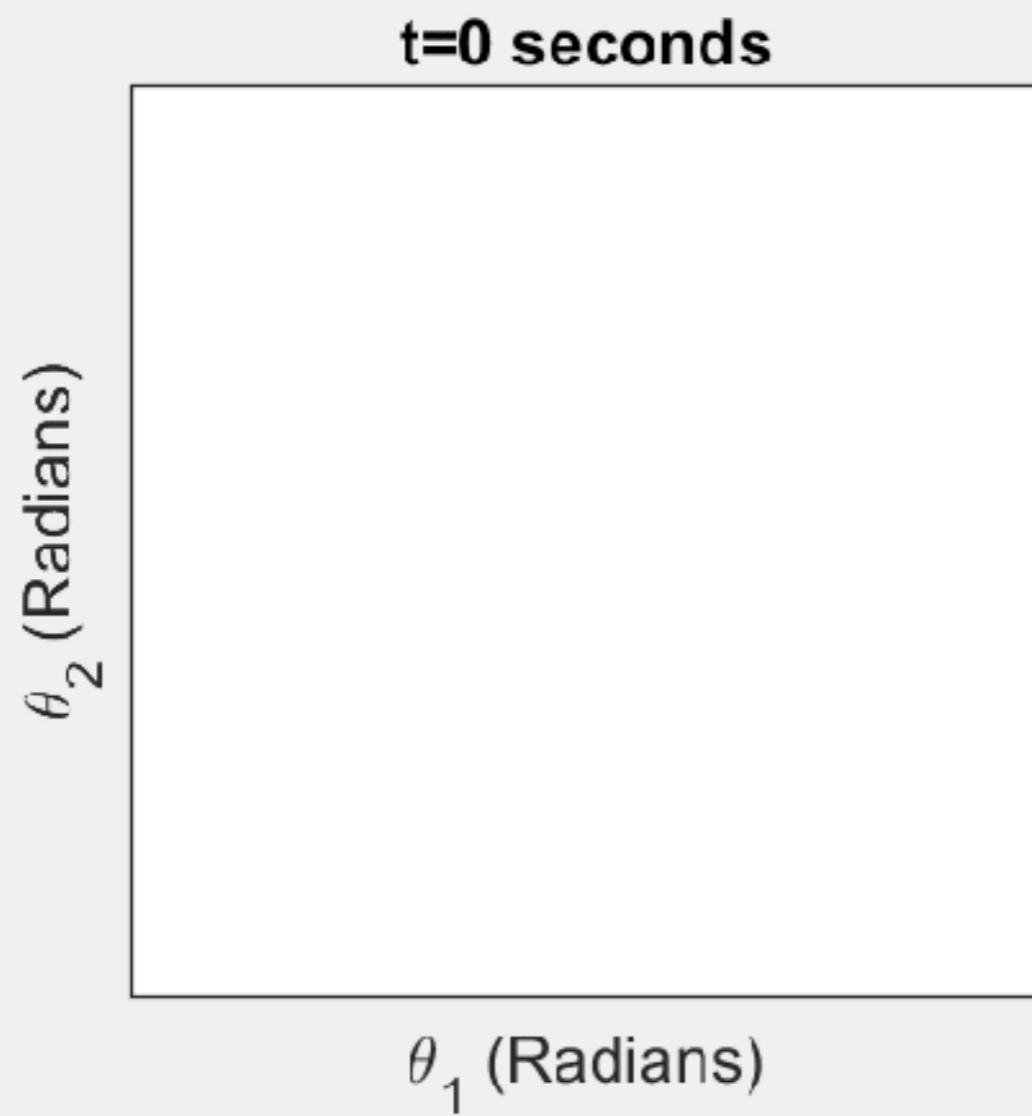
Double Pendulum - Small Displacement



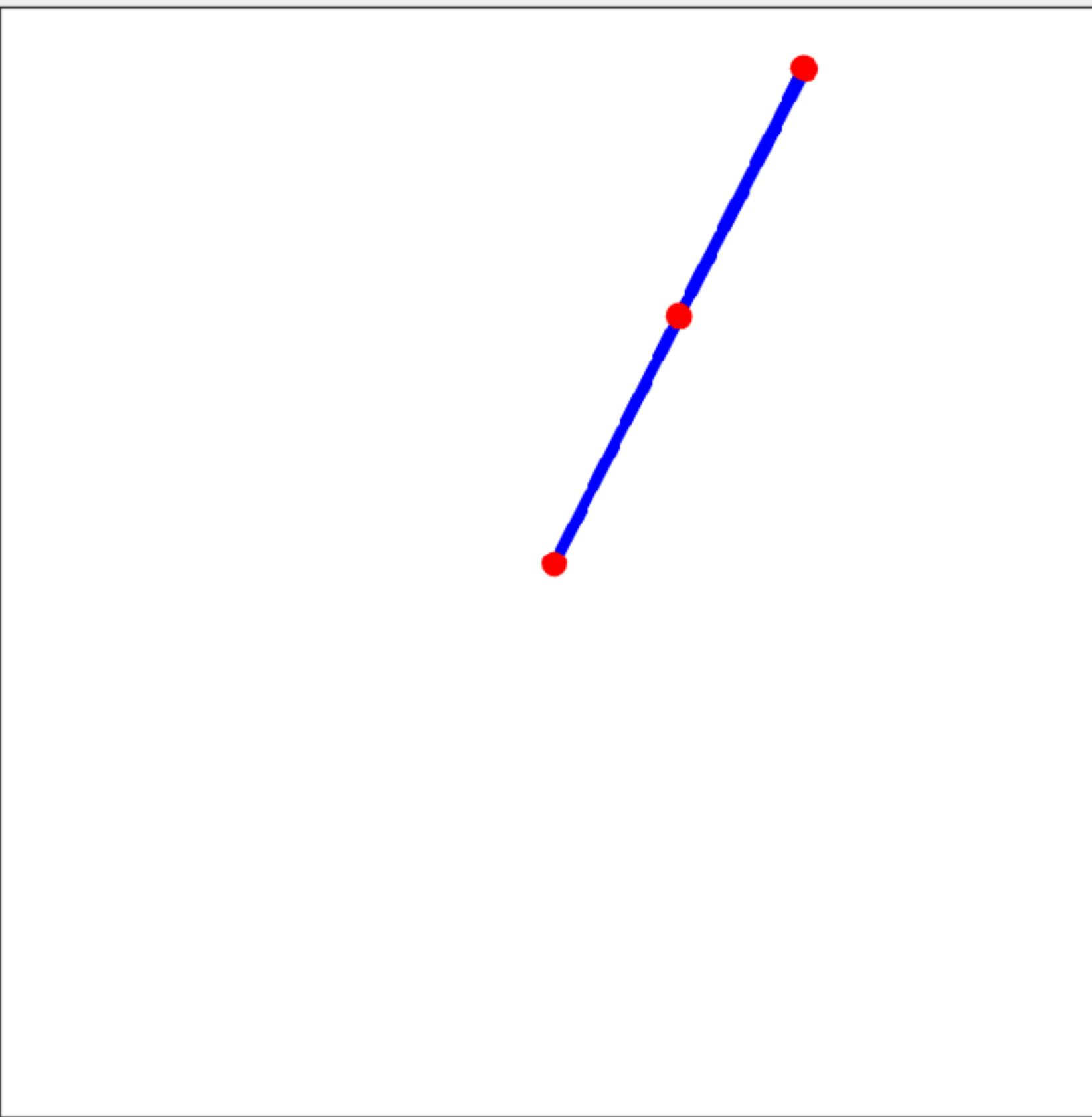
Double Pendulum - Medium Displacement



Double Pendulum - Large Displacement



Double Pendulum at t=0 seconds



<https://youtu.be/PrPYeu3GRLg?t=68>

