③

1: Hints: $G_X(k) = \langle e^{ikx} \rangle = \int dx\, e^{ikx} f_X(x)$

Theorem 2.5.4: $G_Z(k) = G_X(k) \cdot G_Y(k)$

For minimal var. use Lagrange multiplier

First show $X' = wX \sim N_{w\mu, w\sigma}$

$$\mathbb{P}(X' < x) = \mathbb{P}\left(X < \frac{x}{w}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x/w} e^{-(y-\mu)^2/2\sigma^2}\, dy$$

$y \to wy'$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-(y'-w\mu)^2/(2w^2\sigma^2)}\, \frac{dy'}{w}$$

$\mu' = w\mu \;;\; \sigma' = w\sigma'$

$$= \frac{1}{\sqrt{2\pi\sigma'^2}} \int_{-\infty}^{x} e^{-(y'-\mu')^2/2\sigma'^2}\, dy' \qquad \square$$

$$G_Z(k) = \prod_{j=1}^{n} G_{w_j x_j}(k) = \exp\Big(i \underbrace{\sum_{j=1}^{n} w_j \mu}_{\mu'} - \frac{k^2}{2} \underbrace{\sum_{j=1}^{n} w_i^2 \sigma_j^2}_{\sigma'^2}\Big)$$

$$\mathcal{L} = \sum_{j=1}^{n} w_j^2 \sigma_j^2 + \lambda \Big(\underbrace{\sum_{j=1}^{n} w_j - 1}_{=0 \text{ in principle}}\Big)$$

$$\frac{\partial \mathcal{L}}{\partial w_k} = 0 \Rightarrow 2 w_k \sigma_k^2 + \lambda = 0 \Leftrightarrow w_k = -\frac{\lambda}{2\sigma_k^2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow \sum_{j=1}^{n} w_j = 1 = -\frac{\lambda}{2} \sum_{j=1}^{n} \frac{1}{\sigma_j^2}$$

$$\Leftrightarrow \lambda = -2 \left(\sum_{j=1}^{n} \frac{1}{\sigma_j^2} \right)^{-1} \Rightarrow w_k = \frac{\frac{1}{\sigma_k^2}}{\sum_{j=1}^{n} \frac{1}{\sigma_j^2}}$$

2: X, Y iid with $\mu = 0$, σ^2

cumulants are additiv.

$$z = X + Y = 1 \cdot X + 1 \cdot Y$$

$$\Rightarrow \mu_z = 1 \cdot \mu_x + 1 \cdot \mu_y = 0$$