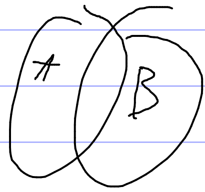
Defs: $K1$: $1 \geq \mathbb{P}(A) \geq 0 \quad \forall A \in \Sigma$

$K2$: $\mathbb{P}(\Omega) = 1$

$K3$: $A, B \in \Sigma$ with $\mathbb{P}(A \cap B) = 0$ (not correlated)

$\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

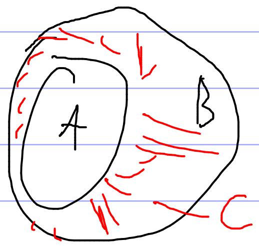


Exercise 1

(1)

$$1 \overset{K2}{=} \mathbb{P}(\Omega) = \mathbb{P}(A \cup \bar{A}) \overset{A3 \text{ since } A \cap \bar{A} = \emptyset}{=} \mathbb{P}(A) + \mathbb{P}(\bar{A}) \iff \mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A)$$

(2) $A \neq 0 \quad A \subset B$



$C = B \setminus A$

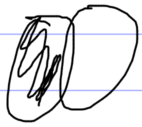
So Define C

such that $A \cup C = B$

$$\mathbb{P}(B) = \mathbb{P}(A \cup C) = \mathbb{P}(A) + \underbrace{\mathbb{P}(C)}_{\geq 0}$$

$\Rightarrow \mathbb{P}(B) \geq \mathbb{P}(A)$

(3)



$A \setminus (A \cap B) \qquad A \setminus B \cup (A \cap B)$

$\Rightarrow \boxed{A = A \setminus B \cup (A \cap B)}$

$$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \setminus B \cup (A \cap B)) + \mathbb{P}(B \setminus A \cup (A \cap B))$$

$$= \underbrace{\mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)}_{= P(A \cup B)} + \mathbb{P}(A \cap B)$$

$\Leftrightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)$

(4) $\mathbb{P}(B) = \mathbb{P}(B \cap \Omega) = \mathbb{P}(B \cap (A \cup \bar{A}))$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$= \mathbb{P}((B \cap A) \cup (B \cap \bar{A}))$ Not ready

$\mathbb{P}(B \setminus A) = \mathbb{P}(B \cap \bar{A}) \overset{!}{=} \mathbb{P}(B) - \mathbb{P}(B \cap A)$

$\Rightarrow \mathbb{P}(B) = \mathbb{P}((B \setminus A) \cup (B \setminus \bar{A})) = \mathbb{P}(B \setminus A) + \mathbb{P}(B \setminus \bar{A})$

5) Proof of the theorem of Bayes:

$$\mathbb{P}(A|B) \overset{\text{Def}}{=} \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \qquad \mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$$

$\Rightarrow A \cap B = B \cap A \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(B \cap A)$

$$\Leftrightarrow \mathbb{P}(A|B) \cdot \mathbb{P}(B) = \mathbb{P}(B|A) \cdot \mathbb{P}(A)$$

$$\Leftrightarrow \boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}} \qquad \square$$