Exercises for Computational Physics (physik760)
WS 2019/2020

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Exercises for the week from 4th to 8th of November 2019.

Central Limit Theorem

1: In this exercise we will study the convergence properties of the central limit theorem (CLT). Consider $X_1^\alpha, X_2^\alpha, \ldots$ i.i.d. random variables with p.d.f.

$$f_\alpha(x) = \frac{1}{2\alpha} |x|^{-1-\frac{1}{\alpha}} \mathbb{1}_{\{|x|\geq 1\}} ,$$

where $\alpha \in (0, 1)$.

a) Determine the cumulative distribution function, F_α, and its inverse analytically.

b) Set up the *generalized inverse transform method* for sampling $X \sim f_\alpha$. Verify your sampling visually by comparing your observed distribution via the known pdf and cdf.

c) Consider the partial sums

$$S_K^\alpha = \frac{1}{\sqrt{K \operatorname{Var}(X_1^\alpha)}} \sum_{n=1}^{K} X_n^\alpha , \quad X_n \sim f_\alpha \, \forall \, n .$$

Implement the sampling of S_K^α.

d) Sample S_K^α for $\alpha = 0.1$, 0.25, 0.45 and $K = 100$ and compare your observed distribution to $\mathcal{N}_{0,1}$ in a Q-Q-plot. Use $N = 10^6$ samples. You can employ *Sturges' formula* $N_{\text{bin}} = \log_2(N) + 1$ for the binning.

e) How does the distribution of S_{100}^α behave as a function of α and why? *Bonus:* Derive the variance as a function of α analytically. Which constraint on α does this suggest for the validity of the CLT?

3 Estimators

2: Show that

$$\Sigma_N^2 = \frac{1}{N-1} \sum_i \left(X_i - \bar{X}_N\right)^2 \tag{1}$$

is an unbiased estimator for the variance σ^2.