

計算科学における情報圧縮

Information Compression in Computational Science

**2018.10.11**

**#3:情報圧縮の数理1 (線形代数の復習)**

**Review of linear algebra**

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理学系研究科 物理学専攻 大久保 毅

Department of Physics, **Tsuyoshi Okubo**

# Outline

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- Vector space- Abstract vectors-
  - General vector space (with inner product)
  - Basis and relation to coordinate vector space
  - Vector subspace and spanned vector subspace
- Matrix and linear map
  - Relation between matrices and linear maps
  - Important properties and operations for matrices
  - Relation to simultaneous linear equations
- Eigenvalue problem and diagonalization

Vector space -Abstract vectors-

# Geometric vector

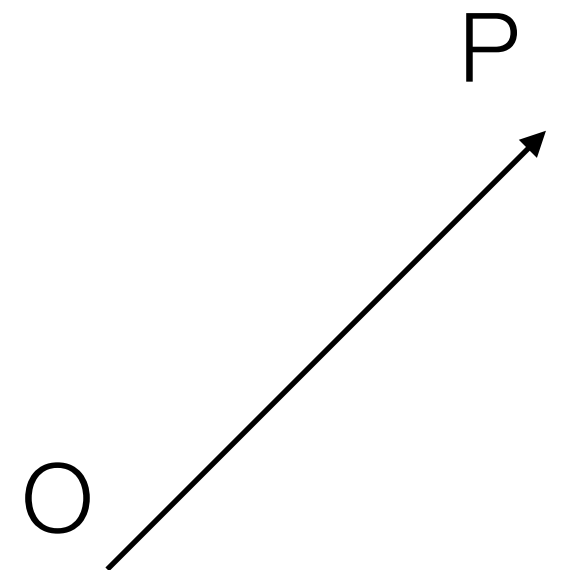
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Geometric vector: Arrow on the plane (or the space) ,  
which has "Direction" and "Length"

$$\vec{v} \equiv \overrightarrow{OP}$$

We can express a vector by its component:

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} x_p - x_o \\ y_p - y_o \\ z_p - z_o \end{pmatrix}$$



# Properties of vector

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Properties of addition:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

Commutative property (交換法則)

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

Associative property (結合法則)

$$\vec{a} + \vec{0} = \vec{a}$$

zero vector

$$\vec{a} + (-\vec{a}) = \vec{0}$$

inverse vector



Multiplication of scalar  $c \in \mathbb{R}$  (実数) :

$$c(\vec{a} + \vec{b}) = c\vec{b} + c\vec{a}$$

Distributive property (分配法則)

$$(c + d)\vec{a} = c\vec{a} + d\vec{a}$$

$$(cd)\vec{a} = c(d\vec{a})$$

# Inner product of vector

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Inner product:

$$\begin{aligned}(\vec{a}, \vec{b}) &\equiv \vec{a} \cdot \vec{b} \\ &= a_x b_x + a_y b_y + a_z b_z\end{aligned}$$

Example:

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \vec{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

Properties:

$$(\vec{a}, \vec{a}) \geq 0$$

$$(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})$$

$$(\vec{a} + \vec{b}, \vec{c}) = (\vec{a}, \vec{c}) + (\vec{b}, \vec{c})$$

$$(c\vec{a}, \vec{b}) = c(\vec{a}, \vec{b}) \quad c \in \mathbb{R}$$

Norm (length):

$$\|\vec{a}\| \equiv \sqrt{(\vec{a}, \vec{a})}$$

# Vector space (linear space)

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Vector space  $\mathbb{V}$  : generalization of geometric vector

Set of elements (vectors) satisfying following **axioms** (公理)

## Properties of addition:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

Commutative property (交換法則)

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

Associative property (結合法則)

$$\vec{a} + \vec{0} = \vec{a}$$

Existence of **unique** zero vector

$$\vec{a} + (-\vec{a}) = \vec{0}$$

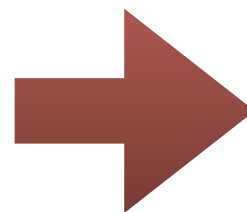
Existence of **unique** inverse vector

## Multiplication of scalar $c$ :

$$c(\vec{a} + \vec{b}) = c\vec{b} + c\vec{a}$$

$$(c + d)\vec{a} = c\vec{a} + d\vec{a}$$

$$(cd)\vec{a} = c(d\vec{a})$$



$c \in \mathbb{R}$  : Real vector space

$c \in \mathbb{C}$  : Complex vector space

# Inner product space (metric vector space)

(計量空間)

Inner product space:

Vector space + definition of **inner product**

Inner product:  $(\vec{a}, \vec{b})$

**Axiom:**

$$(\vec{a}, \vec{a}) \geq 0$$

$$(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})^*$$

$$(\vec{a} + \vec{b}, \vec{c}) = (\vec{a}, \vec{c}) + (\vec{b}, \vec{c})$$

$$(c\vec{a}, \vec{b}) = c(\vec{a}, \vec{b})$$

\*If a norm defined from the inner product is "complete" (完備) ,  
that space is called **Hilbert space**.



# Examples of vector spaces

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(1) Coordinate space (数ベクトル空間)  $\mathbb{R}^n, \mathbb{C}^n$

Vector:  $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad v_i \in \mathbb{R} \text{ or } \mathbb{C}$

Inner product:  $(\vec{a}, \vec{b}) \equiv \vec{a} \cdot \vec{b}^*$

(2) Wave vectors in quantum physics

Vector:  $|\Psi\rangle$

Inner product:  $(|a\rangle, |b\rangle) = \langle b|a\rangle$

Linearly independent or dependent

————— (線形独立) ————

(線形従属) —————

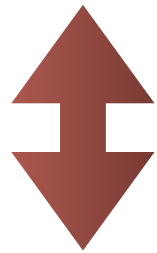
Linear combination:

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots c_k \vec{v}_k$$

$$\vec{v}_i \in \mathbb{V} \quad c_i \in \mathbb{R} \text{ or } \mathbb{C}$$

A set  $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k\}$  is **linearly independent** when

$\vec{x} = \vec{0}$  is satisfied **if and only if**  $c_1 = c_2 = \cdots = c_k = 0$



A set  $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k\}$  is **linearly dependent** when

it is not linearly independent.

# Basis of vector space

(基底)

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A set  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  ( $\vec{e}_i \in \mathbb{V}$ ) is a basis (基底) of  $\mathbb{V}$  when

$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  is linearly independent.

and

Any vectors in  $\mathbb{V}$  are represented by its linear combination.

$\vec{e}_i$  : basis vector

# of basis vectors ( $n$ ) is called **dimension** (次元) of  $\mathbb{V}$  .

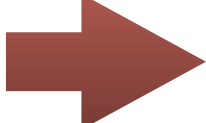
$$n = \dim \mathbb{V}$$

# Relation (map) to coordinate vector space

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By using a basis  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ ,  $\vec{v} \in \mathbb{V}$  is **uniquely represented** as  
(\* From linear independency)

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_n \vec{e}_n$$

 We can represent  $\vec{v}$  as a coordinate vector

$$\vec{v} \rightarrow \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix} \in \mathbb{C}^n \text{ ( or } \mathbb{R}^n \text{ )}$$

By selecting a basis, we obtain a "**concrete**" coordinate vector  
for an "**abstract**" vector

## Orthonormal basis (正規直交基底)

When a vector space has an inner product,

$\vec{a}, \vec{b}$  is **orthogonal** (直交) if  $(\vec{a}, \vec{b}) = 0$ .

### Orthonormal basis

A basis  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  is an orthonormal basis when

$$\|\vec{e}_i\| = 1 \quad (i = 1, 2, \dots, n)$$

$$(\vec{e}_i, \vec{e}_j) = 0 \quad (i \neq j; i, j = 1, 2, \dots, n)$$

\*A basis can be transformed into an orthonormal basis.

**cf. Gram-Schmidt orthonormalization**

# Vector subspace (linear subspace)

## **Vector subspace** (ベクトル部分空間) :

A subset  $\mathbb{W}$  of a vector space  $\mathbb{V}$  is a **vector subspace** of  $\mathbb{V}$  when  $\mathbb{W}$  satisfies the same axioms of vector space with  $\mathbb{V}$ .

The following conditions are necessary and sufficient.

$$\begin{array}{ll} \vec{a}, \vec{b} \in \mathbb{W} & \Rightarrow \vec{a} + \vec{b} \in \mathbb{W} \\ \vec{a} \in \mathbb{W}, c \in \mathbb{C} & \Rightarrow c\vec{a} \in \mathbb{W} \end{array}$$

(In the case of **complex** vector space)

# Spanned vector subspace

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## Spanned subspace:

For a subset  $\mathbb{S}$  of a vector space  $\mathbb{V}$ , a set of linear combinations

$$\{c_1\vec{s}_1 + c_2\vec{s}_2 \cdots + c_k\vec{s}_k \mid c_i \in \mathbb{C}, \vec{s}_i \in \mathbb{S}\}$$

becomes a vector subspace of  $\mathbb{V}$ .

We often use

$$\text{Span}\{\vec{s}_1, \vec{s}_2, \cdots, \vec{s}_k\}$$

to represents a vector subspace spanned by a set of vectors

$$\{\vec{s}_1, \vec{s}_2, \cdots, \vec{s}_k\}$$

(This representation may appear in Krylov subspace method)

Matrix and linear map



# Matrix (行列)

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**Matrix:** "Table" of (complex) numbers in a rectangular form

$M \times N$  matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,N} \\ A_{21} & A_{22} & \cdots & A_{2,N} \\ \vdots & \vdots & & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{M,N} \end{pmatrix}$$

Product of matrices:  $C = AB$   $A_{ij} \in \mathbb{C} \text{ ( or } \mathbb{R} \text{ )}$

$$C_{ij} = \sum_{k=1}^K A_{ik} B_{kj}$$

$$A : M \times K$$

$$B : K \times N$$

$$C : M \times N$$

In general:  $XY \neq YX$

\*We also know addition, multiplication of scalar.

# Identity matrix (単位行列)

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## Identity matrix:

$N \times N$  matrix  
(Square matrix)

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Product:

$$IA = A \quad A : N \times M$$

$$BI = B \quad B : K \times N$$

\* Element of the identity matrix:  $I_{ij} = \delta_{ij}$  (Kronecker delta)

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

# Transpose, complex conjugate and adjoint

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Transpose:  
(転置)

$$A^t \quad (A^t)_{ij} = A_{ji}$$

Complex conjugate:  
(複素共役)

$$A^* \quad (A^*)_{ij} = A_{ij}^*$$

Adjoint:  
(随伴)

$$A^\dagger = (A^t)^* = (A^*)^t$$

or

$$(A^\dagger)_{ij} = A_{ji}^*$$

Hermitian conjugate:  
(エルミート共役)

("Dagger" is convention in physics)

# Multiplication to coordinate vector

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$$\begin{array}{ccc} A : M \times N & \vec{v} \in \mathbb{C}^N & \vec{v}' \in \mathbb{C}^M \\ \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,N} \\ A_{21} & A_{22} & \cdots & A_{2,N} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{M,N} \end{pmatrix} & \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} & = \begin{pmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_M \end{pmatrix} \end{array}$$

$M \times N$  matrix **transforms** a  $N$ -dimensional coordinate vector to a  $M$ -dimensional coordinate vector.

**$M \times N$  matrix**  **Linear map:**  $\mathbb{C}^N \rightarrow \mathbb{C}^M$   
**1 to 1** (線形写像)

# General linear map

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Map:  $f : \mathbb{V} \rightarrow \mathbb{V}'$

$$f(\vec{v}) = \vec{v}' \quad (\vec{v} \in \mathbb{V}, \vec{v}' \in \mathbb{V}')$$

Linear map:

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$$

$$f(c\vec{x}) = cf(\vec{x})$$

$$(\vec{x}, \vec{y} \in \mathbb{V}, c \in \mathbb{C})$$

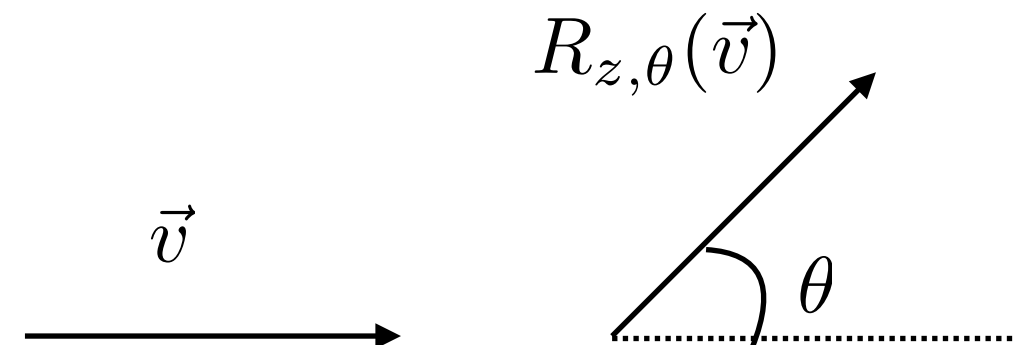
Examples:

**Rotation** (e.g.  $\theta$  rotation around z-axis)

$$R_{z,\theta} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$$

**Hamiltonian operator**

$$\mathcal{H} : \mathbb{V} \rightarrow \mathbb{V}$$



$$|\Psi\rangle \quad \rightarrow \quad \mathcal{H}|\Psi\rangle$$

# Matrix representation of linear map

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By using a basis, we can represent a linear map in a matrix.

$$f : \mathbb{V} \rightarrow \mathbb{V}'$$

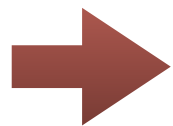
**Vector space**  $\mathbb{V} : \dim \mathbb{V} = N$   
**Basis**  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_N\}$



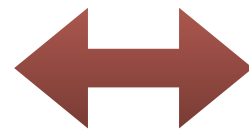
$\mathbb{V}' : \dim \mathbb{V}' = M$   
 $\{\vec{e}'_1, \vec{e}'_2, \dots, \vec{e}'_M\}$

Transformation of basis vectors:

$$f(\vec{e}_j) = f_{1j}\vec{e}'_1 + f_{2j}\vec{e}'_2 + \dots + f_{Mj}\vec{e}'_M$$



$$f : \mathbb{V} \rightarrow \mathbb{V}'$$



**1 to 1**

(if we fix basis)

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1,N} \\ f_{21} & f_{22} & \cdots & f_{2,N} \\ \vdots & \vdots & & \vdots \\ f_{M1} & f_{M2} & \cdots & f_{M,N} \end{pmatrix}$$

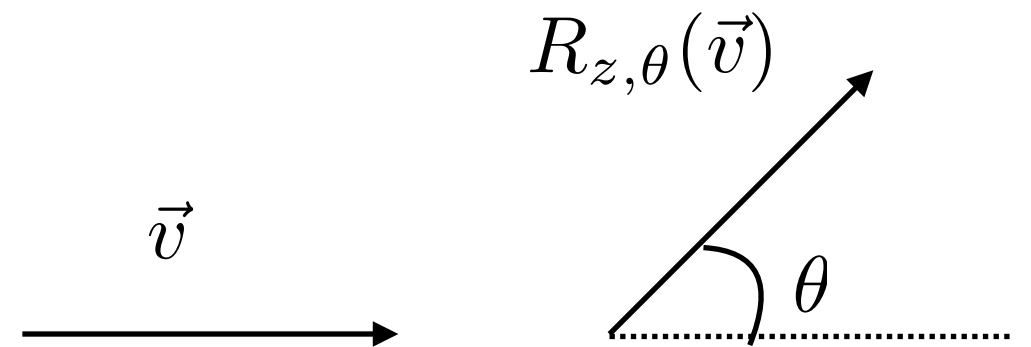
# Examples of matrix

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**Rotation** (e.g.  $\theta$  rotation around z-axis)

$$R_{z,\theta} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$$

$$R_{z,\theta} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



**Hamiltonian operator**

$$\mathcal{H} : \mathbb{V} \rightarrow \mathbb{V} \quad \mathcal{H} \rightarrow \begin{pmatrix} H_{0,0;0,0} & H_{0,0;0,1} & H_{0,0;1,0} & H_{0,0;1,1} \\ H_{0,1;0,0} & H_{0,1;0,1} & H_{0,1;1,0} & H_{0,1;1,1} \\ H_{1,0;0,0} & H_{1,0;0,1} & H_{1,0;1,0} & H_{1,0;1,1} \\ H_{1,1;0,0} & H_{1,1;0,1} & H_{1,1;1,0} & H_{1,1;1,1} \end{pmatrix}$$

**Matrix element:**  $H_{\alpha,\beta;\alpha',\beta'} \equiv \langle \alpha\beta | \mathcal{H} | \alpha'\beta' \rangle$   
(行列要素)

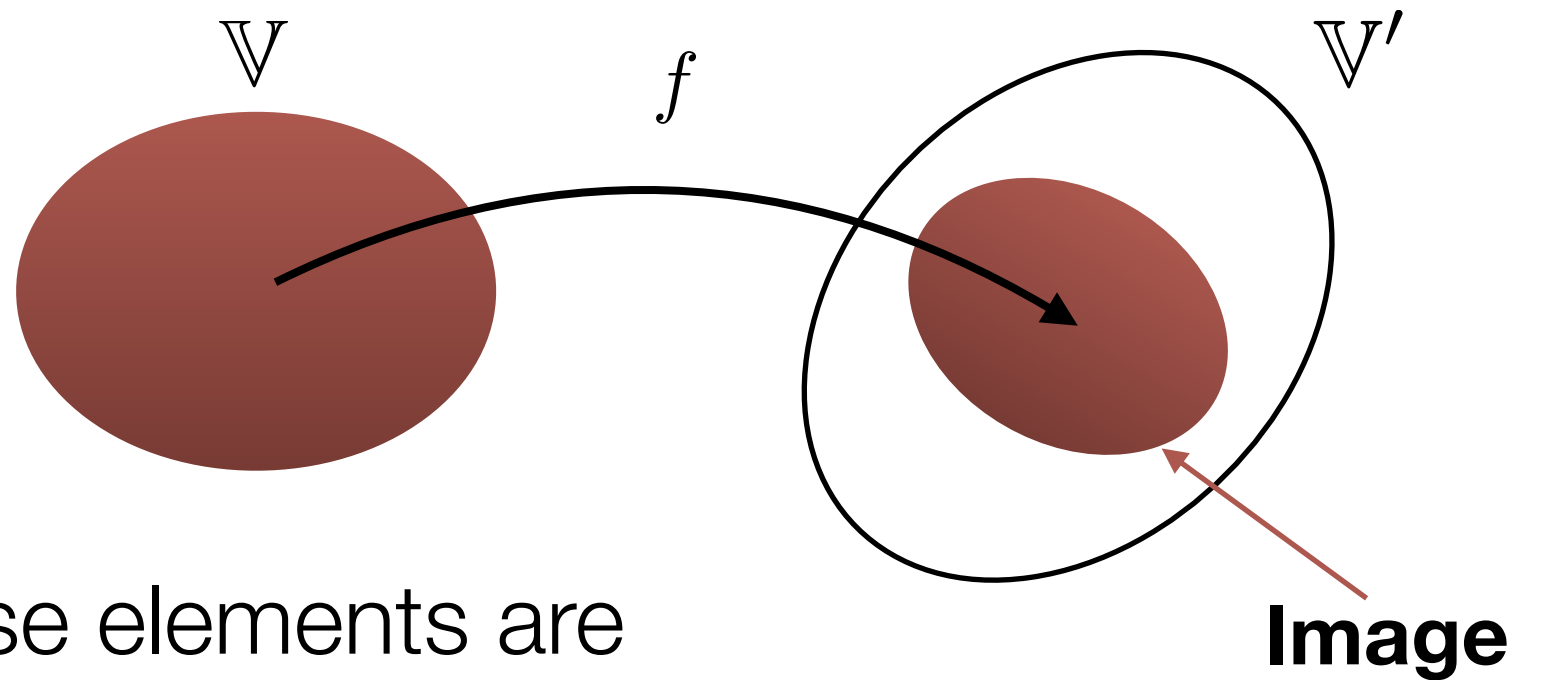
\* In this notation, **basis should be orthonormal.**

# Image of a map

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$$f : \mathbb{V} \rightarrow \mathbb{V}'$$

Image of  $f$ :  
(像)



Vector subspace whose elements are mapped from  $\mathbb{V}$  by  $f$ .

$$\text{img}(f) = \{\vec{v}' \mid \vec{v} \in \mathbb{V}, \vec{v}' = f(\vec{v})\}$$

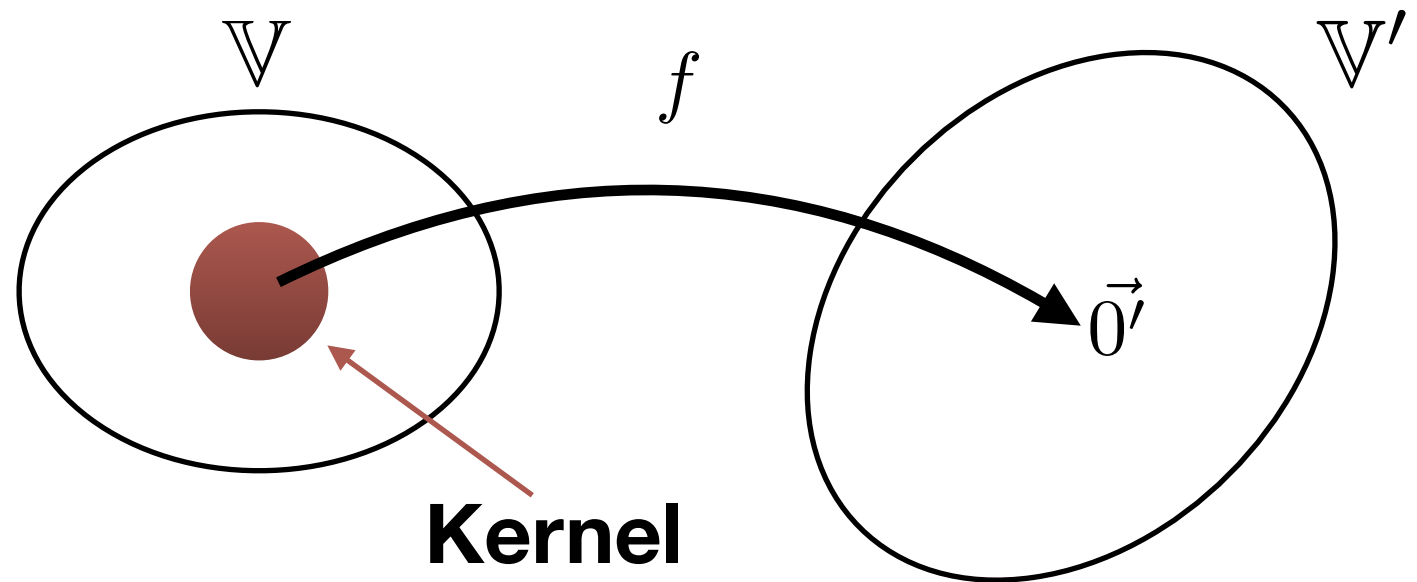


# Kernel of a map

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$$f : \mathbb{V} \rightarrow \mathbb{V}'$$

Kernel of  $f$ :  
(核)



Vector subspace whose elements are mapped into zero vector by  $f$ .

$$\ker(f) = \{\vec{v} | \vec{v} \in \mathbb{V}, f(\vec{v}) = \vec{0}'\}$$

## Theorem:

$$\dim(V) = \dim(\ker(f)) + \dim(\text{img}(f))$$

# Rank of matrix

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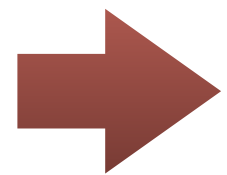
**Rank** (ランク or 階数) of a matrix  $A$ :

$$\text{rank}(A) \equiv \dim(\text{img}(A))$$

**Rank** is identical with

Maximum # of linearly independent column vectors (列ベクトル) in  $A$

Maximum # of linearly independent row vectors (行ベクトル) in  $A$



$$\text{rank}(A) \leq \min(M, N)$$

for a  $N \times M$  matrix  $A$ .

$A_{11}$	$A_{12}$	$\cdots$	$A_{1,N}$
$A_{21}$	$A_{22}$	$\cdots$	$A_{2,N}$
$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
$A_{M1}$	$A_{M2}$	$\cdots$	$A_{M,N}$

# Regular matrix and its inverse matrix

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A square matrix  $A$  is a **regular matrix** (正則) if a matrix  $X$  satisfying

$$AX = XA = I$$

exists. The matrix  $X$  is called inverse matrix (逆行列) of  $A$  and it is written as  $X = A^{-1}$ .

**Properties:**  $A^{-1}$  is unique.

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$A$  is a regular matrix  $\longleftrightarrow \text{rank}(A) = N$

Can we consider an "inverse matrix" of a non-regular matrix (including a rectangular matrix) ?

 **Next week!**

# Simultaneous linear equation

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## **Simultaneous linear equation** (連立一次方程式)

can be represented by a matrix and a vector as

$$A\vec{x} = \vec{b} \quad A : M \times N, \vec{x} \in \mathbb{C}^N, \vec{b} \in \mathbb{C}^M$$

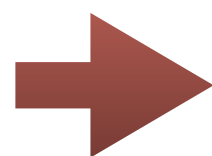
If  $A$  is a **square matrix** ( $N=M$ ), and it **has a inverse matrix** ( $\text{rank}(A) = N$ ),  
we can solve the equation as

$$\vec{x} = A^{-1}\vec{b}$$

$N > M$ : Underdetermined problem (劣決定問題)

$N < M$ : Overdetermined problem (劣決定問題)

How can we find a "solution" when  $A$  does not have the "inverse"?



It is related to the topic "sparse modeling".  
(Especially for **underdetermined problems**.)

# Determinant of matrix

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For a square matrix  $A$  its **determinant** (行列式) is defined as

$$\begin{aligned}\det A = |A| &= \sum_{\sigma} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{N\sigma(N)} \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(N)N}\end{aligned}$$

$\sigma$  : permutation (置換) of  $\{1, 2, \dots, N\}$

$$\sigma = \begin{cases} 1 & \text{even permutation (偶置換)} \\ -1 & \text{odd permutation (奇置換)} \end{cases}$$

Examples:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ab - cd$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh \\ - afg - bdi - ceg$$

# Determinant and inverse matrix

---

By using the determinant of  $A$ , we can represent its inverse matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} & \cdots & \tilde{A}_{N1} \\ \tilde{A}_{12} & \tilde{A}_{22} & \cdots & \tilde{A}_{N2} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{1N} & \tilde{A}_{2N} & \cdots & \tilde{A}_{NN} \end{pmatrix}$$

$\tilde{A}_{ij}$  : cofactor (余因子)

We can see that

$$\det(A) = 0 \quad \Rightarrow \quad A^{-1} \text{ diverges}$$

Indeed,

$$A \text{ is a regular matrix.} \quad \longleftrightarrow \quad \det(A) \neq 0$$

**necessary and sufficient**

# Eigenvalue problems and diagonalization

# Eigenvalue and Eigenvector

---

For a square matrix  $A$

$$A\vec{v} = \lambda\vec{v}$$

$\vec{v} \neq \vec{0}$  :eigenvector (固有ベクトル)

$\lambda \in \mathbb{C}$  :eigenvalue (固有値)

Properties:

If  $\vec{v}$  is an eigenvector,  $c\vec{v}$  is also an eigenvector.

Eigenspace (固有空間) :

The set of eigenvectors corresponds an eigenvalue  $\lambda$ .

Eigenvectors corresponding to different eigenvalues are  
linearly independent.



# Right and left eigenvectors

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In general, **left eigenvectors** can be different from the right eigenvectors.

$$A\vec{v} = \lambda\vec{v}$$

$$(\vec{u}^*)^t A = \lambda(\vec{u}^*)^t$$

$\vec{v}$  :Right eigenvector

$(\vec{u}^*)^t$  :Left eigenvector

## Properties:

Set of **eigenvalues are identical** between the right and the left eigenvectors.

A left eigenvector and a right eigenvector are **orthogonal** when they correspond to different eigenvalues.

$$\vec{u}_i^* \cdot \vec{v}_j = 0 \quad (\lambda_i \neq \lambda_j)$$

# Diagonalization

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Diagonalization (対角化) :

$$A : N \times N$$
$$P^{-1}AP = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_N \end{pmatrix}$$

$A$  can be diagonalized.   $A$  has  $N$  linearly independent eigenvectors.

**necessary  
and  
sufficient**

$$\alpha_i = \lambda_i$$

$$P = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N)$$

$$(P^{-1})^t = (\vec{u}_1^*, \vec{u}_2^*, \dots, \vec{u}_N^*)$$

Normalization:  $\vec{u}_i^* \cdot \vec{v} = 1$

# Meaning of diagonalization

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General transform using a regular matrix:  $P^{-1}AP$

It is a transform of the basis:

$$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_N\} \rightarrow \{P\vec{e}_1, P\vec{e}_2, \dots, P\vec{e}_N\}$$

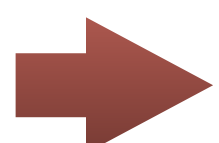
Diagonalization:

By using **eigenvectors as a basis**,  
we can obtain a simple linear map  
represented by a diagonal matrix.

$$A \rightarrow P^{-1}AP$$

\* The determinant of  $A$  is invariant under this transformation:

$$\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(A)\det(P^{-1}P) = \det(A)$$

  $\det(A) = \prod_i^N \lambda_i$  (This relation is true **even if  $A$  cannot be diagonalized**)

# Unitary matrix

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**Unitary matrix** (ユニタリ行列) :  $U^\dagger = U^{-1}$

**Real Orthogonal matrix** (実直交行列) :  $P^t = P^{-1}, (P_{ij} \in \mathbb{R})$

When we consider a unitary matrix as a set of vectors:

$$U = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N)$$

it is an orthonormal basis:  $\vec{v}_i^* \cdot \vec{v}_j = \delta_{i,j}$

 The linear map represented by a unitary matrix  
(**unitary transformation**) does not change

- the norm of a vector

$$\|U\vec{v}\| = \|\vec{v}\|$$

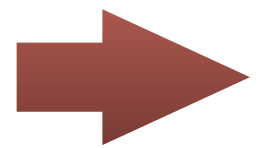
- "distance" between two vectors

$$\|U\vec{v}_1 - U\vec{v}_2\| = \|\vec{v}_1 - \vec{v}_2\|$$

# Normal matrix

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**Normal matrix** (正規行列) :  $A^\dagger A = AA^\dagger$



We can **always diagonalize it** by a unitary matrix

$$U^\dagger = U^{-1}$$

as

$$U^\dagger A U = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} \quad \lambda_i \in \mathbb{C}$$

Its eigenvalues could be **complex**.  
(even if  $A$  is a real matrix)

# Hermitian matrix and its eigenvalue

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**Hermitian matrix** (エルミート行列) :  $A^\dagger = A$

**Real symmetric matrix** (実対称行列) :  $A^t = A$ ,  $(A_{ij} \in \mathbb{R})$

➡ It is a special **normal matrix**.  $A^\dagger A = AA^\dagger = AA$   
Its eigenvalues are **real**.

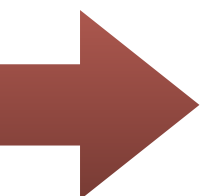
We can **always diagonalize it** by a unitary matrix

$$U^\dagger AU = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} \quad \lambda_i \in \mathbb{R}$$

**Hermitian (or real symmetric) matrices often appear in physics.**

# Generalization of diagonalization

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- Eigenvalue problems and diagonalizations are **defined for a square matrix**.
  - Even if  $A$  is a square matrix, it **may not be diagonalized**.
- 
- Is it possible **to transform all square matrixes** into diagonal forms by **generalizing the diagonalization**?
  - Is it possible to generalize it to **a rectangular matrices**?

Yes. **The singular value decomposition**  
(特異値分解) is an generalization of the diagonalization.

(We can also consider a **decomposition of a tensor**.)

# Next week

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第1回： 現代物理学における巨大なデータ

第2回： 現代物理学と情報圧縮

第3回： 情報圧縮の数理1 (線形代数の復習)

**第4回： 情報圧縮の数理2 (特異値分解と低ランク近似)**

**(Singular value decomposition and low rank approximation)**

第5回： 情報圧縮の数理3 (スパース・モデリングの基礎)

第6回： 情報圧縮の数理4 (クリロフ部分空間法の基礎)

第7回： 物質科学における情報圧縮

第8回： データ解析の高速化：スパース・モデリングの物質科学への応用

第9回： データ空間の圧縮：クリロフ部分空間法の物質科学への応用

第10回： 高度なデータ圧縮：情報のエンタングルメントと行列積表現

第11回： 行列積表現の固有値問題への応用

第12回： テンソルネットワーク表現への発展

第13回： テンソルネットワーク繰り込みによる情報圧縮