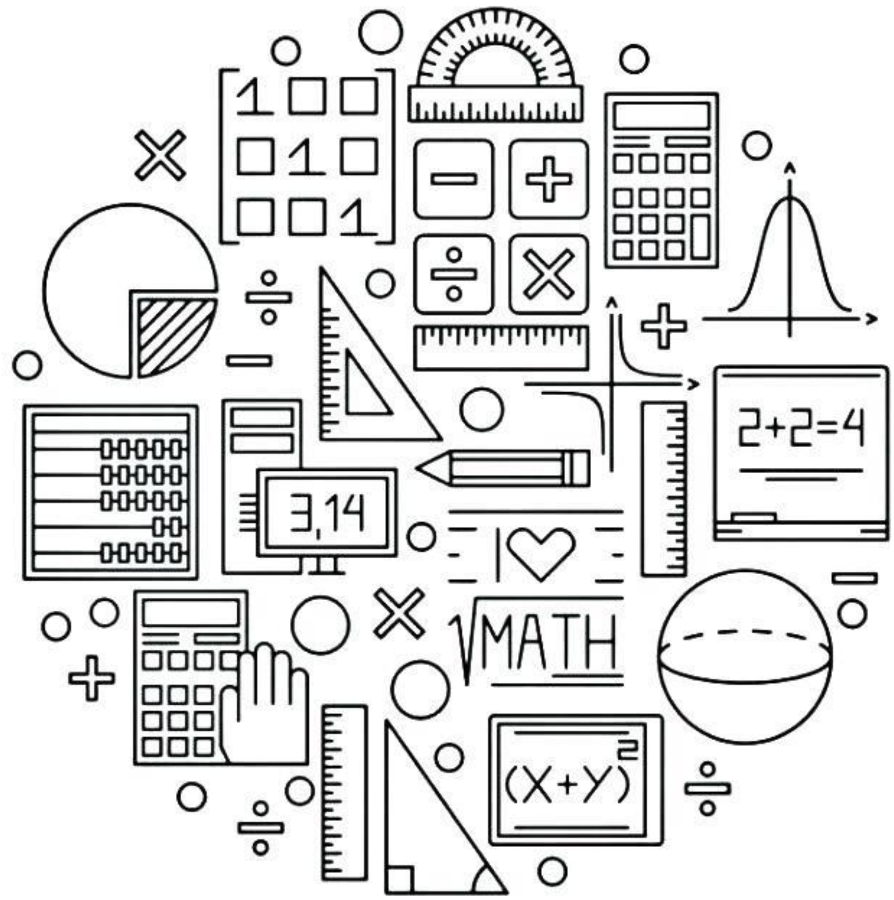
Department of Mathematics and Applied Mathematics

Mathematics (MAT 312)

Assignment 1

Due Date: 25 May 2025



Rolande Solomons

Student number: 4269355

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Question 1

Descartes' Rule of Signs is a useful method for estimating the number of positive and negative real roots of a polynomial. This rule is based on counting the number of sign changes in the polynomial terms when it is expressed in standard form, that is, with terms ordered by decreasing powers of the variable. A sign change occurs when consecutive non-zero terms of the polynomial have opposite signs.

According to Descartes' Rule:

- The number of positive real zeros of a polynomial $f(x)$ is either equal to the number of sign changes in $f(x)$, or less than that number by an even integer.
- To determine the possible number of negative real zeros, the rule is applied to $f(-x)$, and the number of sign changes in that transformed expression is counted in the same manner.

It is important to ignore any zero coefficients while counting sign changes, as these do not contribute to a change in sign between terms. Additionally, this rule only gives the maximum number of positive or negative real zeros; the actual number could be lower by an even amount. Complex (non-real) roots are accounted for by subtracting the total number of real roots from the polynomial's degree.

Sidebar: The difference between real roots vs real zeros, take the following example of the real root $x = 2$, its real zero is $f(2) = 0$. In the context of Descartes' Rule of Signs, "positive real roots" and "positive real zeros" mean the same thing since the real values of $x > 0$ that satisfy $f(x) = 0$.

To illustrate this concept, consider the following example:

Example 1:

Given the polynomial,

$$f(x) = 2x^4 + x^3 - 6x^2 - 7x + 1$$

1. Estimate the positive real zeros by counting the sign changes in $f(x)$

Examining the coefficients: $+2, +1, -6, -7, +1$

Sign changes occur between:

- $+1 \rightarrow -6$
- $-7 \rightarrow +1$

Thus, there are 2 sign changes, so the maximum number of positive real zeros is 2 or 0.

2. Estimate the negative real zeros by counting the sign changes in $f(x)$

Substitute $-x$ in place of x,

$$f(-x) = \ 2x^4 - \ x^3 - 6x^2 + 7x + 1$$

Examining the coefficients: $+2, +1, -6, -7, +1$

Sign changes occur between:

- $+2 \rightarrow -1$
- $-6 \rightarrow +7$

Thus, there are 2 sign changes, so the maximum number of negative real zeros is 2 or 0.

3. Consider the possible number of imaginary (complex) roots

Since the polynomial is degree 4, there must be 4 total roots (counting multiplicity). Using the possible combinations, see the chart below:

Descartes's rule of signs chart:

Positive Real Zeros	Negative Real Zeros	Imaginary Zeros
2	2	0 (=4-(2+2))
2	0	2 (=4-(2+0))
0	2	2 (=4-(0+2))
0	0	4 (=4-(0+0))

Each combination maintains a total of 4 roots.

Example 3:

The following is an example of Exact Count,

Prove that the exact number of positive real zeros of $f(x) = +x^3 - 6x^2 - 7x - 1$ is 1.

The signs of the coefficients (as circled in equation) are,

$$+, -, -, -$$

There is one sign change, from $+ \rightarrow -$, so Descartes' Rule tells us there is exactly one positive real root, because subtracting 2 from 1 would result in a negative number, which is not valid in this context.

Example 4: (Original Example)

Descartes's Rule Applied to a Degree 5 Polynomial

Consider the polynomial:

$$f(x) = x^5 - 4x^4 + 6x^3 + 4x^2 - x + 2$$

Step 1: Count sign changes in $f(x)$ (for positive real roots)

$$f(x) = x^5 - 4x^4 + 6x^3 + 4x^2 - x + 2$$

- From dark blue term to the orange term there is a sign change
- From the orange term to the purple term there is a sign change
- From the purple term to the light blue term there is no sign change
- From the blue term to the green term there is a sign change
- From the green term to the pink term there is a sign change

Therefore, Number of sign changes = 4

So, the number of positive real roots is either: 4, 2, or 0

Step 2: Count sign changes in $f(-x)$ (for negative real roots)

$$f(x) = (-x)^5 - 4(-x)^4 + 6(-x)^3 + 4(-x)^2 - (-x) + 2$$

$$\therefore f(-x) = -x^5 - 4x^4 - 6x^3 + 4x^2 + x + 2$$

- From dark blue term to the orange term there is a sign change
- From the orange term to the purple term there is a sign change
- From the purple term to the light blue term there is a sign change
- From the blue term to the green term there is a sign change
- From the green term to the pink term there is a sign change

Therefore, Number of sign changes = 1

So, the number of negative real roots is exactly one.

Answer: The maximum number of positive real zeros = 4 and the maximum number of negative real zeros = 1.

Question 2

The Routh-Hurwitz criteria provide a method for determining whether all roots of a real-coefficient polynomial have negative real parts, without solving the polynomial explicitly. This is particularly useful in control theory and stability analysis.

<u>Basic Idea:</u>

Given a polynomial

$$p(x) = a_0 x^n + \ a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

We need to construct a Routh-Hurwitz table. The number of sign changes in the first column of the table equals the number of roots with positive real parts.

For stability (i.e., all roots have negative real parts), all elements in the first column must be positive.

<u>For a degree 4 Polynomial:</u>

Let:

$$p(x) = a_0 x^4 + \ a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

The Routh table looks as follows:

Row	Elements
x^4	$a_0 \quad a_2$
x^3	$a_1 \quad a_3$
x^2	$\frac{a_1 a_2 - a_0 a_3}{a_1} \quad a_4$
x^1	$\frac{b_1 a_3 - a_1 a_4}{b_1}$
x^0	a_4

$$\ldots where\ b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

<u>Conditions for Stability (All Roots Have Negative Real Parts):</u>

- All coefficients $a_i > 0$
- All entries in the first column of the Routh table are positive (no sign changes)

If either condition is violated, the polynomial has at least one root with a non-negative real part (i.e., on or to the right of the imaginary axis), indicating instability.

Write down the Routh table for the polynomial given below:

$$p(x) = x^4 + \ 3x^3 + 4x^2 + 7x + 1$$

Given polynomial Routh table:

$p(x) = x^4 + \; 3x^3 + 4x^2 + 7x + 1$

Row	Elements
x^4	1 4
x^3	$3 \quad 7$
x^2	$\frac{5}{3} \quad 1$
x^1	$\frac{26}{5}$
x^0	1

$\ldots where\ b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{(3)(4)-(1)(7)}{3} = \frac{5}{3}$

The Routh table shows that all the first-column entries are positive, so all roots have negative real parts → the system is stable.

Question 3

Consider the initial-value problem: $\frac{dQ}{dt} = Q(6-Q), \quad Q(0) = 4$ (1)

(a) Find the analytical solution to the problem (1).

Separate the variables:

$$\frac{dQ}{Q(6-Q)} = dt$$

Apply partial fraction decomposition on the left-hand side:

We get,

$$\frac{1}{Q(6-Q)} = \frac{A}{Q} + \frac{B}{6-Q}$$

Multiply both sides by $Q(6-Q)$,

$$1 = A(6-Q) + BQ$$

Now solve for A and B,

$$1 = 6A - AQ + BQ \rightarrow 1 = 6A + (B-A)Q$$

Equate the coefficients:

- Constant term: $6A = 1 \rightarrow A = \frac{1}{6}$
- Coefficient of Q: $B - A = 0 \ \rightarrow B = A = \frac{1}{6}$

Substitute back and integrate

$$\int (\frac{1}{6Q} + \frac{1}{6(6-Q)}) dQ = \int dt$$

Factor out the $\frac{1}{6}$,

$$\frac{1}{6}\int \frac{1}{Q} dQ + \frac{1}{6}\int \frac{1}{(6-Q)} dQ = t + C$$

Note: use u-sub for $\int \frac{1}{(6-Q)} dQ$ and then integrating gives,

$$\frac{1}{6}\ln|Q| - \frac{1}{6}\ln|6-Q| = t + C$$

Combine the logs (lns),

$$\frac{1}{6}\left(\ln\left|\frac{Q}{6-Q}\right|\right) = t + C$$

Take the $\frac{1}{6}$ over,

$$\ln\left|\frac{Q}{6-Q}\right| = 6t + 6C$$

Let $6C = C_1$,

$$\ln\left|\frac{Q}{6-Q}\right| = 6t + C_1$$

Taking exponentials of both sides yields,

$$\frac{Q}{6-Q} = e^{6t} \cdot e^{C_1}$$

Let $e^{C_1} = A$,

$$\frac{Q}{6-Q} = Ae^{6t}$$

Solve for $Q(t)$,

$$Q = (6-Q)Ae^{6t} \rightarrow Q + QAe^{6t} = 6Ae^{6t} \rightarrow Q(1 + Ae^{6t}) = 6Ae^{6t} \therefore Q(t) = \frac{6Ae^{6t}}{1 + Ae^{6t}}$$

Apply initial condition, $Q(0) = 4$ to solve for A,

$$Q(0) = \frac{6Ae^{6(0)}}{1 + Ae^{6(0)}} = 4 \rightarrow 6A = 4(1 + A) \rightarrow 6A = 4 + 4A \rightarrow 2A = 4 \rightarrow A = 2$$

Sub A back in and the final answer is,

$$\boxed{Q(t) = \frac{12e^{6t}}{1 + 2e^{6t}}}$$

(b) Plot the analytical solution (The Octave code should be included in assignment).

Solution Curve:

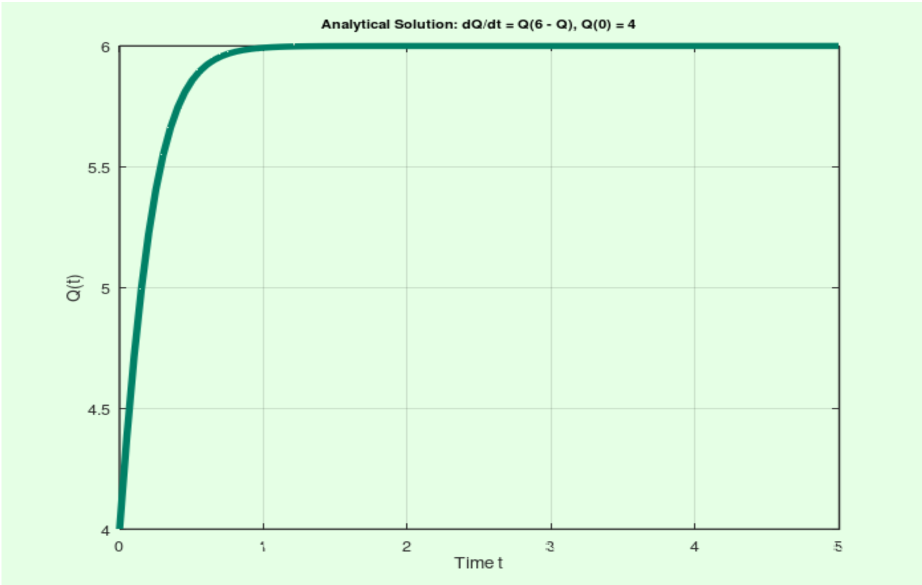

Figure 1: Question 3, Analytical Solution

Code:

% QUESTION 3(b): Analytical solution plot for $\frac{dQ}{dt} = Q(6 - Q), \quad Q(0) = 4$

```
% Define the time range first
t = linspace(0, 5, 100);  % 100 points between 0 and 5

% Define the analytical solution as an inline function
analytic_Q = @(t) 6 ./ (1 + ((6 - 4) / 4) * exp(-6 * t));  % General solution using Q(0) = 4

% Evaluate the function over time
q_vals = analytic_Q(t);

% Plot the results
plot(t, q_vals,'color', [0.7 0 0.4], 'b-', 'LineWidth', 2); %dark pink line
xlabel('Time t');
ylabel('Q(t)');
title('Analytical Solution: dQ/dt = Q(6 - Q), Q(0) = 4');
grid on;

% Set background colour to pink
set(gca, 'Color', [1 0.8 0.9]);  % Light pink axes background
set(gcf, 'Color', [1 0.8 0.9]);  % Light pink figure background
```

(c) Compute and plot a numerical solution using ode45 (Octave code included).

Solution Curve:

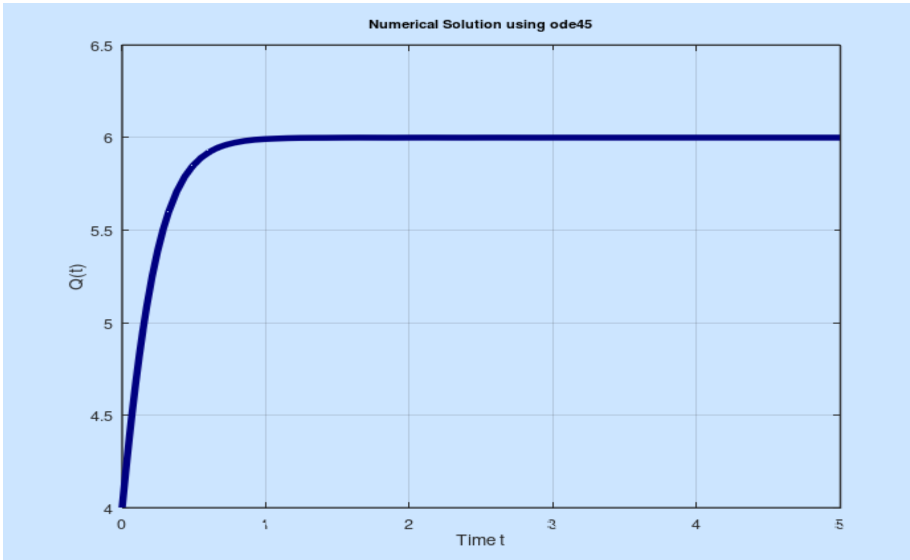

Figure 2: Question 3, Numerical Solution

Code:

% QUESTION 3(c): Numerical solution using ode45 for $\frac{dQ}{dt} = Q(6 - Q), \quad Q(0) = 4$

```
% Define the model as an anonymous function
growth_model = @(t, Q) Q * (6 - Q);

% Define time span and initial condition
tspan = [0 5];
Q0 = 4;

% Solve using ode45
[t, Q] = ode45(growth_model, tspan, Q0);

% Plot the result
plot(t, Q,'color', [0 0 0.5], 'r-', 'LineWidth', 2);
xlabel('Time t');
ylabel('Q(t)');
title('Numerical Solution using ode45');
grid on;

% Set background colour to pink
set(gca, 'Color', [0.8 0.9 1]);   % Light baby blue plot background
set(gcf, 'Color', [0.8 0.9 1]);   % Light baby blue figure background (optional)
```

Question 4

Growth rate r   Carrying capacity K

Given the differential equation $\frac{dQ}{dt} = 0.8Q\left(1 - \frac{Q}{400}\right)$, $Q(0) = 20$, (2) where Q(t) is the represents population at time t, measured in weeks.

(a) Plot the numerical solution curve of ODE (2) in Octave using ode45, clearly showing the equilibrium points and the inflection point. (The Octave code should be included in your assignment)

Solution Curve:

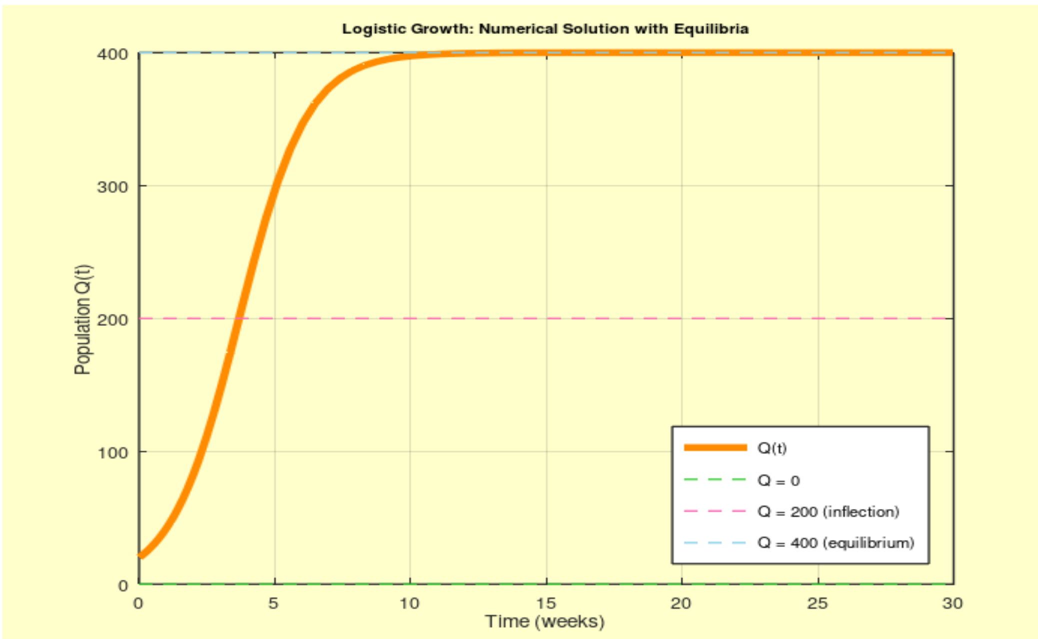

Figure 3: Question 4, Numerical Solution

Code:

```
% QUESTION 4(a): Numerical solution for logistic model using ode45


% Define the logistic model

logistic_model = @(t, Q) 0.8 * Q * (1 - Q / 400);


% Time span and initial population

tspan = [0 30];

Q0 = 20;


% Solve using ode45

[t, Q] = ode45(logistic_model, tspan, Q0);
```

```
% Plot the solution
plot(t, Q, 'color', [1 0.55 0], 'LineWidth', 2);
hold on;

% Plot equilibrium and inflection points manually
plot([0 30], [0 0], 'color', [0.2 0.8 0.2], 'k--');          % Q = 0 (unstable)
plot([0 30], [200 200], 'color', [1 0.41 0.71],'r--');       % Inflection point
plot([0 30], [400 400], 'color', [0.53 0.81 0.92], 'g--');       % Carrying capacity

% Labels
xlabel('Time (weeks)');
ylabel('Population Q(t)');
title('Logistic Growth: Numerical Solution with Equilibria');
legend('Q(t)', 'Q = 0', 'Q = 200 (inflection)', 'Q = 400 (equilibrium)', ...
    'Location', 'southeast');
grid on;

% Set background colour to pink
set(gca, 'Color', [1 1 0.8]);   % Light yellow plot background
set(gcf, 'Color', [1 1 0.8]);
```

(b) Find the analytical solution of system (2).

Separate the variables:

$$\frac{dQ}{Q\left(1-\frac{Q}{400}\right)} = 0.8dt$$

Simplify the denominator,

$$1- \frac{Q}{400} = \frac{400-Q}{400} \to Q\left(1-\frac{Q}{400}\right) = \frac{Q(400-Q)}{400} \therefore \frac{dQ}{Q\left(1-\frac{Q}{400}\right)} = \frac{400}{Q(400-Q)}dQ$$

Now we have,

$$\int \frac{400}{Q(400-Q)}dQ = \int 0.8\, dt$$

Apply partial fraction decomposition on the left-hand side:

We get,

$$\frac{400}{Q(400-Q)} = \frac{A}{Q} + \frac{B}{400-Q}$$

Multiply both sides by $Q(400-Q)$,

$$400 = A(400-Q) + BQ$$

Now solve for A and B,

$$400 = 400A - AQ + BQ \rightarrow 400 = 400A + (B-A)Q$$

Equate the coefficients:

- Constant term: $400A = 400 \rightarrow A = 1$
- Coefficient of Q: $B - A = 0 \rightarrow B = A = 1$

Substitute back and integrate

$$\int (\frac{1}{Q} + \frac{1}{400-Q}) dQ = \int 0.8\, dt$$

Separate the integrals on the left-hand side,

$$\int \frac{1}{Q} dQ + \int \frac{1}{400-Q} dQ = \int 0.8\, dt$$

Note: use u-sub for $\int \frac{1}{400-Q} dQ$ and then integrating gives,

$$\ln|Q| - \ln|400-Q| = 0.8t + C$$

Combine the logs (lns),

$$\ln(\frac{Q}{400-Q}) = 0.8t + C$$

Taking exponentials of both sides yields,

$$\frac{Q}{400-Q} = e^{0.8t} \cdot e^{C}$$

Let $e^{C} = A$,

$$\frac{Q}{400-Q} = Ae^{6t}$$

Solve for $Q(t)$,

$$Q = (400-Q)Ae^{0.8t} \rightarrow Q + QAe^{0.8t} = 400Ae^{0.8t} \rightarrow Q(1 + Ae^{0.8t}) = 400Ae^{0.8t} \therefore Q(t) = \frac{400Ae^{0.8t}}{1 + Ae^{0.8t}}$$

Apply initial condition, $Q(0) = 20$ to solve for A,

$$Q(0) = \frac{400Ae^{0.8(0)}}{1 + Ae^{0.8(0)}} = 20 \rightarrow 400A = 20(1 + A) \rightarrow 400A = 20 + 20A \rightarrow 380A = 20 \rightarrow A = \frac{1}{19}$$

Sub A back in and we have,

$$Q(t) = \frac{\left(\frac{400e^{0.8t}}{19}\right)}{\left(\frac{1 + e^{0.8t}}{19}\right)}$$

After simplifying, the final answer is,

$$\boxed{Q(t) = \frac{400e^{0.8t}}{19 + e^{0.8t}}}$$

Question 5

Consider the system of differential equations below:

$$\frac{dx}{dt} = x - x^2 - xy$$
$$\frac{dy}{dt} = y - y^2 - \frac{1}{2}xy$$

These terms represent the negative effect each species has on the other when they interact

where you may assume that x ≥ 0 and y ≥ 0.

(a) What type of interaction is this?

The above system of differential equations represents a Competition model.

(b) Draw the nullclines on Octave and locate all the steady states graphically (The Octave code should be included in your assignment).

Octave Graph of Nullclines and Steady States:

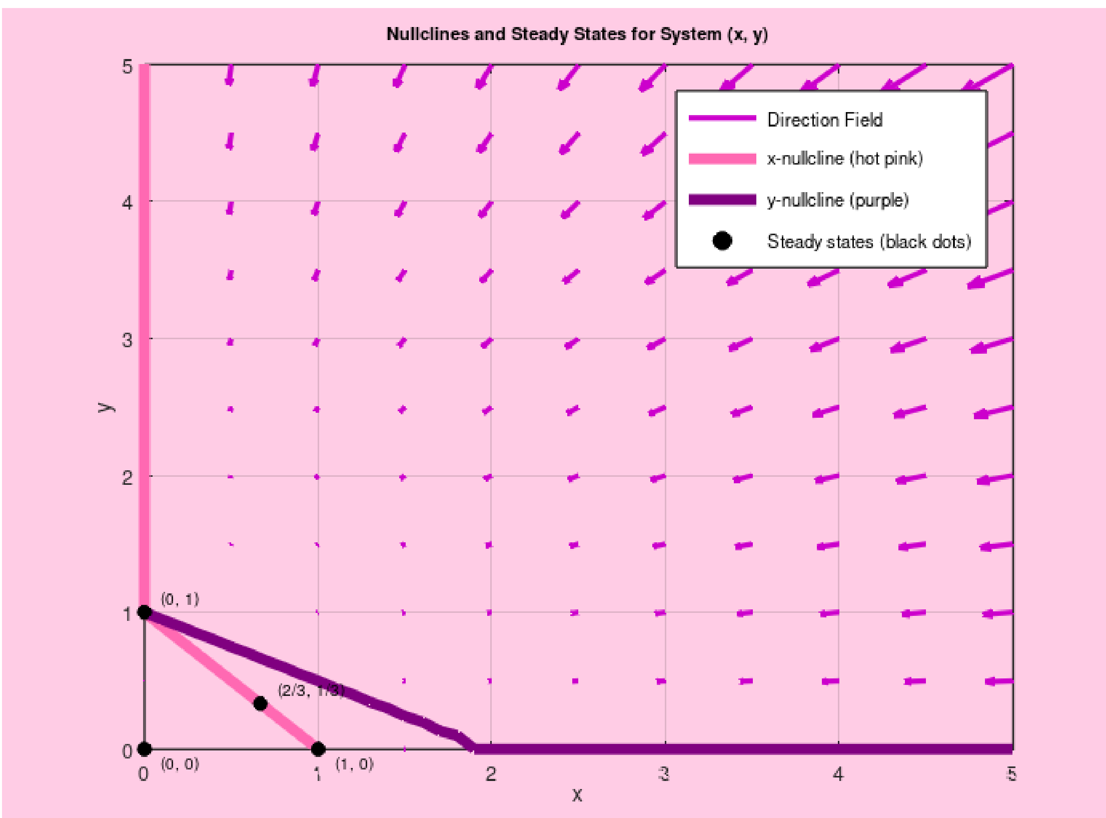

Figure 4: Question 5, Nullclines & Steady States (Direction Field for own contribution)

Sidebar: Included directional field arrows as own contribution (Thompson, 2021)*. Had an issue with plotting the nullclines and directional field on the same 'meshgrid', see code below.*

Code:

```
% Clear figure
clf;

% Define fine grid for nullclines
[x1, y1] = meshgrid(0:0.1:5, 0:0.1:5);
dx1 = x1 - x1.^2 - x1 .* y1;
dy1 = y1 - y1.^2 - 0.5 * x1 .* y1;

% Define coarse grid for directional field arrows
[x2, y2] = meshgrid(0:0.5:5, 0:0.5:5);
dx2 = x2 - x2.^2 - x2 .* y2;
dy2 = y2 - y2.^2 - 0.5 * x2 .* y2;

% Plot Direction Field arrows (magenta)
quiver(x2, y2, dx2, dy2, 0.6, 'color', [0.8 0 0.8], 'LineWidth', 1.2);
hold on;

% Plot Nullclines
contour(x1, y1, dx1, [0 0], 'LineColor', [1 0.41 0.7], 'LineWidth', 2.5); % x-nullcline (hot pink)
contour(x1, y1, dy1, [0 0], 'LineColor', [0.5 0 0.5], 'LineWidth', 2.5); % y-nullcline (purple)

% Plot steady states as black dots
plot([0 1 0 2/3], [0 0 1 1/3], 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 3);

% Add coordinate labels
text(0.1, -0.1, '(0, 0)', 'FontSize', 10, 'Color', 'k');
text(1.1, -0.1, '(1, 0)', 'FontSize', 10, 'Color', 'k');
text(0.1, 1.1, '(0, 1)', 'FontSize', 10, 'Color', 'k');
text(2/3 + 0.1, 1/3 + 0.1, '(2/3, 1/3)', 'FontSize', 10, 'Color', 'k');

% Dummy lines for legend
h1 = plot(NaN, NaN, '-', 'Color', [0.8 0 0.8], 'LineWidth', 1.2);    % direction
h2 = plot(NaN, NaN, '-', 'Color', [1 0.41 0.7], 'LineWidth', 2.5);  % x-nullcline
h3 = plot(NaN, NaN, '-', 'Color', [0.5 0 0.5], 'LineWidth', 2.5);   % y-nullcline
```

```
h4 = plot(NaN, NaN, 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 4);% steady state

legend([h1, h2, h3, h4], 'Direction Field', 'x-nullcline (hot pink)', 'y-nullcline (purple)', 'Steady states (black dots)', 'Location',
'northeast');

% Axes labels & title
xlabel('x');
ylabel('y');
title('Nullclines and Steady States for System (x, y)');
axis([0 5 0 5]);
grid on;

% Background color
baby_pink = [1 0.8 0.9];
set(gca, 'Color', baby_pink);
try
    set(gcf, 'Color', baby_pink);
catch
end
```

Question 6

Consider the system of differential equations below:

$$\frac{dx}{dt} = x - 0.5xy$$

$$\frac{dx}{dt} = -0.75y + 0.25xy$$

(a) Does the given system represent a predator-prey or competing species model? Justify your answer.

The given system represents a predator-prey model. This is apparent from the structure of the equations. One species (the prey) exhibits natural growth in the absence of interaction but is negatively affected by the presence of the other species through the term '−0.5xy'. In contrast, the predator species experiences natural decay without prey, but benefits from their interaction, as indicated by the term '+0.25xy'. This asymmetric relationship, where one species benefits and the other species loses from their interaction, is characteristic of predator-prey model rather than competition.

(b) Plot a solution curve for the given system of differential equations using an Octave. Ensure that the Octave code used for generating the graph is included in your assignment.

Octave Solution Curve:

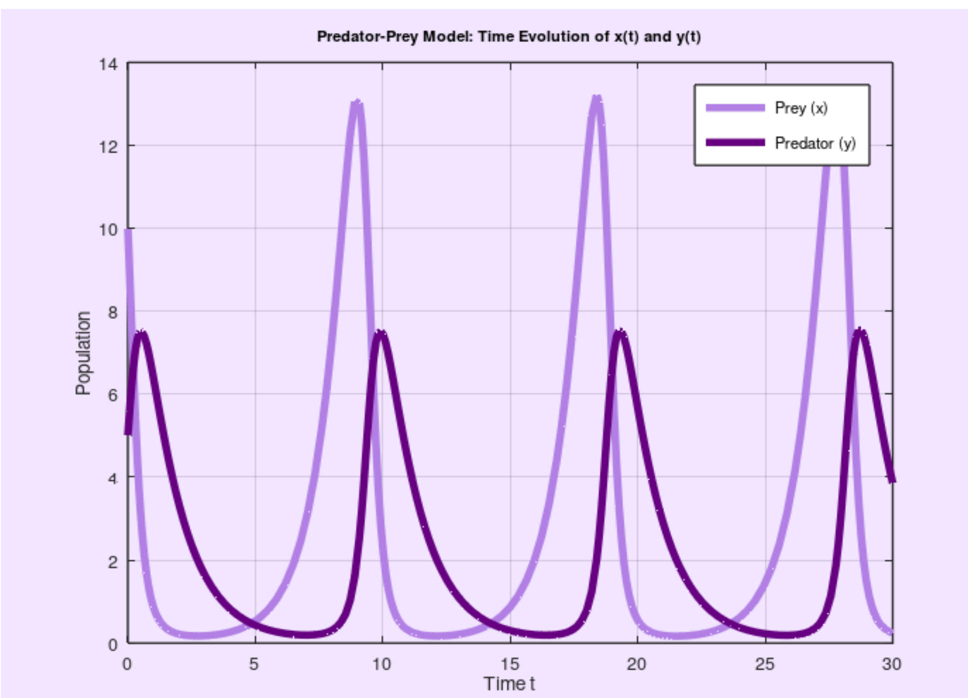

Figure 5: Question 6, Predator-Prey Model

Code:

```
% QUESTION 6(b): Predator-Prey model simulation using ode45

% Define the system of equations as a function
predator_prey = @(t, Y) [
    Y(1) - 0.5 * Y(1) * Y(2);       % dx/dt = x - 0.5xy
    -0.75 * Y(2) + 0.25 * Y(1) * Y(2) % dy/dt = -0.75y + 0.25xy
];

% Initial populations: [Prey x0, Predator y0]
Y0 = [10; 5];

% Time span
tspan = [0 30];

% Solve the system
[t, sol] = ode45(predator_prey, tspan, Y0);

% Define purple color tones
light_purple = [0.7 0.5 0.9];     % for prey
deep_purple = [0.4 0 0.5];        % for predator
background_purple = [0.95 0.9 1]; % background

% Plot results using purple tones
plot(t, sol(:,1), '-', 'Color', light_purple, 'LineWidth', 2); hold on;
plot(t, sol(:,2), '-', 'Color', deep_purple, 'LineWidth', 2);

xlabel('Time t');
ylabel('Population');
title('Predator-Prey Model: Time Evolution of x(t) and y(t)');

legend('Prey (x)', 'Predator (y)');

% Add grid and background color
grid on;
set(gca, 'Color', background_purple);
try
    set(gcf, 'Color', background_purple);
catch
    warning('Could not set figure background color');
end
```

(c)Do the species coexist in the long run? Justify your answer.

Yes, the species coexist in the long run. As seen from the graph, neither species is becoming extinct and their populations are still fluctuating. As the characteristics of the predator-prey model predict, this prolonged interaction suggests long-term coexistence.

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