Computational Psychiatry Course – Zürich 2016

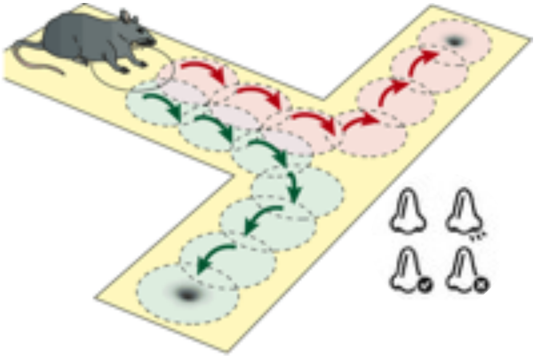
Partially Observable Markov Decision Processes

Lionel Rigoux & Frederike Petzschner

Introduction

- MDP >> Full observability: the agent always knows the state of the world

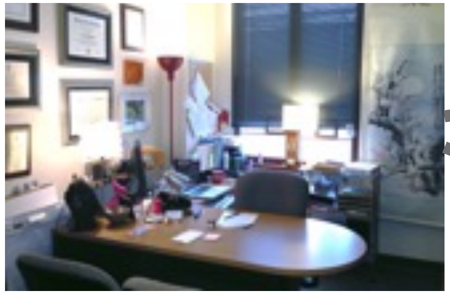
- This might often not be true in real life
  - *Imperfect memory*
    // navigation: “turn left on the seventh street”
    > what if you loose track of the number of streets already passed?
  - *Changing environment*
    // reward selection in a T-maze
    > reward location changes every trials, as cued by a smell



leave

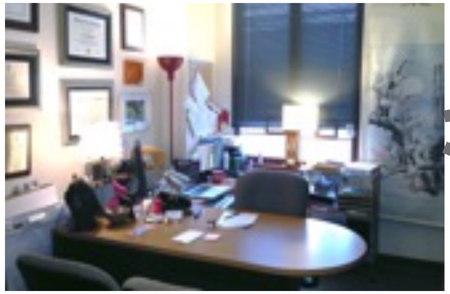
leave

stay



leave

stay



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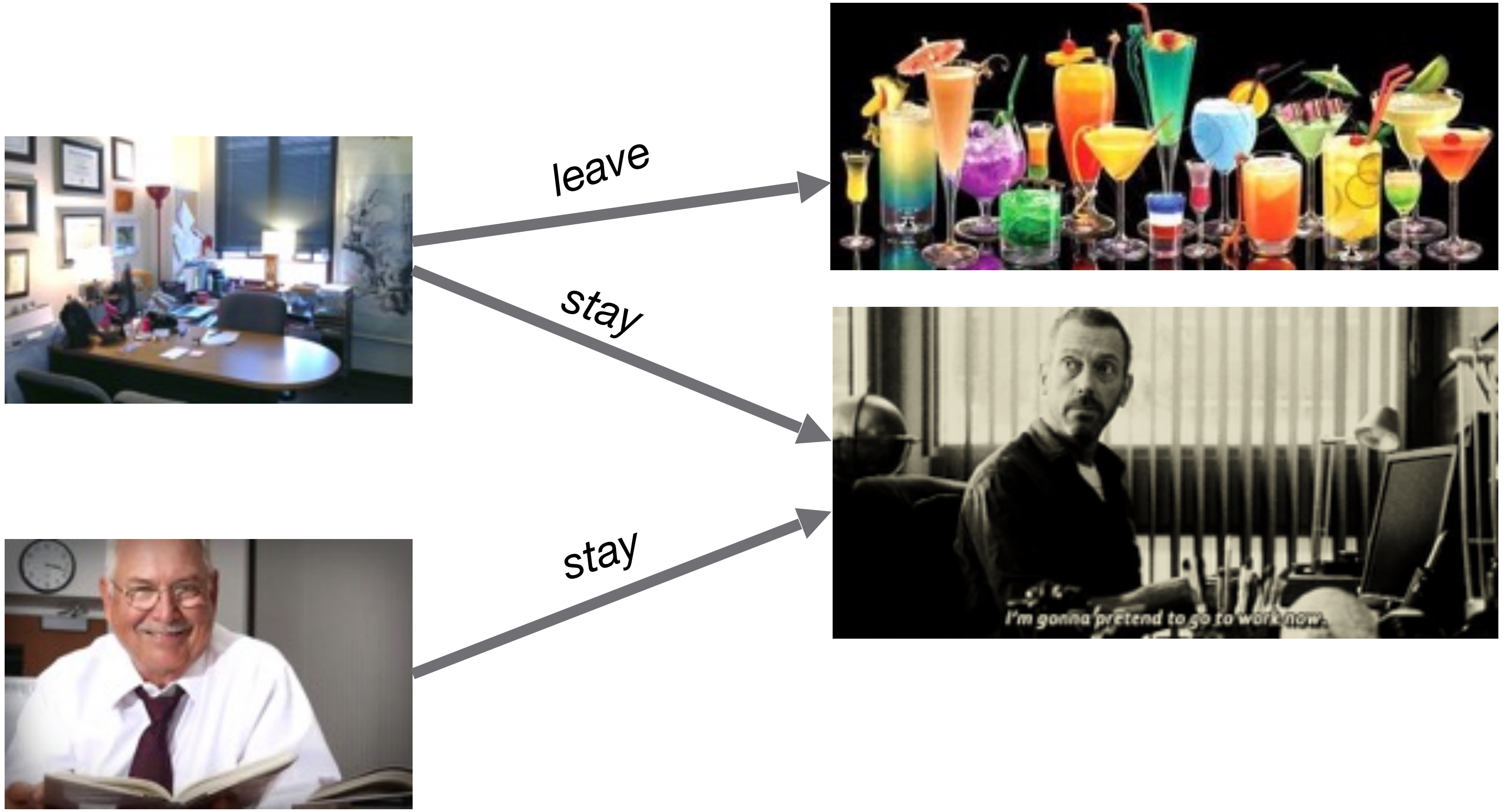
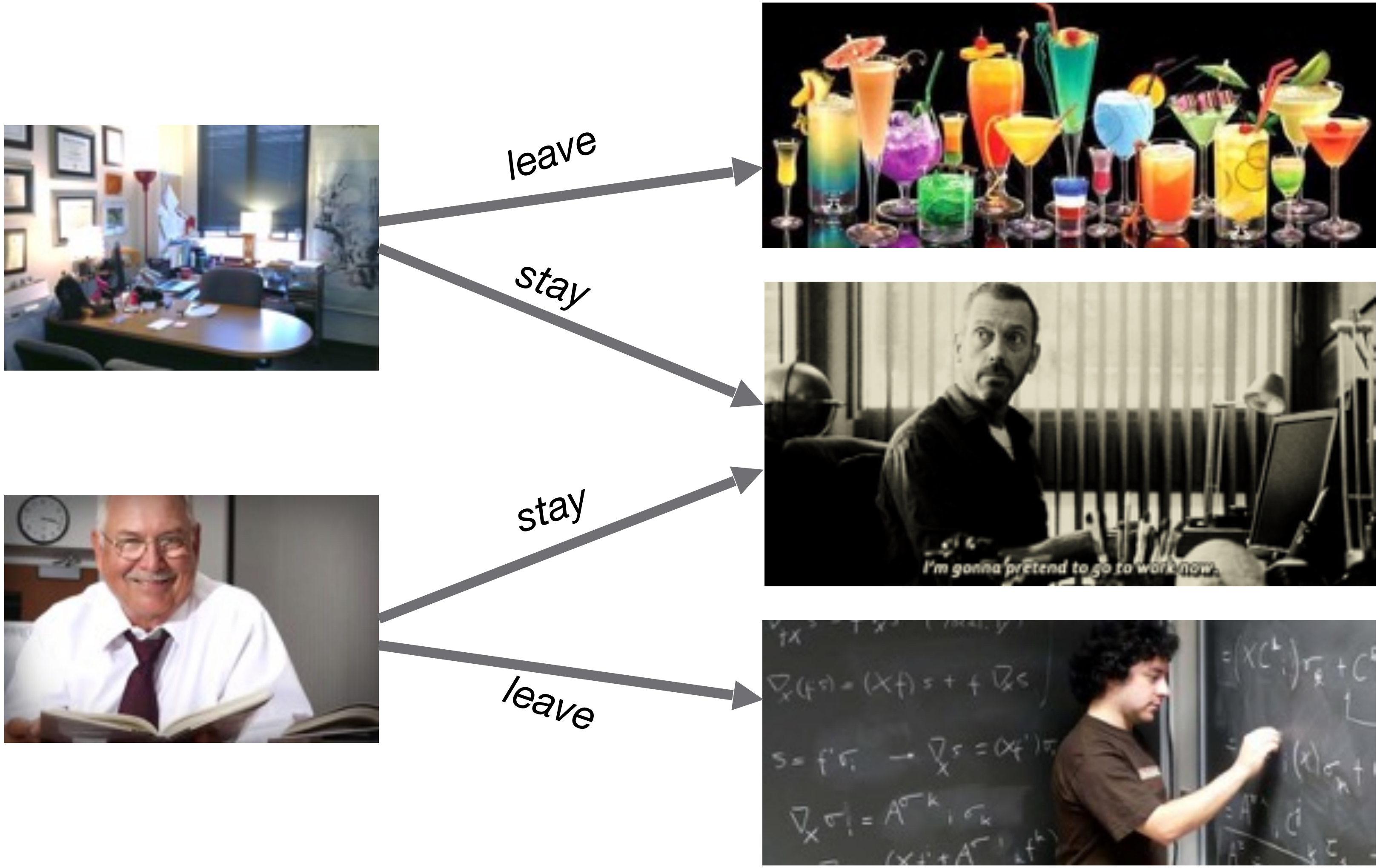
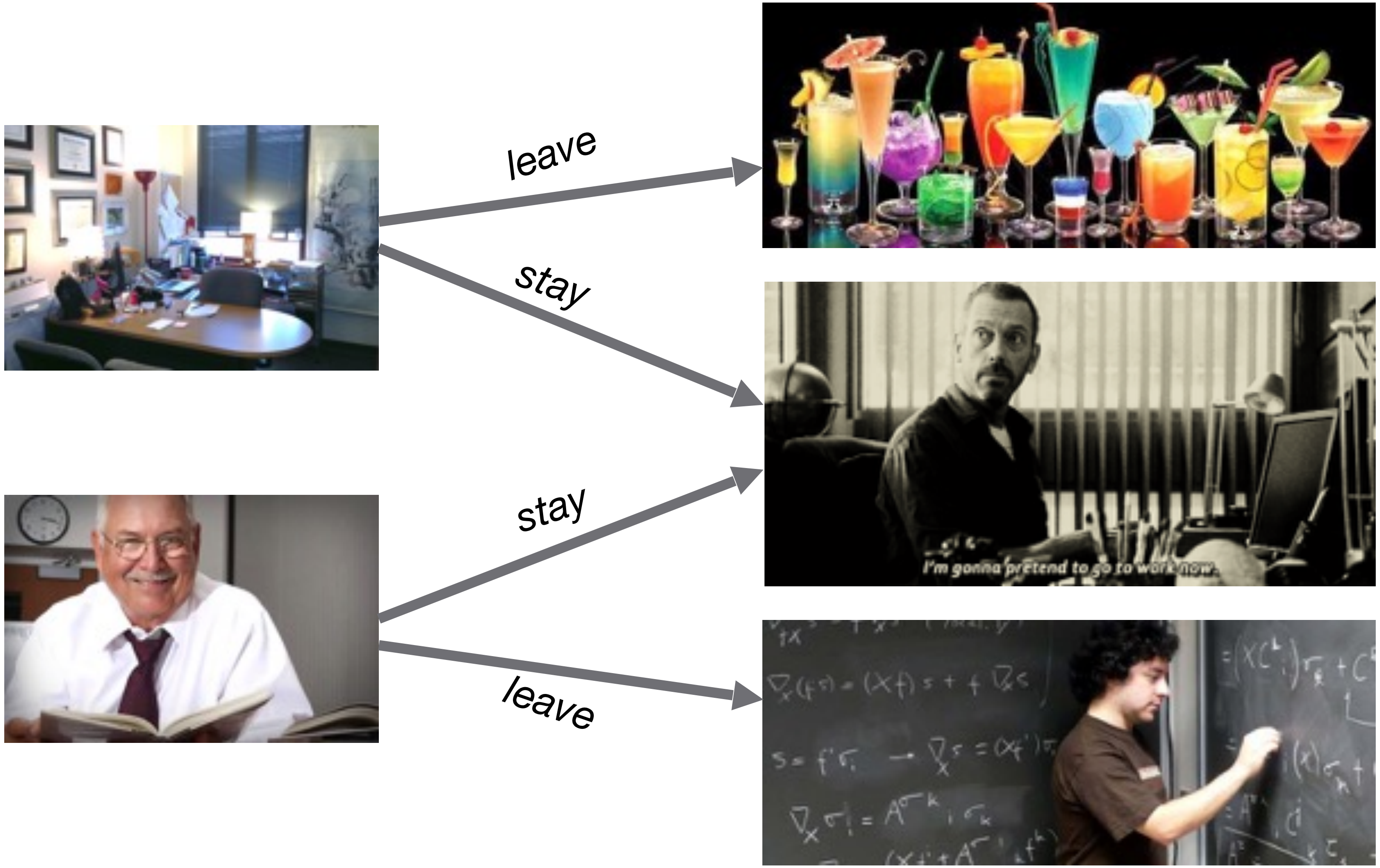
stay

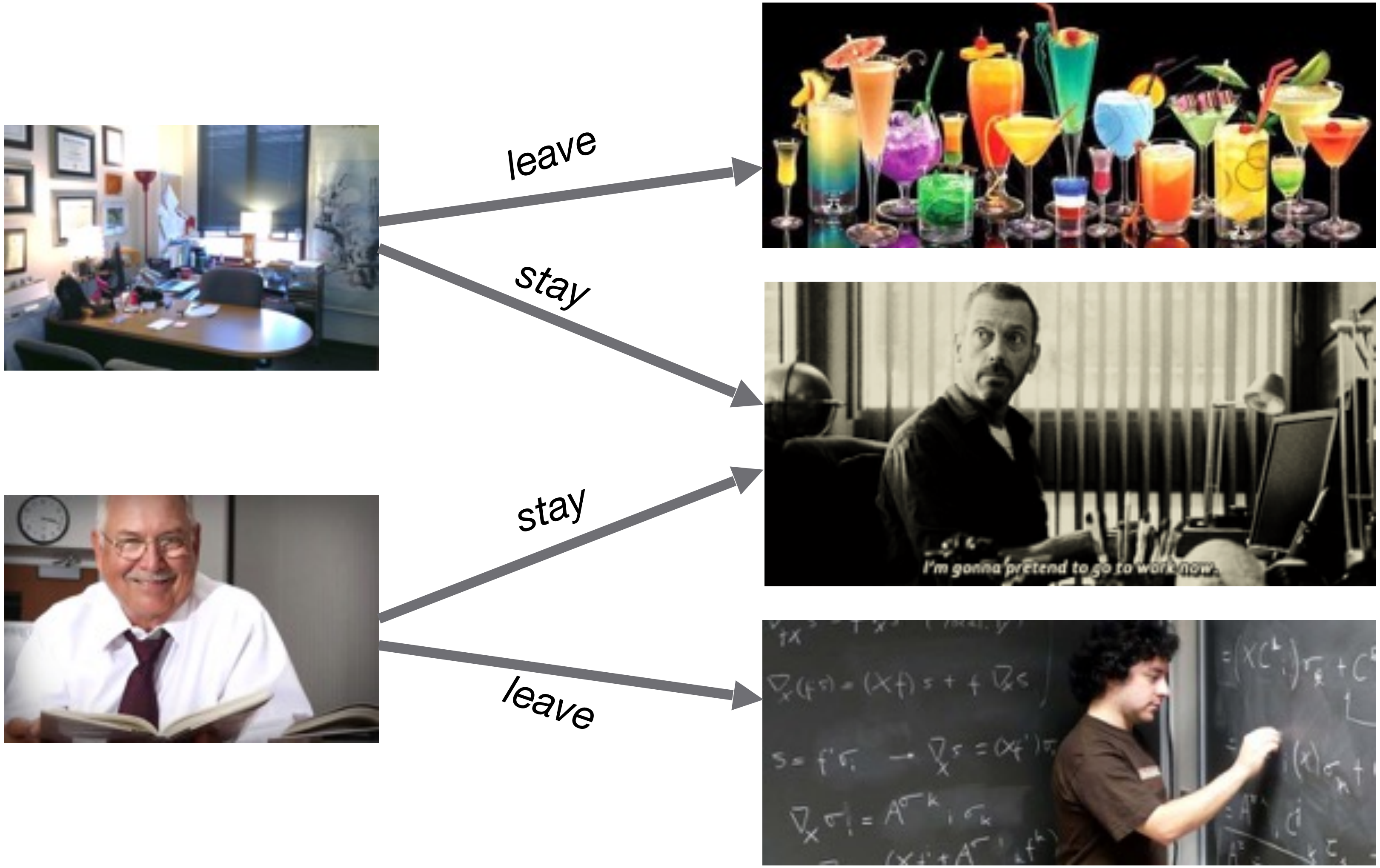

leave
stay
stay
I'm gonna pretend to go to work now.


leave
stay
I'm gonna pretend to go to work now.
stay
leave

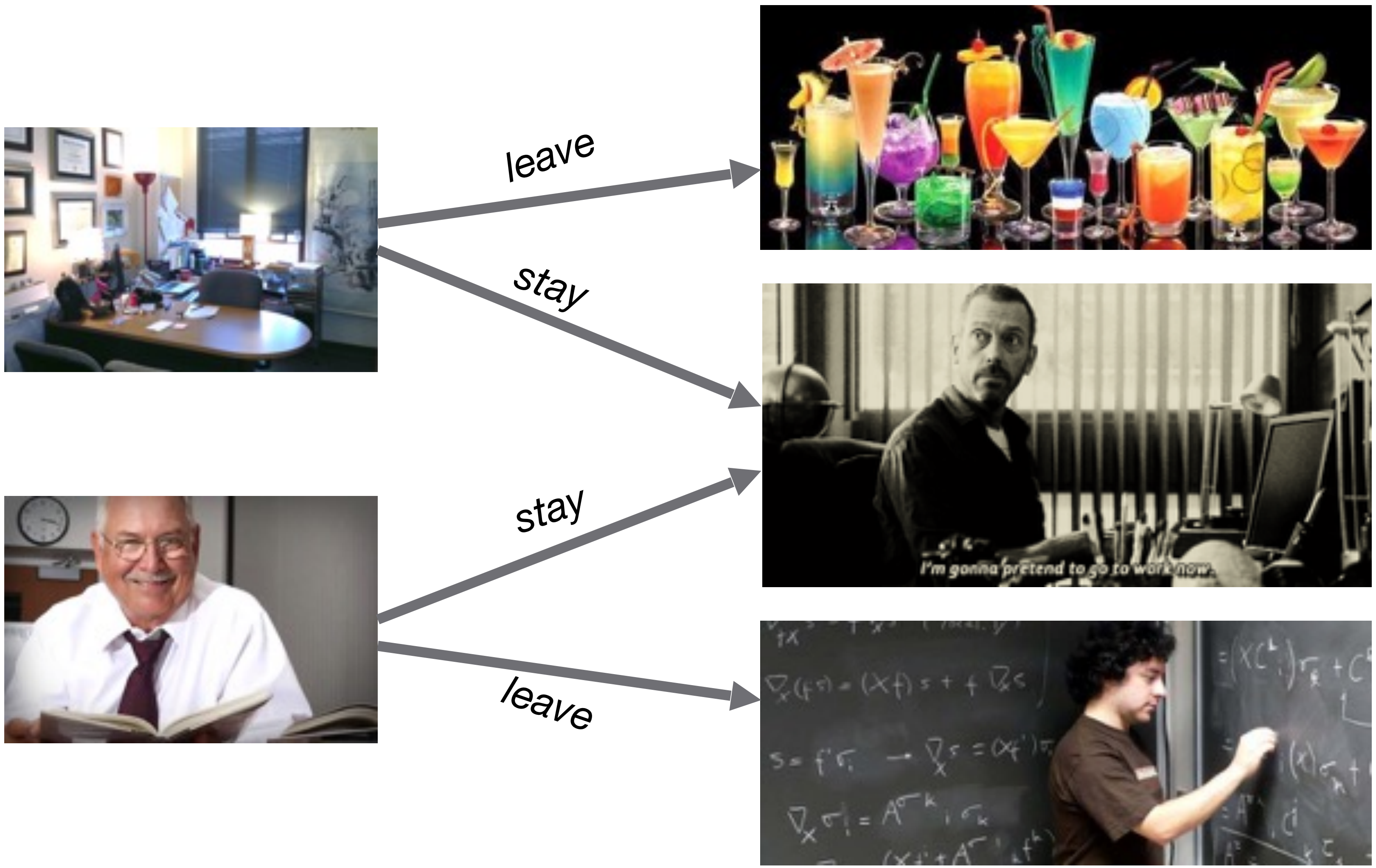
state

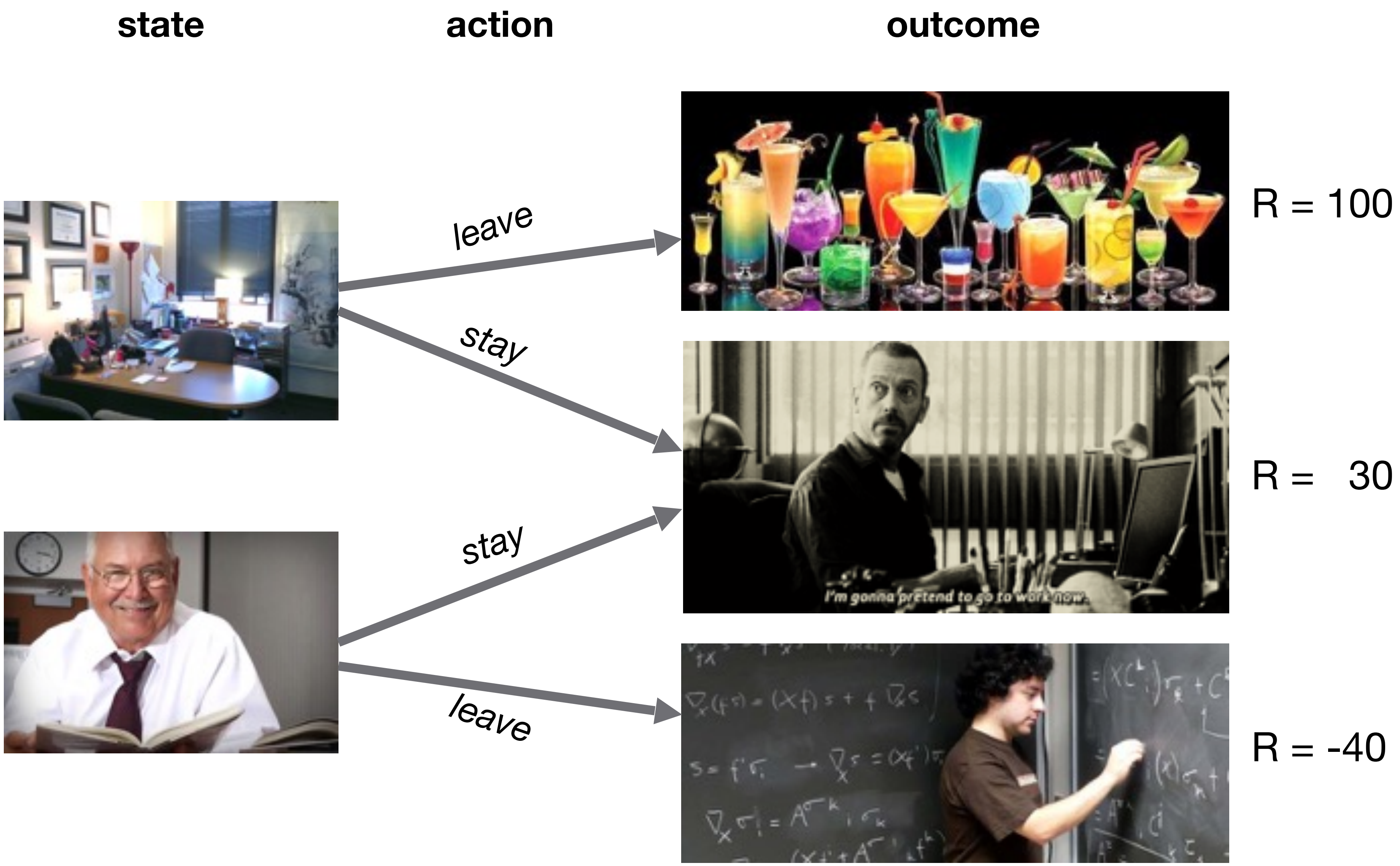


state **action**

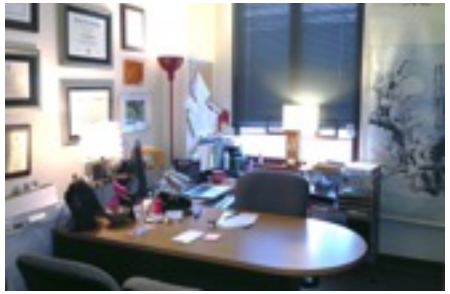


state **action** **outcome**




state
action
outcome
leave
stay
stay
leave
R = 100
R = 30
R = -40
I'm gonna pretend to go to work now.

state **action** **outcome**



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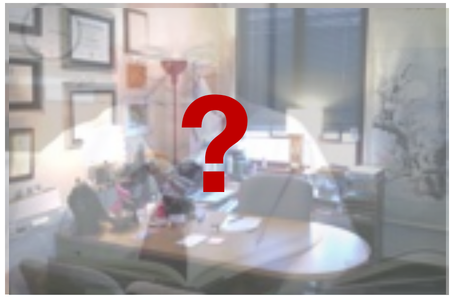
state **action** **outcome**



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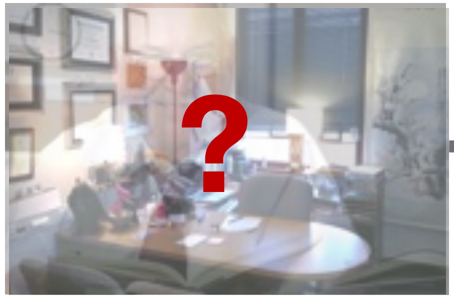
state **action** **outcome**

R = 100



stay

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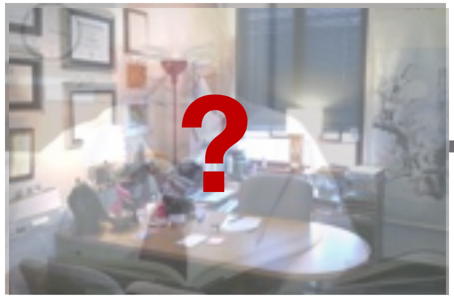
state **action** **outcome**



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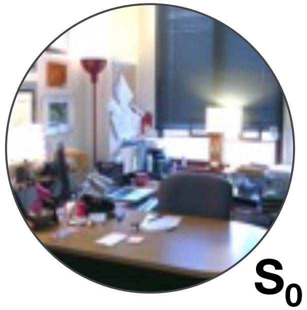
leave

state



S_0

S_1

state

not known

S_0

S_1

state

not known

belief

$b=p(s=S_1)$

S_0

$p(s=S_1) = 0$

$p(s=S_1) = 1$

S_1

state

not known

belief

$b = p(s = S_1)$

actions and payoff function

S_0

$p(s=S_1) = 0$

$p(s=S_1) = 1$

S_1

state

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$b=p(s=S_1)$

actions and payoff function

S_0

$p(s=S_1) = 0$

S_1

$p(s=S_1) = 1$

$$E[R](a) = p(x=0)\, R_0(a) + p(x=1)\, R_1(a)$$

state

not known

belief

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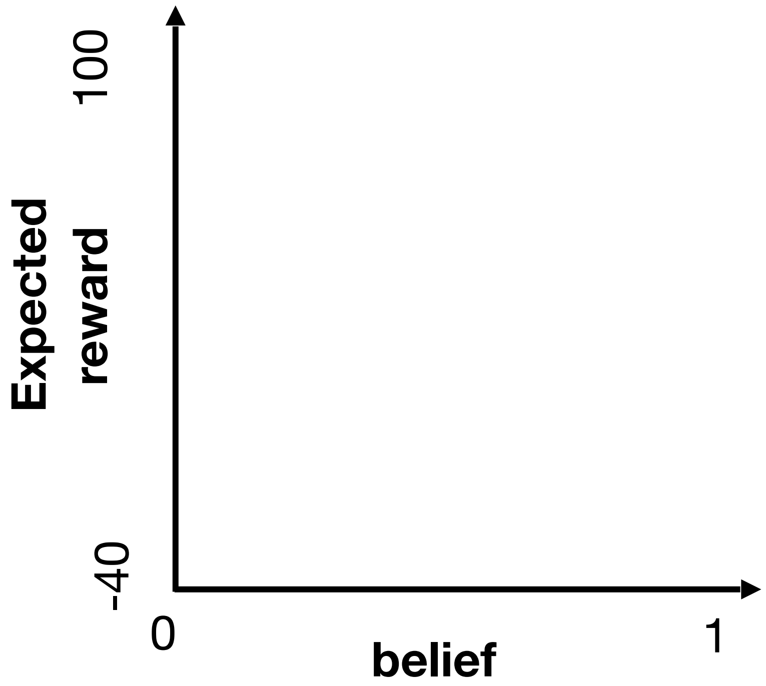
actions and payoff function

S_0

$p(s=S_1) = 0$

$p(s=S_1) = 1$

S_1



$$E[R](a) = p(x=0)\, R_0(a) + p(x=1)\, R_1(a)$$

state

not known

S_0

S_1

belief

$b=p(s=S_1)$

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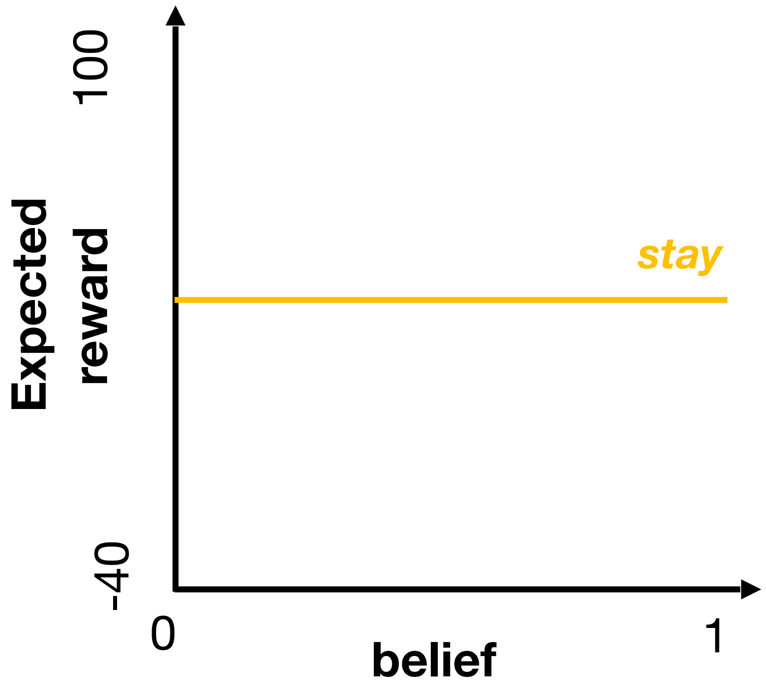
actions and payoff function



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actions and payoff function

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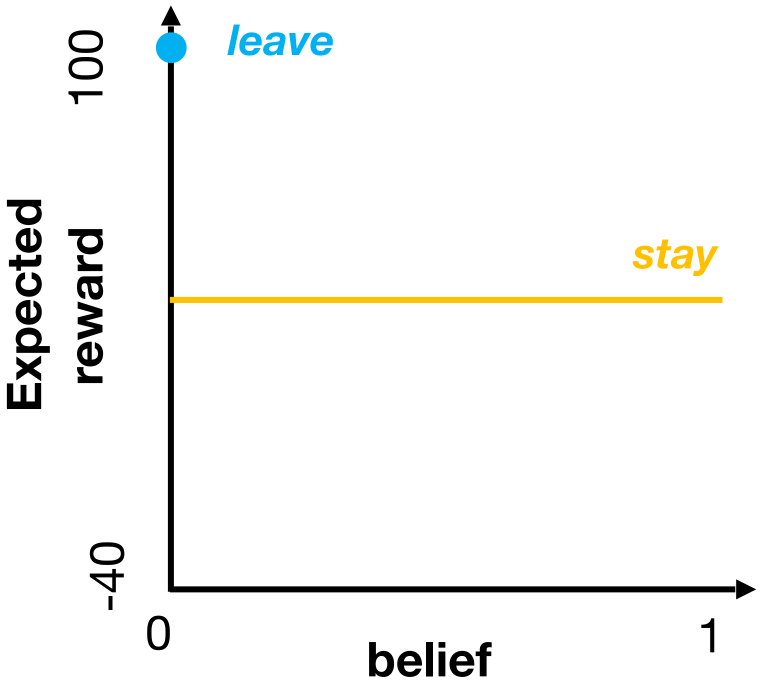

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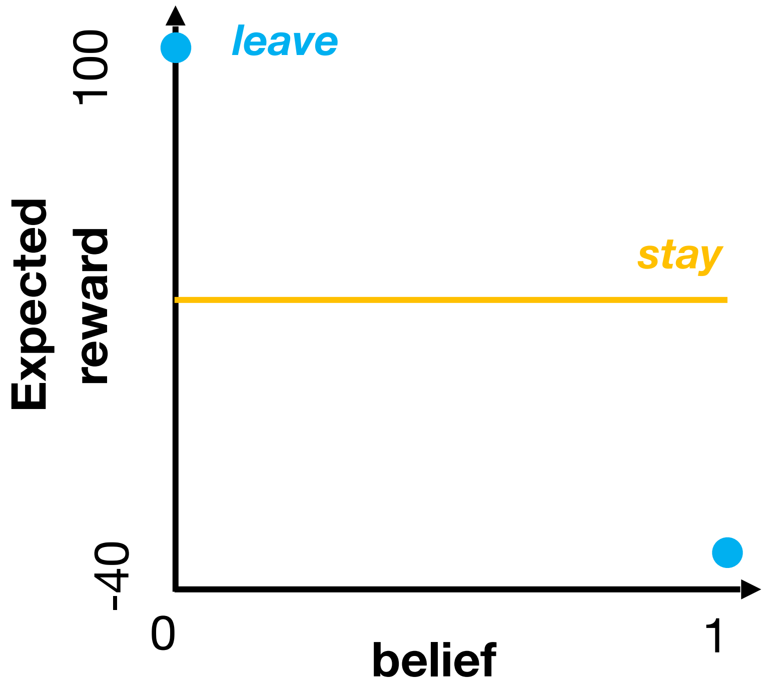

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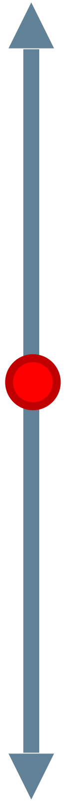

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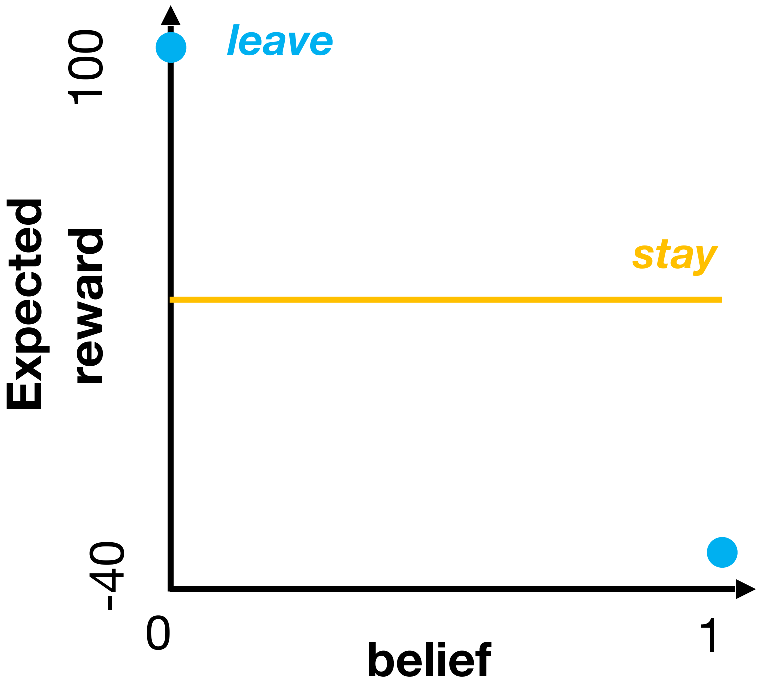
actions and payoff function



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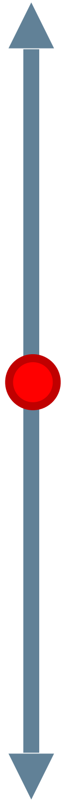

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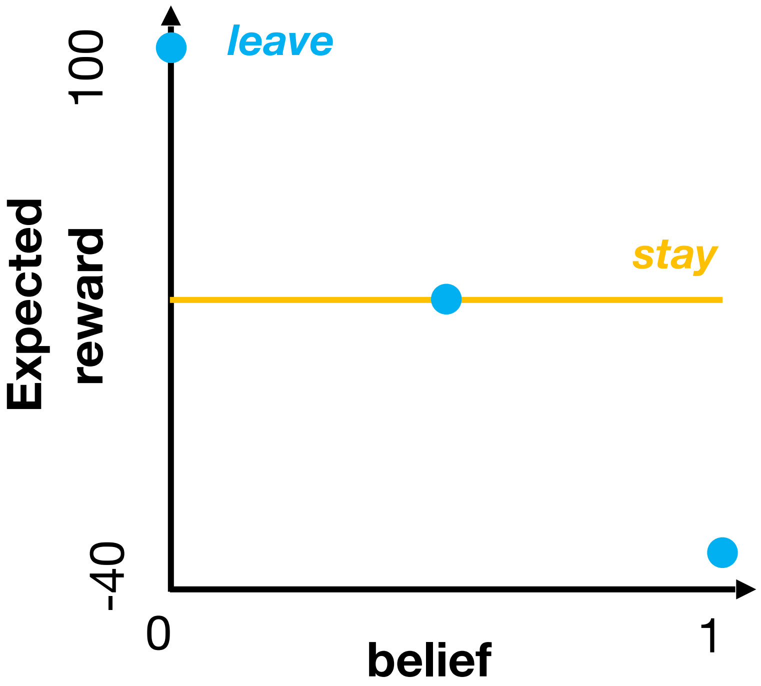
actions and payoff function



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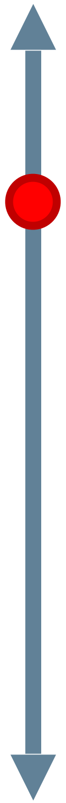

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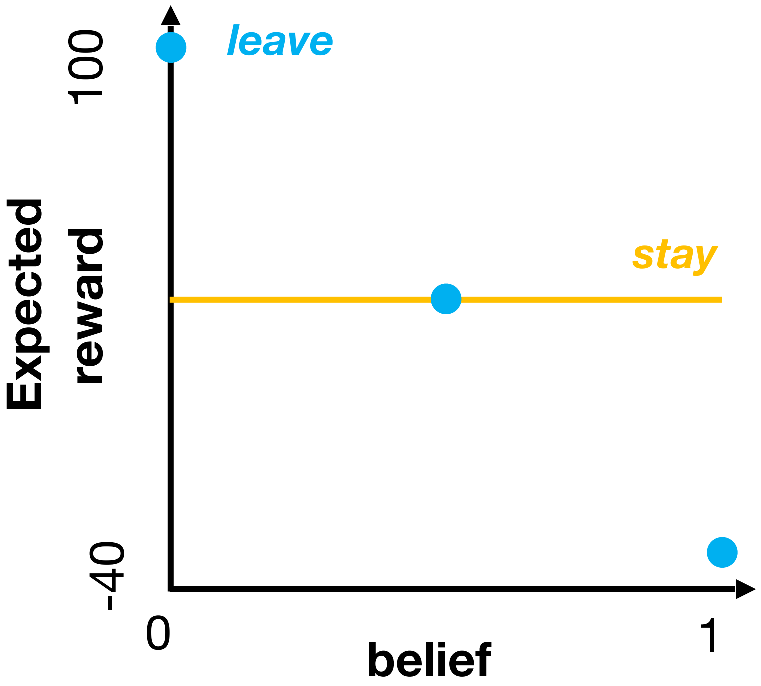

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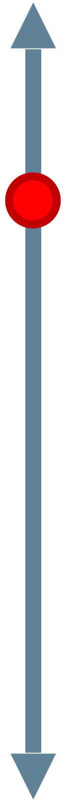

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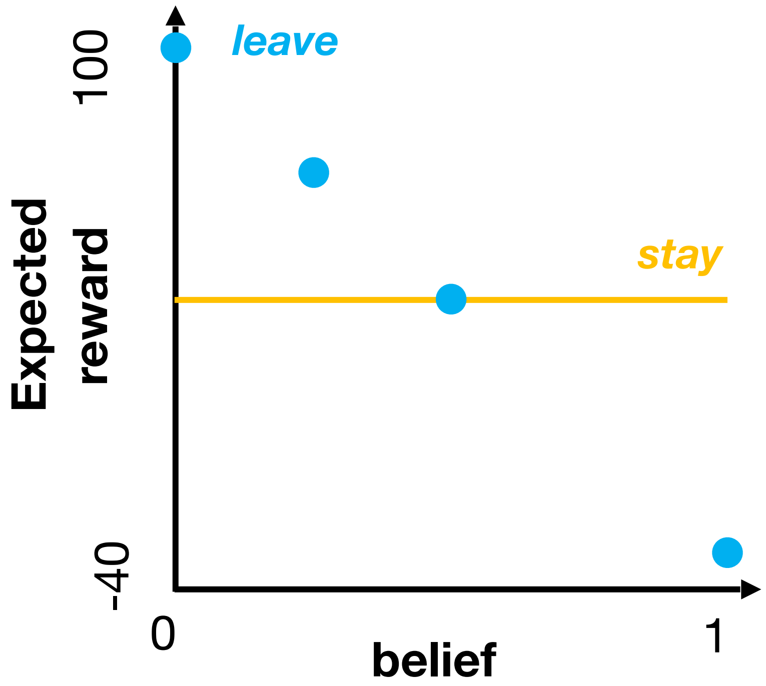

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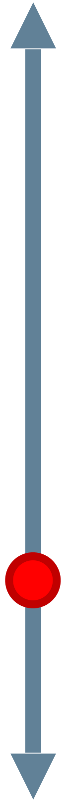

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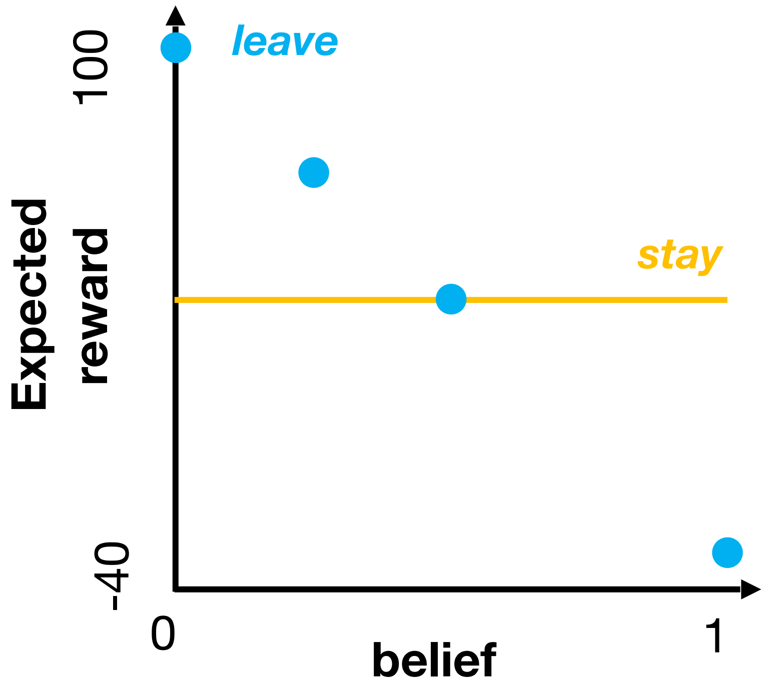

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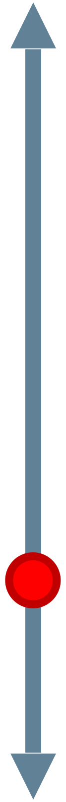

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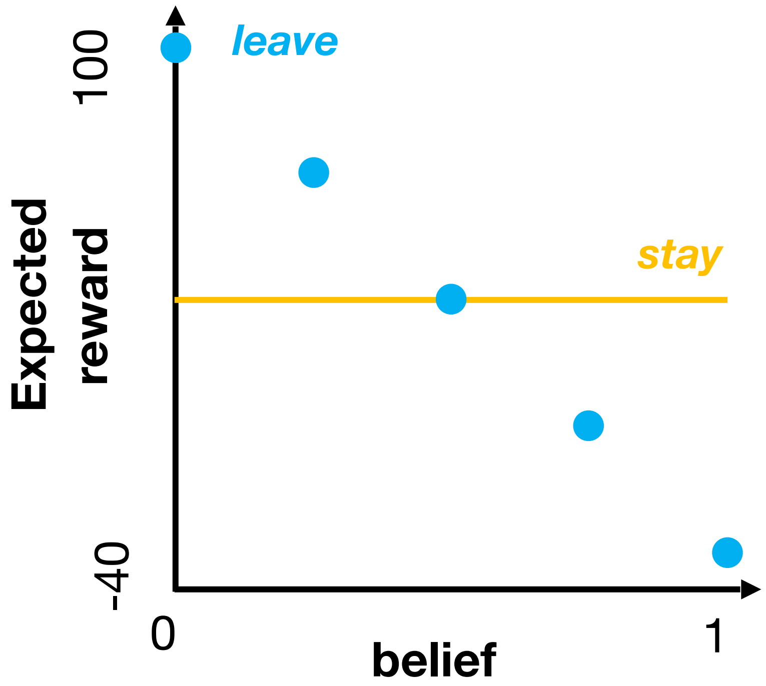

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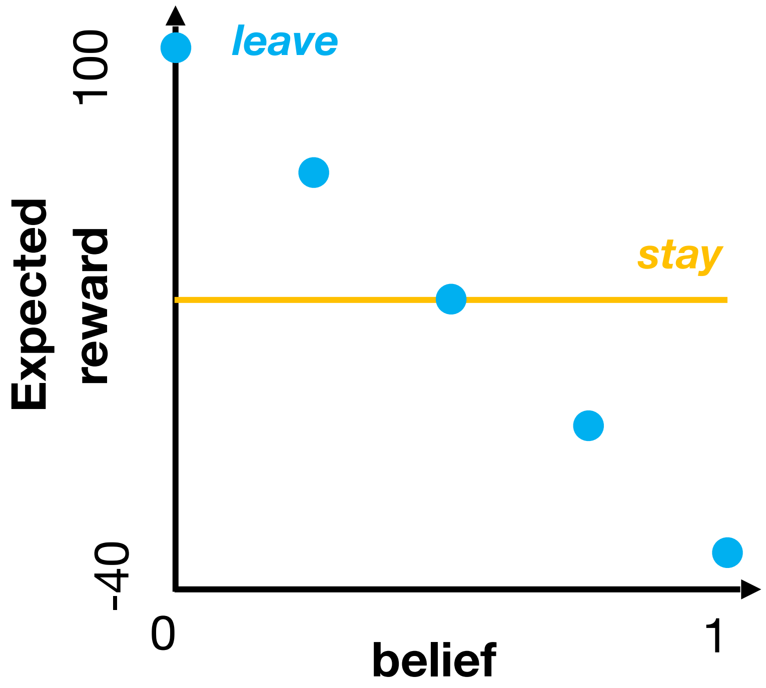

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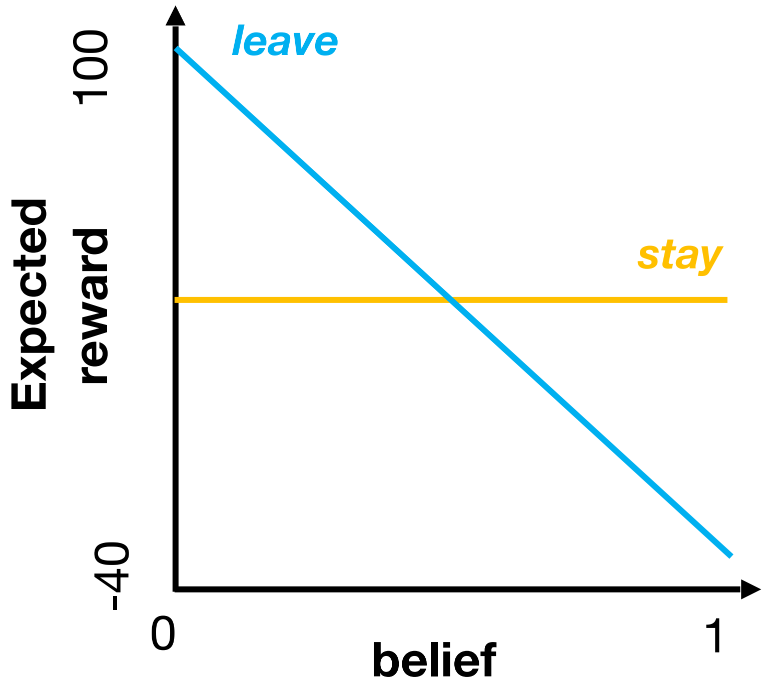
actions and payoff function



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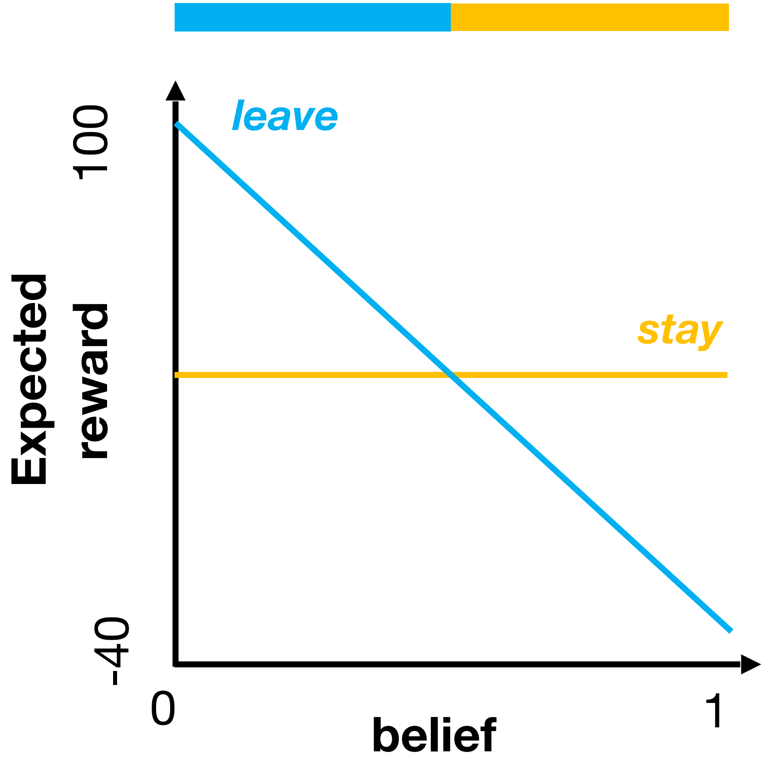
actions and payoff function

optimal policy



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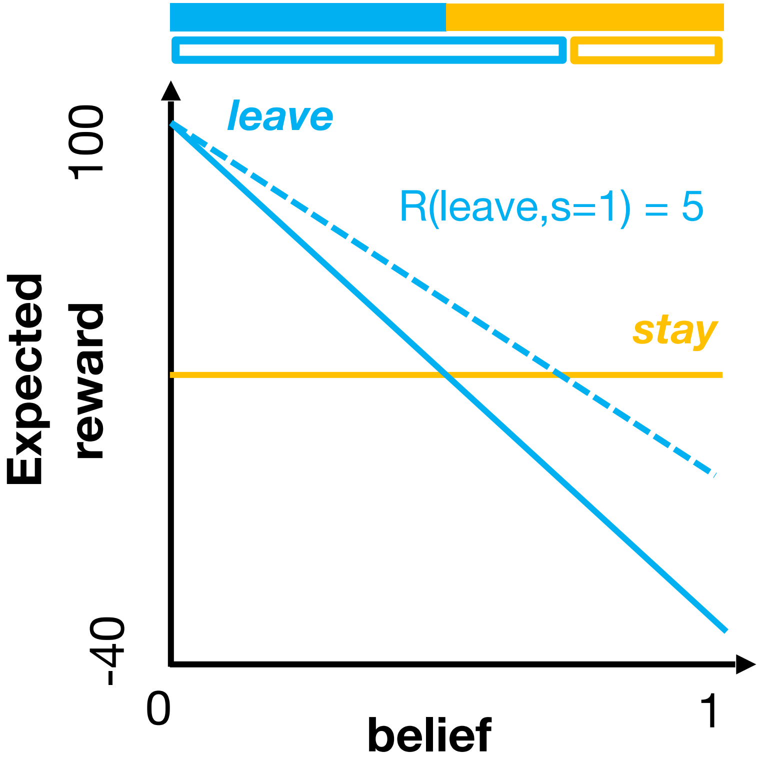
actions and payoff function

optimal policy



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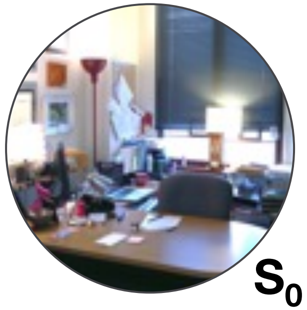
state



S_0

S_1

belief

$p(s=S_1) = 0$

$p(s=S_1) = 1$

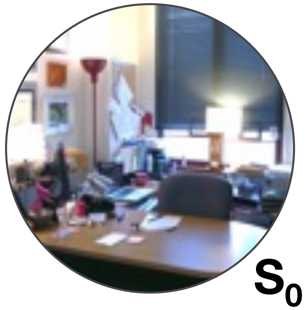
state

observation function

provide information about state

belief

$p(s=S_1) = 0$



S_0

S_1

$p(s=S_1) = 1$

state

observation function

provide information about state

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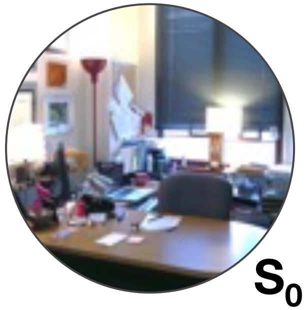


S_0

	leave	*stay*	*listen*
noises	0	0.5	0.15
no one	1	0.5	0.85

S_1

$p(s=S_1) = 1$

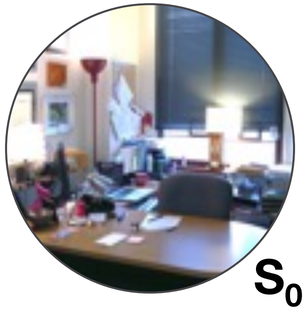
state

observation function

provide information about state

belief

$p(s=S_1) = 0$



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state

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provide information about state

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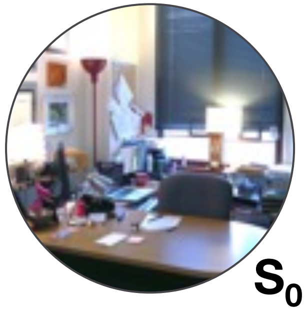

S_0

	leave	*stay*	*listen*
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$$b' \sim p(o|s', a) \sum_s p(s'|s, a) b(s)$$

S_1

	leave	*stay*	*listen*
noises	1	0.5	0.85
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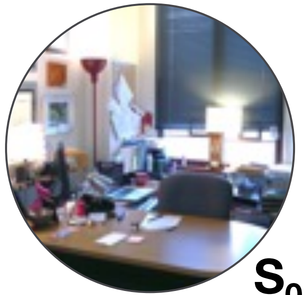
state

observation function

provide information about state

belief



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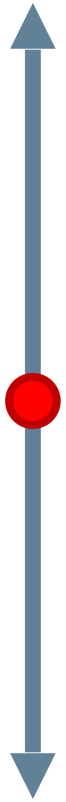

$p(s=S_1) = 1$

state

observation function

provide information about state

belief

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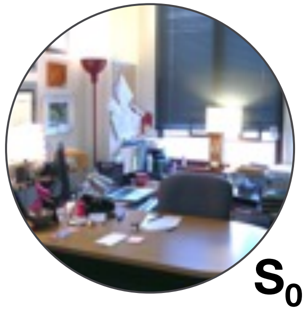

S_0

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$$b' \sim p(o|s', a) \sum_{s} p(s'|s, a) b(s)$$

listen
no one

S_1

	leave	*stay*	*listen*
noises	1	0.5	0.85
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$p(s=S_1) = 1$

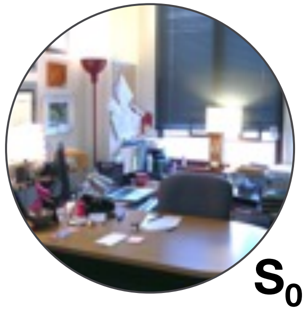
state

observation function

provide information about state

belief

$p(s=S_1) = 0$



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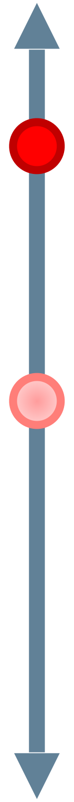

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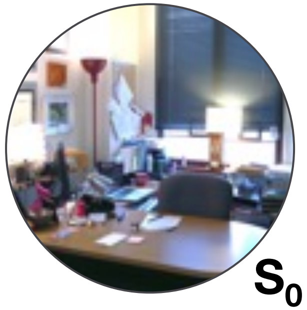

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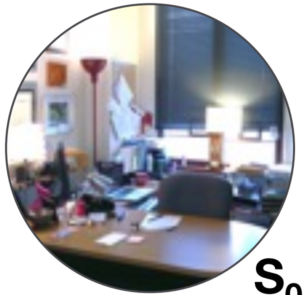

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$$b' \sim p(o|s', a) \sum_{s} p(s'|s, a) b(s)$$

S_1

	leave	stay	listen
noises	1	0.5	0.85
no one	0	0.5	0.15

belief

$p(s=S_1) = 0$

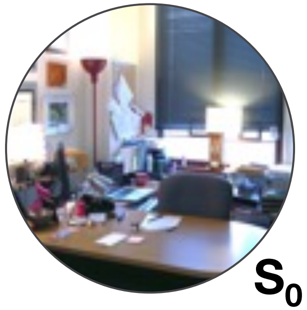
$p(s=S_1) = 1$

state

observation function

provide information about state

belief



S_0

	leave	*stay*	*listen*
noises	0	0.5	0.15
no one	1	0.5	0.85

$$b' \sim p(o|s', a) \sum_s p(s'|s, a) b(s)$$

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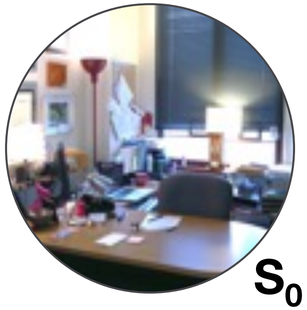

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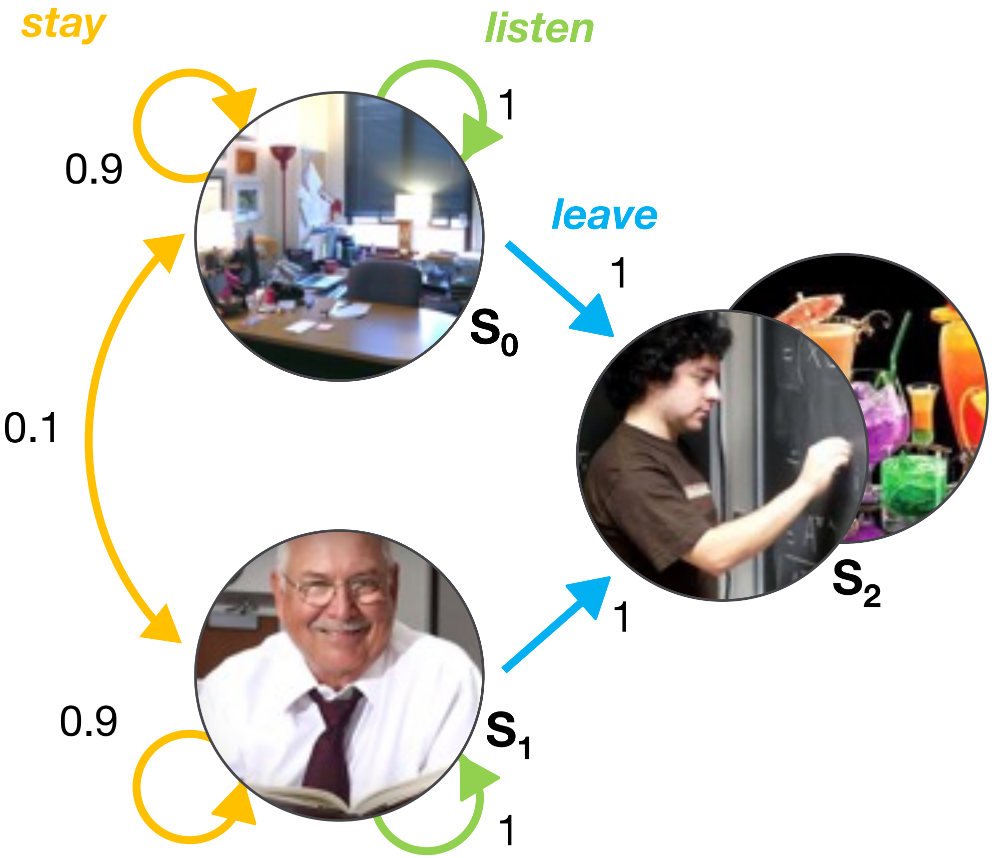
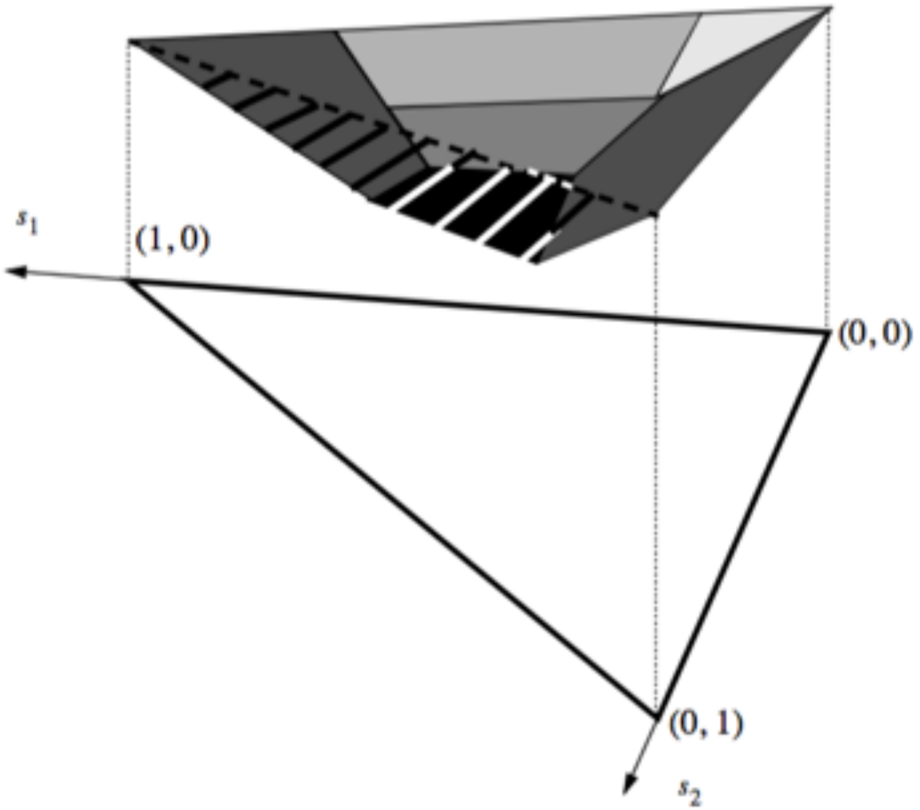
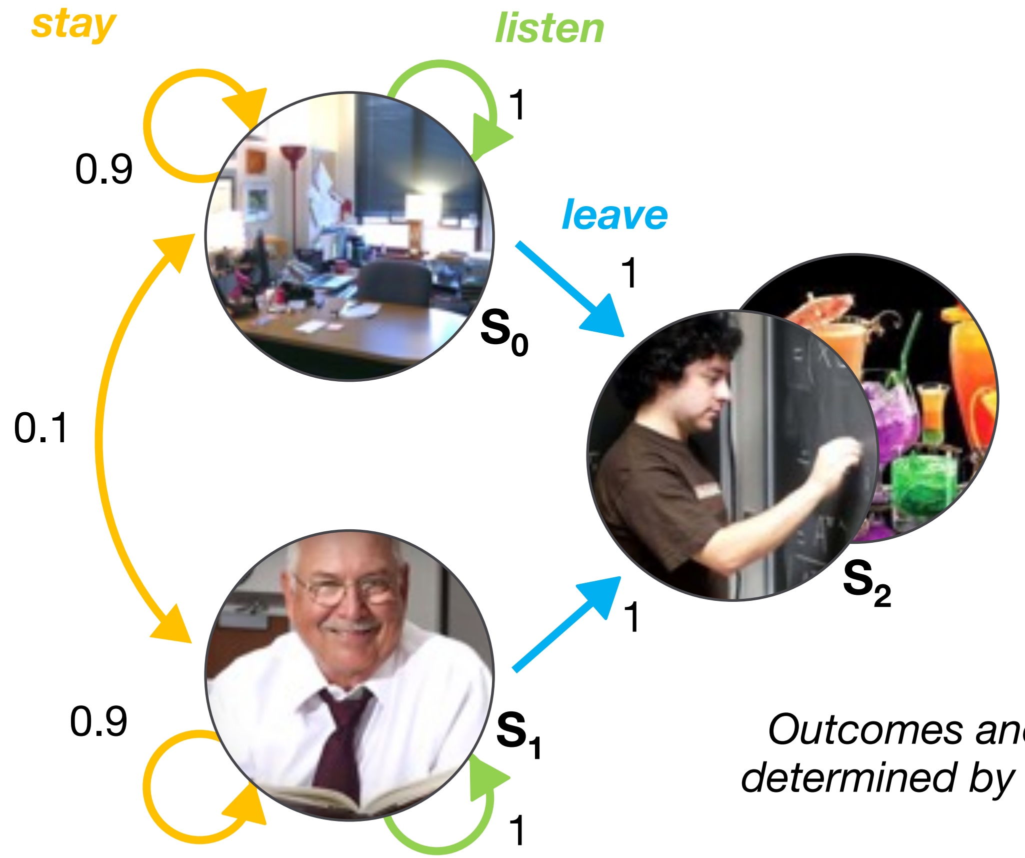
state space



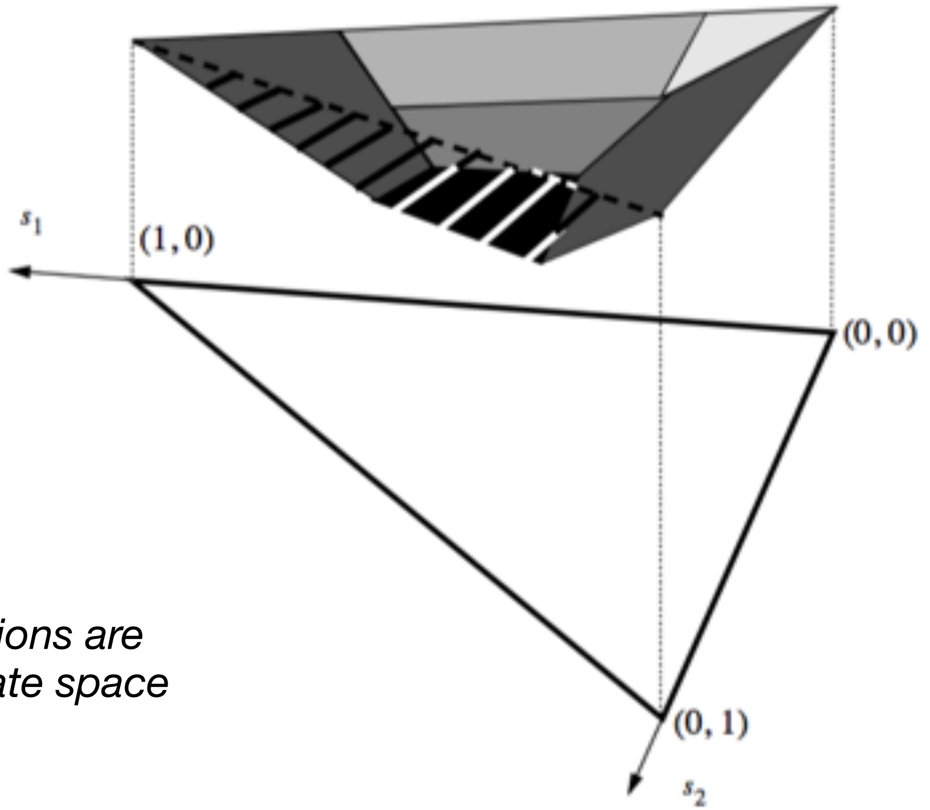
belief space



state space



belief space



Outcomes and observations are determined by the real state space

Policy relies on the belief state

POMDP Formalism

- *MDP*
  - S set of states
  - A set of actions
  - T transition matrix $S \times A \rightarrow S$
  - R reward function $S \times A \rightarrow \mathbb{R}$
  - γ discount factor

  POMDP extension
  - Ω set of observations
  - O observation probabilities
    $S \times A \times \Omega \rightarrow [0, 1]$
  - B belief space
  - r reward function $B \times A \rightarrow \mathbb{R}$
  - τ belief update function $B \times A \times \Omega \rightarrow B$

$$V^\pi(b) = \sum_{t=0}^{\infty} \gamma^t \, r(b_t, a_t)$$

$$\pi^\star = \underset{\pi}{\operatorname{argmax}} \, V^\pi$$

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Simulation workflow

Initial state (s, b)
- Select action $a = \pi(b)$
- Update state $s' = T(s, a)$
- Receive outcome $R(s, a)$
- Get observation o $= O(s', a)$
- Update belief $b' = \tau(b, a, o)$
- Start over

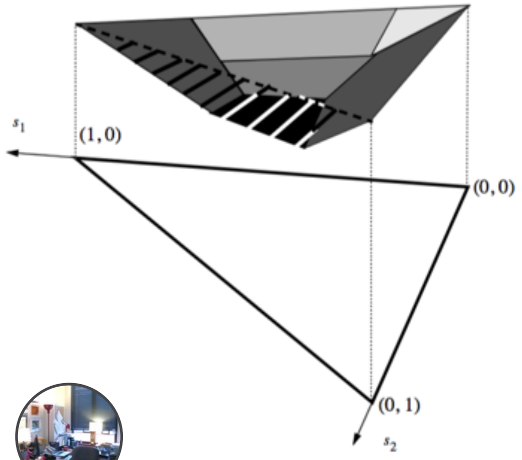
$$V^{\pi}(b) = \sum_{t=0}^{\infty} \gamma^t \, r(b_t, a_t)$$

$$\pi^{\star} = \underset{\pi}{\operatorname{argmax}} V^{\pi}$$

Resolution

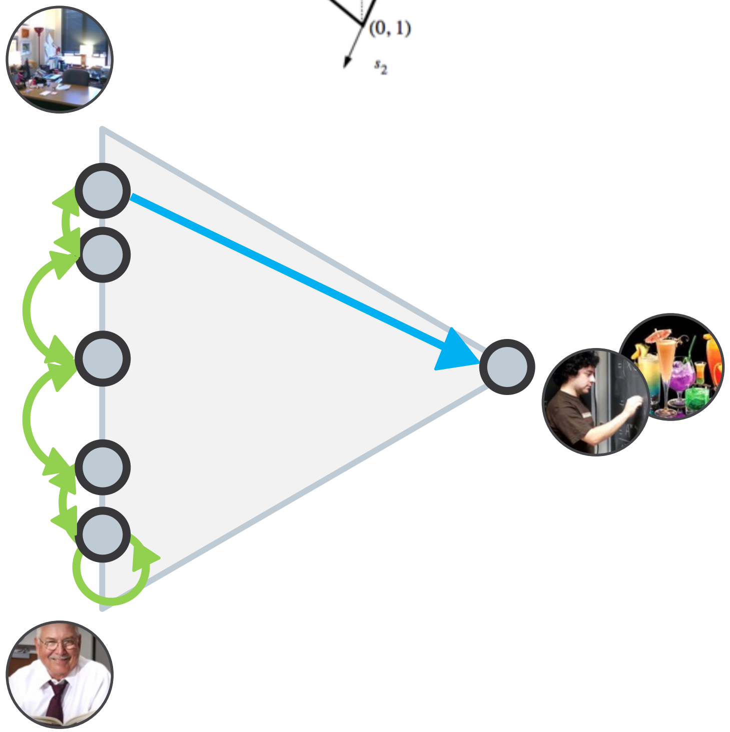
The value function is always convex

- Certainty is preferable to uncertainty
- Gathering information is valuable



The solution can be discretized

- Optimal solution often visit a finite number of belief states
- The POMDP can then be reformulated as a (fully observable) MDP



Take home message

POMDPs allow to model:

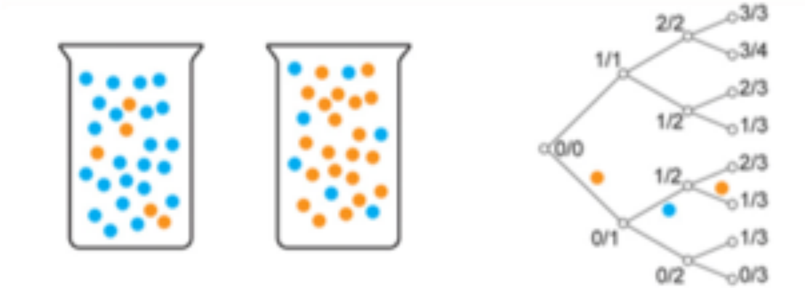
- sequential decision making in a complex, evolving environment (MDP)
- subjectivity about the state of the world (PO)

POMDPs can capture:

- information gathering as an economic decision
- irrational behaviour as an optimal policy based on wrong representations

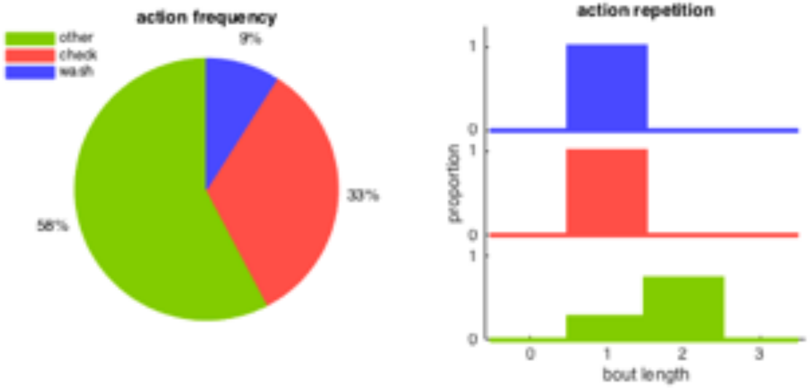
Perspectives

Information sequential sampling with varying payoffs



Errors as exploratory behaviour in reversal learning tasks

Checking behaviours in OCD



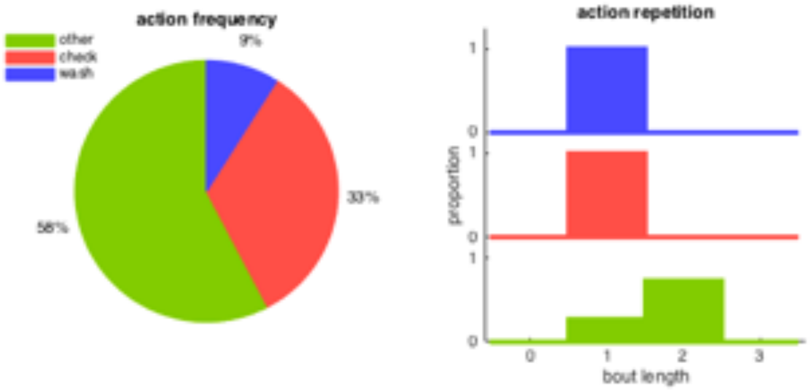
Perspectives

Information sequential sampling with varying payoffs

[Averbeck 2015, PCB]

Errors as exploratory behaviour in reversal learning tasks

Checking behaviours in OCD



Questions?

The story, characters, and incidents portrayed in this presentation are fictitious. No identification with actual persons, places, and buildings is intended or should be inferred.

Thank you for your attention