A beginner's guide to Active Inference

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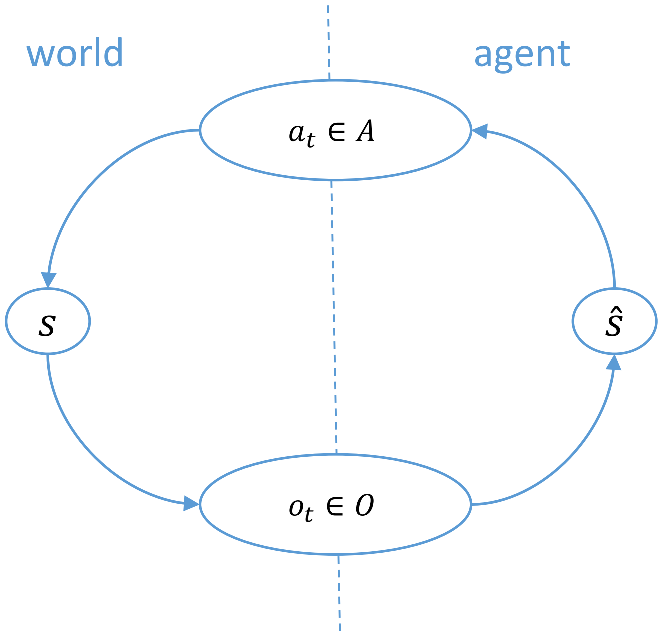
Inferring the latent structure of the world

To make sense of the world, we need to infer its latent structure (hidden causes)

v. Helmholtz, 1867

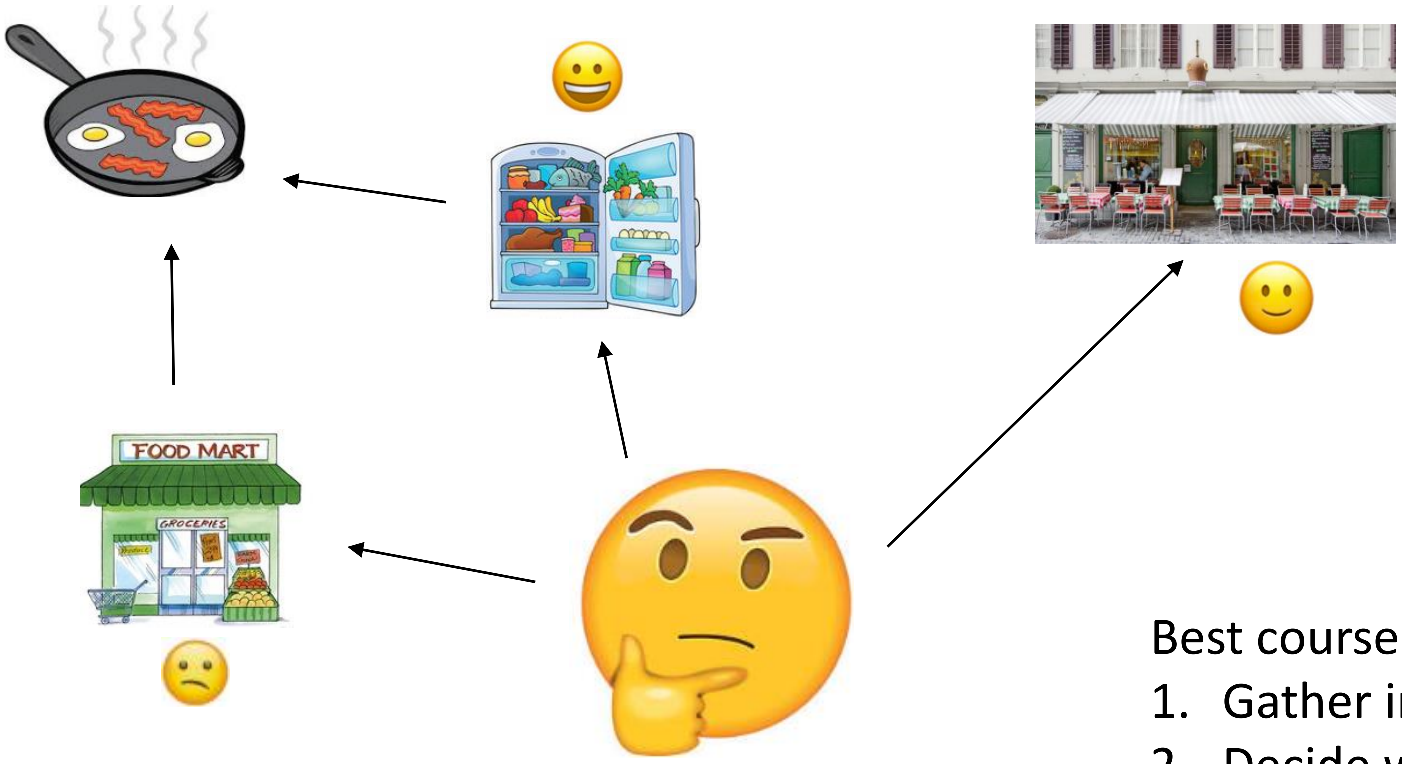


Bayesian Brain, Good Regulator Theorem (Conant & Ashby, 1947)..

"…[a human mental model] is a working physical model which works in the same way as the process it parallels"
(Kenneth Craik, *The Nature of Explanation*, 1943)

Inferring the latent structure of the world - to behave adaptively!

More contemporary example: make breakfast in Airbnb in Zurich



Best course of action:

1. Gather information (in the fridge)
2. Decide what to do

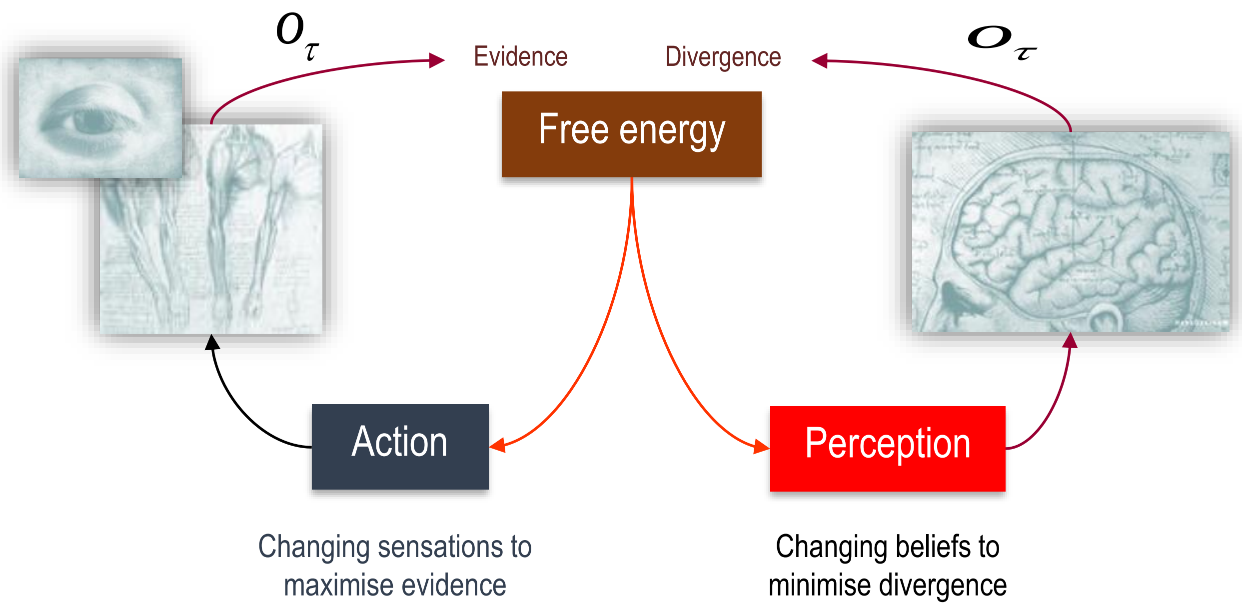
Inferring the latent structure of the world
- to behave adaptively!

$$F = \ln P(o_t) - D[Q(s_t)||P(s_t|o_t)]$$

Central idea:

Action fulfils prior expectations
= preferences

⇔ minimise *surprise*



Outline

1. Active Inference: basic building blocks
   i. Markov Decision Processes
   ii. Generative models
   iii. Information theory
   iv. Variational Bayes/free energy

2. Active Inference: computational framework

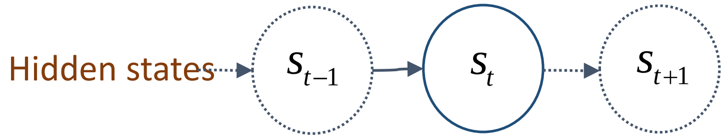
Active inference: basic building blocks

I. Markov Decision Processes

Background: **partially observable Markov decision processes (POMDP)**

Key ingredients:

- $1, \ldots, \mathrm{T}$ discrete time-steps



I. Markov Decision Processes

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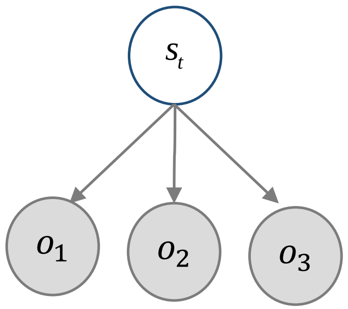
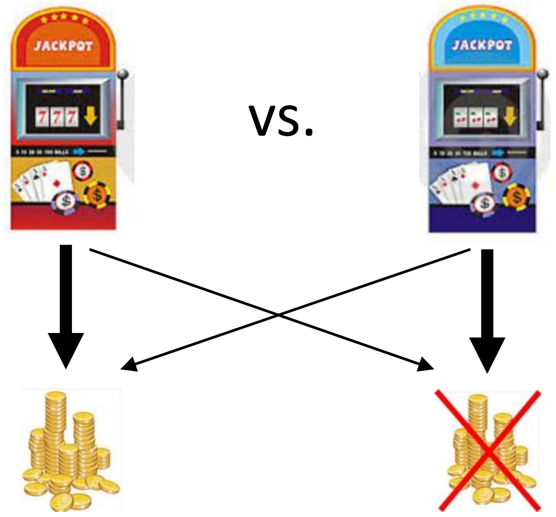
- $1, \ldots, T$ discrete time-steps
- $P(o_t|s_t)$ not trivial





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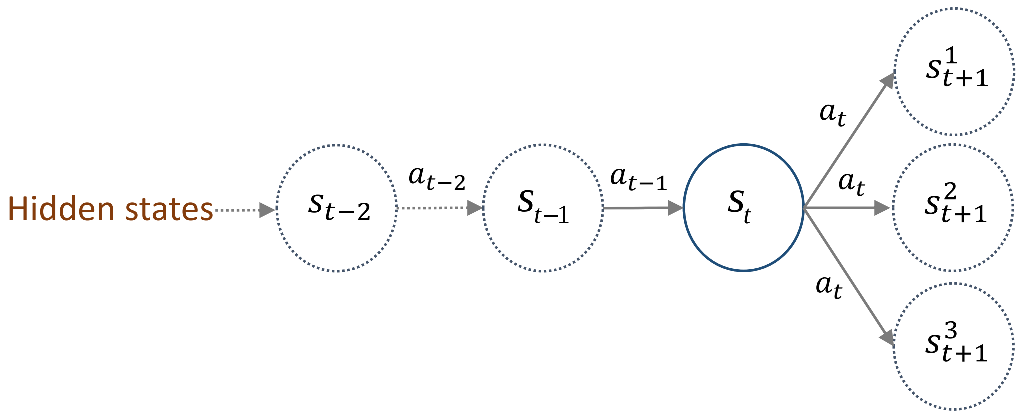


- Markov-property: $P(s_{t+1}|s_t, a_t) = P(s_{t+1}|s_t, a_t, s_{t-1}, a_{t-1}, s_{t-2}, a_{t-2}, \ldots)$

II. Generative models and hidden states

Generative models are probabilistic mappings based on a **likelihood function** and a **prior density**:

$$p(y, \theta | m) = p(y | \theta, m) p(\theta, m) \qquad [\Leftrightarrow p(o_t, s_t) = p(o_t | s_t) p(s_t)]$$

- Allows to *generate* data by sampling from prior and plugging into likelihood (‘**forward model**’, Stephan & Mathys, 2014)

Perform inference on hidden states by applying Bayes rule (‘**model inversion**’):

$$p(\theta | y, m) = \frac{p(y | \theta, m) p(\theta | m)}{p(y | m)}$$

Active inference emphasises that **agents build generative models** of the world

- Use them to perform inference on perception and behaviour

III. Information theory

Information theory quantifies the information content of a signal

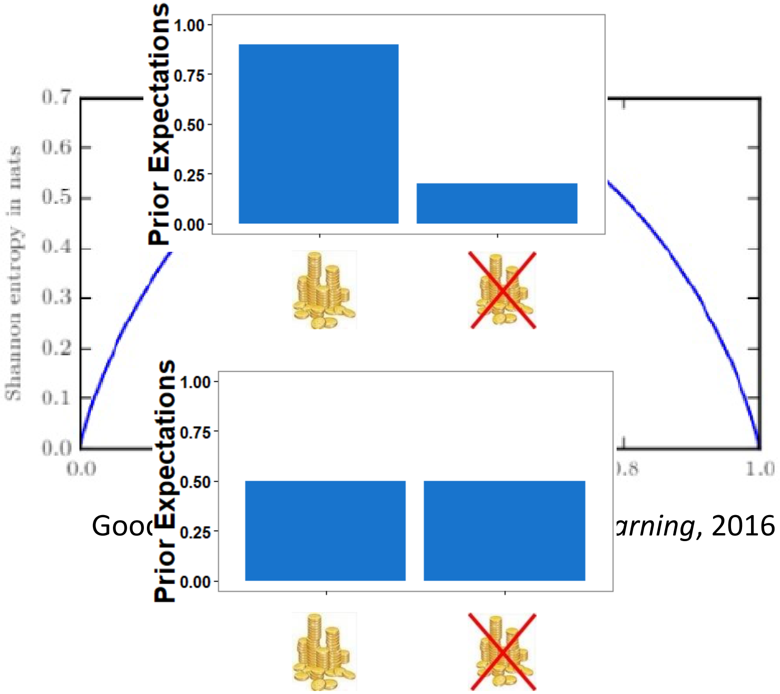
- Unlikely events are more informative than likely events

This can be quantified as the **self-information** or **surprise** of a signal *(Shannon, 1948)*:

$$I(y) = -logP(y|m)$$

Expected value of surprise is called (**Shannon**) **entropy**:

$$H(y) = \mathbb{E}[I(y)] = -\sum_i p(y_i|m) \cdot \log p(y_i|m)$$



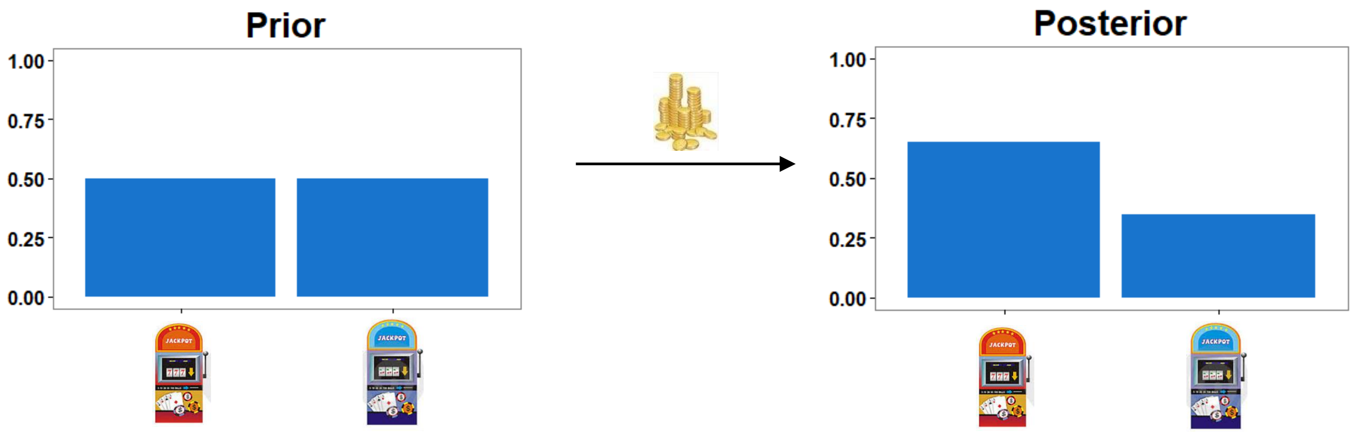
- Expected amount of information of an event (important later!)
- Also: measure of uncertainty

III. Information theory

Kullback-Leibler divergence measures difference between two distributions:

$$D_{KL}(P||Q) = \mathbb{E}[\log P(y|m) - \log Q(y|m)] = \sum_i P(y_i|m) \cdot \log\left[\frac{P(y_i|m)}{Q(y_i|m)}\right]$$

Example: quantify shift in beliefs from prior to posterior



$$D_{KL}(Posterior||Prior) = \sum Posterior \cdot \log\left[\frac{Posterior}{Prior}\right]$$

Active inference is heavily **grounded in information theory**, and argues that it has a lot to say about how brains work and how we behave.

IV. Variational Bayes

Exact inference is generally intractable

$$p(\theta|y\,,m) = \frac{p(y|\theta,m)\cdot p(\theta|m)}{p(y|m)} = \frac{p(y|\theta,m)\cdot p(\theta|m)}{\int p(y|\theta,m)\cdot p(\theta|m)}$$

Variational Bayes allows to cast an inference problem (difficult) as a bound optimisation problem (easier) *(cf., Beal, 2003; Bogacz 2017)*

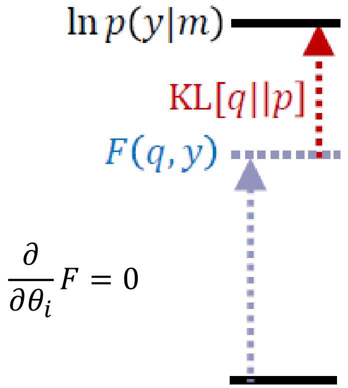
- Aim: approximate model evidence in Bayesian inference

Allows to derive specific variational update equations for a given problem

- Maths is not trivial – but only has to be done once! (By mathematicians, evolution,…)
- Then implement resulting update equations

Provides hypotheses about neuronal implementation of Bayesian inference

IV. Variational Bayes: free energy



Negative variational free energy provides a **lower bound on model evidence**

General formula:

$$F = \log p(y|m) - D_{KL}[q(\theta), p(\theta|y,m)]$$

Note: **negative log-model evidence** is the same as **information-theoretic surprise** $-\log P(y|m)$!

Active inference: basics

I. Choice behaviour is modelled as **partially observable Markov Decision Process**
   - Highlights importance of inferring hidden states *and* adequate choice behaviour

II. Inference on hidden states and choices is based on an agent's **generative model** of the world

III. **Information theory** allows to quantify how informative/expected an observation is, and how much agents shift their beliefs
   - Surprise = negative (log-) model evidence!

IV. Inference is **approximated** based on (**negative**) **variational free energy**
   - Approximates model evidence (surprise!)

Active Inference: model structure

A Markovian generative model - the ABC of choice behaviour

$$p(y,\theta) = p(y|\theta)p(\theta)$$

$$P(\tilde{o},\tilde{s},\pi,\gamma) = P(\tilde{o}|\tilde{s})P(\tilde{s}|\pi)P(\pi|\gamma)P(\gamma)$$

$P(\tilde{o}|\tilde{s}) = P(o_0|s_0)P(o_1|s_1)\ldots P(o_t|s_t)$

$P(o_t|s_t) =: \mathbf{A}$

Likelihood

$P(\tilde{s}|\pi) = P(s_t|s_{t-1},\pi)\ldots P(s_1|s_0,\pi)\ P(s_0)$

$P(s_{t+1}|s_t,\pi) =: \boldsymbol{B}(u = \pi(t))$

$P(s_0) =: \boldsymbol{D}$

$P(o_t) =: \sigma(\boldsymbol{C})$

Empirical priors – hidden states

$P(\pi|\gamma) = \sigma(-\gamma \cdot \boldsymbol{Q})$

– control states

$$\boldsymbol{Q} = \sum_t \mathbb{E}[\ln P(o_t|s_t) - \ln Q(o_t|\pi) + \ln P(o_t)]$$

$P(\boldsymbol{A}) = Dir(a)$ $\quad P(\boldsymbol{D}) = Dir(d)$

$P(\boldsymbol{B}) = Dir(b)$ $\quad P(\gamma) = \Gamma(1,\beta)$

Full priors

$Q(\tilde{s},\pi,A,B,D,\gamma) = Q(s_1|\pi)\ldots Q(s_T|\pi)Q(\pi)Q(A)Q(B)Q(D)Q(\gamma)$

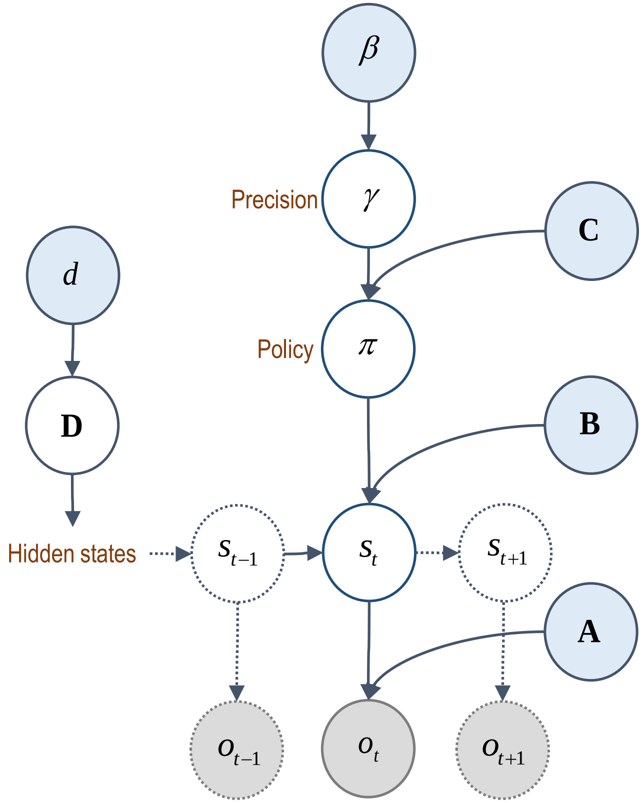
$Q(s_t|\pi) = Cat(\boldsymbol{s_t})$

$Q(\pi) = Cat(\boldsymbol{\pi})$

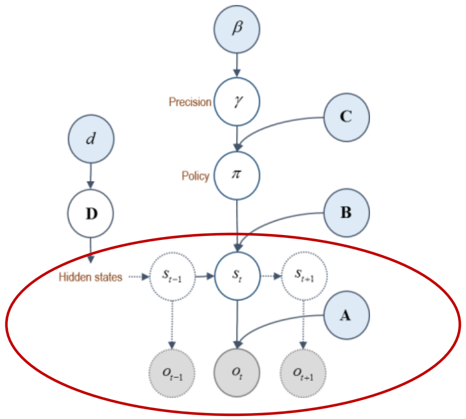
$Q(\boldsymbol{A}) = Dir(\boldsymbol{a})$ $\quad Q(\boldsymbol{D}) = Dir(\boldsymbol{d})$

$Q(\boldsymbol{B}) = Dir(\boldsymbol{b})$ $\quad Q(\gamma) = \Gamma(1,\boldsymbol{\beta})$

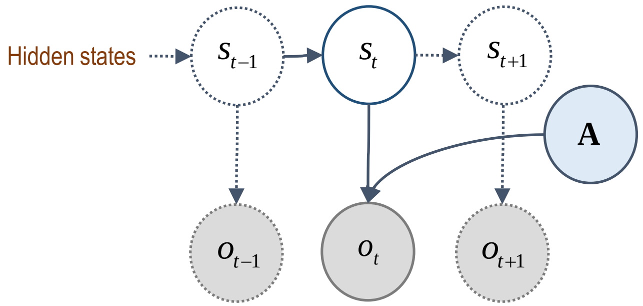
Approximate posterior



A-matrix: observation function



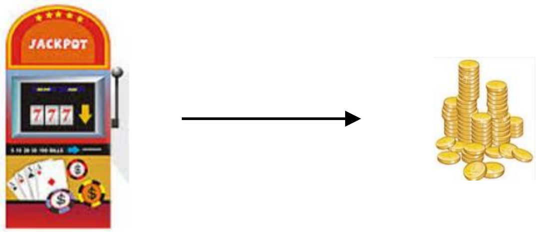
$$P(\tilde{o}, \tilde{s}, \pi, \gamma) = P(\tilde{o}|\tilde{s})P(\tilde{s}|\pi)P(\pi|\gamma)P(\gamma)$$



$P(\tilde{o}|\tilde{s})= P(o_0|s_0)P(o_1|s_1) \dots P(o_t|s_t)$

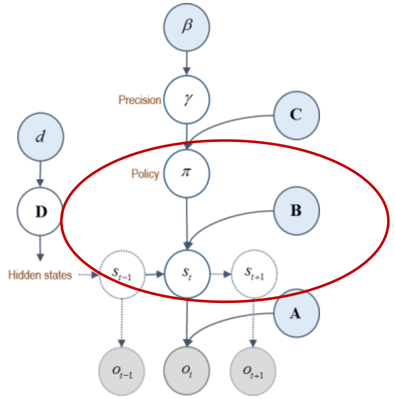
$P(o_t|s_t) =: \mathbf{A}$

Likelihood



B-Matrix: Markovian transition probabilities

$$P(\tilde{o}, \tilde{s}, \pi, \gamma) = P(\tilde{o}|\tilde{s})P(\tilde{s}|\pi)P(\pi|\gamma)P(\gamma)$$



$P(\tilde{s}|\pi) = P(s_t|s_{t-1}, \pi) \ldots P(s_1|s_0, \pi)\ P(s_0)$

$P(s_{t+1}|s_t, \pi) =: \boldsymbol{B}(u = \pi(t))$

$P(s_0) =: \boldsymbol{D}$

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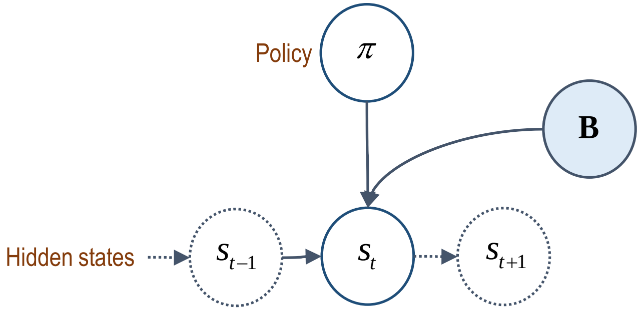
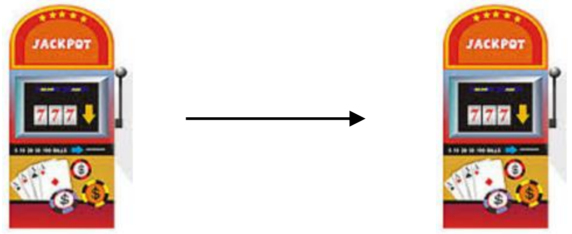
Empirical priors – hidden states

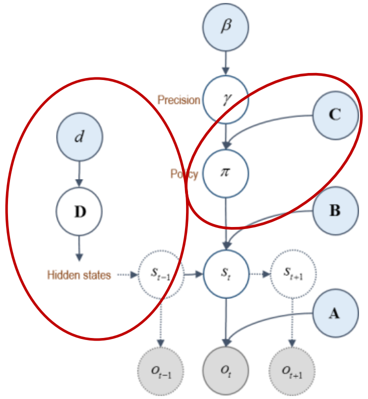




t t+1

Prior over outcomes (**c**-vector) and initial states (**d**-vector)

$$P(\tilde{o}, \tilde{s}, \pi, \gamma) = P(\tilde{o}|\tilde{s})P(\tilde{s}|\pi)P(\pi|\gamma)P(\gamma)$$



$P(\tilde{s}|\pi) = P(s_t|s_{t-1}, \pi) \dots P(s_1|s_0, \pi)\ P(s_0)$
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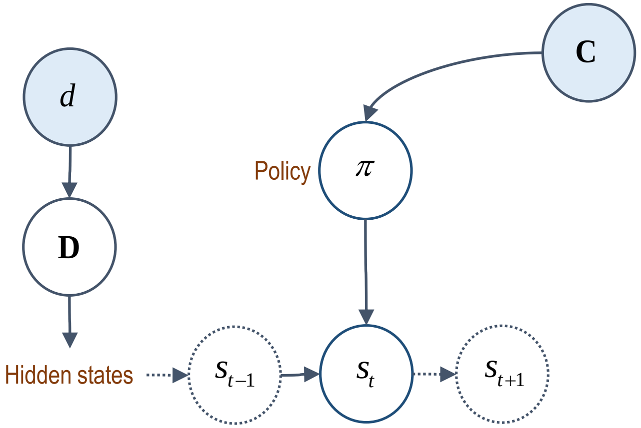

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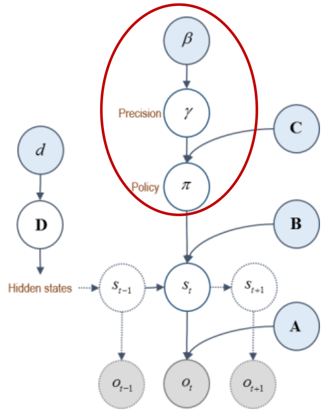
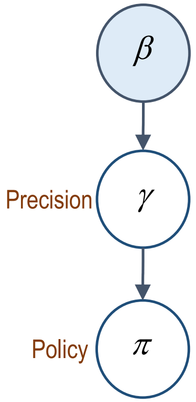
Precision γ

$$P(\tilde{o}, \tilde{s}, \pi, \gamma) = P(\tilde{o}|\tilde{s})P(\tilde{s}|\pi)P(\pi|\gamma)P(\gamma)$$





$P(\pi|\gamma) = \sigma(-\gamma \cdot \boldsymbol{Q})$ – control states

$$\boldsymbol{Q} = \sum_t -\mathbb{E}\,[\mathrm{H}[P(o_t|s_t)]] + H[Q(o_t|\pi)] + \mathbb{E}[lnP(o_t)]$$

$P(\boldsymbol{A}) = Dir(a)$ $\quad P(\boldsymbol{D}) = Dir(d)$

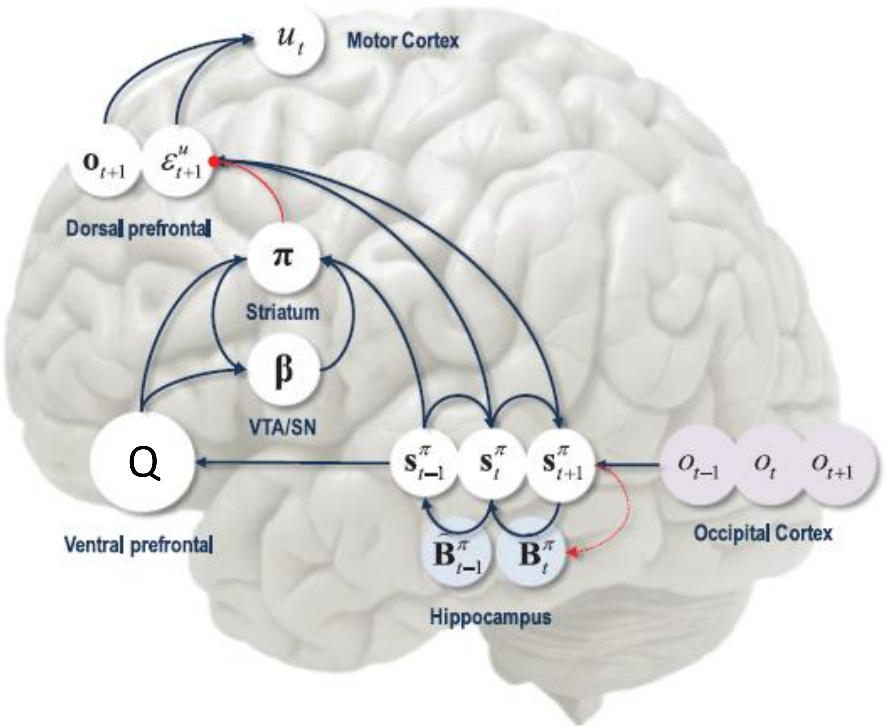
$P(\boldsymbol{B}) = Dir(b)$ $\quad P(\gamma) = \Gamma(1, \beta)$ Full priors

$gamble?\ don't\ gamble?$

Inference

Belief updating

Perception	$\hat{s}_t = \sigma(\log A \cdot o_t + \log B^{\pi}_{t-1} \cdot \hat{s}_{t-1})$	$\frac{\partial}{\partial_s} F$
Policy selection	$\hat{\pi} = \sigma(-\gamma \cdot Q)$	$\frac{\partial}{\partial_\pi} F$
Precision	$\hat{\beta} = \beta - \hat{\pi} \cdot Q$	$\frac{\partial}{\partial_\gamma} F$



Derive via variational free energy with respect to hidden states $x = \{s_t, \pi, \beta\}$:

$$Q(x) = \underset{Q(x)}{\arg\min} F(\tilde{o}, x, Q) \approx P(x|\tilde{o})$$

$$F(\tilde{o}, x, Q) = -E_{Q(\tilde{s}_t,\pi,\beta)}[\ln P(\tilde{o}, \tilde{s}_t, \pi, \beta | m)] - E_{Q(\tilde{s}_t,\pi,\beta)}[\ln Q(\tilde{s}_t, \pi, \beta)]$$

$$= \hat{s}_t \cdot (\ln \hat{s}_t - \ln A \cdot o_t - \ln B^{\pi}_{t-1} \cdot \hat{s}_{t-1}) + \cdots$$

cf., Friston et al., 2015, 2017; Bogacz, 2017

Inference: a closer (more conceptual) look

Belief updating

Perception

$$\hat{s}_t = \sigma(\log A \cdot o_t + \log B^{\pi}_{t-1} \cdot \hat{s}_{t-1})$$

"Likely **hidden states** given what I **observe**" → $\log A \cdot o_t$

"Based on what I believe about the **previous state**" → $\log B^{\pi}_{t-1} \cdot \hat{s}_{t-1}$

Policy selection

$$\hat{\pi} = \sigma(-\gamma \cdot Q)$$

"**Confidence**" → γ

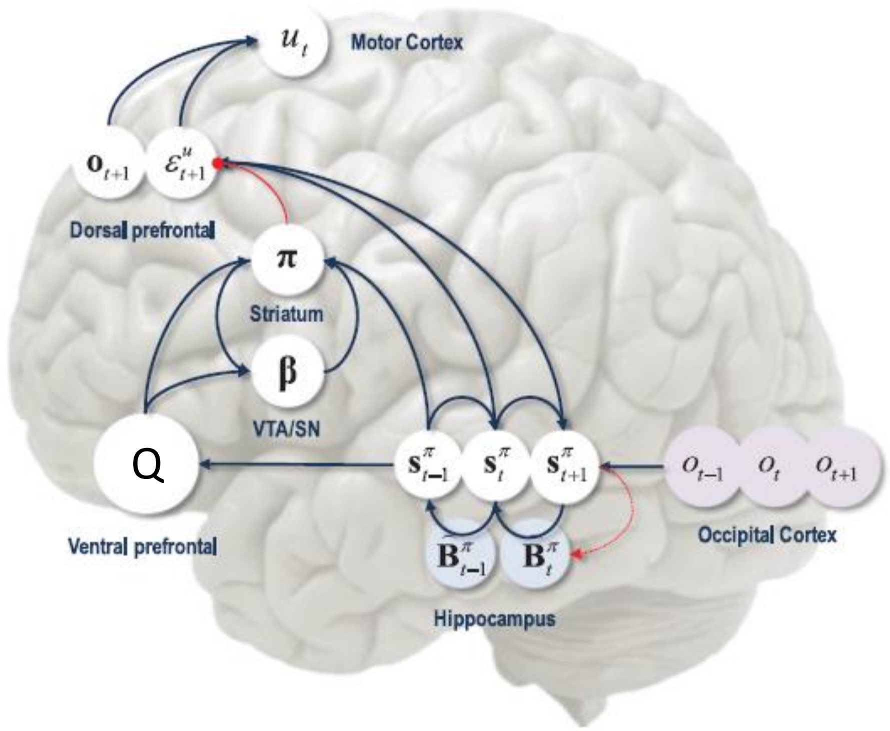
"**Value** of policies" → Q

Precision

$$\hat{\beta} = \beta - \hat{\pi} \cdot Q$$

"**Prior** on precision" → β

"**Value** of **inferred** policy" → $\hat{\pi} \cdot Q$



cf., Friston et al., 2015, 2017

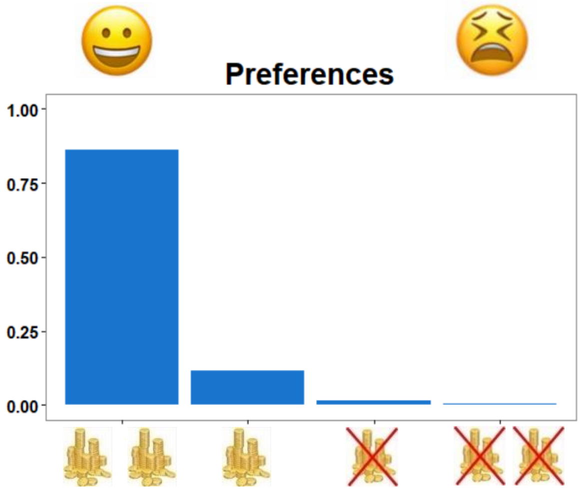
Inference: a closer look on policy selection

Policies become valuable if they

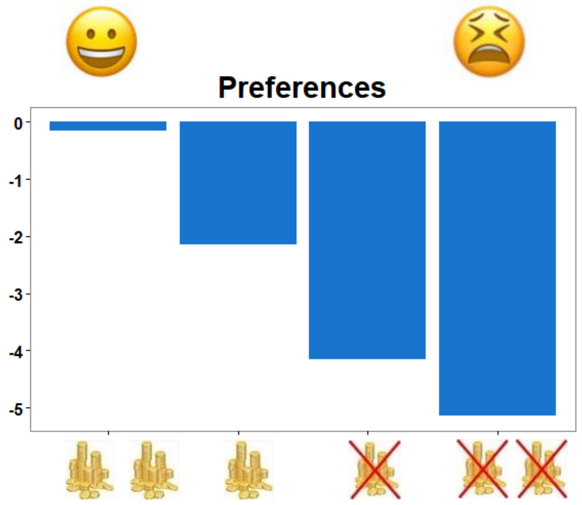
- Maximise reward/utility
- Allow us to minimise uncertainty about the world
  - Keep options open
  - Learn about hidden states in the world

Utilities or preferences are defined as log-expectations over outcomes:

$P(o_t|m)$



$\longrightarrow \log P(o_t|m)$



Fulfilling these preferences minimises surprise $-\log P(o_t|m)$!

- This can be approximated with variational free energy!

Inference: a closer look on policy selection

Values of policies $\boldsymbol{Q(\pi)}$ defined as expected free energy:

$Q(\tilde{s}, \pi, A, B, D, \gamma) = Q(s_1|\pi) \ldots Q(s_T|\pi)\boldsymbol{Q(\pi)}Q(A)Q(B)Q(D)Q(\gamma)$

Approximate posterior

$$\boldsymbol{Q(\pi)} = \sum_t \mathbb{E}_Q[\ln P(o_\tau, s_\tau\, |\pi) - \ln Q(s_\tau|\pi)]$$
$$= \cdots$$
$$= \sum_t -\mathbb{E}_Q[\mathrm{H}[P(o_t|s_t)]] - D_{KL}[Q(o_t|\pi)||\, P(o_t)]$$

Friston et al., 2015 Appendix A

$$\boldsymbol{Q(\pi)} = \sum_t \underbrace{\underbrace{-\mathbb{E}\,[\mathrm{H}[P(o_t|s_t)]]}_{\substack{\text{Expected uncertainty} \\ (H\,=0\,\Leftrightarrow o_t\,=s_t)}} + \underbrace{H[Q(o_t|\pi)]}_{\substack{\text{Entropy over} \\ \text{outcomes}}}}_{\text{Epistemic or intrinsic value}} + \underbrace{\underbrace{\mathbb{E}[lnP(o_t)]}_{\text{Expected utility}}}_{\text{Extrinsic value}}$$

Friston et al., 2015; 2017; Parr & Friston, 2017

Inference: a closer look on policy selection

Imagine you are a Tennis player and your opponent serves…

You want to find a position where…

Expected utility: $\mathbb{E}[lnP(o_t)]$

...there is a high chance of getting the ball, …

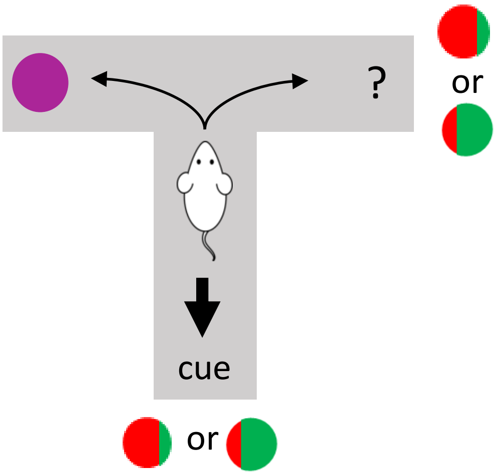
Entropy over outcomes: $H[Q(o_t|\pi)]$

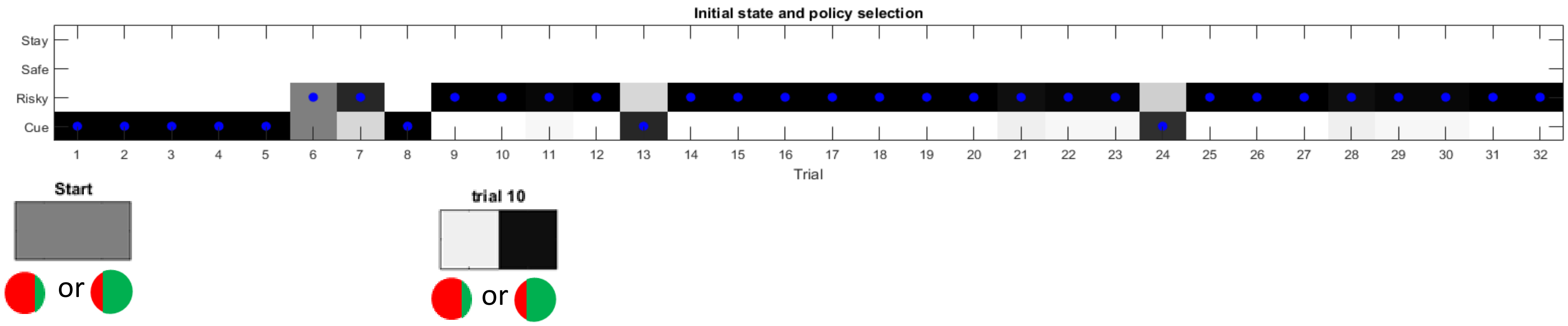
... you have a good chance of getting a ball that was unexpected, …

...and where you can minimise uncertainty about the character of your opponent.

Ambiguity: $-\mathbb{E}[\mathrm{H}[P(o_t|s_t)]]$

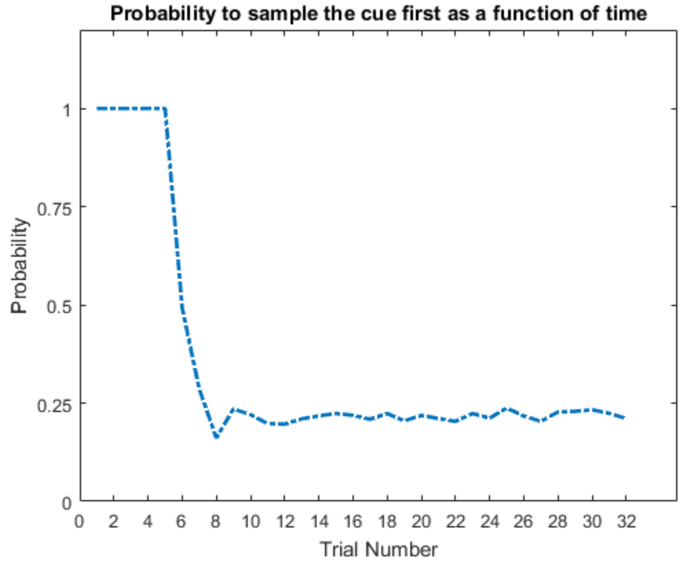
Active Inference: ‘hidden sate exploration’





$$\boldsymbol{Q}(\boldsymbol{\pi}) = \underbrace{-\mathbb{E}_Q[\mathrm{H}[P(o_t|s_t)]]}_{\text{Minimising uncertainty}} - \underbrace{D_{KL}[Q(o_t|\pi)||\,P(o_t)]}_{\text{Realising preferences}}$$

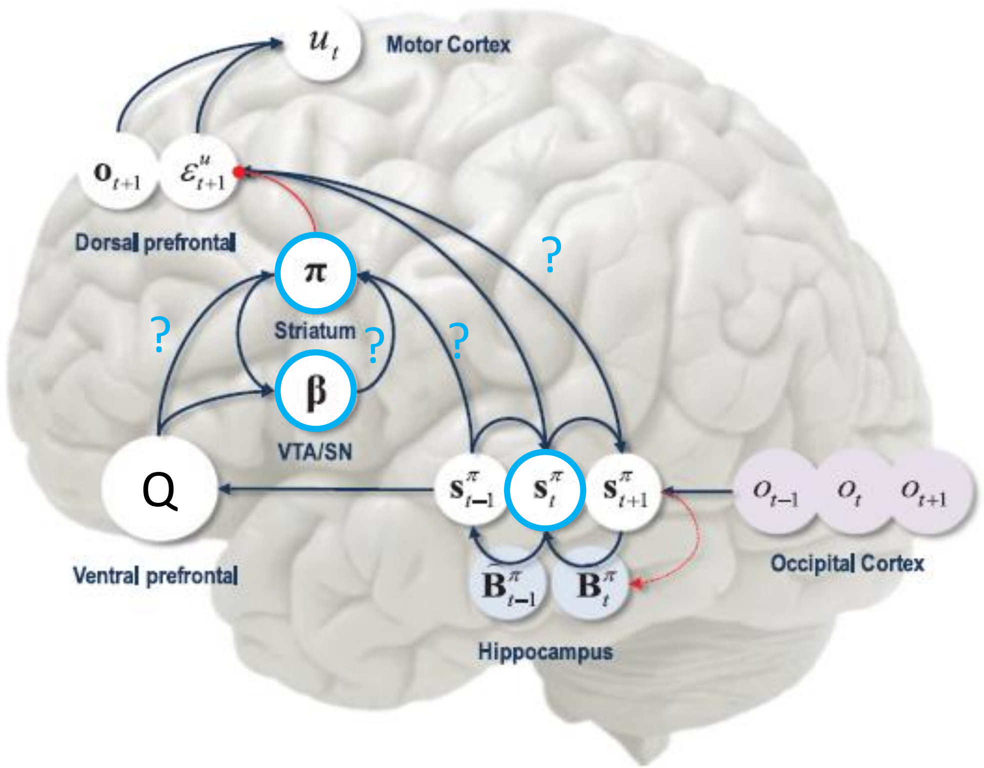
Exploring hidden states



Schwartenbeck et al., 2018, bioRxiv

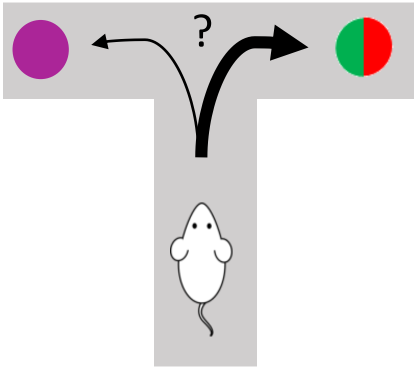
Active inference and *active learning*

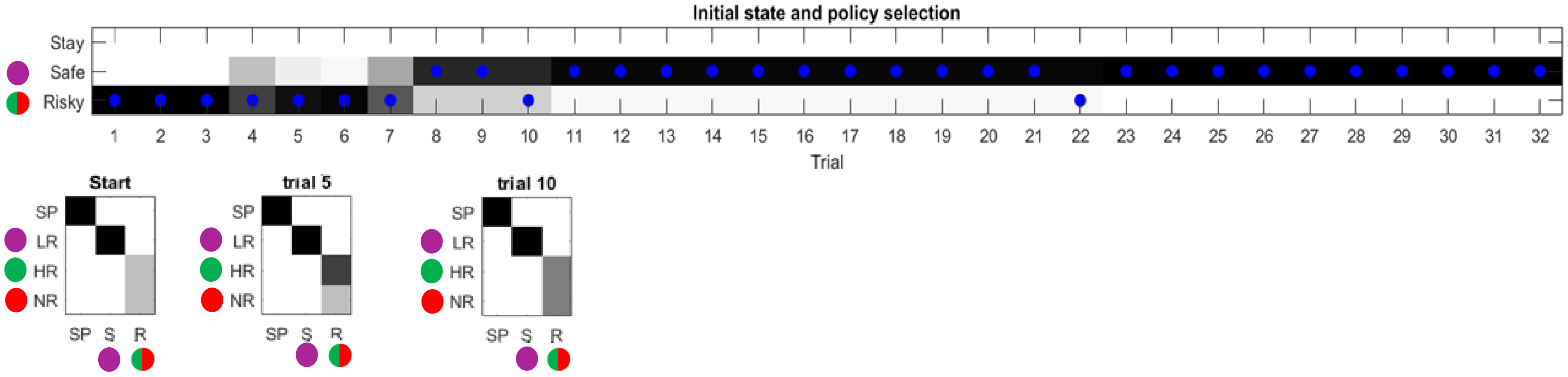
We use generative models to perform inference on hidden states.



But we also learn and update our models.

Active learning: 'model parameter exploration'





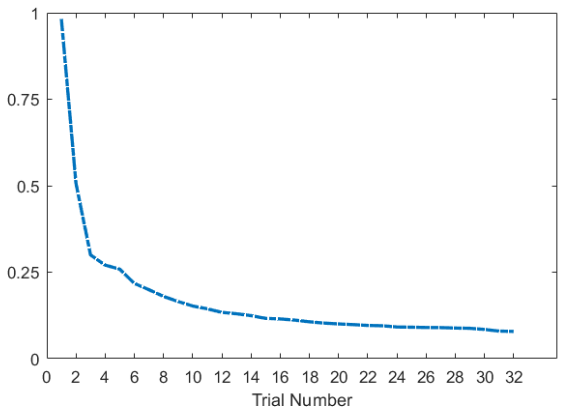
$$Q(\pi) = \mathbb{E}_{Q(o_\tau, s_\tau, A|\pi)}[\ln Q(s_\tau, A|\pi) - \ln P(o_\tau, s_\tau, A|\pi)]$$

$$= \cdots$$

$$= \underbrace{-\mathbb{E}_{Q(o_\tau, s_{\tau,A}|\pi)}[H[P(o_\tau|s_\tau)]]}_{\text{Minimising uncertainty}} - \underbrace{D_{KL}[Q(o_\tau|\pi)||P(o_\tau)]}_{\text{Obtaining preferred outcomes}} + \underbrace{\mathbb{E}_{Q(o_\tau, s_\tau, A|\pi)}[\ln Q(A) - \ln Q(A|s_\tau, o_\tau, \pi)]}_{\text{Expected model-update}}$$

Exploring model parameters

Prob choose risky as a function of time



Schwartenbeck et al., 2018, bioRxiv

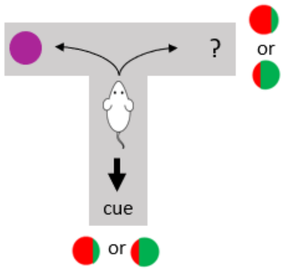
Active inference and *active learning*

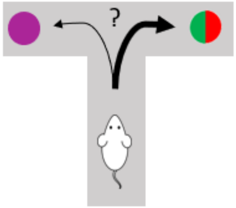
Goal-directed exploration can reflect active inference or active learning *(cf., Wilson et al., 2014; Gershman, 2018)*

$$
\begin{aligned}
Q(\pi) &= \mathbb{E}_{Q(o_\tau, s_\tau, A|\pi)}[\ln Q(s_\tau, A|\pi) - \ln P(o_\tau, s_\tau, A|\pi)] \\
&= \cdots \\
&= \underbrace{-\mathbb{E}_{Q(o_\tau, s_\tau, A|\pi)}[H[P(o_\tau|s_\tau)]]}_{\text{Minimising uncertainty}} - \underbrace{D_{KL}[Q(o_\tau|\pi)||P(o_\tau)]}_{\text{Obtaining preferred outcomes}} + \underbrace{\mathbb{E}_{Q(o_\tau, s_\tau, A|\pi)}[\ln Q(A) - \ln Q(A|s_\tau, o_\tau, \pi)]}_{\text{Expected model-update}}
\end{aligned}
$$

Exploring hidden states

Exploring model parameters





Schwartenbeck et al., 2018, bioRxiv

Computational Phenotyping in active inference

All models are wrong, but some are useful - for understanding how things can break

Failures in inference

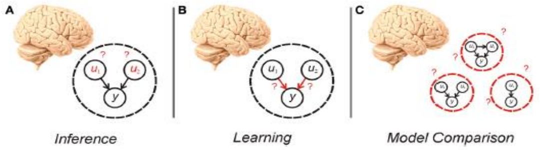
- 'Suboptimal' priors induce pathological behaviour based on 'optimal' inference
- E.g., 'suboptimal' preferences, transition probabilities, observation model, …, underlying inference

Failures in learning

- Computational basis of information gain about hidden states and model parameters
- E.g., inability to update after bad experiences, inability to optimise one's model

Failures in model building

- State space underlying inference and learning



cf., FitzGerald, Dolan & Friston, 2014; Huys, Guitart-Masip, Dolan & Dayan, 2015; Dayan, 2014

Other Resources

Blog by Oleg Solopchuk: https://medium.com/@solopchuk/tutorial-on-active-inference-30edcf50f5dc

Rafal Bogacz 2017 tutorial on free energy

Buckley, Kim, McGregor, & Seth 2018 tutorial on free energy

Parr, Rees, & Friston, 2018; Parr & Friston, 2017; Schwartenbeck and Friston, 2016

Take home messages

Active inference combines **probabilistic inference**, **Markov Decision Processes** and **information theory**

- Actions fulfil expectations ⇔ minimise surprise ⇔ maximise model evidence

Approximate inference takes place based on variational Bayes

- Inference on the current state, policy and confidence

Defining the value of policies as expected free energy predicts that agents try to

- Realise preferences (maximise utility)
- Solicit information from the world

Provides a computational framework for *active inference* and *active learning* - and how this might break

- Exploration of hidden states and model parameters

Thank you!

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