Linear Algebra Review

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Created from Kolter's Linear Algebra Review for Stanford CS229

Linear Equations, Notation, Matrix Multiplication

See blackboard

Identity/ Diagonal Matrix, Transpose, Symmetric Matrices, Trace, Norms

See blackboard

Linear Independence

- A set of vectors $\{x_1, x_2, \ldots x_n\} \subset \mathbb{R}^m$ is <u>linearly independent</u> if no vector can be written as a linear combination of the other vectors.

Linear Independence

- Conversely, the set is <u>linearly dependent</u> if *one vector can* be written as a linear combination of the remaining vectors, ie. if

$$x_n = \sum_{i=1}^{n-1} \alpha_i x_i$$

for some scalar values $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$.

- Example on board

Rank

- Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$.
  - The column rank of A is the number of linearly independent columns of A.
  - The row rank of a A is the number of linearly independent rows of A.
  - The row rank of A equals the column rank of A, and is called the rank of A.

Properties of Rank

- Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$.
  - rank(A) ≤ min(m,n)
  - If rank(A) = min(m,n), A has full rank.
  - rank(A) = rank(A^{T})
  - rank(AB) ≤ min(rank(A), rank(B))
  - rank(A+B) ≤ rank(A) + rank(B)

Inverse

- The <u>inverse</u> of a square matrix $A \in \mathbb{R}^{n \times n}$ is denoted by A^{-1} and is the unique matrix such that
$$A^{-1}A = I = AA^{-1}$$
- Not all matrices have inverses.
- A is <u>invertible</u> (or <u>non-singular</u>) if A^{-1} exists and is called <u>non-invertible</u> or <u>singular</u> otherwise.

Inverse

- A is invertible if it is square and has full rank.

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^T = (A^T)^{-1}$

- If A non-singular, $Ax=b$ has unique solution $x=A^{-1}b$

Orthogonality and Normalization

- Recall two vectors $x, y \in \mathbb{R}^n$ are orthogonal if

$$x^T y = 0$$

- A vector $x \in \mathbb{R}^n$ is normalized if

$$||x||_2 = 1$$

Orthogonal Matrices

- A square matrix $U \in \mathbb{R}^{n \times n}$ is orthogonal if all of its columns are orthogonal to each other and are normalized, or equivalently,

$$U^T U = I = U U^T$$

- Orthogonal matrices are norm (or distance) preserving, ie,

$$\|Ux\|_2 = \|x\|_2$$

for any $x \in \mathbb{R}^n$, $U \in \mathbb{R}^{n \times n}$.

Span and Basis

- The span of a set of vectors $\{x_1, x_2, \ldots x_n\} \subset \mathbb{R}^m$ is the set of all vectors that can be expressed as a linear combination of $\{x_1, \ldots, x_n\}$, ie,

$$\operatorname{span}(\{x_1, \ldots x_n\}) = \left\{ v : v = \sum_{i=1}^{n} \alpha_i x_i, \;\; \alpha_i \in \mathbb{R} \right\}$$

- If $\{x_1, \ldots, x_n\}$ is a set of *n* linear independent vectors then .

$$\operatorname{span}(\{x_1, \ldots x_n\}) = \mathbb{R}^n$$

- $\{x_1, \ldots, x_n\}$ is called a basis of $\mathbb{R}^n$.

Range and Nullspace

- The <u>range</u> of $A \in \mathbb{R}^{m \times n}$ is the span of the columns of A, ie,

$$\mathcal{R}(A) = \{v \in \mathbb{R}^m : v = Ax, x \in \mathbb{R}^n\}$$

- The <u>nullspace</u> of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all vectors that equal 0 when multiplied by A, ie,

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

Range and Nullspace

- The range of A^{T} and the nullspace of A are orthogonal complements in $\mathbb{R}^n$.

- This means they are

  - disjoint subsets which span the entire space, and

  - every vector in the range of A^{T} is orthogona
    every vector in the nullspace of A.

Projection

- The projection of a vector $y \in \mathbb{R}^m$ onto the span of $\{x_1, \ldots, x_n\}$ (with $x_i \in \mathbb{R}^m$) is the vector $v \in \text{span}(\{x_1, \ldots x_n\})$ closest to y as measured by Euclidean Distance (ie $\|v - y\|_2$).

- Formally,

$$\text{Proj}(y; \{x_1, \ldots x_n\}) = \text{argmin}_{v \in \text{span}(\{x_1, \ldots, x_n\})} \|y - v\|_2$$

Projection

- Assuming A is full rank and $n<m$, the projection of $y \in \mathbb{R}^m$ onto the range of A is

$$\text{Proj}(y; A) = \text{argmin}_{v \in \mathcal{R}(A)} \|v - y\|_2 = A(A^T A)^{-1} A^T y$$

- When A contains a single column, ie, $A = a \in \mathbb{R}^m$, we see the familiar case of a projection of a vector onto a line:

$$\text{Proj}(y; a) = \frac{aa^T}{a^T a} y$$

Determinant

- The determinant of a square matrix is a function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$, denoted $|A|$ or $\det A$.

- Let $a_1, \ldots, a_n \in \mathbb{R}^n$ denote the columns of A. Consider S the the restriction of $\mathrm{span}(\{a_1, \ldots, a_n\})$ to linear combinations whose coefficients satisfy

$$0 \leq \alpha_i \leq 1,\ i = 1, \ldots, n.$$

Determinant

- The absolute value of the determinant of A is a measure of the 'volume' of S.

- For example, for 2 x 2 matrices A, $|\det A|$ is a measure of the area enclosed by the parallelogram with edges the columns of A.

Properties of Determinant

- $|I| = 1$.

- If B is formed by multiplying a single row of A by a real number t, then
$$|B| = t|A|.$$

- If B is formed by switching ay two rows of A, then
$$|B| = -|A|.$$

Properties of Determinant

- $|A| = |A^{\mathrm{T}}|$.
- $|AB| = |A||B|$
- $|A| = 0$ if and only if A is singular (non-invertible).
- If A is invertible, then $|A^{-1}| = 1/|A|$.

Eigenvalues and Eigenvectors

- Given a square matrix $A \in \mathbb{R}^{n \times n}$, we say $\lambda \in \mathbb{C}$ is an eigenvalue of A and $x \in \mathbb{C}^n$ is the corresponding eigenvector if

$$Ax = \lambda x, \quad x \neq 0.$$

- Intuitively, this means multiplying A by x results in a new vector in the same direction as x but scaled by λ.

Eigenvalues and Eigenvectors

- Note that if *x* is an eigenvector, then *cx* is an eigenvector for any complex *c*.

- We can rewrite the equation above as

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

- $(\lambda I - A)x = 0$ has a non-zero solution iff $(\lambda I - A)$ has a non-empty nullspace, which only happens if $(\lambda I - A)$ is singular, ie,

$$|(\lambda I - A)| = 0.$$

E-values and E-vectors Properties

- The trace of A is equal to the sum of its eigenvalues,

$$\mathrm{tr} A = \sum_{i=1}^{n} \lambda_i .$$

- The determinant of A is equal to the product of its eigenvalues

$$|A| = \prod_{i=1}^{n} \lambda_i .$$

- The rank of A is equal to the number of non-zero eigenvalues of A.
- The eigenvalues of a diagonal matrix are just the diagonal entries.

Diagonalization

- We can write all the eigenvector equations simultaneously as
$$AX = X\Lambda\,.$$
with $X \in \mathbb{R}^{n\times n}$ the eigenvectors of A and a diagonal matrix Λ whose entries are the eigenvalues A, ie,

$$X \in \mathbb{R}^{n\times n} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Diagonalization

- If the eigenvectors of A are linearly independent, then X will be invertible, so

$$A = X\Lambda X^{-1}.$$

We say that A is <u>diagonalizable</u>.

What Just Happened?

- Linear Equations
- Matrices / Special Matrices
- Subspaces and Projection
- Eigenvalues and Eigenvectors

Halfway done with linear algebra!