UNIT - 2 COMBINATORICS

Topic 1:MATHEMATICAL INDUCTION

Working Rule of the Principle of Mathematical Induction:

Let P(n) be a statement .

To prove that a statement P(n) is true for all-natural numbers,
we must go through three steps.

Step 1: We must prove that P (1) is true.

Step 2: assume P (k) is true

Step 3: By assuming P (k) is true we must prove that P (k + l) is also true.

Note:
The condition (i) is known as the **Basic step** and
The condition (ii) (iii) are known as **Inductive step**

Problems:

1) Show that by mathematical induction 1+2+3+4+... n $= \frac{n(n+1)}{2}$

Solution: STEP (i): To prove P (1) is true.

$$P(1) = \frac{1(1+1)}{2} = 1 \text{ which is true.}$$

STEP (ii)

Assume that the result is true for P(k).That is

$$p(k) = 1+2+3+4+\ldots +k = \frac{k(k+1)}{2}$$

STEP (iii)

To prove p (k+1) is true

$$1+2+3+\ldots\ldots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k\,(k+1)+2(k+1)}{2}$$

$$p(k+1) = \frac{(k+1)(k+2)}{2}$$

Hence the result is derived using Mathematical induction method.

2) For all n ≥ 1, prove that $1^2+2^2+3^2+4^2+\ldots\ldots +n^2= \frac{n\,(n+1)\,(2n+1)}{6}$

Solution:

P(n): $1^2+2^2+3^2+4^2+\ldots\ldots+n^2= \frac{n\,(n+1)\,(2n+1)}{6}$

(i) To prove P (1) is true.

P (1) = $\frac{1(1+1)\,(2\times1+1)}{6}$ = 1 which is true.

(ii) Assume that P(k) is true for some positive integer k, i.e.,

$1^2+2^2+3^2+4^2+\ldots\ldots+k^2= \frac{k\,(k+1)\,(2k+1)}{6}$ --------- (1)

(iii) To prove P(k+1) is also true.

$(1^2+2^2+3^2+4^2+\ldots\ldots +k^2) + (k+1)= \frac{k\,(k+1)\,(2k+1)}{6} + (k+1)^2$ (Using (1))

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$$

Thus P(k+1) is true, wherever P(k) is true. Hence, from the principle of mathematicalinduction, the statement is true for all-natural number n.

3). Show that $n^3 + 2n$ is divisible by 3.

Solution:

Let P(n): n^3+2n is divisible by 3

(i) Toprove P (1) is true.

P (1): $1^3 + 2.1 = 3$ is divisible by 3 is true……………(1)

(ii) Assume

P(k): k^3+2k is divisible by 3

(iii) To prove: P(k+1) is true

Now, P(k+1): $(k+1)^3+2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= k^3 + 3k^2 + 3k + 2k + 3$

$= (k^3+2k) + 3(k^2+k+1)$-----------------------------------(2)

(using (l)) $k^3 + 2k$ is divisible by 3 and $3(k^2+k + l)$ is a multiple of 3, so

we have equation (2) is divisible by 3.

$\therefore$ P (k + l) is true.

By the principle of Mathematical induction, $n^3 + 2n$ is divisible by 3

4. Prove that $8^n - 3^n$ is a multiple of 5.

Solution:

Let P(n): $8^n - 3^n$ is a multiple of 5.

(i) To prove P (1) is true.

P (1) = $8^1 - 3^1 = 5$ is a multiple of 5 which is true.

(ii) Assume P (k) = $8^k - 3^k$ is a multiple of 5 is true.

i.e., $8^k - 3^k = 5m$ where $m \in Z^+$

$\Rightarrow 8^k = 5m+3$ ....(1)

(iii) Claim: P (k+1) is true.

Now, P(k+1) = $8^{k+1} - 3^{k+1}$

$= 8^k 8 - 3^k 3$

$= (5m + 3^k) .8 - 3^k.3$ (Using (1))

$= 5.\ 8m + 8.\ 3^k - 3.\ 3^k$

$= 5.\ 8m + 5.\ 3^k$

$= 5\ (8m + 3^k)$ Which is a multiple of 5 for all 'm'.

$\therefore$ P (k + 1) is true.

Hence, $8^n — 3^n$ is a multiple of 5 for all n.

5. Using mathematical induction prove that (3^n+7^n-2) is divisible by 8, for $n \geq 1$.

Solution:

Let P(n): (3^n+7^n-2) is a multiple of 8.

(i) To prove P (1) is true.

P (1) = $(3^1+7^1-2) \equiv 8$ which is divisible by 8 is true.

(ii) Assume P (k) = (3^k+7^k-2) is divisible by 8 is true.------------------- (1)

(iii)Claim: P (k+1) is true.

Now, P (k+1) = $3^{k+1}+ 7^{k+1}-2$

$=3^{k}.3 +7^{k}.7 - 2$

$=3(3^{k}+7^{k}- 2) +4(7^{k}+1)$ --- (2)

Now, 7^k+1 is an even number, fork ≥ 1.

$\therefore$ 4 (7^k + l) is divisible by 8.

Since 3 (3^k+7^k-2) is divisible by 8 (Using (l)) and 4 (7^k+ l) is divisible by 8, the RHS of (2) is divisible by 8.

$\therefore$P (k + l) is true.

Hence, P(n): (3^n+7^n-2) is a multiple of 8.

6. Show that a^n-b^n is divisible by (a-b).

Solution:

Let P(n): a^n-b^n is divisible by (a-b).

(i) To prove P (1) is true
P (1) = $a^1- b^1$ is divisible by (a-b) is true.

(ii) Assume P (k) = a^k-b^k is divisible by (a-b) is true………… (1)

(iii) Claim: P (k+1) is true.

Now, P (k+1) = $a^{k+1}-b^{k+1}$

$=a^k.a-b^k.b$

$= [m (a-b) +b^k] a-b^k.b$

$=am (a-b) +ab^k-bb^k$

$= (a-b) ma+ (a-b)b^k$

$= (a-b) [ma +b^k]$

Which is a multiple of (a-b).

$\therefore$ P (k+1) is true.

Hence, P(n): a^n-b^n is divisible by (a-b).

7) Prove that $2^n > n$ for all positive integers n.

Solution:

Let $P(n): 2^n > n$

a. To prove p(1) is true
when n=1, $p(1) = 2^1 > 1$.
Hence, P (1) is true

b. Assume
P (k) is true for any positive integer k,
i.e., $2^k > k$ ----------------------- (1)

(c) Claim: P (k+1) is true whenever P (k) is true.

Multiplying both sides of (1) by 2,

We get $2.2^k > 2k$
i.e., $2^{k+1} > 2k$
$2^k + k > k+1$
$\therefore$ P (k+1) is true when P (k) is true.

Hence, P(n): $2^n > n$ is true for every positive integer n.

8) Show that $2^n < n!$ for all $n \geq 4$.

Solution:

Let $P(n): 2^n < n!$

(i) To prove p (1) is true. Since $n \geq 4$.
P (4): $2^4 < 4!$ is true

(ii) Assume P (k): $2^k < k!$ is true. ------------------ (1)

(iii) Claim: P (k+1) is true whenever P(k) is true.
From (1) $2^k < k!$

Multiplying both sides of (1) by 2,

We get $2.2^k < 2.k!$

i.e., $2^{k+1} < (k+1)\, k!$

$= (k+1)!$

$2^k + k < k+1!$

$\therefore$ P (k+1) is true when P (k) is true.

Hence, P(n): $2^n < n!$ is true for all $n \geq 4$.

9) Using mathematical induction,Prove that $2+2^2+2^3+\ldots. +2^n=2^{n+1}-2$

Solution:

Let P(n): $2+2^2+2^3+\ldots. +2^n$

(i) To prove p(1) is true

P (1): $2^1=2^{1+1}-2$ is true.

(ii) Assume P(k): $2+2^2+2^3+\ldots. +2^k=2^{k+1}-2$ is true. ------------- (1)

(iii) Claim: P(k+1) is true.

P(k+1): $2+2^2+2^3+\ldots. +2^k+2^{k+1}$

$=2^{k+1} -2 +2^{k+1}$ ....(Using(1))

$=2.\, 2^{k+1} -2$

$=2^{k+2} -2$

$\therefore$ P(k+1) is true.

Hence, $2+2^2+2^3+\ldots. +2^n=2^{n+1}-2$ is true for all n.

10)Using mathematical induction,Prove that $2+2^2+2^3+\ldots. +2^n=2^{n+1}-2$

Solution:

Let P(n): $2+2^2+2^3+\ldots. +2^n$

(i) To prove p(1) is true

P (1): $2^1=2^{1+1}-2$ is true.

(ii) Assume P(k): $2+2^2+2^3+.... +2^k=2^{k+1}-2$ is true. ------------- (1)

(iii) Claim: P(k+1) is true.

P(k+1): $2+2^2+2^3+ .... +2^k+2^{k+1}$

$=2^{k+1} -2 +2^{k+1}$ ....(Using(1))

$=2.\ 2^{k+1} -2$

$=2^{k+2} -2$

$\therefore$ P(k+1) is true.

Hence, $2+2^2+2^3+.... +2^n=2^{n+1}-2$ is true for all n.

11) Show that $\frac{1}{1.2}+\frac{1}{2.3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

Solution:

Let P(n): $\frac{1}{1.2}+\frac{1}{2.3}+ ....+\frac{1}{n(n+1)}$

(i) To prove p(1) is true

P (1): $\frac{1}{1.2}=\frac{1}{1.(1+1)}$ is true.

(ii) Assume P(k): $\frac{1}{1.2}+\frac{1}{2.3}+ ....+\frac{1}{k(k+1)}$

$=\frac{k}{(k+1)}$

(iii) Claim: P(k+1) is true.

$$P(k+1) = \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= = \frac{k^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$\therefore$ P(k+1) is true.

Hence, $\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ is true for all n.

12. For all n ≥ 1, prove that $1^2+2^2+3^2+4^2+\ldots\ldots.+n^2= \frac{n(n+1)(2n+1)}{6}$

Solution:

P(n): $1^2+2^2+3^2+4^2+\ldots\ldots+n^2= \frac{n(n+1)(2n+1)}{6}$

(i) To prove p(1) is true For n=1,

P(1)= $\frac{1(1+1)(2\times1+1)}{6} = \frac{(1\times2\times3)}{6}$ =1 which is true

therefore, P(n) is true. Where n = 1

(ii) Assume that P(k) is true for some positive integer k ,i.e.,

$1^2+2^2+3^2+4^2+\ldots\ldots.+k^2= \frac{k(k+1)(2k+1)}{6}$ ………….(1)

(iii) prove that P(k+1) is also true.

$$(1^2+2^2+3^2+4^2+\ldots\ldots+k^2)+(k+1)^2$$

$= \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$ (Using (1))

$= \dfrac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$= \dfrac{(k+1)(2k^2+7k+6)}{6}$

$= \dfrac{(k+1)(k+1+1)\{2(k+1)+1\}}{6}$

Thus P(k+1) is true, wherever P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural number n.

13. Show that $1+3+5+\ldots\ldots + (2n-1) = n^2$

Solution:

(i) To prove p(1) is true

$p(1) = 1 = 1^2$ is true

(ii) Assume that P(k) is true for some positive integer n=k

$1+3+5+\ldots+(2k-1)=k^2$ is true

(iii) To prove for p(k+1)

$1+3+5+\ldots..+(2k-1)+(2(k+1)-1)=(k+1)^2$

We know that $1+3+5+\ldots+(2k-1) = K^2$ so,

$1+3+5+\ldots+(2k-1)+(2(k+1)-1=K^2+(2(k+1)-1)$

Expanding $=k^2+2k+2-1$

$=k^2+2k+1$

$=(k+1)^2$

They are same! So, it is true.

14. Show that if $n \geq 1$, then $1.1!+2.2!+3.3!+\ldots+n.n!= (n+1)!-1$

Solution:

Let P(n): $1.1!+2.2!+3.3!+\ldots+n.n!=(n+1)!-1$

(i) To prove p(1) is true
P(1): $1.1!=(1+1)!-1$ is true.

(ii) Assume (k): $1.1!+2.2!+3.3!+\ldots+k.k! = (k+1)!-1$ is true.

(iii) Claim: P(k+1) is true .

To prove: $1.1!+2.2!+3.3!+\ldots+k.k!+ (k+1)(k+1)!$

$= (k+1)! -1+ (k+1)(k+1)!$ ------------------------- (Using (1)

$= (k+1)! [(1+k+1)]-1$

$= (k+1)! (k+2)-1= (k+2)! - 1$

$= ((k+1) +1)! - 1$

$\therefore$ P(k+1) is true.By mathematical induction we have,

P(n): $1.1! +2.2! + 3.3! + \ldots +n.n! = (n+1)!-1, n \geq 1$.

15)Use mathematical Induction, Prove that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\text{.......}+\frac{1}{\sqrt{n}}=\sqrt{n}$

for n $\geq$ 2

Solution: Let: P(n): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\text{.....}+\frac{1}{\sqrt{n}}=\sqrt{n}$

(i) Assume P (2): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}$ = (1.707)> $\sqrt{2}$ is true.

(iii) Assume P(k): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\text{.......}+\frac{1}{\sqrt{k}}>\sqrt{k}$ is true ------ (1)

Claim: P(k+1) is true.

i.e., To prove $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\text{.......}+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}$

Consider, $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\text{.......}+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$

$=\sqrt{k}+\frac{1}{\sqrt{k+1}}$ (Using(1))

$=\frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}}$

$=\frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}$

$>\frac{\sqrt{k^2}+1}{\sqrt{k+1}}$

$>\frac{k+1}{\sqrt{k+1}}$

$=\sqrt{k+1}$

i.e.., $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\text{.......}+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}\ \therefore$ P(k+1) is true.

By mathematical induction we have, P(n) : $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\text{.....}+\frac{1}{\sqrt{n}}>\sqrt{n}$

Strong Induction:

- **Base step**: $P(n)$ is true for n = 1; 2.
- **Induction step**: Let $k \in \mathbb{N}$ with $k \geq 2$ be given andassume $P(n)$ holds for $n = k$ and $n = k-1$. Therefore $P(n)$ holds for $n = k+1$.
- **Conclusion:** By the principle of strong induction, $P(n)$ holds for all $n \in \mathbb{N}$.

Problems

1. **Fundamental theorem of Arithmetic**
   Prove that every integer $n \geq 2$ is prime or a product of primes".

Proof: We will prove by strong induction that the following statement holds for all integers $n \in \mathbb{N}$

N

Base Step: 2 is a prime number, so the propertyholds for $n = 2$.

Inductive step:
Assume that if $2 \leq k \leq n$,then k is a prime number or a product of primes.
To prove $n+1$ is a prime number or it is not.

If it is a prime number then it is proved.

If it is not a prime number, $n+1 = k_1 k_2$, such that $1 < k_{1,}\ k_2 < n+1$.
The induction hypothesis can be applied to k_1 and k_2 shows that k_1 and k_2 can be represented as products of one or more primes.

By induction hypothesis each of $n+1$ is a product of primes.
Hence, every integer $n \geq 2$ is prime or a product of primes".

2. If $u_1 = 1$, $u_2 = 5$ and $u_{n+1} = 5u_n - 6u_{n-1}, \forall n \geq 2$, then prove that $u_n = 3^n - 2^n, \forall n \geq 1$.

Solution: Given $u_1 = 1$, $u_2 = 5$ and $u_{n+1} = 5u_n - 6u_{n-1}, \forall n \geq 2$.

Let $P(n): u_n = 3^n - 2^n, \forall n \geq 1$. (1)

Base step: Put $n = 1$ in (1); $P(1): u = 3^1 - 2^1 \Rightarrow 1 = 1$ is true. Therefore, P(1) is true.
Put $n = 2$ in (1); $P(2): u_1 = 3^2 - 2^2 \Rightarrow 5 = 5$ is true. Therefore, P(2) is true.

Therefore, P(1) and P(2) are true.

Inductive step: Assume P(k) is true for all integer up to k, k>2.
That is, P(3), P(4), P(5), …, P(k) are true.

To Prove: P(k+1) is True. That is $u_{k+1} = 3^{k+1} - 2^{k+1}$

$$
\begin{aligned}
u_{k+1} &= 5u_k - 6u_{k-1} \\
&= 5\left(3^k - 2^k\right) - 6\left(3^{k-1} - 2^{k-1}\right) \\
&= \left(5.3^k - 5.2^k - 6.3^k 3^{-1} + 6.2^k 2^{-1}\right) \\
&= 5.3^k - 5.2^k - 2.3^k + 3.2^k \\
&= 3^k (5-2) - 2^k (5-3) \\
u_{k+1} &= 3^{k+1} - 2^{k+1}
\end{aligned}
$$

Therefore, P(k+1) is true
Hence by strong induction, P(n) is true $\forall n \geq 1$.

3. Using strong induction prove that $a - b$ is a factor of $a^n - b^n$ for all positive integer n.

Solution: Given P(n) : $a - b$ is a factor of $a^n - b^n$. (1)
To prove: P(n) is true for all $\forall n \geq 1$.

Base Step: Verify P(1) and P(2) are true.

Put $n = 1$ in (1), we get

P(1)= $a - b$ is a factor of $a^n - b^n$
Put $n = 2$ in (1), we get

P(2)= $a^2 - b^2 = (a - b)(a + b)$ is a factor of $a^n - b^n$.
Thus P(1), P(2) is true.

Inductive Step: Assume that P(k) is true for all integer up to k ($k \geq 3$). That is P(3), P(4), . . . , P(k-1), P(k) are true. (2)

To prove: P(k+1) is true.
That is, to prove P(k+1)= $a - b$ is a factor of $a^{k+1} - b^{k+1}$.
Now,

$$a^{k+1} - b^{k+1} = (a - b)(a^k - b^k) + ab^k - ba^k$$

$$= (a+b)(a^k - b^k) - ab(a^{k-1} - b^{k-1})$$
$$= (a+b)(a-b)x - ab(a-b)y$$
$$= (a-b)\left((a+b)x - aby\right)$$

Therefore, $a-b$ is a factor of $a^{k+1} - b^{k+1} \Rightarrow$ P(k+1) is true.

Thus, (P(1) $\wedge$ P(2) $\wedge$ ... $\wedge$ P(k-1) $\wedge$ P(k)) $\Rightarrow$ P(k+1) is true.

Therefore, P(n) is true for all $\forall n \geq 1$.

x...x

Topic 2: **Linear homogeneous recurrence relation**

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Definition: A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k} \ldots (1)$$

Where $c_1, c_2, \ldots c_k$ are real numbers and $c_{k \neq} 0$ is called a **linear homogeneous recurrence** relation of degree k with constant coefficients.

Working Rule:

1. Put $a_n = r^n$ to get the characteristic equation .

2. Solve the characteristic equation to get roots r_1 , r_2 ,..........,,r_n

3. To find the general solution according to the nature of the roots

Case (i) If r_1 *and* r_2 are real and different, then the general solution of the recurrence relation (or difference equation) is $a_n = Ar_1^n + Br_2^n$, where A and B are arbitrary constants.

Case (ii) If r_1 *and* r_2 are real and equal, say say r, then the general solution of the recurrence relation is $a_n = (A + Bn)r^n$, where A and B are arbitrary constants.

Case (iii) If r_1 *and* r_2 are complex,

Therefore the general solution of the recurrence relation will be

$a_n = r^n\,(A\, cos\mathrm{n}\theta \;+ B\, sin\mathrm{n}\,\theta)$, where $\mathrm{r} = \sqrt{\alpha^2} + \beta^2$ and $\tan\theta = \frac{\beta}{\alpha}$

Problems:

1. Solve the recurrence relation $a_n = 3a_{n-1} + 4\ a_{n-2,} \geq 2\ and a_0 = 0, a_1 = 5.$

Solution: Given $a_n = 3a_{n-1} + a_{n-2,} \geq 2\ and\ a_0 = 0, a_1 = 5.$

characteristic equation is $r^2 - 3r - 4 = 0$

$$\Rightarrow \mathrm{r} = 4\ , r\ = -1$$

The roots are real and distinct.

general solution is $= A4^n\ + \mathrm{B}\ (-1)^{\ n}$ ....(1)

To find the values of A and B, using $a_0 = 0, a_1 = 5.$

Put $n = 0 \Rightarrow a_0 = A + B \Rightarrow A + B = 0\ \dots .\ (2)$

Put $n = 1 \Rightarrow a_1 = 4A - B \Rightarrow 4A - B = 5\ \dots .\ \ (3)$

$(2) + (3\ ) \Rightarrow 5A = 5 \Rightarrow A = 1$ therefore

B= -1. Therefore $a_n = 4^n - (-1)^{\ n}$

2. Solve the recurrence relation $a_n = 6a_{n-1} - 9\ a_{n-2,} \geq 2\ and a_0 = 2, a_1 = 3.$

Solution: Given $a_n = 6a_{n-1} - 9\ a_{n-2,} \geq 2\ and a_0 = 2, a_1 = 3.$

characteristic equation is $r^2 - 6r + 9 = 0 \Rightarrow r =\ 3{,}3$

The roots are real and equal.

Therefore the general solution is $a_n = (A + Bn)\ 3^n \dots (1)$

To find the values of A and B using $a_0 = 2, a_1 = 3.$

Put $n = 0\ in\ (1)\ weget\ A = 2.$

Put $n = 1\ in\ (1)\ weget\ a_1 = (A + B)\ 3 \Rightarrow B = -1$

Therefore general solution is $a_n = (2 - \mathrm{n}\ )\ 3^n$

3. Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$ where $n \geq 2$ and $a_0 = 1$, $a_1 = 2$.

Solution :

Given : $a_n = 2(a_{n-1} - a_{n-2})$

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic equation is

$r^2 - 2r + 2 = 0$

$r = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

Therefore $\alpha = 1$ and $\beta = 1$

$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{2}$ and $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

General solution is given by $a_n = r^n (A\cos n\theta + B\sin n\theta)$

$$a_n = \sqrt{2}^{\,n}\left(A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To find the values of A and B ,

Given $a_0 = 1$, $a_1 = 2$

Put n = 0 in (1) ,we get

$a_0 = \sqrt{2}\,(A\cos 0 + B\sin 0)$

$\Rightarrow 1 = A$

Put n = 1 in (1) ,we get

$a_1 = \sqrt{2}\,\left(A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}\right)$

$\Rightarrow 2 = \sqrt{2}\,\left(A\,\frac{1}{\sqrt{2}} + B\,\frac{1}{\sqrt{2}}\right)$

$\Rightarrow 2 = A + B$

$\Rightarrow 2 = 1 + B$

$B = 1$

(1) $\Rightarrow a_n = \sqrt{2}^{\,n}\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right)$

x………………………………………x

Topic 3 & 4 : **Non-Homogeneous linear recurrence relation with constant coefficient**

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Definition: A recurrence relation is of the form

$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k\, a_{n-k} = \mathrm{f(n)}$ ----------(1)

Where $c_0, c_1, c_2, \ldots .\, c_k$ are constants with $c_0 \neq 0, c_k \neq 0$ is called a **linear non-homogeneous recurrence relation** with constant coefficient.

The recurrence relation $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$-----------(2)

is called the associated homogeneous recurrence relation.

Let $a_n^{(h)}$ be the general solution of (2).

Suppose $a_n^{(p)}$ is a particular solution of (1), then the general solution of (1) is $a_n = a_n^{(h)} + a_n^{(p)}$

S.No	$\mathrm{f}(n)$	Particular Integral
1.	**Type 1:** c [constant]	C
2.	**Type 2:** Polynomial $\mathrm{P}(n)$ of degree m	$\mathrm{C_0 + C_1 n + C_2 n^2 + \cdots + C}_m\, \mathrm{n}^m$
	$\mathrm{a}^n\, \mathrm{P}(n)$ [if a is not a root of the characteristic equation]	$\mathrm{a}^n\, (\mathrm{C_0 + C_1 n + C_2 n^2 + \cdots + C}_m\, \mathrm{n}^m)$
	$\mathrm{a}^n\, \mathrm{P}(n)$ [if a is a root of the characteristic equation with multiplicity t]	$\mathrm{a}^n\, \mathrm{n}^t\, (\mathrm{C_0 + C_1 n + C_2 n^2 + \cdots + C}_m\, \mathrm{n}^m)$
5.	**Type 3:** a^n [if a is a root of characteristic equation with multiplicity s]	$\mathrm{Cn}^s \mathrm{a}^n$
	a^n [if a is not a root of the characteristic equation]	Ca^n

Problems

1. Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 2$, $a_0 = 1$ and $a_1 = -1$

Solution:

Given recurrence relation is $a_n - 5a_{n-1} + 6a_{n-2} = 2$

The homogeneous recurrence relation is $a_n - 5a_{n-1} + 6a_{n-2} = 0$

The characteristic equation is $r^2 - 5r + 6 = 0$

$\Longrightarrow r = 2,3$

Therefore the solution of homogeneous recurrence relation is $a_n^{(h)} = A.2^n + B.3^n$

Given $f(n) = 2$, where 2 is a constant **(type 1)**

Let $a_n = C$ is the particular solution and the constant c is to be determined using

$$a_n - 5a_{n-1} + 6a_{n-2} = 2$$

$$c - 5c + 6c = 2$$

$$c = 1$$

$$a_n^{(p)} = c = 1$$

Hence the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$\Longrightarrow a_n = A.2^n + B.3^n + 1$$

We have $a_0 = 1$, $\therefore$ putting $n = 0$ in (1) we get $a_0 = A.2^0 + B.3^0 + 1$

$$\Longrightarrow 1 = A + B + 1 \quad \Longrightarrow A + B = 0$$

We have $a_1 = -1$, $\therefore$ putting $n = 1$ in (1) we get $a_1 = A.2^1 + B.3^1 + 1$

$$\Longrightarrow -1 = 2A + 3B + 1 \qquad \Longrightarrow 2A + 3B = -2$$

Solving we get $A = 2$ and $B = -2$

$\therefore$ The general solution is $a_n = 2.2^n - 2.3^n + 1$ where $n \geq 0$

2. Solve $a_n - 3a_{n-1} = 2n$, $a_1 = 3$

Solution: Given $a_n - 3a_{n-1} = 2n$

∴ The homogeneous recurrence relation is $a_n - 3a_{n-1} = 0$

The characteristic equation is $r - 3 = 0 \Longrightarrow r = 3$

Therefore the solution of homogeneous recurrence relation is $a_n^{(h)}$

$$a_n^{(h)} = \mathrm{A}\ 3^n$$

Given $\mathrm{f(n)} = 2n$, which is a polynomial of degree 1 **(type 2)**

∴ The particular solution $a_n = \mathrm{cn} + \mathrm{d}$ where c and d are constants

We find c and d satisfying given equation

$$\mathrm{Cn} + \mathrm{d} - 3\,[\mathrm{c}\,(\mathrm{n}-1) + \mathrm{d}\,] = 2n$$

$$\mathrm{cn} + \mathrm{d} - 3\mathrm{cn} + 3\mathrm{c} - 3\mathrm{d} = 2n$$

$$-2\mathrm{cn} - 2\mathrm{d} + 3\mathrm{c} = 2n$$

equating like coefficients on both sides we get

$$-2\mathrm{c} = 2 \implies \mathrm{c} = -1$$

and $-2\mathrm{d} + 3\mathrm{c} = 0$

$$a_n^{(p)} = \mathrm{cn} + \mathrm{d} = -3/2 + (-1)\mathrm{n}$$

The general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = \mathrm{A}\,.3^n - n - \frac{3}{2} \ldots\ldots\ldots\ldots.(1)$$

Given $a_1 = 3$, put n =1 in (1), we get

$$a_1 = \mathrm{A}\,.\,3^1 - 1 - \frac{3}{2}$$

$3 = 3A - \frac{5}{2}$

$3A = \frac{11}{2}$

$A = \frac{11}{6}$

Substituting $A = \frac{11}{6}$ in (1) , we get

$\therefore$ General solution $a_n = \frac{11}{6} \cdot 3^n - n - \frac{3}{2}$

3. Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 8n^2$, $a_0 = 4$, $a_1 = 7$

Solution: Given $- 5a_{n-1} + 6a_{n-2} = 8n^2$ ------------------------(1)

The homogeneous recurrence relation is $a_n - 5a_{n-1} + 6a_{n-2} = 0$

$\therefore$ The characteristic equation is $r^2 - 5r + 6 = 0$

$$r - 3 \quad r - 2 \ = 0$$

$$= 2, 3$$

The solution of the homogeneous recurrence relation is

$a_n^{(h)} = A\, 2^n + B\, 3^n$

Also given $f(n) = 8n^2$, which is a polynomial of degree 2 **(type2)** and 2 is a root of the characteristic equation.

$\therefore$ particular solution is $a_n = A_0 + A_1 n + A_2 n^2$ [(or) you can assume as $a_n = cn^2 + dn + e$] , where A_0, A_1, A_2 are constants to be determined using the given equation (1)

(1) $\Longrightarrow a_n - 5a_{n-1} + 6a_{n-2} = 8n^2$

Substitute in (1)

$$\Longrightarrow \quad A_0 + A_1 n + A_2 n^2 - 5\,[A_0 + A_1\,(n-1) + A_2\,(n-1)^{\,2}]$$
$$+\,6\,[A_0 + A_1\,(n-2) + A_2\,(n-2)^{\,2} = 8n^2$$

$$A_0 + A_1 n + A_2 n^2 - 5A_0 - 5A_1 n + 5A_1 - 5A_2 n^2 + 10A_2 n - 5A_2 + 6A_0 + 6A_1 n - 12A_1 + 6A_2 n^2 + 24A_2 - 24nA_2 = 8n^2$$

$$n^2\,A_2 - 5A_2 + 6A_2 \;+ n\,A_1 - 5A_1 + 10A_2 + 6A_1 - 24A_2$$
$$+\;A_0 - 5A_0 + 5A_1 - 5A_2 + 6A_0 - 12A_1 + 24A_2 \;= 8n^2$$
$$2A_2\,n^2 + 2A_1 - 14A_2\,n + 2A_0 - 7A_1 + 19A_2 = 8n^2$$

equating like coefficients, we get

$2A_2 = 8 \Longrightarrow A_2 = 4$

$2A_1 - 14A_2 = 0 \Longrightarrow 2A_1 = 14\;4 \Longrightarrow A_1 = 28$

$$2A_0 - 7A_1 + 19A_2 = 0 \Longrightarrow 2A_0 - 7(28) + 19(4) = 0$$
$$A_0 - 7(14) + 19(2) = 0$$
$$A_0 - 98 + 38 = 0$$
$$A_0 = 60$$
$$a_n^{(p)} = 60 + 28n + 4n^2$$

∴ The general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A\,2^n + B\,3^n + 60 + 28n + 4n^2$ -------------------------------- (2)

To find A and B we use $a_0 = 4,\ a_1 = 7$

Putting $n = 0$ in (2)

$a_0 = A + B + 60 \Longrightarrow A + B + 60 = 4$ (3)

$\Longrightarrow A + B = -56$

Putting $n = 1$ in (2)

$$a_1 = 2A + 3B + 60 + 28 + 4$$
$$7 = 2A + 3B + 92$$
$$7 - 92 = 2A + 3B$$

$-85 = 2A + 3B$ (4)

Solving (3) and (4) we get, $A = -83$ and $B = 27$

∴ general solution is $a_n = -83\,.\,2^n + 27.\,3^n + 60 + 28n + 4n$

4. Solve the recurrence relation $a_n - 2a_{n-1} = 2^n$, $a_0 = 2$

Solution: Given recurrence relation is $a_n - 2a_{n-1} = 2$, $a_0 = 2$

The homogeneous recurrence relation is $a_n - 2a_{n-1} = 0$

The characteristic equation is $r - 2 = 0 \Longrightarrow r = 2$

Therefore the solution of homogeneous recurrence relation is $a_n^{(h)} = A.2^n$

Given $f(n) = 2^n$ **(Type3)** , where 2 is a root of characteristic equation

$a_n = c\,n2^n$ is the particular solution and the constant c is to be determined using

$a_n - 2a_{n-1} = 2^n$

$cn2^n - 2c\,(n - 1)2^{n-1} = 2^n$

$2^n\,(c\,n - n - 1\,) = 2^n$

$c\,n - n - 1 = 1$

$c = 1$

$a_n^{(p)} = cn.\,2^n = n.\,2^n$

Hence the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$\Longrightarrow a_n = A\,.\,2^n + n.\,2^n$

We have $a_0 = 2$, ∴ putting $n = 0$ in (1) we get $a_0 = \mathrm{A}\ 2^0 + 0$

$$\Longrightarrow 2 = \mathrm{A} \quad \Longrightarrow \mathrm{A} = 2$$

∴ The general solution is $a_n = 2.\,2^n + n.\,2^{\mathrm{n}}$, $n \geq 0$

$\Longrightarrow a_n = (n + 2)\ 2^{\mathrm{n}}$, $n \geq 0$

5. Find the general solution of the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 4^n$, $n \geq 2$

Solution:

Given $- 5a_{n-1} + 6a_{n-2} = 4^n$ -- (1)

The homogeneous recurrence relation is $a_n - 5a_{n-1} + 6a_{n-2} = 0$

∴ The Characteristic equation is $r^2 - 5r + 6 = 0$

$r - 2\,r - 3 = 0$

$r = 2, 3$

∴ The solution of homogeneous recurrence relation is $\mathrm{a}_n{}^{(h)} = \mathrm{A}\ 2^n + \mathrm{B}\ 3^n$

Given $\mathrm{f}\,(\mathrm{n}) = 4^{\mathrm{n}}$ where 4 is not a root of the characteristic equation

∴ particular solution is $a_n^{(p)} = \mathrm{c}.\,4^n$ (**Type 3**)

Substituting in (1), we get $\mathrm{c}.\,4^n - 5\mathrm{c}.\,4^{n-1} + 6\mathrm{c}.\,4^{n-2} = 4^n$

$$4^{n-2}\ [\mathrm{c}\ 16 - 20 + 6\ ] = 4^n$$

Divide by 4^{n-2}, $2\mathrm{c} = 16 \quad \Longrightarrow \mathrm{c} = 8$

∴$\mathrm{a}_n{}^{(p)} = 8\ .\ 4^n$

So, the general solution is $a_n = \mathrm{a}_n{}^{(h)} + \mathrm{a}_n{}^{(p)}$

$$= \mathrm{A}\ 2^n + \mathrm{B}\ 3^n + 8.\,4^n$$

Note: Initial conditions are not given so we can’t find A and B

6. .Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 3.\,2^n + 7.\,3^n$, where $n \geq 0$ and $a_0 = 1, a_1 = 4$

Solution: Given $a_{n+2} - 6a_{n+1} + 9a_n = 3.\,2^n + 7.\,3^n$ ....(1)

The homogeneous recurrence relation is $a_{n+2} - 6a_{n+1} + 9a_n = 0$

$\therefore$ The Characteristic equation is $r^2 - 6r + 9 = 0$

$$r = 3, 3$$

$\therefore$ The solution of the homogeneous recurrence relation is $a_n^{(h)} = (A + Bn)\, 3^n$

Given $f(n) = 3.\,2^n + 7.\,3^n$, where 3 is a double root of the characteristic equation.

So, we assume the particular solution as $a_n = A_0\, 2^n + A_1\, n^2\, 3^n$ (**Type 3**)

where A_0 and A_1 are constants to be determined using the given equation (1)

$$\therefore A_0\, 2^{n+2} + A_1\, n + 2\,^2.\, 3^{n+2} - 6\, A_0\, 2^{n+1} + A_1\, n + 1\,^2.\, 3^{n+1} + 9\, A_0\, 2^n + A_1 n^2.\, 3^n$$
$$= 3.\,2 + 7.\,3^n$$

$$A_0\, 2^{n+2} - 6.\, 2^{n+1} + 9\, 2^n + A_1\, n + 2\,^2.\, 3^{n+2} - 6\, n + 1\,^2.\, 3^{n+1} + 9n^2.\, 3^n$$
$$= 3.\,2 + 7.\,3^n$$

$\Longrightarrow 2^n\, A_0\, 4 - 12 + 9 + 3^n\, A_1\, 9\, n + 2\,^2 - 18\, n + 1\,^2 + 9n^2 = 3.\,2^n + 7.\,3^n$

$\Longrightarrow 2^n\, A_0 + 3^n\, A_1\, 9\, n^2 + 4n + 4 - 18\, n^2 + 2n + 1 + 9n^2 = 3.\,2^n + 7.\,3^n$

$\Longrightarrow 2^n\, A_0 + 18\, A_1 3^n = 3.\,2^n + 7.\,3^n$

equating like coefficients we get

$A_0 = 3$ and $18\, A_1 = 7 \implies A_1 = \frac{7}{18}$

$\therefore$ particular solution is $a_n^{(p)} = 3.\,2^n + \frac{7}{18}\, n^2\, 3^n$

Hence the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = (A + Bn)\, 3^n + 3.\,2^n + \frac{7}{18}\, n^2\, 3^n$ -- (2)

To find A and B, we use the initial conditions $a_0 = 1, a_1 = 4$

$\therefore$ putting $n = 0$ in (2) we get, $a_0 = A + 3 \implies 1 = A + 3 \implies A = -2$

putting $n = 1$ in (2) we get, $a_1 = (A + B)3 + 3.2 + \frac{7}{18}.\,3$

$$4 = (-2 + B)\;3 + 6 + \frac{7}{6}$$

$$\implies B = \frac{17}{18}$$

$a_n = (-2 + 17/18n)3^n + 3.\,2^n + (7/18)n^2\, 3^n$ is the general solution

Topic 5: Generating function method

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Definition: The **generating function** of the sequence $a_0, a_1, a_2 \dots, a_n, \dots$ of real numbers is the infinite series $\mathrm{G}(\mathrm{x}) = a_0 + a_1\mathrm{x} + a_2\mathrm{x}^2 + \cdots + a_n\,\mathrm{x}^n + \cdots$

$\mathrm{G}(\mathrm{x})$ is called the generating function for the finite sequence $a_0, a_1, a_2 \dots, a_n$

Note:

For example

(i)The generating function for the sequence 1, 1, 1, …… is

$$\mathrm{G}(\mathrm{x}) = 1 + \mathrm{x} + \mathrm{x}^2 + \cdots = \frac{1}{1-\mathrm{x}}$$

(ii) The generating function for the sequence 1, 2, 3, 4, …. is

$$\mathrm{G}(\mathrm{x}) = 1 + 2\mathrm{x} + 3\,\mathrm{x}^2 + 4\,\mathrm{x}^3 + \cdots = (1 - \mathrm{x})^{-2} = \frac{1}{(1-\mathrm{x})^2}$$

(iii)The generating function of the sequence 1, 2, 2^2, 2^3, …. is

$$\mathrm{G}(\mathrm{x}) = 1 + 2\mathrm{x} + 2^2\,\mathrm{x}^2 + 2^3\,\mathrm{x}^3 + \cdots = (1 - 2\mathrm{x})^{-1} = \frac{1}{1 - 2\mathrm{x}}$$

(iv) $\frac{1}{1-a\mathrm{x}} = 1 + a\mathrm{x} + a^2\,\mathrm{x}^2 + a^3\,\mathrm{x}^3 + \cdots + a^n\,\mathrm{x}^n + \cdots$

Problems

1. Using the method of generating function, solve the recurrence relation
   $a_n = 3a_{n-1} + 1$ where $n \geq 1$ given that $a_0 = 1$.

   Solution :

   Given : $a_n = 3a_{n-1} + 1$

   $a_n - 3a_{n-1} - 1 = 0$

   $a_n x^n - 3a_{n-1} x^n - x^n = 0$

   $\sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=1}^{\infty} x^n = 0$

   $\sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} - \sum_{n=1}^{\infty} x^n = 0$

$[a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\infty] - 3x[a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\infty]$

$- [x + x^2 + x^3 + x^4 + \ldots\ldots\infty] = 0$

We know $G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$

$[G(x) - a_0] - 3x\,G(x) - x[1 + x + x^2 + x^3 + \ldots\ldots\infty] = 0$

$G(x) - 3x\,G(x) = a_0 + x(1-x)^{-1}$

$G(x)(1 - 3x) = a_0 + x(1-x)^{-1}$

$G(x)(1 - 3x) = 1 + \frac{x}{1-x}$ [since $a_0 = 1$]

$G(x)(1 - 3x) = \frac{1}{1-x}$

$G(x) = \frac{1}{(1-x)(1-3x)}$

Apply partial fraction method

$$\frac{1}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$\frac{1}{(1-x)(1-3x)} = \frac{A(1-3x)+B\,(1-x)}{(1-x)(1-3x)}$$

$$1 = A(1-3x) + B\,(1-x)$$

Putting $x = 1/3$, $1 = 0 + B\,(1 - 1/3)$

$=> B = \frac{3}{2}$

Putting $x = 1$, $1 = A(1 - 3) + 0$

$=> A = -\frac{1}{2}$

$$G(x) = \frac{1}{(1-x)(1-3x)} = \frac{-1/2}{1-x} + \frac{3/2}{1-3x}$$

$$G(x) = -\frac{1}{2}(1-x)^{-1} + \frac{3}{2}(1-3x)^{-1}$$

$$G(x) = -\frac{1}{2}[1 + x + x^2 + x^3 + \ldots\ldots\infty] + \frac{3}{2}[1 + 3x + (3x)^2 + (3x)^3 + \ldots\ldots\infty]$$

General solution given by a_n = coefficient of x^n in G (x)

Therefore $a_n = -\frac{1}{2}(1) + \frac{3}{2}(3^n)$

2. Using the method of generating function, solve the recurrence relation $a_{n+1} - 2a_n - 4^n = 0$ where $n \geq 0$ given that $a_0 = 1$.

Solution :

$$a_{n+1} - 2a_n - 4^n = 0$$

$$a_{n+1}x^n - 2a_n x^n - 4^n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+1}x^n - 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 4^n x^n = 0$$

$$x^{-1}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 4^n x^n = 0$$

$$x^{-1}[a_1x + a_2x^2 + a_3x^3 + \ldots\infty] - 2[a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\infty] - [1 + 4x + (4x)^2 + (4x)^3 + (4x)^4 + \ldots\infty] = 0$$

We know $G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$x^{-1}[G(x) - a_0] - 2G(x) - (1-4x)^{-1} = 0$

$x^{-1}G(x) - 2G(x) = a_0 x^{-1} + (1-4x)^{-1}$

$\frac{G(x)}{x} - 2G(x) = \frac{a_0}{x} + \frac{1}{1-4x}$

$G(x)\left(\frac{1}{x} - 2\right) = \frac{1}{x} + \frac{1}{1-4x}$ [since $a_0 = 1$]

$G(x)\left(\frac{1-2x}{x}\right) = \frac{1-4x+x}{x(1-4x)}$

$G(x) = \frac{(1-3x)\ x}{x(1-4x)(1-2x)}$

$G(x) = \frac{(1-3x)}{(1-4x)(1-2x)}$

Apply partial fraction method

$\frac{1-3x}{(1-4x)(1-2x)} = \frac{A}{1-4x} + \frac{B}{1-2x}$

$\frac{1-3x}{(1-4x)(1-2x)} = \frac{A(1-2x)+B(1-4x)}{(1-4x)(1-2x)}$

$1-3x = A(1-2x) + B(1-4x)$

Putting $x = 1/2$, $1 - \frac{3}{2} = A(0) + B(1 - 4/2)$

$=> B = \frac{1}{2}$

Putting $x = 1/4$, $1 - \frac{3}{4} = A(1 - 2/4) + 0$

$=> A = \frac{1}{2}$

$$G(x) = \frac{1}{(1-4x)(1-2x)} = \frac{1/2}{1-4x} + \frac{1/2}{1-2x}$$

$$G(x) = \frac{1}{2}(1-4x)^{-1} + \frac{1}{2}(1-2x)^{-1}$$

$$G(x) = \frac{1}{2}[1 + 4x + (4x)^2 + (4x)^3 + \ldots\ldots\ldots\infty] + \frac{1}{2}[1 + 2x + (2x)^2 + (2x)^3 + \ldots\ldots\ldots\infty]$$

General solution given by a_n = coefficient of x^n in G (x)

Therefore $a_n = \frac{1}{2}(4^n) + \frac{1}{2}(3^n)$

3. Using the method of generating function, solve the recurrence relation $a_{n+2} - a_{n+1} - 6a_n = 0$ where $n \geq 0$ given that $a_0 = 2$ and $a_1 = 1$.

Solution :

$$a_{n+2} - a_{n+1} - 6a_n = 0$$

$$a_{n+2}x^n - a_{n+1}x^n - 6a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}x^n - \sum_{n=0}^{\infty} a_{n+1}x^n - 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$x^{-2}\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - x^{-1}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} - 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$x^{-2}[a_2x^2 + a_3x^3 + \ldots\ldots\infty] - x^{-1}[a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots\infty]$$

$$-6[a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots\infty] = 0$$

We know $G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots$

$$x^{-2}[G(x) - a_0 - a_1x] - x^{-1}[G(x) - a_0] - 6G(x) = 0$$

$$x^{-2}G(x) - x^{-1}G(x) - 6G(x) = a_0x^{-2} + a_1x^{-1} - a_0x^{-1}$$

$$\frac{G(x)}{x^2} - \frac{G(x)}{x} - 6G(x) = \frac{a_0}{x^2} + \frac{a_1}{x} - \frac{a_0}{x}$$

$G(x)\left(\frac{1}{x^2} - \frac{1}{x} - 6\right) = \frac{2}{x^2} + \frac{1}{x} - \frac{2}{x}$ [since $a_0 = 2$, $a_1 = 1$]

$G(x)\left(\frac{1-x-6x^2}{x^2}\right) = \frac{2-x}{x^2}$

$G(x) = \frac{(2-x)\ x^2}{x^2(1-x-6x^2)}$

$G(x) = \frac{(2-x)}{(1-3x)(1+2x)}$

Apply partial fraction method

$\frac{2-x}{(1-3x)(1+2x)} = \frac{A}{1-3x} + \frac{B}{1+2x}$

$\frac{2-x}{(1-3x)(1+2x)} = \frac{A(1+2x)+B\ (1-3x)}{(1-3x)(1+2x)}$

$2 - x = A(1+2x) + B\ (1-3x)$

Putting $x = -1/2$, $2 + \frac{1}{2} = 0 + B\ (1 + 3/2)$

$\Rightarrow B = 1$

Putting $x = 1/3$, $2 - \frac{3}{4} = A(1 + 2/3) + 0$

$\Rightarrow A = 1$

$G(x) = \frac{2-x}{(1-3x)(1+2x)} = \frac{A}{1-3x} + \frac{B}{1+2x}$

$G(x) = \frac{2-x}{(1-3x)(1+2x)} = \frac{1}{1-3x} + \frac{1}{1+2x}$

$G(x) = (1-3x)^{-1} + (1+2x)^{-1}$

$G(x) = [1 + 3x + (3x)^2 + (3x)^3 + \ldots\ldots\infty] +$
$[1 - 2x + (2x)^2 - (2x)^3 + \ldots\ldots\infty]$

General solution given by a_n = coefficient of x^n in G (x)

Therefore $\mathbf{a_n = 3^n + (-1)^n 2^n}$

Topic 6: Principle of Inclusion and Exclusion:

View the video on ponjesly app

Statement:

If A and B are finite subsets of a finite universal set U, then
$\mathbf{n}(A \cup B) = \mathbf{n}(A) + \mathbf{n}(B) - \mathbf{n}(A \cap B)$ where A denotes the cardinality of the set (i.e., the number of elements in the set A)

Note:

- For any finite sets A,B and C, we have

$$\mathbf{n}(A \cup B \cup C) = \mathbf{n}(A + \mathbf{n}(B) + \mathbf{n}(C) - \mathbf{n}(A \cap B) - \mathbf{n}(A \cap C) - \mathbf{n}(B \cap C) + \mathbf{n}(A \cap B \cap C)$$

- For any finite sets A,B ,C and D, we have

$$\mathbf{n}(A \cup B \cup C \cup D) = \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) + \mathbf{n}(D) - \mathbf{n}(A \cap B) - \mathbf{n}(A \cap C) - \mathbf{n}(A \cap D) - \mathbf{n}(B \cap C) - \mathbf{n}(B \cap D) - \mathbf{n}(C \cap D) + \mathbf{n}(A \cap B \cap C) + \mathbf{n}(A \cap B \cap D) + \mathbf{n}(A \cap C \cap D) + \mathbf{n}(B \cap C \cap D) - \mathbf{n}(A \cap B \cap C \cap \mathrm{D})$$

Problems:

1. Determine the number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2, 3, 5 and 7.

Solution:

Let A be set of integers divisible by 2,
B be the set of integers divisible by 3,
C be the set of integers divisible by 5 and
D be the set of integers divisible by 7 between 1 and 250

To find :n($A \cup B \cup C \cup D$)

We know that

$$\mathbf{n}(A \cup B \cup C \cup D) = \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) + \mathbf{n}(D) - \mathbf{n}(A \cap B) - \mathbf{n}(A \cap C) - \mathbf{n}(A \cap D) - \mathbf{n}(B \cap C) - \mathbf{n}(B \cap D) - \mathbf{n}(C \cap D) + \mathbf{n}(A \cap B \cap C) + \mathbf{n}(A \cap \mathrm{B} \cap D) + \mathbf{n}(A \cap \mathrm{C} \cap D) + \mathbf{n}(B \cap C \cap D) - \mathbf{n}(A \cap B \cap C \cap D)$$

Then

n (A) = $[\frac{250}{2}]$ = 125 n (B) = $[\frac{250}{3}]$ = 83 n (C) = $[\frac{250}{5}]$ = 50 n (D) = $[\frac{250}{7}]$ =35

$$\mathbf{n}\,(\mathbf{A}\cap \boldsymbol{B}) = [\tfrac{\mathbf{250}}{\mathbf{2\times 3}}] = \mathbf{41} \quad \mathbf{n}\,(\mathbf{A}\cap \boldsymbol{C}) = [\tfrac{\mathbf{250}}{\mathbf{2\times 5}}] = \mathbf{25} \quad \mathbf{n}\,(\mathbf{A}\cap \boldsymbol{D}) = [\tfrac{\mathbf{250}}{\mathbf{2\times 7}}] = \mathbf{17}$$

$$\text{n}\,(\text{B}\cap C) = [\tfrac{250}{3\times 5}] = 16 \quad \text{n}\,(\text{B}\cap D) = [\tfrac{250}{3\times 7}] = 11 \quad \text{n}\,(\text{C}\cap D) = [\tfrac{250}{5\times 7}] = 7$$

$$\text{n}\,(\text{A}\cap\text{B}\cap C) = [\tfrac{250}{2\times 3\times 5}] = 8 \quad \text{n}\,(\text{A}\cap\text{B}\cap D) = [\tfrac{250}{2\times 3\times 7}] = 5 \quad \text{n}\,(\text{A}\cap\text{C}\cap D) = [\tfrac{250}{2\times 5\times 7}] = 3$$

$$\text{n}\,(\text{B}\cap\text{C}\cap D) = [\tfrac{250}{3\times 5\times 7}] = 2 \quad \text{n}\,(\text{A}\cap\text{B}\cap C\cap D) = [\tfrac{250}{2\times 3\times 5\times 7}] = 1$$

By the principle of Inclusion - Exclusion, the number of integers between 1 and 250 that are divisible by at least one of 2, 3, 5 and 7 is given by

$$\begin{aligned}\mathbf{n}(A\cup B\cup C\cup D) = {} & \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) + \mathbf{n}(D) - \mathbf{n}(A\cap B) - \mathbf{n}(A\cap C) - \\ & \mathbf{n}(A\cap D) - \mathbf{n}(B\cap C) - \mathbf{n}(B\cap D) - \mathbf{n}(C\cap D) \\ & + \mathbf{n}(A\cap B\cap C) + \mathbf{n}(A\cap B\cap D) + \\ & \mathbf{n}(A\cap C\cap D) + \mathbf{n}(B\cap C\cap D) - \mathbf{n}(A\cap B\cap C\cap D)\end{aligned}$$

$= (125 + 83 + 50 + 35) - (41 + 25 + 17 + 16 + 11 + 7) + (8 + 5 + 3 + 2) - 1$

$= 293 - 117 + 18 - 1 = 193$

$\therefore$ n $(A\cup B\cup C\cup D) = 193.$

.2. Find the numbers of integers between 1 and 500 that are not divisible by any of The integers 2, 3, 5 and 7.

Solution:

Let A, B, C, D be the sets of integers that lie between 1 and 500 and that are divisible by 2, 3, 5 and 7 respectively.

To find :n($A\cup B\cup C\cup D$)

We know that

$$\begin{aligned}\mathbf{n}(A\cup B\cup C\cup D) = {} & \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) + \mathbf{n}(D) - \mathbf{n}(A\cap B) - \mathbf{n}(A\cap C) - \\ & \mathbf{n}(A\cap D) - \mathbf{n}(B\cap C) - \mathbf{n}(B\cap D) - \mathbf{n}(C\cap D) + \\ & \mathbf{n}(A\cap B\cap C) + \mathbf{n}(A\cap \text{B}\cap D) + \\ & \mathbf{n}(A\cap \text{C}\cap D) + \mathbf{n}(B\cap C\cap D) - \mathbf{n}(A\cap B\cap C\cap D)\end{aligned}$$

Then

$$\text{n}\,(\text{A}) = [\tfrac{500}{2}] = 250 \quad \text{n}\,(\text{B}) = [\tfrac{500}{3}] = 166 \quad \text{n}\,(\text{C}) = [\tfrac{500}{5}] = 100 \quad \text{n}\,(\text{D}) = [\tfrac{500}{7}] = 71$$

$$\text{n}\,(\text{A}\cap B) = [\tfrac{500}{2\times 3}] = 83 \quad \text{n}\,(\text{A}\cap C) = [\tfrac{500}{2\times 5}] = 50 \quad \text{n}\,(\text{A}\cap D) = [\tfrac{500}{2\times 7}] = 35$$

$n(B \cap C) = [\frac{500}{3 \times 5}] = 33$ $n(B \cap D) = [\frac{500}{3 \times 7}] = 23$ $n(C \cap D) = [\frac{500}{5 \times 7}] = 14$

$n(A \cap B \cap C) = [\frac{500}{2 \times 3 \times 5}] = 16$ $n(A \cap B \cap D) = [\frac{500}{2 \times 3 \times 7}] = 11$ $n(A \cap C \cap D) = [\frac{500}{2 \times 5 \times 7}] = 7$

$n(B \cap C \cap D) = [\frac{500}{3 \times 5 \times 7}] = 4$ $n(A \cap B \cap C \cap D) = [\frac{500}{2 \times 3 \times 5 \times 7}] = 2$

By the principle of Inclusion - Exclusion, the number of integers between 1 and 500 that are divisible by at least one of 2, 3, 5 and 7 is given by

$$\mathbf{n}(A \cup B \cup C \cup D) = \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) + \mathbf{n}(D) - \mathbf{n}(A \cap B) - \mathbf{n}(A \cap C) - \mathbf{n}(A \cap D) - \mathbf{n}(B \cap C) - \mathbf{n}(B \cap D) - \mathbf{n}(C \cap D) + \mathbf{n}(A \cap B \cap C) + \mathbf{n}(A \cap B \cap D) + \mathbf{n}(A \cap C \cap D) + \mathbf{n}(B \cap C \cap D) - \mathbf{n}(A \cap B \cap C \cap D)$$

= (250 + 166 + 100 + 71) − (83 + 50 + 35 + 33 + 23 + 14) + (16 + 11 + 7+ 4)- 2
= 587 − 238 + 38 − 2 = 385

$$\therefore n(A \cup B \cup C \cup D) = 385.$$

Number of integers between 1 and 500 that are not divisible by any of the integers 2, 3, 5 and 7

$$= \text{Total} - n(A \cup B \cup C \cup D) = 500 - 385 = 115.$$

3.. Find the numbers of positive integers ≤ 1000 and not divisible by any of 3, 5, 7 and 22.

Solution: Let A, B, C, D be the sets of integers that lie between 1 and 1000 and that are divisible by 3, 5, 7 and 22 respectively.

To find :n($A \cup B \cup C \cup D$)

We know that

$$\mathbf{n}(A \cup B \cup C \cup D) = \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) + \mathbf{n}(D) - \mathbf{n}(A \cap B) - \mathbf{n}(A \cap C) - \mathbf{n}(A \cap D) - \mathbf{n}(B \cap C) - \mathbf{n}(B \cap D) - \mathbf{n}(C \cap D) + \mathbf{n}(A \cap B \cap C) + \mathbf{n}(A \cap B \cap D) + \mathbf{n}(A \cap C \cap D) + \mathbf{n}(B \cap C \cap D) - \mathbf{n}(A \cap B \cap C \cap D)$$

$n(A) = [\frac{1000}{3}] = 333$ $n(B) = [\frac{1000}{5}] = 200$ $n(C) = [\frac{1000}{7}] = 142$ $n(D) = [\frac{1000}{22}] = 45$

$n(A\cap B) = [\frac{1000}{3\times 5}] = 66$ $n(A\cap C) = [\frac{1000}{3\times 7}] = 47$ $n(A\cap D) = [\frac{1000}{3\times 22}] = 15$

$n(B\cap C) = [\frac{1000}{5\times 7}] = 28$ $n(B\cap D) = [\frac{1000}{5\times 22}] = 9$ $n(C\cap D) = [\frac{1000}{7\times 22}] = 6$

$n(A\cap B\cap C) = [\frac{1000}{3\times 5\times 7}] = 9$ $n(A\cap B\cap D) = [\frac{1000}{3\times 5\times 22}] = 3$ $n(A\cap C\cap D) = [\frac{1000}{3\times 7\times 22}] = 2$

$n(B\cap C\cap D) = [\frac{1000}{5\times 7\times 22}] = 1$ $n(A\cap B\cap C\cap D) = [\frac{1000}{3\times 5\times 7\times 22}] = 0$

By the principle of Inclusion - Exclusion, the number of integers between 1 and 1000 that are divisible by at least one of 3, 5, 7 and 22 is given by

$$\begin{aligned} \mathbf{n}(A\cup B\cup C\cup D) = {} & \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) + \mathbf{n}(D) - \mathbf{n}(A\cap B) - \mathbf{n}(A\cap C) - \\ & \mathbf{n}(A\cap D) - \mathbf{n}(B\cap C) - \mathbf{n}(B\cap D) - \mathbf{n}(C\cap D) \\ & + \mathbf{n}(A\cap B\cap C) + \mathbf{n}(A\cap B\cap D) + \\ & \mathbf{n}(A\cap C\cap D) + \mathbf{n}(B\cap C\cap D) - \mathbf{n}(A\cap B\cap C\cap D) \end{aligned}$$

$= (333 + 200 + 142 + 45) - (66 + 28 + 47 + 15 + 9 + 6) + (9 + 3 + 2 + 1) - 0$
$= 720 - 171 + 15 - 0 = 564$

$\therefore n(A\cup B\cup C\cup D) = 564.$

Number of integers between 1 and 1000 that are not divisible by any of the integers 3, 5, 7 and 22

$= \text{Total} - n(A\cup B\cup C\cup D) = 1000 - 564 = 436.$

4. Among the first 1000 positive integers: Determine the number of integers which are not divisible by 5, nor by 7, nor by 9.

Solution: Let A, B, C be the sets of integers that lie between 1 and 1000 and that are divisible by 5, 7 and 9 respectively.

To find : $n(A \cup B \cup C)$

$$\mathbf{n}(A \cup B \cup C) = \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) - \mathbf{n}(A \cap B) - \mathbf{n}(A \cap C) - \mathbf{n}(B \cap C) + \mathbf{n}(A \cap B \cap C)$$

$n(A) = [\frac{1000}{5}] = 200$ $\quad n(B) = [\frac{1000}{7}] = 142$ $\quad n(C) = [\frac{1000}{9}] = 111$

$n(A\cap B) = [\frac{1000}{5\times 7}] = 28$ $\quad n(A\cap C) = [\frac{1000}{5\times 9}] = 22$ $\quad n(B\cap C) = [\frac{1000}{7\times 9}] = 15$

$n(A\cap B\cap C) = [\frac{1000}{5\times 7\times 9}] = 3$

The number of integers divisible by 5, 7 and 9 is

$$\mathbf{n}(A \cup B \cup C) = \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) - \mathbf{n}(A \cap B) - \mathbf{n}(A \cap C) - \mathbf{n}(B \cap C) + \mathbf{n}(A \cap B \cap C)$$

$$= (200 + 142 + 111) - (28 + 22 + 15) + 3 = 391.$$

$\therefore\ n(A \cup B \cup C) = 391.$

Hence, the number of integers not divisible by 5, nor by 7, nor by 9

$= \text{Total} - n(A \cup B \cup\ ) = 1000 - 391 = 609.$

5. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French and 114 have taken a course in Russian. Further, 103 have taken a course in both Spanish and French, 23 have taken courses in both Spanish and ussian, and 14 have taken courses in both French and Russian. if 2092 students

have taken at least one of Spanish, French and Russian, how many students have taken a course in all three languages?

Solution: Let A, B and C be the set of students who have taken course in Spanish, French and Russian language respectively.

Given : n $(A) = 1232$,n $(B) = 879$,
n$(C) = 114$ $\mathbf{n}(A \cap \mathrm{B}) = 103$,
n $(A \cap C) = 23$,n $(B \cap C) = 14$,
n $(A \cup B \cup C) = 2092$

To find : n $(A \cap B \cap C)$

We know that

$$\mathbf{n}\,(A \cup B \cup C) = \mathrm{n}\,(A) + \mathrm{n}\,(B) + \mathrm{n}\,(C) - \mathrm{n}\,(A \cap B) - \mathrm{n}\,(A \cap C) - \mathrm{n}\,(B \cap C) + \mathrm{n}\,(A \cap B \cap C)$$

$$\therefore \mathrm{n}\,(A \cap B \cap C) = \mathrm{n}\,(A \cup B \cup C) - \mathrm{n}\,(A) - \mathrm{n}\,(B) - \mathrm{n}(C) + \mathrm{n}\,(A \cap B) + \mathrm{n}\,(A \cap C) + \mathrm{n}\,(B \cap C)$$

$$\therefore \mathrm{n}\,(\mathrm{A} \cap B \cap C) = 2092 - 1232 - 879 - 114 + 103 + 23 + 14 = 7.$$

6. In a survey of 300 students, 64 had taken a mathematics course, 94 had taken a English course, 58 had taken a computer course, 28 had taken both a Mathematics and a computer science course, 26 had taken both a English and Mathematics course, 22 had taken both a English and a computer science course,14 had taken all three courses. How many students were surveyed who had taken none of the three courses?

Solution: Let A, B and C be the set of students who have taken course Mathematics, English and Computer science respectively.

To find : n $(A \cup B \cup C)$

$$\mathbf{n}\,(A \cup B \cup C) = \mathrm{n}\,(A) + \mathrm{n}\,(B) + \mathrm{n}\,(C) - \mathrm{n}\,(A \cap B) - \mathrm{n}\,(A \cap C) - \mathrm{n}\,(B \cap C) + \mathrm{n}\,(A \cap B \cap C)$$

: n $(A) = 64$,n $(B) = 94$,
n$(C) = 58$ $\mathbf{n}(A \cap \mathrm{B}) = 26$
n $(A \cap C) = 28$,n $(B \cap C) = 22$ n $(A \cap B \cap C) = 14$

The number of integers divisible by 5, 7 and 9 is

∴

= (64 + 94 + 58) − (26 + 28 + 22) + 14

= 216 − 76 + 14 = 154.

$\therefore \mathrm{n}\,(A \cup B \cup C) = 154.$

Hence, the number of integers not divisible by 5, nor by 7, nor by 9

$= \mathrm{Total}\ - \mathrm{n}\,(A \cup B \cup C) = 300 - 154 = 146.$

7. There are 250 students in an engineering college. Of these 188 have taken a course in Fortran,100 have taken a course in C and 35 have taken a course in Java. Further 88 have taken courses in both Fortran and C, 23 have taken coursesin both C and Java and 29 have taken courses in both Fortran and Java. If 19 of these students have taken all the three courses, how many of these 250 students have not taken a course in any of these three programming languages?

Solution: Let A, B and C be the set of students who have taken the programming languages Fortran, C and Java respectively.

To find : $n(A \cup B \cup C)$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

: $n(A) = 188$, $n(B) = 100$,
$n(C) = 35$, $n(A \cap B) = 88$
$n(A \cap C) = 29$, $n(B \cap C) = 23$ $n(A \cap B \cap C) = 19$

By the principle of Inclusion - Exclusion, we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$= (188 + 100 + 35) - (88 + 29 + 23) + 19$
$= 323 - 140 + 19 = 202.$
$\therefore n(\cup B \cup C) = 202.$

Number of students who have not taken any courses is
$= \text{Total} - n(\cup B \cup C) = 250 - 202 = 48.$

8) A, B, C and D are subsets of a set U containing 75 elements with the following 0000-properties. Each subset contains 28 elements, the intersection of any two of the subsets contains 12 elements, the intersection of any three of the subsets contains 5 elements, the intersection of all four subsets contains 1 element. How many elements belong to none of the four subsets?

Solution:

Given : A, B, C and D are subsets of a set U containing 75 elements with the following properties.

$n(A) = n(B) = n(C) = n(D) = 28$

$n(A \cap B) = n(A \cap C) = n(A \cap D) = n(B \cap C) = n(B \cap D) = n(C \cap D) = 12$

$n(A \cap B \cap C) = n(A \cap B \cap D) = n(A \cap C \cap D) = n(B \cap C \cap D) = 5$, $n(A \cap B \cap C \cap D) = 1$

To find : $n(A \cup B \cup C \cup D)$

By the principle of Inclusion - Exclusion, we have

$$\mathbf{n}(A \cup B \cup C \cup D) = \mathbf{n}(A) + \mathbf{n}(B) + \mathbf{n}(C) + \mathbf{n}(D) - \mathbf{n}(A \cap B) - \mathbf{n}(A \cap C) - \mathbf{n}(A \cap D) - \mathbf{n}(B \cap C) - \mathbf{n}(B \cap D) - \mathbf{n}(C \cap D) + \mathbf{n}(A \cap B \cap C) + \mathbf{n}(A \cap \mathrm{B} \cap D) + \mathbf{n}(A \cap \mathrm{C} \cap D) + \mathbf{n}(B \cap C \cap D) - \mathbf{n}(A \cap B \cap C \cap D)$$

= (28 + 28 + 28 + 28) − (12 + 12 + 12 + 12 + 12 + 12) + (5 + 5 + 5 + 5) − 1

= 112 − 72 + 20 − 1 = 59.

$\therefore \mathrm{n}\,(A \cup B \cup C \cup D) = 59.$

No. of elements that belongs to none of the four subjects

= Total number of elements - Number of elements in the union of 4 sets

∴ No. of elements that belongs to none of the four subjects

$= \quad \text{Total} - \mathrm{n}\,(\ \cup B \cup C \cup D) = 75 - 59 = 16.$

9. There are 2500 students in a college, of these 1700 have taken a course in C, 1000 have taken a course Pascal and 550 have taken a course in Networking. Further 750 have taken courses in both C and Pascal. 400 have taken courses in both C and Networking, and 275 have taken courses in both Pascal and Networking. If 200 of these students have taken courses in C, Pascal and Networking, then (i) How many of these 2500 students have taken a course in any of these three courses C, Pascal and Networking? (ii) How many of these 2500 students have not taken a course in any of these three courses C, Pascal and Networking?

Solution:

Let A, B and C be set of students who have taken a course in C,Pascal and Networking respectively.

To find : $\mathrm{n}\,(A \cup B \cup C)$

$$\mathbf{n}\,(A \cup B \cup C) = \mathrm{n}\,(A) + \mathrm{n}\,(B) + \mathrm{n}\,(C) - \mathrm{n}\,(A \cap B) - \mathrm{n}\,(A \cap C) - \mathrm{n}\,(B \cap C) + \mathrm{n}\,(A \cap B \cap C)$$

$\mathrm{n}\,(A) = 1700$, $\mathrm{n}\,(B) = 1000$,

$\mathrm{n}(C) = 550$, $\mathbf{n}(A \cap \mathrm{B}) = 750$

$\mathrm{n}\,(A \cap C) = 400$, $\mathrm{n}\,(B \cap C) = 275$ $\mathrm{n}\,(A \cap B \cap C) = 200$

i Number of students who have taken a course in any of these three program is given by

By the principle of Inclusion - Exclusion, we have

$$\therefore \mathbf{n}\,(A \cup B \cup C) = \mathrm{n}\,(A) + \mathrm{n}\,(B) + \mathrm{n}\,(C) - \mathrm{n}\,(A \cap B) - \mathrm{n}\,(A \cap C) - \mathrm{n}\,(B \cap C) + \mathrm{n}\,(A \cap B \cap C)$$

$= (1700 + 1000 + 550) - (750 + 400 + 275) + 200$

$= 3250 - 1425 + 200 = 2025$

$\therefore n(A \cup B \cup C) = 2025.$

ii Number of students who have not taken a course in any of these three courses is

$= \text{Total} - n(A \cup B \cup C) = 2500 - 2025 = 475.$

x...x

Topic 7 & 8:

PERMUTATIONS & COMBINATIONS

PERMUTATIONS:

Definition: A permutation is an arrangement of a given collection of objects in a definite order taking some of the objects (or) all at a time.

Example: Formation of words with the given letters

Results:

1) $nP_n = n!$
2) $nP_r = 0$ if $\mathrm{r} > n$
3) $nP_0 = 1.$

PERMUTATION WITHOUT REPETITION:

PROBLEMS:

1. Find the value of 'r' if $5P_r = 60$.

Solution: $5P_r = 60$

$$5P_r = 5 \times 4 \times 3$$
$$5P_r = 5P_3$$
$$r = 3.$$

2. Find the value of 'n' if $nP_3 = 5nP_2$

Solution: $\mathrm{np}_3 = 5nP_2$

$$(\mathrm{n} - 1)\ (\ - 2) = 5(n - 1)$$
$$n - 2 = 5$$
$$n = 7.$$

3. How many ways are there to select a first-prize winner, a second-prize winner and a third-prize winner from 100 different people who have entered a contest?

Solution:

The number of 3-permutations of a set of 100 elements. Consequently, the answer is $P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200$.

4. How many ways can the word 'MONDAY' be arranged. How many of them begins with M and ends with Y.

Solution:

Number of letters in 'MONDAY' = 6 [No Repetitions]

Total number of ways to arrange = $6\mathrm{P}_6 = 6! = 720$ ways

The word begins with 'M' and ends with 'Y'

The number of ways the remaining 4 letters are arranged= $4\mathrm{P}_4 = 4!$=24ways.

5. Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily.
Consequently, there are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways for the saleswoman to choose her tour.

6. How many permutations of the letters *ABCDEFGH* contain
   a. the string *ABC*
   b. the string *CFGA*
   c. the string *BA & GF*

Solution:
i) number of permutations of six objects, namely, the block *ABC* and the individual letters *D*, *E*, *F*, *G* and *H*.
These six objects can be permuted in $6P_6 = 6! = 720$ permutations

(ii) Let us consider the string CFGA as an object.
∴ We have the following 5 objects : CFGA, B, D, E and H
These 5 objects can be permuted in $5P_5 = 5! = 120$ ways.

(i) Let us consider the string BF and GF as an object.
∴ We have the following 6 objects : BA, C, D, E, GF and H
These 6 objects can be permuted in $6P_6 = 6! = 720$ ways.

PERMUTATION WITH REPETITION:

Suppose a given collection of n–objects.Then the total number of permutations of the $'n'$ things is $= \frac{n!}{n_1!\,n_2!....n_r!}$.

7. In how many ways can the letters of the following word could be arranged
   a. COMPUTER ii) ENGINEERING

Solution: i) COMPUTER
Number of letters = 8 {Since, no repetitions of letters}
Total number of arrangements of all the letters = $8P_8 = 8! = 40320$.

ii) ENGINEERING
Number of letters = 11

Number of times 'E' occur = 3
Number of times 'N' occur = 3
Number of times 'G' occur = 2
Number of times 'I' occur = 2
∴ The total number of arrangements of all the letters

$$= \frac{11!}{3!3!2!2!} = 277200 \text{ ways.}$$

8. There are 5 red, 4 white and 3 blue marbles in the bag. They are drawn one by one and arranged in a row. Find the number of different arrangements.

Solution: Total number of marbles = 5+4+3=12
Number of red marbles = 5, Number of white marbles = 4
Number of blue marbles = 3

Total number of ways the marbles could be arranged = $\frac{12!}{5!4!3!} = 27720$ ways.

9. Find the number of permutations of the letters of the word "MATHEMATICS". Also find the number of arrangements beginning and ending with the same letter.

Solution: Given word is "MATHEMATICS". Total number of letters = 11
Number of times 'M' occur = 2
Number of times 'T' occur = 2
Number of times 'A' occur = 2
∴ Total number of ways the letters are arranged = $\frac{11!}{2!2!2!} = 4989600$ ways.

M, A, T are the letter that could be placed in the beginning and ending.
Suppose M's are the ends, (i.e., beginning and ending) then the other 9-places are to be filled up with 2-A's, 2-T's, 1-H, 1-E, 1-I, 1-C, 1-S.
∴Number of ways this can be done = $\frac{9!}{2!2!1!1!1!1!1!} = \frac{9!}{2!2!}$
Similarly, If A's are the ends , then number of ways this can be done = $\frac{9!}{2!2!}$
If T's are the ends , then number of ways this can be done = $\frac{9!}{2!2!}$
∴ The total number of arrangements = $\frac{9!}{2!2!} + \frac{9!}{2!2!} + \frac{9!}{2!2!} = 272160$ ways.

10. Five boys and 5 girls are to be arranged around a circular table for a discussion so that the boys and girls alternate. In how many ways can they be seated?

Solution:

First arrange the boys around the table.

This can be done in $(5-1)! = 4! = 24$ ways

In each of these arrangements, there are 5 gaps in which the 5 girls can be arranged in $5!$ ways.

∴Total number of ways $= 4!\,5! = 2880$ ways.

COMBINATIONS

Definition: A combination is a selection of objects from a given collection of objects taking some or all of them at a time. The order of selection is immaterial.

- The number of combinations of 'n' different things taken 'r' at a time is denoted by nC_r $\therefore nC_r = \frac{n!}{r!(n-r)!}$.

PROBLEMS:

1. How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

Solution:

5 members can be selected from 10 members in $10C_5$ ways

$$10C_5 = \frac{10!}{5!(10-5)!} = \frac{10!}{5!5!} = 252 \text{ways.}$$

2. A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Solution: The number of such combinations is

$$30C_6 = \frac{30!}{6!24!} = 593{,}775 \text{ ways}$$

.

3. Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule,

The number of ways to select the committee is

= (the number of 3-combinations of a set with nine elements)

× (the number of 4-combinations of a set with 11 elements)

$= 9C_3 \times 11C_4$

$= \frac{9!}{3!6!} \times \frac{11!}{4!7!} = 84\ 330 = 27720.$

4. How many bit strings of length 10 contain

a. Exactly four 1's

b. Atmost four 1's

iii)Atleast four 1's

iv)An equal number of 0's and 1's.

Solution:

i) Exactly four 1's:

A bit string of length 10 can be considered to have 10 positions. These 10 positions should be filled with four 1's and six 0's.

∴ Number of required bit strings of length 10 contains exactly four 1's

$$= 10C_4 = \frac{10!}{4!6!} = 210 \text{ ways.}$$

ii) Atmost four 1's:

The 10 positions should be filled with atmost four 1's

- No 1's and Ten 0's

- One 1's and Nine 0's
- Two 1's and Eight 0's
- Three 1's and seven 0's
- Four 1's and Six 0's

  ∴ Number of bit strings of length 10 contains atmost four 1's

  $=10C_0 + 10C_1 + 10C_2 + 10C_3 + 10C_4 = 386$ ways

iii) Atleast four 1's:

The 10 positions are to be filled up with atleast four 1's

- Four 1's and Six 0's
- Five 1's and Five 0's
- Six 1's and Four 0's

  …………………………..
- Ten 1's and Zero 0's

∴ Number of bit strings of length 10 contains atleast four 1's

$=10C_4 + 10C_5 + 10C_6 + 10C_7 + 10C_8 + 10C_9 + 10C_{10} = 848$ ways

iv) An equal number of 0's and 1's:

The 10 positions are to be filled up with five 1's and five 0's

∴ Required number of bit strings $= 10C_5 = 252$ ways.

5. From a club consisting of six men and seven women. In how many ways we select a committee of

   a. 3 men and 4 women
   b. 4 person which has atleast 1 women
   c. 4 person which has atlmost 1 man
   d. 4 person that has children of both sexes.

Solution: i) A committee of 3 men and 4 women can be selected in

$= 6C_3 \times 7C_4$ ways = 700 ways

i. A committee of 4 person which has atleast 1 women

$= 7C_1 \times 6C_3 + 7C_2 \times 6C_2 + 7C_3 \times 6C_1 + (7C_4 \times 6C_0)$

$= 140 + 315 + 210 + 35 = 700$ ways

ii. For the committee of atmost 1 man

$= 6C_1 \times 7C_3 + 6C_0 \times 7C_4 = 245$ ways

iii. For the committee of both sexes

$= 6C_1 \times 7C_3 + 6C_2 \times 7C_2 + 6C_3 \times 7C_1$

$= 665$ ways.

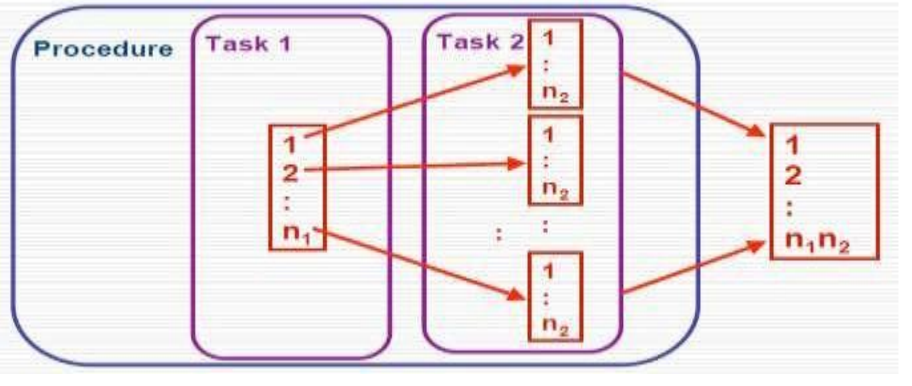
x…………………………………………………………..x

The basics of Counting Principle

- Two basic counting principles
    - The product rule
    - The sum rule

The Product Rule:

Suppose that a procedure can be broken into asequence of two tasks.Assume there are n_1 ways to do the first task.Assume for each of these ways of doing the firsttask, there are $n_2$ways to do the second task.So, there are $n_1 x n_2$ ways to do the procedure.



- For example, if there are n_1 different courses offered in the morning and n_2 courses offered in the afternoon. There will be n_1 xn_2 choices for a student to enroll 1 in morning and 1 in afternoon.

Problems (Two marks)

1. A new company with just two employees rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

Task 1: assigning an office to employee 1
Task 2: assigning an office to employee 2

Task 1 can be done in 12 different ways and
Task 2 can be done in 11 different ways.
By product rule, There are 12 (11) = 132 ways to assign offices to two employees.

2. The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. How many chairs can be labeled differently?

Solution:

Task 1: assigning one of the 26 letters

Task 2: assigning one of the 100 possible integers

Task 1 can be done in 26 different ways and Task 2 can be done in 100 different ways.

By product rule, There are 26(100) = 2600 ways to assign labels to the chairs.

3. How many different license plates are available if each plate contains a sequence of three letters followed by 3 digits?

Solution:

a. Task 1: choose letter 1
b. Task 2: choose letter 2
c. Task 3: choose letter 3
d. Task 4: choose digit1
e. Task 5: choose digit2
f. Task 6: choose digit3

g. By product rule, There are 26 . 26 . 26 .10 .10 . 10 = 17576000 different license plates.

4. If a child can draw 2 kinds of faces and 3 kinds of hates, how many cartoon can she produce?

Solution:

Number of ways faces is drawn=2 ways

Number of ways hats is drawn=3 ways.

By product rule, no. of ways cartoon can be drawn = 2x3=6 ways.

5. (i) How many 3 digit numbers can be formed using 1, 3, 4, 5, 6, 8, 9.

(ii) How many can be formed if no digits can be repeated?

Solution:

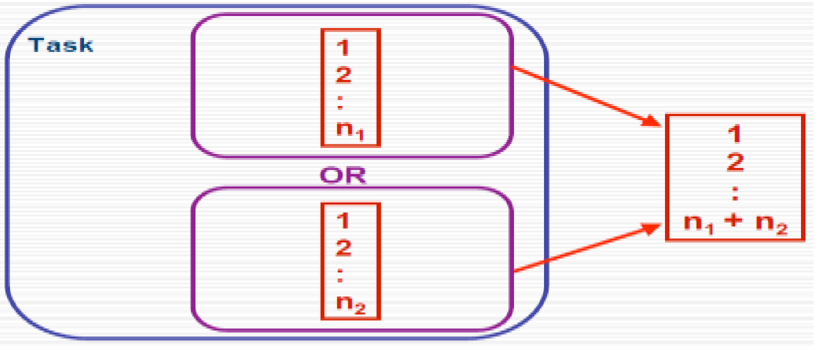
(ii) Number of ways each of 3 digits is filled (using 1, 3, 4, 5, 6, 8, 9) =7 ways.

Number of ways 3 digit numbers formed =7x7x7 ways.

(iii) Number of ways 3 digit numbers formed without repetitions=7x6x5 ways.

The Sum Rule

Assume a task can be done either in one of n_1 ways or in one of n_2 ways. Assume none of the set of n_1 ways is the same as any of the set n_2 ways. So, there are $n_1 + n_2$ ways to do the task.



For example, if there are n_1 different courses offered in the morning and n_2 courses offered in the afternoon. There will be n_1+n_2 choices for a student to enroll in only one course.

6. A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects. No project is on more than one list. How many possible projects are there to choose from?
Solution:
The student can choose a project from the first list or the second list or the third list.
By the sum rule, there are 23+15+19=57 ways to choose a project.

7. Suppose either a CS faculty or CS student must be chosen as representative for a committee. There are 14 faculty, and 50 majors. How many ways are there to choose the representative?
Solution:
By the sum rule, 50 + 14 = 64 ways.

8. In how many ways can we draw a heart or a shade from an ordinary deck of playing cards? a heart or an ace? an ace or a king?
Solution:
a. Number of heart=13, number of shade=13.
Sum rule=Ways to draw heart or shade=13+13=26 ways.
b. Number of heart=13, number of ace =3
Sum rule=Ways to draw heart or ace=13+3=16 ways.
c. Number of ace=4, number of king =4
Sum rule=Ways to draw ace or king=4+4=8 ways.

9. There are 5 Chinese books, 7 English books, 10 French books How many ways to choose two books of different languages from them?
Solution: 5x7+5x10+7x10=155 ways.

TOPIC :9

View the video on ponjesly app.

THE PIGEONHOLE PRINCIPLE

Statement: If *n* is a positive integer and *n +1* or more objects are placed into *n* boxes, then there is atleast one box containing two or more of the objects.
i.e If m pigeons are assigned to n pigeonholes, m>n, then there must be a pigeonhole containing atleast 2 pigeons.

Generalised Pigeonhole Principle:
Statement: If m pigeons are assigned to n pigeonholes, then there must be a pigeonhole containing at $[\frac{m-1}{n}]+1$ pigeons

Problems:

1. If 9 colors are used to paint 100 houses, such that at least 12 houses will be the same color.

   Solution: Number of Pigeons, m= Number of Houses=100
   Number of Pigeonholes, n= Number of colors=9

   By Generalized Pigeonhole principle,

   No. of houses having same color=$[\frac{m-1}{n}]+1=[\frac{100-1}{9}]+1=12$
   Hence at least 12 houses will have the same color.

2. If we select any group of 1000 students on campus. Show that at least three of them must have same birthday.

   Solution: Number of Pigeons, m= Number of students=1000
   Number of Pigeonholes, n= Number of days in a year=366 days

   By Generalized Pigeonhole principle,
   Number of students having same birthday=$[\frac{m-1}{n}]+1=[\frac{1000-1}{366}]+1=3$

   Hence at least 3 students will have the same birthday.

3. What is the maximum number of students required in a class to be sure that atleast 6 will receive the same grade, if there are five possible grades A,B,C,D,F.

Solution: Number of Pigeons , m= Number of students=?

Number of Pigeonholes, n= Number of possible grades=5

Given : At least 6 will receive the same grade.

By Generalized Pigeonhole principle,

$$\Rightarrow [\frac{m-1}{5}] + 1 = 6 \Rightarrow m - 1 + 5 = 30 \Rightarrow m = 26$$

Thus, minimum no. of students needed to ensure that atleast 6 students will receive the same grade is 26.

4. **State and prove pigeon hole principle.**

(or)

If n+1 pigeonholes are occupied by m pigeons, , prove that at least one pigeonhole is occupied by more than one pigeons.

Solution:

Number of Pigeons =n+1

Number of Pigeonholes =n

To prove: At least one Pigeonhole is occupied by more than one pigeon.

Assume that, if at least one pigeonhole is not occupied by more than one pigeon.

⇒Each pigeonhole contains exactly one pigeon.

So n pigeonhole contain n pigeons
Which is a contradiction because there are n+1 pigeons

Hence our assumption is wrong.

x...x