Navier-Stokes Equations and Turbulence Modelling

Assignment 1

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1 Kinematic Viscosity and Bulk Reynolds Number

The friction Reynolds number is defined as $Re_\tau = \frac{u_\tau h}{\nu}$, where ν is kinematic viscosity, u_τ is the wall friction velocity and h is the channel half-width. Therefore, the kinematic viscosity is given by

$$\nu = \frac{u_\tau h}{Re_\tau} \tag{1}$$

The wall friction velocity is then determined using the following relation.

$$u_\tau^2 = \nu \frac{d\bar{u}}{dy}|_{wall} \tag{2}$$

where $\frac{d\bar{u}}{dy}|_{wall}$ is the velocity gradient in the y-direction evaluated at the wall.

Combining equations (1) and (2) gives the following relation.

$$\nu = \frac{h^2}{Re_\tau^2}\frac{d\bar{u}}{dy}|_{wall} \tag{3}$$

Considering the channel half-width is given $h = 1$ *(measured in cm)* and the friction Reynolds number is given $Re_\tau = 181.17$, estimating the kinematic viscosity just becomes a matter of determining $\frac{d\bar{u}}{dy}|_{wall}$.

The mean velocity at various points in the turbulent channel is included in the data file given, and so using this information one can determine an estimate for this value.

The kinematic viscosity was determined to be approximately $2.354 \times 10^{-8} \frac{m^2}{s}$.

The bulk Reynolds number is defined as

$$Re_b = \frac{U_b h}{\nu} \tag{4}$$

where U_b is the bulk velocity, and for turbulent channel flow is given by

$$U_b = \frac{1}{2h}\int_0^{2h} \bar{u} dy$$

.

Using the given data and the *trapz* function, the bulk velocity was determined to be $6.667 \times 10^{-3} \frac{m}{s}$.

Inserting these values into equation (4) gives $Re_b = 2832.3$.

2 Plot Mean Flow Profile and the von Kármán Constant

The y-coordinate can be normalised by dividing by δ_ν, where $\delta_\nu = \frac{\nu}{u_\tau}$ is the wall unit which characterises the size of a viscous layer at the wall. The u velocity can also be normalised by dividing by u_τ.

Furthermore, the y-coordinate can also be normalised by the wall half-width h, and u velocity can be normalised by dividing by bulk velocity U_b.

A plot of these normalised values can be seen in figure 1.

Near the centre of the channel $\frac{y}{h} = \mathcal{O}(1)$, the mean velocity is at its greatest, and the mean flow profile has a strong inherent dependency on the channel half-width h. However, far from the centre, the flow contains no information regarding the channel half-width and therefore there is no dependency on h. This region, $y \ll h$, is called the inner layer, and in this region $\frac{d\bar{u}_+}{dy_+} = \frac{1}{y_+}F_i(y_+)$, where F_i is some functional relationship.

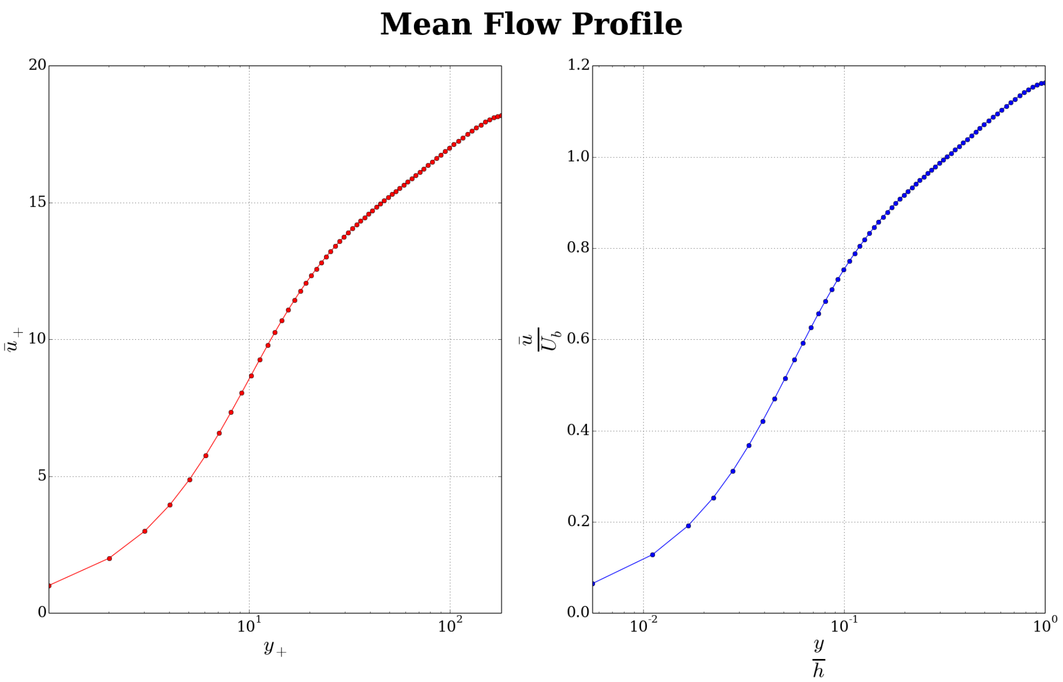

Figure 1: Mean flow profile from y = 0 to y = h, where h is channel half-width

Conversely, near the wall $y_+ = \mathcal{O}(1)$, the mean flow profile is heavily affected by the shear stress. In this viscous region, the mean flow profile is strongly dependent on the viscosity ν. However, far from the wall the flow is not affected by viscous shear stresses and therefore the mean flow profile does not have any dependency on ν. This region is called the outer layer, and in this region $\frac{d\bar{u}_+}{d(y/h)} = \frac{1}{(y/h)}F_o(\frac{y}{h})$, where F_o is some functional relationship.

The above imply a region of overlap, $\delta_\nu \ll y \ll h$, where both of these functional relationships hold, implying the following functional relationship between $\bar{u}_+$ and y_+ in this region.

$$\bar{u}_+ = \frac{1}{\kappa}\ln y_+ + B \tag{5}$$

By taking the gradient of the above curve between two points in the aforementioned region, one can find the value of κ, referred to as the von Kármán constant.

In this coursework it was found that $\kappa \approx 0.39$, which is reasonably close to the empirically found value of $\kappa = 0.41$ [1].

3 Plots of Reynolds Stresses

The full turbulent velocities are given by the relation

$$u = \bar{u} + u' \tag{6}$$

where u' is the fluctuating velocity. The Reynolds stresses $\langle u_i' u_i' \rangle$ can be determined by squaring equation (6) and averaging.

$$\langle u_i^2 \rangle = \langle \bar{u_i}^2 \rangle + 2\langle \bar{u}_i u_i' \rangle + \langle u_i'^2 \rangle \tag{7}$$

Note that $\bar{u}$ is an average and so $\langle \bar{u_i}^2 \rangle = \bar{u_i}^2$.

Furthermore, the second term in equation (7) goes to zero because $\langle \bar{u}_i u_i' \rangle = \bar{u}_i \langle u_i' \rangle$, and the average of a fluctuating velocity is zero.

Therefore, the Reynolds stresses are given by

$$\langle u_i'^2 \rangle = \langle u_i^2 \rangle - \bar{u_i}^2 \tag{8}$$

The Reynolds stresses for data set one are then normalised by u_τ^2. Plots of these normalised values compared to the Reynolds stresses from data set 2 can be found in figures 2 3 and 4.

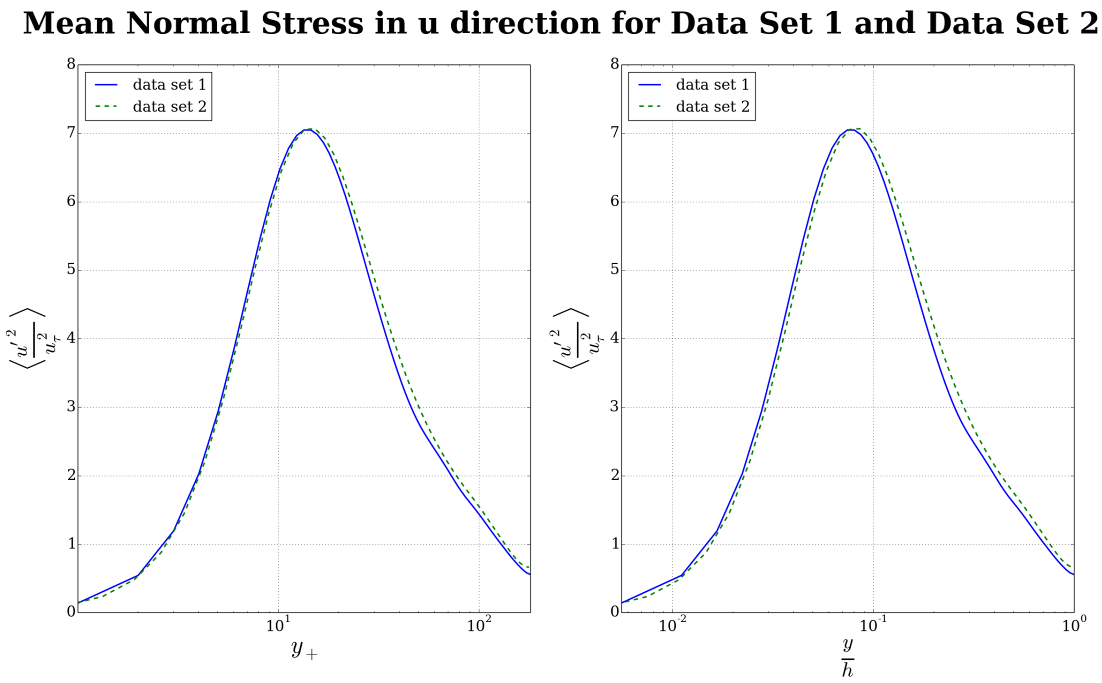

Figure 2: Normalised Reynolds stresses $\langle \frac{u'^2}{u_\tau^2} \rangle$ for turbulent channel flow, where the channel has width 2cm. Time-averaged friction Reynolds of $Re_\tau \approx 181.17$ for data set 1 and $Re_\tau \approx 178$ for data set 2.

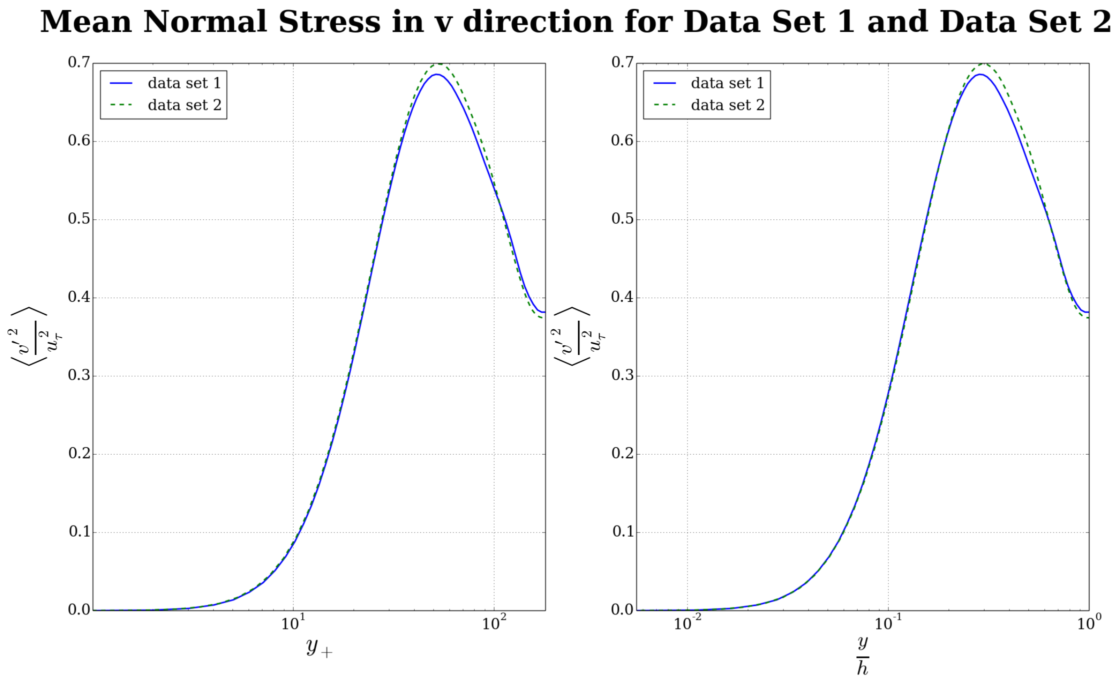

Figure 3: Normalised Reynolds stresses $\langle \frac{v'^2}{u_\tau^2} \rangle$ for turbulent channel flow.

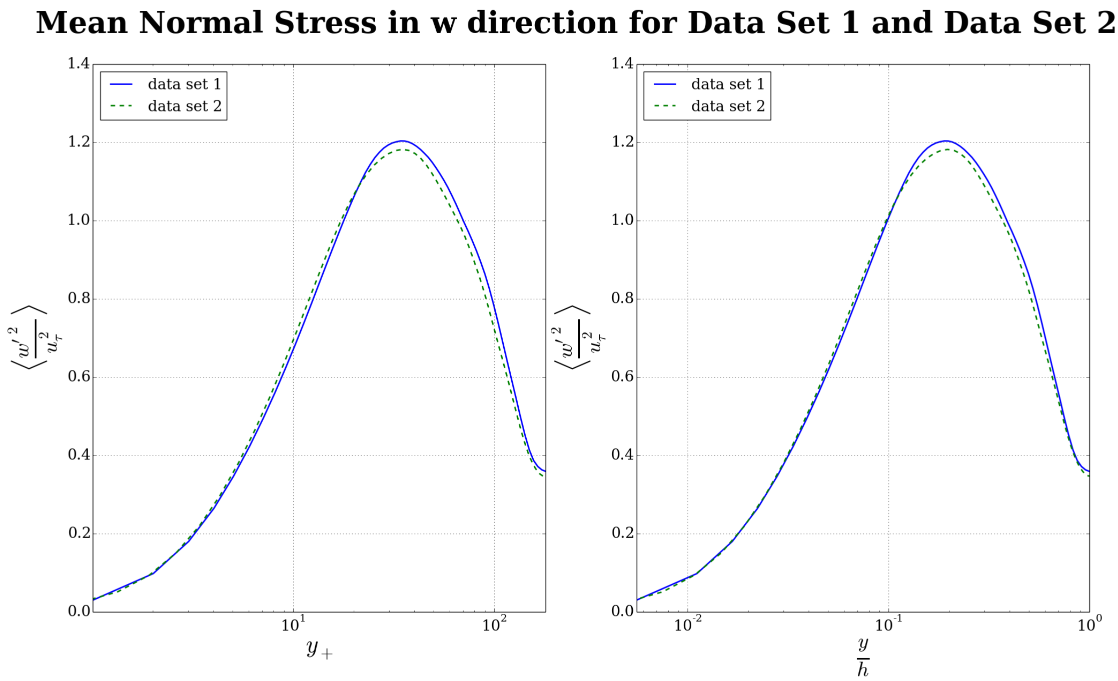

Figure 4: Normalised Reynolds stresses $\langle \frac{w'^2}{u_\tau^2} \rangle$ for turbulent channel flow.

As can be seen from the plots above, the mean normal Reynolds stresses of both sets of data are very similar. This is to be expected as the time-averaged friction Reynolds numbers from both sets of data are also very similar, with $Re_\tau \approx 181.17$ for data set 1 and $Re_\tau \approx 178$ for data set 2.

However, the normalised Reynolds stresses in the inner layer $y \ll h$ are more or less equivalent for both sets of data. This is a by-product of Prandtl's law of the wall, which states that $\bar{u}_+$ is dependent on y_+ only. Therefore, it is not dependent on any other properties of the flow and so the plots of $\bar{u}_+$ of y_+ will produce the same curve, even if the flow properties are markedly different [1].

There is a slight difference between the values $\langle \frac{u'^2}{u_\tau^2} \rangle$ for the two data sets, as it is slightly larger for the second set of data. This is also true for $\langle \frac{v'^2}{u_\tau^2} \rangle$. This slight discrepancy might be explained by the slight difference in Reynolds number, which corresponds to a difference in inertial forces. This would lead to differences in the Reynolds stresses between the data.

Furthermore, the average fluctuating kinetic energy per unit mass is defined as the sum of the mean normal stresses divided by two.

$$k \equiv \frac{1}{2} \langle u_i' u_i' \rangle \tag{9}$$

So that by summing the Reynolds mean normal stresses plotted above, one can find the time-averaged fluctuating kinetic energy (non-dimensionalised by u_τ^2). Furthermore, the mean normal stress for the u velocity component dominates this term, as it directly receives energy from the mean flow [2].

It might then be expected that $\langle u'^2 \rangle$ would be at it's greatest at the mixing layer, where the energy dissipation from the large scale to the small scale is occurring. Indeed, the maximum occurs at $y_+ \approx 12$, which corresponds to the middle of the buffer layer [3].

4 Profile of Reynolds Stresses

The y-integrated axial Reynolds averaged momentum balance can be shown to be given by the following.

$$\nu \frac{d\bar{u}}{dy} - \langle u'v' \rangle = u_\tau^2 (1 - \frac{y}{h}) \tag{10}$$

Therefore, upon reordering equation (10) one can deduce the normalised Reynolds stress.

$$\langle \frac{u'v'}{u_\tau^2} \rangle = \frac{\nu}{u_\tau^2} \frac{d\bar{u}}{dy} - 1 + \frac{y}{h} \tag{11}$$

Plots of the normalised Reynolds stress from data set 1 compared to data set 2 can be found in figures 5 and 6.

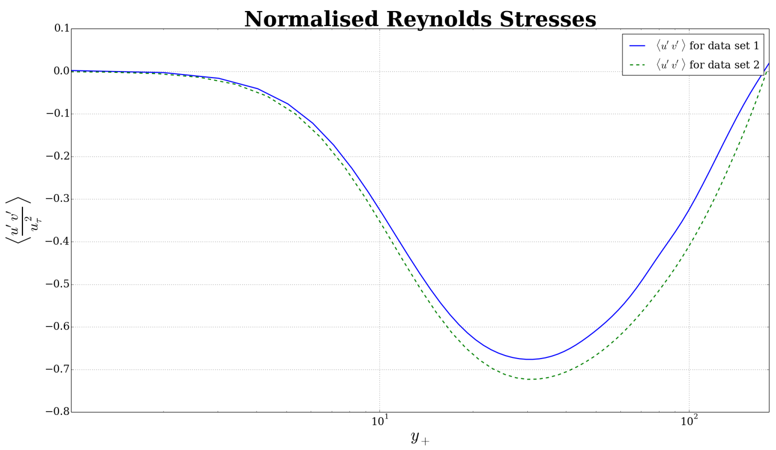

Figure 5: Normalised Reynolds stresses $\langle \frac{u'v'}{u_\tau^2} \rangle$ for two turbulent channel flow experiments of similar friction Reynolds number. Time-averaged friction Reynolds of $Re_\tau \approx 181.17$ for data set 1 and $Re_\tau \approx 178$ for data set 2.

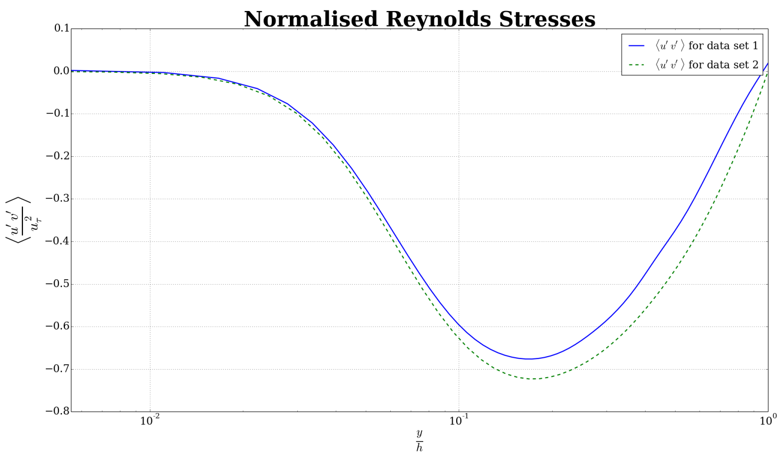

Figure 6: Normalised Reynolds stresses $\langle \frac{u'v'}{u_\tau^2} \rangle$ for turbulent channel flow, where the channel has width 2cm. Time-averaged friction Reynolds of $Re_\tau \approx 181.17$ for data set 1 and $Re_\tau \approx 178$ for data set 2.

First thing one may notice is the fact that the Reynolds stresses are approximately zero for $0 \leq y_+ \leq 5$. This region is referred to as the viscous sub-layer. In this region viscous stresses dominate the Reynolds stresses [3], and so we see minimal growth in the magnitude of the Reynolds stress here.

However, as we move away from the wall the magnitude Reynolds stress grows until it reaches its peak around $y_+ = 12$. As stated in task 3, this region corresponds to the middle of the buffer layer [3]. In this region the Reynolds stress and viscous stress are of approximately equal magnitude, acting in opposite directions i.e. $-u_\tau^2 = \langle u'v' \rangle$.

This result was proven in the lecture notes [1] by considering the y-integrated axial Reynolds averaged momentum balance i.e. equation (10) in this region, and considering the dependencies of $\frac{d\bar{u}}{dy}$ (briefly mentioned in task 2).

The slight discrepancy between the values of Reynolds stresses for the two sets of data might be explained by the difference in the friction Reynolds number, which is proportional to u_τ. Therefore, as Re_τ increases, one may expect the ratio to turbulent stresses and viscous stresses to also increase.

Appendices

In this assignment, Python was used to read the data file and perform calculations. The plots were developed using the *matplotlib* package. In order for others to be able to reproduce the results presented in this assignment, all of the source code used has been included.

```
import matplotlib.pyplot as plt
plt.rcParams["font.family"] = 'serif'
import math as m
import numpy as np
import matplotlib.font_manager as font_manager
import pylab as plot
params = {'legend.fontsize': 20,
          'legend.linewidth': 2}
plot.rcParams.update(params)


################################# Upload Data ####################################
##################################################################################

data1 = np.loadtxt("CHANNEL_DATA.txt")
y1 = data1[:,0]
u1 = data1[:,1]
v1 = data1[:,2]
w1 = data1[:,3]
uu1 = data1[:,4]
vv1 = data1[:,5]
ww1 = data1[:,6]

data2 = np.loadtxt("MOSER_KIM_MANSOUR.txt")
y2 = data2[:, 0]
yplus2 = data2[:, 1]
uuprime2 = data2[:, 2]
vvprime2 = data2[:, 3]
wwprime2 = data2[:, 4]
uvprime2 = data2[:, 5]
uwprime2 = data2[:, 6]
vwprime2 = data2[:, 7]

################################# Data Set 1 #####################################
##################################################################################

h = 1
Re_tau1 = 181.17

grad = np.gradient(u1, 0.005556)
Ub1 = np.trapz(u1, y1)/(2*h)

nu1 = (grad[0]*h**2)/Re_tau1**2
Rb1 = Ub1*h/nu1

u_tau = m.sqrt(nu1*grad[0])
delta_nu = nu1/u_tau
```

```
uplus1 = (1/u_tau)*u1
yplus1 = (1/delta_nu)*y1

u_ub = (1/Ub1)*u1

lny = np.zeros(64)
lob = np.zeros(64)

for i in range (0,64):
        lny[i] = m.log(yplus1[i+1])

[ka, B] = np.polyfit(lny[38:49], uplus1[39:50], 1)

for i in range (0,64):
        lob[i] = (lny[i] - B)/ka

print 1/ka

############################ Plots for Question 2 ###################################
#####################################################################################

fig = plt.figure(1)
fig.suptitle('Mean Flow Profile', fontsize=40, fontweight='bold')
label_size = 20
plt.rcParams['xtick.labelsize'] = label_size
plt.rcParams['ytick.labelsize'] = label_size
plt.subplot(121)
plt.plot(yplus1[0:65], uplus1[0:65], 'r-o')
plt.xscale('log')
plt.xlabel( r'$y_+$', fontsize=28)
plt.ylabel(r'$\bar{u}_+$', fontsize=28)
axes = plt.gca()
axes.set_xlim([min(yplus1[0:65]),max(yplus1[0:65])])
plt.grid(True)

plt.subplot(122)
plt.plot(y1[0:65], u_ub[0:65], 'b-o')
plt.xscale('log')
plt.xlabel( r'$\frac{y}{h}$', fontsize=40)
plt.ylabel(r'$\frac{\bar{u}}{U_b}$', fontsize=40)
axes = plt.gca()
axes.set_xlim([min(y1[0:65]),max(y1[0:65])])
plt.grid(True)

plt.show()

################################## Question 3 ######################################
#####################################################################################

uuprime1 = np.zeros_like(uuprime2)
vvprime1 = np.zeros_like(vvprime2)
wwprime1 = np.zeros_like(wwprime2)
```

```
for i in range(len(uuprime2)):
        uuprime1[i] = (uu1[i]-u1[i]**2)/(u_tau**2)
        vvprime1[i] = (vv1[i]-v1[i]**2)/(u_tau**2)
        wwprime1[i] = (ww1[i]-w1[i]**2)/(u_tau**2)

############################## uu stresses #####################################
################################################################################

fig = plt.figure(1)
fig.suptitle('Mean Normal Stress in u direction for Data Set 1 and Data Set 2'
        , fontsize=40, fontweight='bold')
label_size = 20
plt.rcParams['xtick.labelsize'] = label_size
plt.rcParams['ytick.labelsize'] = label_size

plt.subplot(121)
plt.plot(yplus1[0:65], uuprime1, linewidth=2,
        label = 'data set 1')
plt.xscale('log')
plt.hold(True)
plt.plot(yplus2, uuprime2,linewidth=2, linestyle='--',
        label = 'data set 2')
plt.xlabel( r'$y_+$', fontsize=32)
plt.ylabel(r'$\langle \frac{{{u}^\prime}^2}{u_\tau^2} \rangle$', fontsize=40)
axes = plt.gca()
axes.set_xlim([min(yplus1[1:65]),max(yplus1[1:65])])
plt.legend(loc="upper left")
plt.grid(True)

plt.subplot(122)
plt.plot(y1[0:65], uuprime1, linewidth=2,
        label = 'data set 1')
plt.xscale('log')
plt.hold(True)
plt.plot(y2, uuprime2,linewidth=2, linestyle='--',
        label = 'data set 2')
plt.xscale('log')
plt.xlabel( r'$\frac{y}{h}$', fontsize=40)
plt.ylabel(r'$\langle \frac{{{u}^\prime}^2}{u_\tau^2} \rangle$', fontsize=40)
axes = plt.gca()
axes.set_xlim([min(y1[1:65]),max(y1[1:65])])
plt.legend(loc="upper left")
plt.grid(True)

plt.show()

########################## vv stresses #########################################
################################################################################

fig = plt.figure(1)
fig.suptitle('Mean Normal Stress in v direction for Data Set 1 and Data Set 2'
        , fontsize=40, fontweight='bold')
```

```
label_size = 20
plt.rcParams['xtick.labelsize'] = label_size
plt.rcParams['ytick.labelsize'] = label_size

plt.subplot(121)
plt.plot(yplus1[0:65], vvprime1, linewidth=2,
        label = 'data set 1')
plt.xscale('log')
plt.hold(True)
plt.plot(yplus2, vvprime2,linewidth=2, linestyle='--',
        label = 'data set 2')
plt.xlabel( r'$y_+$', fontsize=32)
plt.ylabel(r'$\langle \frac{{{v}^\prime}^2}{u_\tau^2} \rangle$', fontsize=40)
axes = plt.gca()
axes.set_xlim([min(yplus1[1:65]),max(yplus1[1:65])])
plt.legend(loc="upper left")

plt.grid(True)

plt.subplot(122)
plt.plot(y1[0:65], vvprime1, linewidth=2,
        label = 'data set 1')
plt.xscale('log')
plt.hold(True)
plt.plot(y2, vvprime2,linewidth=2, linestyle='--',
        label = 'data set 2')
plt.xscale('log')
plt.xlabel( r'$\frac{y}{h}$', fontsize=40)
plt.ylabel(r'$\langle \frac{{{v}^\prime}^2}{u_\tau^2} \rangle$', fontsize=40)
axes = plt.gca()
axes.set_xlim([min(y1[1:65]),max(y1[1:65])])
plt.legend(loc="upper left")

plt.grid(True)

plt.show()

################################# ww stresses ####################################
##################################################################################
fig = plt.figure(1)
fig.suptitle('Mean Normal Stress in w direction for Data Set 1 and Data Set 2',
 fontsize=40, fontweight='bold')
label_size = 20
plt.rcParams['xtick.labelsize'] = label_size
plt.rcParams['ytick.labelsize'] = label_size

plt.subplot(121)
plt.plot(yplus1[0:65], wwprime1, linewidth=2,
        label = 'data set 1')
plt.xscale('log')
plt.hold(True)
plt.plot(yplus2, wwprime2,linewidth=2, linestyle='--',
        label = 'data set 2')
```

```
plt.xlabel( r'$y_+$', fontsize=32)
plt.ylabel(r'$\langle \frac{{{w}^\prime}^2}{u_\tau^2} \rangle$', fontsize=40)
axes = plt.gca()
axes.set_xlim([min(yplus1[1:65]),max(yplus1[1:65])])
plt.legend(loc="upper left")

plt.grid(True)

plt.subplot(122)
plt.plot(y1[0:65], wwprime1, linewidth=2,
         label = 'data set 1')
plt.xscale('log')
plt.hold(True)
plt.plot(y2, wwprime2,linewidth=2, linestyle='--',
         label = 'data set 2')
plt.xscale('log')
plt.xlabel( r'$\frac{y}{h}$', fontsize=40)
plt.ylabel(r'$\langle \frac{{{w}^\prime}^2}{u_\tau^2} \rangle$', fontsize=40)
axes = plt.gca()
axes.set_xlim([min(y1[1:65]),max(y1[1:65])])
plt.legend(loc="upper left")

plt.grid(True)

plt.show()


# ################################ Question 4 ####################################
# ##############################################################################################
uvprime1 = np.zeros_like(uvprime2)

for i in range(len(uvprime2)):
        uvprime1[i] = (nu1*grad[i])/u_tau**2 - (1-y1[i])

plt.figure()
label_size = 20
plt.rcParams['xtick.labelsize'] = label_size
plt.rcParams['ytick.labelsize'] = label_size
plt.semilogx(yplus1[0:65], uvprime1, linewidth=2,
        label = r'$\langle u^\prime v^\prime \rangle$' ' for data set 1')
plt.title('Normalised Reynolds Stresses',
        fontsize=40, fontweight='bold')
plt.xlabel( r'$y_+$', fontsize=32)
plt.ylabel(r'$\langle \frac{u^\prime v^\prime}{u_\tau^2} \rangle$', fontsize=40)
plt.hold(True)
plt.semilogx(yplus2, uvprime2,linewidth=2, linestyle='--',
        label = r'$\langle u^\prime v^\prime \rangle$' ' for data set 2')
plt.legend()
axes = plt.gca()
axes.set_xlim([min(yplus1[1:65]),max(yplus1[1:65])])
plt.grid(True)
plt.show()
```

```
plt.figure()
label_size = 20
plt.rcParams['xtick.labelsize'] = label_size
plt.rcParams['ytick.labelsize'] = label_size
plt.semilogx(y1[0:65], uvprime1, linewidth=2,
         label = r'$\langle u^\prime v^\prime \rangle$' ' for data set 1')
plt.title('Normalised Reynolds Stresses', fontsize=40, fontweight='bold')
plt.xlabel( r'$\frac{y}{h}$', fontsize=32)
plt.ylabel(r'$\langle \frac{u^\prime v^\prime}{u_\tau^2} \rangle$', fontsize=40)
plt.hold(True)
plt.semilogx(y2, uvprime2,linewidth=2, linestyle='--',
         label = r'$\langle u^\prime v^\prime \rangle$' ' for data set 2')
plt.legend()
axes = plt.gca()
axes.set_xlim([min(y1[1:65]),max(y1[1:65])])
plt.grid(True)
plt.show()
```

References

[1] Vassilicos, J.C. *A Brief Introduction to Turbulence* [*Navier Stokes Equations and Turbulence Modelling*] Imperial College London, 2016

[2] George, W.K. *Lectures in Turbulence for the 21st Century* January, 2016 http://www.turbulenceonline.com/Publications/Lecture_Notes/xTurbulence_Lille/TB_16January2013.pdf

[3] Sreenivasan, K.R. *Frontiers in Experimental Fluid Mechanics* [*Lecture Notes in Engineering*] Springer Berlin Heidelberg 1986 Volume 46, pages: 159-209