chapter 5

Exercise 5.1

1) $\lim_{x\to 2}(x^2-3) = 4-3 = 1$

2) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\left(3-\frac{1}{x^3}\right) = 1\times 3 = 3$

3) $\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}} = \frac{\lim_{x\to 2} x-2}{\lim_{x\to 2}\sqrt{x+2}} = \frac{2-2}{2} = 0$

4) $\lim_{x\to\frac{1}{2}}\frac{8x^3-1}{6x^2-5x+1} =$

$= \lim_{x\to\frac{1}{2}}\frac{(2x-1)(4x^2+1+2x)}{(2x-1)(3x-1)}$

$= \lim_{x\to\frac{1}{2}}\frac{4x^2+1+2x}{3x-1} = \frac{4\times\frac{1}{4}+1+2\times\frac{1}{2}}{3\times\frac{1}{2}-1} = \frac{3}{\frac{1}{2}} = 6$

5) $\lim_{x\to 0}\frac{4x^3-2x^2+x}{3x^2+2x} = \lim_{x\to 0}\frac{4x^2-2x+1}{3x+2} = \frac{4\lim_{x\to 0}x^2-2\lim_{x\to 0}x+1}{3\lim_{x\to 0}x+2}$

$= \frac{1}{2}$

Exercise 5.2

(1) $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+2x\cdot\Delta x+\Delta x^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta x^2}{\Delta x}$

$= \lim_{\Delta x\to 0} 2x+\Delta x = 2x$

if $x=0$

$f'(x=0) = 2\times 0 = 0$

(2) $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2+(x+\Delta x)-x^2-x}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+\Delta x^2+2x\Delta x+\Delta x-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x^2+2x\Delta x+\Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0}\Delta x+2x+1 = 2x+1$

if $x=0$, $f'(x=0) = 2\times 0+1 = 1$

Exercise 5.3

(1) $f'(x)=1$

(2) $f'(x)=6x^5$

(3) $f'(x)=10e^x$

(4) $f'(x)=\frac{5}{x}$

(5) $f'(x)=\frac{1}{x}\sin x+\ln(x)\cos x$

(6) $f'(x)=\dfrac{\cos x\cdot e^x+e^x\cdot\sin x}{(\cos x)^2}$

(7) $f'(x)=10e^{10x+1}$

(8)

$f(x)=4e^{2x}+5$

$f'(x)=8e^{2x}$

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(9) $f'(x)=e^{3x}\cdot 5\cos 5x+3e^{3x}\sin 5x$

(10) $f'(x)=\dfrac{e^{x^2}\cdot\frac{8}{8x}-\ln(8x)\cdot e^{x^2}\cdot 2x}{(e^{x^2})^2}=\dfrac{1}{xe^{x^2}}-\dfrac{2x\ln(8x)}{e^{x^2}}$

Exercise 5.4

(1) $y'=3x^2\ln x+x^3\frac{1}{x}=3x^2\ln+x^2$

$y''=6x\ln x+3x^2\frac{1}{x}+2x=6x\ln x+5x$

(2) $y'=ae^{-ax}\cdot(-a)=-a^2e^{-ax}$

$y''=-a^2e^{-ax}\cdot(-a)=a^3e^{-ax}=a^2\cdot y$

(3) $y'=-a^2e^{-ax}+abe^{ax}$

$y''=a^3e^{-ax}+a^2be^{ax}=a^2y$

Exercise 5.5

(1) $f'(x) = 3x^2 - 6x - 24 = 0$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x_1 = 4 \quad x_2 = -2$

$f''(x) = 6x - 6$

$f''(4) = 6\times4 - 6 = 18 > 0$

at $x_1 = 4$, $f(x)$ has the minimum value:

$f(x=4) = 4^3 - 3\times4^2 - 24\times4 + 3 = -77$

$f''(-2) = 6\times(-2) - 6 = -18 < 0$

at $x_2 = -2$, $f(x)$ has the maximum value:

$f(x=-2) = 31$

(2) $f'(x) = 12x^2 - 6x = 0$

$2x^2 - x = 0$

$x(2x-1) = 0$

$x_1 = 0, \ x_2 = \frac{1}{2}$

$f''(x) = 24x - 6$

$f''(0) = -6 < 0$

at $x_1 = 0$, $f(x)$ has the maximum value: $f(x=0) = 1$

$f''(\frac{1}{2}) = 6 > 0$

at $x_2 = \frac{1}{2}$, $f(x)$ has the minimum value: $f(x=\frac{1}{2}) = 4(\frac{1}{2})^3 - 3\times(\frac{1}{2})^2 + 1 = \frac{3}{4}$

(3) $f'(x) = 24 - 6x^2 = 0$

$4 - x^2 = 0$

$x = \pm 2$

$f''(x) = -12x$

at $x_1 = 2$

$f''(2) = -24 < 0$

$\therefore$ at $x_1 = 2$, $f(x)$ has the local maximum value: $24\times2 - 2\times8 = 32$.

at $x_2 = -2$

$f''(-2) = 24 > 0$

$\therefore$ at $x_2 = -2$, $f(x)$ has the local minimum value: $24\times(-2) + 16 = -32$.

(4) $f'(x) = 8x + x^{-2} = 0$

$8x^3 + 1 = 0$

$x^3 = -\frac{1}{8}$

$x = -\frac{1}{2}$

$f''(x) = 8 - 2x^{-3}$

$f''(-\frac{1}{2}) = 8 - 2\times\frac{1}{(-\frac{1}{2})^3} = 24 > 0$

at $x = -\frac{1}{2}$, $f(x)$ has the local minimum value:

$4\times(-\frac{1}{2})^2 + \frac{1}{\frac{1}{2}} = 3.$

Exercize 5.5

5 $f'(x)=4x^3-12x^2-4x+12=0$

$x^3-3x^2-x+3=0$

$x^2(x-3)-(x-3)=0$

$(x-3)(x-1)(x+1)=0$

$x_1=3,\ x_2=1,\ x_3=-1$

$f''(x)=12x^2-24x-4$

at $x_1=3$

$f''(x_1=3)=12\times 9-24\times 3-4=32>0$

f(x) has a minimum value.

$f(3)=3^4-4\times 3^3-2\times 3^2+12\times 3+4=-5$

at $x_2=1$

$f''(x_2=1)=12\times 1-24\times 1-4=-16<0$

f(x) has a maximum value.

$f(1)=1-4-2+12+4=11$.

at $x_3=-1$

$f''(x_3=-1)=12\times(-1)^2+24-4=32>0$

f(x) has a minimum value.

$f(-1)=1+4-2-12+4=-5$.

7 $f'(x)=e^{-x}-xe^{-x}=0$

$e^{-x}(1-x)=0$

$x=1$

$f''(x)=-e^{-x}-e^{-x}+xe^{-x}=-2e^{-x}+xe^{-x}$

at $x=1$

$f''(1)=-2e^{-1}+e^{-1}=-e^{-1}<0$

f(x) has a maximum value

$f(1)=e^{-1}$

6.

$f'(x)=(e^x\cos x+e^x\sin x)'$

$=e^x\cos x-e^x\sin x+e^x\sin x+e^x\cos x$

$=2e^x\cos x=0.$

$x_1=\frac{\pi}{2}\quad x_2=\frac{3\pi}{2}.$

$f''(x)=2e^x(\cos x-\sin x)$

at $x_1=\frac{\pi}{2}$

$f''(\frac{\pi}{2})=2e^{\frac{\pi}{2}}(0-1)=-2e^{\frac{\pi}{2}}<0$

f(x) has a maximum value.

$f(\frac{\pi}{2})=e^{\frac{\pi}{2}}.$

at $x_2=\frac{3\pi}{2}$

$f''(\frac{3\pi}{2})=2^{\frac{3\pi}{2}}(0+1)>0$

f(x) has a minimum value.

$f(\frac{3\pi}{2})=-e^{\frac{3\pi}{2}}$

8 $f'(x)=2xe^{-x}-x^2e^{-x}=0$

$xe^{-x}(2-x)=0$

$x_1=0\quad x_2=2$

$f''(x)=2e^{-x}-2xe^{-x}-2xe^{-x}+x^2e^{-x}$

at $x_1=0$.

$f''(x)=2>0$ f(x=0) has a minimum value $=0$

at $x_2=2$

$f''(x)=2e^{-2}-8e^{-2}+4e^{-2}=-2e^{-2}<0$

f(x=2) has a maximum value

$f(x=2)=4e^{-2}.$

Exercise 5.6 ⑤

① $\frac{e^{2x}}{2}+c$

② $-8\frac{x^4}{3+1}+c=-2x^4+c$

③ $6\ln|x|+c$

④ $-3e^{-x}+c$

⑤ $-5\cos x+c$

Exercise 5.7

① $-4\cos x+e^x+c$

② $\frac{\theta^2}{2}+\sin\theta\Big|_0^{\frac{\pi}{3}}=\frac{\pi^2}{18}+\frac{\sqrt{3}}{2}$

③ $\frac{1}{2}e^x-\frac{1}{2}e^{-x}+c$

Exercise 5.8

① set $u=2-x^2$

$du=-2x\,dx$

$dx=\frac{du}{-2x}$

$\int x\sqrt{2-x^2}\,dx=\int x\cdot\sqrt{u}\cdot\frac{du}{-2x}=-\frac{1}{2}\int u^{\frac{1}{2}}\,du=-\frac{1}{3}u^{\frac{3}{2}}+c=-\frac{1}{3}(2-x^2)^{\frac{3}{2}}+c$

② set $u=x-1$

$du=dx.$

$\int\frac{dx}{(x-1)^2}=\int u^{-2}\,du=-\frac{1}{u}+c=-\frac{1}{x-1}+c$

③ set $u=4x.$ when $x=0$, $u=0$, when $x=\frac{\pi}{8}$, $u=\frac{\pi}{2}$.

$du=4\,dx$

$\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin u\,du=-\frac{1}{4}\cos u\Big|_0^{\frac{\pi}{2}}=-\frac{1}{4}\times 0+\frac{1}{4}=\frac{1}{4}$

Exercise 5.8

4 set $u=1+x^3$

$du=3x^2dx$

$$\int \frac{x^2}{u^2}\cdot\frac{du}{3x^2}=\frac{1}{3}\int u^{-2}du=-\frac{1}{3u}+C=-\frac{1}{3(1+x^3)}+C$$

5 set $u=(1+x)^3$, when $x=0$, $u=1$, when $x=1$, $u=8$.

$u^{\frac{1}{3}}=1+x$ $\quad dx=\frac{1}{3}u^{-\frac{2}{3}}du$

$x=u^{\frac{1}{3}}-1$

$$\int_1^8 \frac{u^{\frac{1}{3}}-1}{u}\,\frac{1}{3}u^{-\frac{2}{3}}du$$

$$=\frac{1}{3}\int_1^8 (u^{-\frac{2}{3}}-u^{-1})u^{-\frac{2}{3}}du=\frac{1}{3}\int(u^{-\frac{4}{3}}-u^{-\frac{5}{3}})du=-u^{-\frac{1}{3}}+\frac{u^{-\frac{2}{3}}}{2}=\frac{-1}{1+x}+\frac{1}{2(1+x)^2}\Big|_0^1$$

$$=\frac{1}{8}$$

6 set $u=\sin x$ when $x=\frac{\pi}{6}$, $u=\frac{1}{2}$, when $x=\frac{\pi}{2}$, $u=1$

$du=\cos x\,dx$

$$\int_{\frac{1}{2}}^1 \frac{\cos x}{u^3}\,\frac{du}{\cos x}=\int_{\frac{1}{2}}^1 u^{-3}du=\frac{u^{-2}}{-3+1}\Big|_{\frac{1}{2}}^1=-\frac{u^{-2}}{2}\Big|_{\frac{1}{2}}^1=-\frac{1}{2}+\frac{1}{2\times\frac{1}{4}}$$

$$=\frac{3}{2}$$

7 set $u=1+x^2$

$du=2x\,dx$

$$\int \frac{x}{u}\cdot\frac{du}{2x}=\frac{1}{2}\int u^{-1}du=\frac{1}{2}\ln|1+x^2|+C$$

8 set $u=x^2$

$du=2xdx \Rightarrow dx=\frac{du}{2x}$

$$\int xe^{x^2}dx=\int x\cdot e^u\cdot\frac{du}{2x}=\frac{1}{2}\int e^u du$$

$$=\frac{1}{2}e^{x^2}+C$$

Exercise 5.9

① set $u = x$ $\quad \frac{dv}{dx} = e^x$

$du = dx$

$v = \int dv = \int e^x\, dx = e^x$

$\int u \frac{dv}{dx} = x \cdot e^x - \int e^x\, dx = \underline{\quad\quad} \quad xe^x - e^x + c$

$= (x-1)e^x + c.$

② set $u = x^2$ $\quad \frac{dv}{dx} = \underline{\quad} \sin x$

$du = 2x\,dx$ $\quad v = \int \sin x\, dx = -\cos x$

$\int u\,dv = uv - \int v\,du = \underline{\quad\quad} \quad -x^2 \cdot \cos x + 2\int \cos x \cdot x\,dx$

set $u = x$ $\quad dv = \cos x\,dx$

$du = dx$ $\quad v = \int \cos x\,dx = \sin x$

$\int u\,dv = x \cdot \sin x - \int v\,du = x \cdot \sin x - \int \sin x\,dx \underline{\quad} = x\sin x + \cos x + c$

$\therefore \; -x^2 \cdot \cos x + 2x\sin x + 2\cos x + c$

③ set $u = \ln x$ $\quad \frac{dv}{dx} = x^3$

$du = \frac{1}{x}dx$ $\quad v = \int x^3\,dx = \frac{x^4}{4}$

$\int u\,dv = \ln x \cdot \frac{x^4}{4} - \int v\,du = \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x}\,dx$

$= \ln x \cdot \frac{x^4}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + C$

$= \frac{x^4}{4}\left(\ln x - \frac{1}{4}\right) + c$

Exercise 5.9

(4) set $u = \ln x$ $\quad \frac{dv}{dx} = 1$

$du = \frac{1}{x} dx$ $\quad v = x$

$$\int u\,dv = u \cdot v - \int v\,du = x \cdot \ln x - \int_1^2 x \cdot \frac{1}{x} dx = (x \ln x - x)\Big|_1^2$$
$$= 2\ln 2 - 2 + \ln 1 + 1$$
$$= 2\ln 2 - 1$$

(5) set $u = x$ $\quad \frac{dv}{dx} = \sin 4x$

$\frac{du}{dx} = 1$ $\quad v = -\frac{1}{4}\cos 4x$ by substitution

$$\int u\,dv = u \cdot v - \int v \cdot du$$
$$\doteq -\frac{x}{4}\cos 4x + \frac{1}{4}\int \cos 4x\,dx$$

set $u = 4x$

$du = 4dx$

$$\int \cos u \frac{du}{4} = \frac{1}{4}\sin u + C = \frac{1}{4}\sin 4x + C$$

$\therefore$ we have $-\frac{x}{4}\cos 4x + \frac{1}{16}\sin 4x + C$

(6) set $u = \ln 5x$ $\quad dv = 2x\,dx$

$du = 5 \cdot \frac{1}{5x} dx = \frac{1}{x} dx$ $\quad v = x^2$

$$\int u\,dv = u \cdot v - \int v\,du = x^2 \ln 5x - \int x^2 \frac{1}{x} dx = x^2 \ln 5x - \frac{x^2}{2} + C$$

(7) $\because 2\int \sin^2 x\,dx = -\sin x \cos x + \int (\cos^2 x + \sin^2 x)\,dx$

$$= -\sin x \cos x + x + C$$

$$\therefore \int \sin^2 x\,dx = \frac{x - \sin x \cos x}{2} + C$$