Chapter 7

Exercise 7.1

(1) $XU = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$\overline{XU} = 0$

$$Var(XU) = \frac{(-1)^2 + 2^2 + (-1)^2}{5-1} = 1.5$$

$$\Sigma = cov(X) = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1 \end{bmatrix}$$

$$U^T \Sigma U = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.5$$

$\therefore Var(XU) = U^T \Sigma U$.

(2) $XU = \begin{bmatrix} 23 \\ 17 \\ 12 \\ 17 \\ 24 \\ 15 \end{bmatrix}$

$\overline{XU} = 18.0$

$$Var(XU) = \frac{(23-18)^2 + (17-18)^2 + (12-18)^2 + (17-18)^2 + (24-18)^2 + (15-18)^2}{6-1} = 21.6$$

$$\Sigma = cov(X) = \begin{bmatrix} 1.6 & 0 & -0.2 \\ 0 & 1.6 & 0 \\ -0.2 & 0 & 1.6 \end{bmatrix}$$

$$U^T \Sigma U = \begin{bmatrix} 2, & 3, & 1 \end{bmatrix} \begin{bmatrix} 1.6 & 0 & -0.2 \\ 0 & 1.6 & 0 \\ -0.2 & 0 & 1.6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 21.6$$

Exercise 7.2

Proof.

$$\alpha = X^T A X = x_1^2 a_1 + x_1 x_2 a_2 + x_1 x_3 a_3 + x_1 x_2 a_2 + x_2^2 a_4 + x_2 x_3 a_5 + x_3 x_1 a_3 + x_3 x_2 a_5 + x_3^2 a_6$$

$$\frac{\partial \alpha}{\partial X} = \begin{bmatrix} \frac{\partial \alpha}{\partial x_1} \\ \frac{\partial \alpha}{\partial x_2} \\ \frac{\partial \alpha}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 a_1 + 2x_2 a_2 + 2x_3 a_3 \\ 2x_1 a_2 + 2x_2 a_4 + 2x_3 a_5 \\ 2x_1 a_3 + 2x_2 a_5 + 2x_3 a_6 \end{bmatrix}$$

$$= 2 \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 2Ax$$

$\triangle$

Exercise 7.3

$$\because \beta = (x_1, x_2, x_3) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1^2 + x_2^2 + x_3^2$$

$$\therefore \frac{\partial \beta}{\partial X} = \begin{bmatrix} \frac{\partial \beta}{\partial x_1} \\ \frac{\partial \beta}{\partial x_2} \\ \frac{\partial \beta}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} = 2X$$

$$\because \nu = (x_1, x_2, x_3) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = x_1 a_1 + x_2 a_2 + x_3 a_3$$

$$\therefore \frac{\partial \nu}{\partial X} = \begin{bmatrix} \frac{\partial \nu}{\partial x_1} \\ \frac{\partial \nu}{\partial x_2} \\ \frac{\partial \nu}{\partial x_3} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a.$$

Exercise 7.4

(1) Without normalisation.

$\bar{X} = [3, 3]$

$$\Sigma = cov(x) = \frac{1}{4}\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

$$|\Sigma - \lambda I| = \left| \frac{1}{4}\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 4\lambda & 0 \\ 0 & 4\lambda \end{pmatrix} \right| = 0$$

$$\left| \frac{1}{4}\begin{pmatrix} 4-4\lambda & 1 \\ 1 & 4-4\lambda \end{pmatrix} \right| = 0$$

$$\therefore \quad \frac{1}{4}\left[(4-4\lambda)^2 - 1\right] = 0$$

$$16\lambda^2 - 32\lambda + 15 = 0 \qquad (*)$$

$$(4\lambda-5)\ (4\lambda-3) = 0$$

$$\lambda_1 = \frac{5}{4} \qquad \lambda_2 = \frac{3}{4}$$

For $\lambda_1 = \frac{5}{4}$

$$\frac{1}{4}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} = 0 \Rightarrow u_{11} = u_{12}. \quad e.g \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{u}_1^T = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

For $\lambda_2 = \frac{3}{4}$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} = 0 \Rightarrow u_{21} = -u_{22}. \quad e.g \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\hat{u}_2^T = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

With normalisation: $X' = \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 1 \end{pmatrix}$ $\quad Var = \frac{4}{4} = 1$

$$\Sigma = cov(X') = \begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{pmatrix}$$

$$|\Sigma - \lambda I| = \begin{vmatrix} 1-\lambda & \frac{1}{4} \\ \frac{1}{4} & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \frac{1}{16} = 0$$

$$16\lambda^2 - 32\lambda + 15 = 0$$

It's the same as shown in (*).

Exercise 7.4

(2) Without normalisation.

$$\bar{x} = (2,\ 2)$$

$$\operatorname{cov}(X_1 X_1) = \frac{0+1+4+1+0+0}{5} = \frac{6}{5}$$

$$\operatorname{cov}(X_2 X_2) = \frac{0+0+1+4+1+0}{5} = \frac{6}{5}$$

$$\operatorname{cov}(X_1 X_2) = \frac{2+2}{5} = \frac{4}{5}$$

$$\therefore \Sigma = \operatorname{cov}(X) = \frac{1}{5}\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$$

$$|\Sigma - \lambda I| = \left| \frac{1}{5}\begin{bmatrix} 6-5\lambda & 4 \\ 4 & 6-5\lambda \end{bmatrix} \right|$$

$$\therefore \frac{1}{5}\left[(6-5\lambda)^2 - 16\right] = 0$$

$$25\lambda^2 - 60\lambda + 20 = 0 \quad \text{or} \quad 5\lambda^2 - 12\lambda + 4 = 0$$

$$(5\lambda - 2)(\lambda - 2) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = \frac{2}{5}$$

For $\lambda_1 = 2$,

$$\frac{1}{5}\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} = 0 \Rightarrow u_{11} = u_{12} \quad \text{e.g.} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{u}_1^T = \left[\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right]$$

For $\lambda_2 = \frac{2}{5}$

$$\frac{1}{5}\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} = 0 \Rightarrow u_{21} = -u_{22}. \quad \text{e.g.} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\hat{u}_2^T = \left[-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right]$$

Exercise 7.4

(2) with normalisation: $\bar{x} = [2, \ 2]$, $Var(X_1) = Var(X_2) = \frac{6}{5}$

$$X' = \begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{5}}{\sqrt{6}} & 0 \\ \frac{2\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \\ -\frac{\sqrt{5}}{\sqrt{6}} & -\frac{2\sqrt{5}}{\sqrt{6}} \\ 0 & \frac{\sqrt{5}}{\sqrt{6}} \\ 0 & 0 \end{pmatrix}$$

$$Cov(X_1 X_1) = \frac{1}{5}\left(\frac{5}{6} + \frac{20}{6} + \frac{5}{6}\right) = 1$$

$$cov(X_2 X_2) = 1$$

$$cov(X_1 X_2) = \frac{1}{5}\left(\frac{10}{6} + \frac{10}{6}\right) = \frac{2}{3}$$

$$|\Sigma - \lambda I| = \left|\begin{pmatrix} 1 & \frac{2}{3} \\ \frac{2}{3} & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = (1-\lambda)^2 - \frac{4}{9} = 0$$

$$\lambda^2 - 2\lambda + \frac{5}{9} = 0$$

$$9\lambda^2 - 18\lambda + 5 = 0$$

$$(3\lambda - 1)(3\lambda - 5) = 0$$

$$\lambda_1 = \frac{5}{3} \qquad \lambda_2 = \frac{1}{3}$$

For $\lambda_1 = \frac{5}{3}$, we have $\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} \Rightarrow u_{11} = u_{12}$, $\hat{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

For $\lambda_2 = \frac{1}{3}$, we have $\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} \Rightarrow u_{21} = -u_{22}$, $\hat{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Exercise 7.4 (3)

Without normalisation

$$\begin{pmatrix} 2 & 5 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 5 \\ 5 & 4 & 2 \\ 2 & 3 & 2 \end{pmatrix}$$

Mean: 3 3 3

$$\frac{1}{5}\begin{cases} \mathrm{Cov}\, X_1, X_2 = 1+1+1+4+1 = 8 \\ \mathrm{Cov}\, X_1 X_2 = 4+1+1+1+1 = 8 \\ \mathrm{Cov}\, X_3 X_3 = 1+1+4+1+1 = 8 \end{cases}$$

$$\frac{1}{5}\begin{cases} \mathrm{Cov}\, X_1 X_2 = -2-1+1+2 = 0 \\ \mathrm{Cov}\, X_1 X_3 = -1+1-2+1 = -1 \\ \mathrm{Cov}\, X_2 X_3 = 2+1-2-1 = 0 \end{cases}$$

$$\Sigma = \frac{1}{5}\begin{pmatrix} 8 & 0 & -1 \\ 0 & 8 & 0 \\ -1 & 0 & 8 \end{pmatrix}$$

$$\Sigma - \lambda I = \frac{1}{5}\begin{pmatrix} 8-5\lambda & 0 & -1 \\ 0 & 8-5\lambda & 0 \\ -1 & 0 & 8-5\lambda \end{pmatrix} \Rightarrow \frac{1}{5}\left[(8-5\lambda)^3-(8-5\lambda)\right]=0$$

$$(8-5\lambda)\left[(8-5\lambda)^2-1\right]=0$$

$$(8-5\lambda)(25\lambda^2-80\lambda+63)=0$$

$$(8-5\lambda)(5\lambda-7)(5\lambda-9)=0$$

$$\lambda = \frac{9}{5},\ \frac{8}{5} \text{ or } \frac{7}{5}.$$

$$\lambda = \frac{9}{5} \Rightarrow \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} = 0 \Rightarrow \begin{matrix} u_{12}=0 \\ u_{11}=-u_{13} \end{matrix} \quad \text{eg} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = \frac{8}{5} \Rightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} = 0 \Rightarrow \begin{matrix} u_{21}=0,\ u_{23}=0 \\ u_{22} = \text{anything} \end{matrix} \quad \text{eg.} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = \frac{7}{5} \Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} u_{31} \\ u_{32} \\ u_{33} \end{pmatrix} = 0 \Rightarrow \begin{matrix} u_{31}=u_{33} \\ \&\ u_{32}=0 \end{matrix} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \hat{u}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \hat{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Exercise 7.4 (3) With normalisation

$$\begin{pmatrix} 2 & 5 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 5 \\ 5 & 4 & 2 \\ 2 & 3 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & 0 & -1 \end{pmatrix} \quad var = \tfrac{8}{5} \Rightarrow \begin{pmatrix} \frac{-\sqrt5}{2\sqrt2} & \frac{\sqrt5}{\sqrt2} & \frac{\sqrt5}{2\sqrt2} \\ \frac{\sqrt5}{2\sqrt2} & \frac{-\sqrt5}{2\sqrt2} & 0 \\ \frac{-\sqrt5}{2\sqrt2} & \frac{-\sqrt5}{2\sqrt2} & \frac{-\sqrt5}{2\sqrt2} \\ 0 & \frac{-\sqrt5}{2\sqrt2} & \frac{\sqrt5}{\sqrt2} \\ \frac{\sqrt5}{\sqrt2} & \frac{\sqrt5}{2\sqrt2} & \frac{-\sqrt5}{2\sqrt2} \\ \frac{-\sqrt5}{2\sqrt2} & 0 & \frac{-\sqrt5}{2\sqrt2} \end{pmatrix}$$

$\bar{x}$ 3 3 3

$$Cov\, X_1X_1 = \frac{1}{5}\left(\frac{5}{8} + \frac{5}{8} + \frac{5}{8} + \frac{20}{8} + \frac{5}{8}\right) = 1$$

$$Cov\, X_2X_2 = 1 = Cov\, X_3X_3.$$

$$Cov\, X_1X_2 = \frac{1}{5}\left(\frac{-5}{4} - \frac{5}{8} + \frac{5}{8} + \frac{5}{4}\right) = 0$$

$$Cov\, X_1X_3 = \frac{1}{5}\left(\frac{-5}{8} + \frac{5}{8} - \frac{5}{4} + \frac{5}{8}\right) = -\frac{1}{8}. \Rightarrow \begin{pmatrix} 1 & 0 & -\frac18 \\ 0 & 1 & 0 \\ -\frac18 & 0 & 1 \end{pmatrix}$$

$$Cov\, X_2X_3 = \tfrac{1}{5}\left(\frac{5}{4} + \frac{5}{8} - \frac{5}{4} - \frac{5}{8}\right) = 0.$$

$$|\Sigma - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & -\frac18 \\ 0 & 1-\lambda & 0 \\ -\frac18 & 0 & 1-\lambda \end{vmatrix} \Rightarrow (1-\lambda)^3 - \frac{(1-\lambda)}{64} = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 - \frac{1}{64}\right] \quad \lambda = 1 \quad or \quad \lambda^2 - 2\lambda + \frac{63}{64} = 0$$

$$(1-\lambda)(64\lambda^2 - 128\lambda + 63) = 0$$

$$(1-\lambda)(8\lambda - 7)(8\lambda - 9) \quad \lambda = \tfrac{7}{8} \text{ or } \tfrac{9}{8}. \text{ or } \lambda = 1$$

$$\lambda = \tfrac{9}{8} \Rightarrow \begin{pmatrix} -\frac18 & 0 & -\frac18 \\ 0 & -\frac18 & 0 \\ -\frac18 & 0 & -\frac18 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \begin{matrix} u_{11} = -u_{13} \\ u_{12} = 0 \\ u_{11} = -u_{13} \end{matrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \hat{u}_1 \begin{pmatrix} -\frac{1}{\sqrt2} \\ 0 \\ \frac{1}{\sqrt2} \end{pmatrix}$$

$$\lambda = 1 \Rightarrow \begin{pmatrix} 0 & 0 & -\frac18 \\ 0 & 0 & 0 \\ -\frac18 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \begin{matrix} u_{21} = u_{23} = 0 \\ u_{22} = \text{anything}; \text{ eg} \end{matrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = \tfrac{7}{8} \Rightarrow \begin{pmatrix} \frac18 & 0 & -\frac18 \\ 0 & \frac18 & 0 \\ -\frac18 & 0 & \frac18 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \begin{matrix} u_{31} = u_{33} \\ u_{32} = 0 \end{matrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \hat{u}_3 = \begin{pmatrix} \frac{1}{\sqrt2} \\ 0 \\ \frac{1}{\sqrt2} \end{pmatrix}$$

Chapter 8

Exercise 8.1

(1) $\bar{x} = \frac{2+3}{2} = 2.5$, $\bar{y} = \frac{3+5}{2} = 4$

$$\hat{a}_1 = \frac{(-\frac{1}{2})(-1) + \frac{1}{2} \times 1}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\frac{1}{2}} = 2$$

$$\tilde{a}_0 = 4 - 2 \times 2.5 = -1$$

$$\hat{y} = 2x - 1$$

(2) $\bar{x} = \frac{1+2+3}{3} = 2$. $\bar{y} = \frac{3+2+4}{3} = 3$

$$\tilde{a}_1 = \frac{(-1)0 + 0 + 1}{1+0+1} = \frac{1}{2}$$

$$\hat{a}_0 = 3 - \frac{1}{2} \times 2 = 2$$

$$\tilde{y} = \frac{1}{2}x + 2$$

(3) $\bar{x} = \frac{1+2+3}{3} = 2$, $\bar{y} = \frac{4+2+1.5}{3} = 2.5$

$$\hat{a}_1 = \frac{(-1)(1.5) + 0 + 1 \times (-1)}{1+0+1} = -\frac{2.5}{2} = -1.25 = -\frac{5}{4}$$

$$\tilde{a}_0 = 2.5 + 1.25 \times 2 = 5$$

$$\tilde{y} = 5 - \frac{5}{4}x$$

(4) $\bar{x} = \frac{1+3+5}{3} = 3$, $\bar{y} = \frac{1+4+4}{3} = 3.$

$$\hat{a}_1 = \frac{(-2)(-2) + 0 + 2 \times 1}{4+0+4} = \frac{3}{4}$$

$$\hat{a}_0 = 3 - \frac{3}{4} \times 3 = \frac{3}{4}$$

$$\tilde{y} = \frac{3}{4}x + \frac{3}{4}$$

Exercise 8.2

(1) $X=\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ $\qquad Y=\begin{pmatrix} 4 \\ 2 \\ 1.5 \end{pmatrix}$

$$X^TX=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}=\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$$

$$(X^TX)^{-1}=\frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$$

$$X^TY=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ \frac{3}{2} \end{pmatrix}=\begin{pmatrix} 7\frac{1}{2} \\ 12\frac{1}{2} \end{pmatrix}$$

$$\hat{a}=\frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 7\frac{1}{2} \\ 12\frac{1}{2} \end{pmatrix}=\begin{pmatrix} \frac{105-75}{6} \\ \frac{-45+37.5}{6} \end{pmatrix}=\begin{pmatrix} 5 \\ -1.25 \end{pmatrix}$$

$$\tilde{a}_0=5,\quad \hat{a}_1=-1.25$$

$$\tilde{y}=5-1.25x$$

(2) $X=\begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix}$ $\qquad y=\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$

$$X^TX=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix}=\begin{pmatrix} 3 & 9 \\ 9 & 35 \end{pmatrix}$$

$$(X^TX)^{-1}=\frac{1}{24}\begin{pmatrix} 35 & -9 \\ -9 & 3 \end{pmatrix}$$

$$X^Ty=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}=\begin{pmatrix} 9 \\ 33 \end{pmatrix}$$

$$\tilde{a}=\frac{1}{24}\begin{pmatrix} 35 & -9 \\ -9 & 3 \end{pmatrix}\begin{pmatrix} 9 \\ 33 \end{pmatrix}=\frac{1}{24}\begin{pmatrix} 18 \\ 18 \end{pmatrix}$$

$$\therefore\ \tilde{a}_0=\hat{a}_1=\frac{3}{4}$$

$$\hat{y}=\frac{3}{4}+\frac{3}{4}x$$

Exercise 8.3

a_0	a_1	X	Y	Ypred	error	$\frac{\partial error}{\partial a_0}$	$\frac{\partial error}{\partial a_1}$
1	1	1	1	2	$\frac{1}{2}$	1	1
		3	4	4	0	0	0
		5	4	6	2	2	10
					$2\frac{1}{2}$	3	11

First iteration

$$a_0^{new} = 1 - 0.01 \times 3 = 0.97$$

$$a_1^{new} = 1 - 0.01 \times 11 = 0.89$$

After 1 iteration

a_0	a_1	x	Y	Ypred	error	$\frac{\partial error}{\partial a_0}$	$\frac{\partial error}{\partial a_1}$
0.97	0.89	1	1	1.86	0.3698	0.86	0.86
		3	4	3.64	0.0648	-0.36	-1.08
		5	4	5.42	1.0082	1.42	7.1
					1.4428	1.92	6.88

Exercise 8.4

(1) $\hat{y} = 5 - \frac{5}{4}x$ $\qquad\qquad \bar{y} = 2\frac{1}{2}$

$x=1 \Rightarrow \hat{y}_1 = 3\frac{3}{4}$, $res1 = 4 - 3\frac{3}{4} = \frac{1}{4}$

$x=2 \Rightarrow \hat{y}_2 = 2\frac{1}{2}$, $res2 = 2 - 2\frac{1}{2} = -\frac{1}{2}$

$x=3 \Rightarrow \hat{y}_3 = 1\frac{1}{4}$, $res3 = 1.5 - 1\frac{1}{4} = \frac{1}{4}$

$res1 + res2 + res3 = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$

$target1 + target2 + target3 = 4 + 2 + 1.5 = 7.5$

$\hat{y}_1 + \hat{y}_2 + \hat{y}_3 = 3\frac{3}{4} + 2\frac{1}{2} + 1\frac{1}{4} = 7.5$

$$R^2 = 1 - \frac{(\frac{1}{4})^2 + (\frac{1}{2})^2 + (\frac{1}{4})^2}{(1\frac{1}{2})^2 + (\frac{1}{2})^2 + 1^2} = \frac{25}{28} \approx 0.89$$

Exercise 8.4

(2) $\tilde{y} = \frac{3}{4} + \frac{3}{4}x$, $\bar{y} = 3$

$x=1 \Rightarrow \hat{y}_1 = 1\frac{1}{2}$, $res1 = 1 - 1\frac{1}{2} = -\frac{1}{2}$

$x=3 \Rightarrow \tilde{y}_2 = 3$, $res2 = 4 - 3 = 1$

$x=5 \Rightarrow \tilde{y}_3 = 4\frac{1}{2}$ $res3 = 4 - 4\frac{1}{2} = -\frac{1}{2}$

$res1 + res2 + res3 = -\frac{1}{2} + 1 - \frac{1}{2} = 0$

$target1 + target2 + target3 = 1 + 4 + 4 = 9$

$\hat{y}_1 + \tilde{y}_2 + \tilde{y}_3 = 1\frac{1}{2} + 3 + 4\frac{1}{2} = 9$

$$R^2 = 1 - \frac{(-\frac{1}{2})^2 + 1^2 + (-\frac{1}{2})^2}{2^2 + 1^2 + 1^2} = 1 - \frac{\frac{1}{4} + \frac{1}{4} + 1}{6} = \frac{3}{4} = 0.75$$

Chapter 9

Exercise 9.1

$$x_1 = 1, \quad x_2 = 0, \quad W_{11} = W_{12} = 0.5$$

$$W_{21} = W_{22} = 0.25$$

$$t_1 = 0.25$$

$$t_2 = 0.5$$

$$\varepsilon = 0.1$$

$$y_1 = a_1 = 0.5, \qquad y_2 = a_2 = 0.25$$

$$E = \frac{1}{2}\left[(0.25-0.5)^2 + (0.5-0.25)^2\right] = \frac{1}{16} = 0.0625$$

$$\frac{\partial E}{\partial W_{11}} = -\left(-\frac{1}{4}\right) = \frac{1}{4}, \qquad \frac{\partial E}{\partial W_{12}} = 0$$

$$\frac{\partial E}{\partial W_{21}} = -\frac{1}{4}, \qquad \frac{\partial E}{\partial W_{22}} = 0$$

$$W_{11} = 0.5 - (0.1)\times\left(\frac{1}{4}\right) = 0.475$$

$$W_{12} = 0.5 - 0.1 \times 0 = 0.5$$

$$W_{21} = 0.25 - 0.1 \times (-0.25) = 0.275$$

$$W_{22} = 0.25 - 0 = 0.25$$

$$\therefore Y_1 = a_1 = 0.475 \times 1 + 0.5 \times 0 = 0.475$$

$$Y_2 = a_2 = 0.275 \times 1 + 0.25 \times 0 = 0.275$$

$$E = \frac{1}{2}\left[(0.25-0.475)^2 + (0.5-0.275)^2\right] \doteq 0.051$$

Exercise 9.2

$y_1 = \sigma(0.5) = 0.622, \quad y_2 = \sigma(0.25) = 0.562$

$E = \frac{1}{2}\left((0.25 - 0.62)^2 + (0.5 - 0.562)^2\right) = 0.0711$

$\delta_1 = (-0.372)(0.6222)(0.378) = -0.0875$

$\delta_2 = (-0.062)(0.562)(0.438) = -0.0153$

$\frac{\partial E}{\partial w_{11}} = 0.0875$

$\frac{\partial E}{\partial w_{21}} = 0.0153$

$\frac{\partial E}{\partial w_{12}} = \frac{\partial E}{\partial w_{22}} = 0$

$w_{11} = 0.5 - 0.1 \times 0.0875 = 0.491$

$w_{21} = 0.25 - 0.1 \times 0.0153 = 0.248$

$w_{12} = 0.5, \quad w_{22} = 0.25$

$\therefore y_1 = 0.62 \quad y_2 = 0.561$

$E = 0.0703$

Exercise 9.3

$$z_1^1 = a_1^1 = 0.5, \quad z_2^1 = a_2^1 = 0.25$$

$$y_1 = a_1^2 = \frac{1}{8} + \frac{1}{16} = \frac{3}{16} = 0.1875$$

$$y_2 = a_2^2 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} = 0.375$$

$$E = \frac{1}{2}\left((0.25 - 0.1875)^2 + (0.5 - 0.375)^2\right) = 0.0098$$

$$\delta_1^2 = 0.0625, \quad \delta_2^2 = 0.125$$

$$w_{11}^2 = 0.25 + 0.1 \times 0.0625 \times 0.5 = 0.253$$

$$w_{12}^2 = 0.25 + 0.1 \times 0.0625 \times 0.25 = 0.252$$

$$w_{21}^2 = 0.5 + 0.1 \times 0.125 \times 0.5 = 0.506$$

$$w_{22}^2 = 0.5 + 0.1 \times 0.125 \times 0.25 = 0.503$$

$$g_1'(a_1) = 1, \quad g_1'(a_2) = 1$$

$$\delta_1^1 = \delta_2^1 = 1\,(0.25 \times 0.0625 + 0.5 \times 0.125)$$
$$= 0.0781$$

$$w_{11}^1 = 0.5 + 0.1 \times 0.0781 \times 1 = 0.508$$

$$w_{12}^1 = 0.5 + 0.1 \times 0.0781 \times 0 = 0.5$$

$$w_{21}^1 = 0.25 + 0.1 \times 0.0781 \times 1 = 0.258$$

$$w_{22}^1 = 0.25 + 0.1 \times 0.0781 \times 0 = 0.25$$

$$a_1 = z_1 = 0.508, \quad z_2 = a_2 = 0.258$$

$$y_1 = 0.253 \times 0.508 + 0.252 \times 0.258 = 0.194$$

$$y_2 = 0.506 \times 0.508 + 0.503 \times 0.258 = 0.387$$

$$E = \frac{1}{2}\left[(0.25 - 0.194)^2 + (0.5 - 0.387)^2\right] = 0.008$$