Chapter 6.

Exercise 6.1

① $\frac{\partial f}{\partial x} = 3x^2y + 10xy^2 + 2y^3$

$\frac{\partial f}{\partial y} = x^3 + 10x^2y + 6xy^2$

$\frac{\partial f}{\partial x}\Big|_{\substack{x=3\\y=-1}} = 3\times 9\times(-1) + 10\times 3\times 1 + 2\times(-1) = 1$

$\frac{\partial f}{\partial y}\Big|_{\substack{x=3\\y=-1}} = 27 + 10\times 9\times(-1) + 6\times 3\times 1 = -45$

Exercise 6.1

(2) $\frac{\partial f}{\partial x} = 2x\sin y - 3\cos y$ $\quad \frac{\partial f}{\partial x}\Big|_{\substack{x=1\\y=\frac{\pi}{2}}} = 2\times 1\times 1 - 3\times 0 = 2$

$\frac{\partial f}{\partial y} = x^2\cos y + 3x\sin y$ $\quad \frac{\partial f}{\partial y}\Big|_{\substack{x=1\\y=\frac{\pi}{2}}} = 1\times 0 + 3\times 1\times 1 = 3$

(3) $\frac{\partial f}{\partial x} = 2y^3e^{2x} + 3y^2e^{3x} + 4ye^{4x}$ $\quad \frac{\partial f}{\partial x}\Big|_{\substack{x=0\\y=2}} = 2\times 8 + 3\times 4 + 4\times 2 = 36$

$\frac{\partial f}{\partial y} = 3y^2e^{2x} + 2ye^{3x} + e^{4x}$ $\quad \frac{\partial f}{\partial y}\Big|_{\substack{x=0\\y=2}} = 3\times 4 + 2\times 2 + 1 = 17$

Exercise 6.2

height error: $\pm 0.2\%$ of 5 cm is ± 0.01 cm

base error: $\pm 0.1\%$ of 10cm is ± 0.01 cm.

$$A = \text{area} = \frac{1}{2} h \cdot b$$

$$\Delta A \approx dA = \frac{\partial A}{\partial h} dh + \frac{\partial A}{\partial b} db$$

$$= \frac{b}{2} \Delta h + \frac{h}{2} \Delta b.$$

$$= \frac{10}{2} \times 0.01 + \frac{5}{2} \times 0.01$$

$$= 0.05 + 0.025 = 0.075 \text{ cm}^2.$$

The actural area is

$$\frac{1}{2} \times 10 \times 5 = 25 \text{ cm}^2.$$

So the error represents

$$\frac{0.075}{25} \times 100\% = 0.3\%.$$

Exercise 6.3

(1) $\frac{\partial f}{\partial x} = 12x^3y + 18x^2y^2 - 8xy^3 + y^4$

$\frac{\partial^2 f}{\partial x^2} = 36x^2y + 36xy^2 - 8y^3$

$\frac{\partial f}{\partial y} = 3x^4 + 12x^3y - 12x^2y^2 + 4xy^3$

$\frac{\partial^2 f}{\partial y^2} = 12x^3 - 24x^2y + 12xy^2$

$\frac{\partial^2 f}{\partial x \partial y} = 12x^3 + 36x^2y - 24xy^2 + 4y^3$

$\frac{\partial^2 f}{\partial y \partial x} = 12x^3 + \underline{36}\, x^2y - 24xy^2 + 4y^3$

(2) $\frac{\partial f}{\partial x} = 2x\sin y + 18x^2\cos y$

$\frac{\partial^2 f}{\partial x^2} = 2\sin y + 36x\cos y$

$\frac{\partial^2 f}{\partial x \partial y} = 2x\cos y - 18x^2\sin y$

$\frac{\partial f}{\partial y} = x^2\cos y - 6x^3\sin y$

$\frac{\partial^2 f}{\partial^2 y^2} = -x^2\sin y - 6x^3\cos y$

$\frac{\partial^2 f}{\partial y \partial x} = 2x\cos y - 18x^2\sin y$

Exercise 6.3

(3) $\frac{\partial f}{\partial x} = 2xe^{x^2+y^2}$

$\frac{\partial^2 f}{\partial x^2} = 2e^{x^2+y^2} + 4x^2 e^{x^2+y^2}$

$\frac{\partial^2 f}{\partial x \partial y} = \underline{4x \cdot y} e^{x^2+y^2}$

$\frac{\partial f}{\partial y} = 2ye^{x^2+y^2}$

$\frac{\partial^2 f}{\partial y^2} = 2e^{x^2+y^2} + 4y^2 e^{x^2+y^2}$

$\frac{\partial^2 f}{\partial y \partial x} = \underline{4xy} e^{x^2+y^2}$

(4) $\frac{\partial f}{\partial x} = 3x^2 \ln(y^3+x^3) + (x^3+y^3)\frac{3x^2}{y^3+x^3} = 3x^2(\ln(y^3+x^3)+1)$

$\frac{\partial^2 f}{\partial x^2} = 6x(\ln(y^3+x^3)+1) + 3x^2 \frac{3x^2}{y^3+x^3} = 6x(\ln(y^3+x^3)+1) + \frac{9x^4}{y^3+x^3}$

$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 \frac{3y^2}{y^3+x^3} = \frac{9x^2y^2}{y^3+x^3}$

$\frac{\partial f}{\partial y} = 3y^2 \ln(y^3+x^3) + (x^3+y^3)\frac{3y^2}{y^3+x^3} = 3y^2(\ln(y^3+x^3)+1)$

$\frac{\partial^2 f}{\partial y^2} = 6y(\ln(y^3+x^3)+1) + 3y^2 \frac{3y^2}{y^3+x^3} = 6y(\ln(y^3+x^3)+1) + \frac{9y^4}{y^3+x^3}$

$\frac{\partial^2 f}{\partial y \partial x} = 3y^2 \frac{3x^2}{y^3+x^3} = \frac{9x^2y^2}{y^3+x^3}$

Exercize 6.3 ④

$f(x,y) = x^2 + 3\frac{y}{x}$

(5) $\frac{\partial f}{\partial x} = 2x - 3\frac{y}{x^2}$

$\frac{\partial^2 f}{\partial x^2} = 2 + 6\frac{y}{x^3}$

$\frac{\partial^2 f}{\partial x \partial y} = -\frac{3}{x^2}$

$\frac{\partial f}{\partial y} = \frac{3}{x}$

$\frac{\partial^2 f}{\partial y^2} = 0$

$\frac{\partial^2 f}{\partial y \partial x} = -\frac{3}{x^2}$

Exercise 6.4

① $\frac{dz}{dt} = e^{\sin t} \ln V \cdot \cos t - e^{\sin t} \frac{1}{\cos t} \sin t$

② $\frac{\partial z}{\partial s} = (u^2 \cdot \sin v + \sin v)' = 4s(s^2+t^2)\sin(st^2) + (s^2+t^2)^2 \cos(st^2) \cdot t^2 + \cos(st^2) \cdot t^2$

$\frac{\partial z}{\partial t} = 4t(s^2+t^2)\sin(st^2) + (s^2+t^2)^2 \cos(st^2) \cdot 2st + \cos(st^2) \cdot 2st$

Exercise 6.5

(1) $\frac{\partial f}{\partial x} = 3x^2y^3 \quad \frac{\partial f}{\partial y} = 3x^3y^2$

The gradient vector at the point (1, 1) is [3, 3].

" at the point (2, 1) is [12, 24].

" at the point (1, 2) is [24, 12].

(2) $\frac{\partial f}{\partial x} = 2x\sin y - y^2\sin x, \quad \frac{\partial f}{\partial y} = x^2\cos y + 2y\cos x$

The gradient vector at the point (0, 1) is [0, 2],

" at the point (1, 0) is [0, 1],

" at the point $(\frac{\pi}{2}, \frac{\pi}{2})$ is [0.674, 0]:

" at the point $(\frac{\pi}{4}, \frac{\pi}{4})$ is [0.675, 1.547].

Exercise 6.6

(1) $J = \begin{bmatrix} 2x\sin y, & x^2\cos y \\ y^3\cos x, & 3y^2\sin x \end{bmatrix}$

(2) $J = \begin{bmatrix} \cos\theta, & -r\sin\theta \\ \sin\theta, & r\cos\theta \end{bmatrix}$

(3) $J = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$

Exercise 6.7

① $\frac{\partial f}{\partial x} = e^x y^2 + 2xe^y$

$\frac{\partial f}{\partial y} = 2ye^x + e^y x^2$

$$H(x,y) = \begin{bmatrix} e^x y^2 + 2e^y, & 2ye^x + 2xe^y \\ 2ye^x + 2xe^y, & 2e^x + e^y x^2 \end{bmatrix}$$

② $\frac{\partial f}{\partial x} = 3x^2y^2z - 2yz^3$

$\frac{\partial f}{\partial y} = 2x^3yz - 2xz^3$

$\frac{\partial f}{\partial z} = x^3y^2 - 6xyz^2$

$$H(x,y) = \begin{bmatrix} 6xy^2z, & 6x^2yz - 2z^3, & 3x^2y^2 - 6yz^2 \\ 6x^2yz - 2z^3 & 2x^3z & 2x^3y - 6xz^2 \\ 3x^2y^2 - 6yz^2 & 2x^3y - 6xz^2 & -12xyz \end{bmatrix}$$

Exercise 6.8 (2)

Exercise 6.9

(1) $\frac{\partial f}{\partial x} = 6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x_1 = 2,\ x_1 = -1$

$\frac{\partial f}{\partial y} = 6y^2 + 6y - 12 = 0$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y_1 = -2,\ y_2 = 1$

4 pairs of critical values: $(2,-2), (2,1), (-1,-2), (-1,1)$

$A = \frac{\partial^2 f}{\partial x^2} = 12x - 6,\quad B = \frac{\partial^2 f}{\partial x \partial y} = 0.\quad C = \frac{\partial^2 f}{\partial y^2} = 12y + 6$

At

At point $(2,-2)$: $A=18$, $B=0$ $C=-18$. $AC-B^2<0$ no extreme.

At point $(2,1)$: $A=18>0$, $B=0$, $C=18$. $AC-B^2>0$.

Local minimum: $f=-27$

At point $(-1,-2)$: $A=-18<0$, $B=0$, $C=-18$, $AC-B^2=324>0$

Local maximum: $f=27$

At point $(-1,1)$ $A=-18<0$ $B=0$, $C=18$ $AC-B^2<0$ no extreme.

(2) $\frac{\partial f}{\partial x}=-3x^2-4y-1=0$

$\frac{\partial f}{\partial y}=-4x-4y=0 \Rightarrow x=-y$

$3x^2-4x+1=0$

$(3x-1)(x-1)=0$

$x_1=\frac{1}{3}$, $x_2=1$

2 pairs of critical values: $(\frac{1}{3},-\frac{1}{3}),(1,-1)$

$A=\frac{\partial^2 f}{\partial x^2}=-6x$, $B=\frac{\partial^2 f}{\partial x\partial y}=-4$, $C=\frac{\partial^2 f}{\partial y^2}=-4$

At point $(\frac{1}{3},-\frac{1}{3})$, $A=-2$, $B=-4$, $C=-4$, $AC-B^2=-8<0$

no extreme

At point $(1,-1)$, $A=-6<0$, $B=-4$, $C=-4$, $AC-B^2=8>0$

local maximum: $f=6$

(3) $\frac{\partial f}{\partial x}=2x+2(x+y+1)=4x+2y+2=0$

$\frac{\partial f}{\partial y}=2y+2(x+y+1)=2x+4y+2=0$

$\Rightarrow \begin{cases} x=-\frac{1}{3} \\ y=-\frac{1}{3} \end{cases}$

$A=\frac{\partial^2 f}{\partial x^2}=4>0$, $B=\frac{\partial^2 f}{\partial x\partial y}=2$, $C=4$, $AC-B^2=14>0$

Local minimum: $f=\frac{1}{3}$

Exercise 6.9

(4) $\frac{\partial f}{\partial x} = 6x^2 - 18x = 0 \Rightarrow x^2 - 3x = 0$

$x(x-3) = 0$

$x_1 = 0,\ x_2 = 3$

$\frac{\partial f}{\partial y} = 6y^2 + 6y - 36 = 0 \Rightarrow y^2 + y - 6 = 0$

$(y-2)(y+3) = 0$

$y_1 = 2,\ y_2 = -3$

4 pairs of critical points: $(0,2), (0,-3), (3,2), (3,-3)$

$A = \frac{\partial^2 f}{\partial x^2} = 12x - 18,\quad B = 0,\quad C = \frac{\partial^2 f}{\partial y^2} = 12y + 6$

At $(0,2)$, $A = -18 < 0$, $B = 0$, $C = 30$. $AC - B^2 < 0$ no extreme.

At $(0,-3)$, $A = -18 < 0$ $B = 0$, $C = -30$. $AC - B^2 > 0$

Local maximum: 85

At $(3,2)$, $A = 18 > 0$, $B = 0$, $C = 30$. $AC - B^2 > 0$

Local minimum: -67

At $(3,-3)$: $A = 18 > 0$, $B = 0$. $C = -30$ $AC - B^2 < 0$. no extreme.

Exercise 6.10

(1)
$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} &= 2x + \lambda = 0 \\ \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} &= 2y - \lambda = 0 \\ x - y + 1 &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} x + y = 0 \\ x - y = -1 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2} \\ y = \frac{1}{2} \\ \lambda = 1 \end{cases}$$

$f(-\frac{1}{2}, \frac{1}{2}) = (-\frac{1}{2})^2 + (\frac{1}{2})^2 + 1 = 1\frac{1}{2}$

Exercise 6.10

(2) $\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 3x^2 + \lambda = 0$

$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 3y^2 + \lambda = 0$

$x + y - 1 = 0$

$\Rightarrow \begin{cases} x+y=1 \\ x-y=0 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{2} \\ y=\frac{1}{2} \\ \lambda=-\frac{3}{4} \end{cases}$

Substitute $x=\frac{1}{2}$, $y=\frac{1}{2}$ to the function, we have

$f(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2})^3 + (\frac{1}{2})^3 = \frac{1}{4}$

Exercise 6.11

(2)

Exercise 6.12

(1) $\int_{x=0}^{2} \int_{y=0}^{1} xy \, dx \, dy$

$= \int_{x=0}^{2} x \cdot \frac{y^2}{2} \Big|_{y=0}^{1} dx = \frac{1}{2} \int_0^2 x \, dx = \frac{x^2}{4} \Big|_0^2 = 1$

(2) $\int_{x=-1}^{1} \int_{y=-2}^{2} 2x^2+3y^2+1 \;\; dx \, dy$

$= \int_{x=-1}^{1} \left[(2x^2+1)y + y^3 \Big|_{y=-2}^{2} \right] dx = \int_{x=-1}^{1} (8x^2+20) dx$

$= 45\frac{1}{3}$

(3) $\int_{x=0}^{2} \int_{y=0}^{\frac{\pi}{2}} x \cos y \, dx \, dy$

$= \int_{x=0}^{2} x \sin y \Big|_0^{\frac{\pi}{2}} dx = \int_{x=0}^{2} x \, dx = \frac{x^2}{2} \Big|_0^2 = 2$

Exercise 6.13

① $\int_{x=0}^{y}\int_{y=0}^{1} xy\,dx\,dy$

$= \int_{y=0}^{1}\int_{x=0}^{y} xy\,dx\,dy = \int_{y=0} \frac{x^2}{2}y\Big|_{x=0}^{y} dy = \int_{y=0}^{1} \frac{y^3}{2}\,dy = \frac{y^4}{8}\Big|_0^1 = \frac{1}{8}$

② $\int_{y=0}^{1}\int_{x=y}^{1} 1+x^2+y^2\,dx\,dy$

$= \int_{y=0}^{1} x+\frac{x^3}{3}+y^2x\Big|_{x=y}^{1}\,dy$

$= \int_{y=0}^{1} 1+\frac{1}{3}+y^2-y-\frac{y^3}{3}-y^3\,dy = \int_{y=0}^{1} \frac{4}{3}+y^2-y-\frac{4}{3}y^3\,dy$

$= \frac{4}{3}y+\frac{y^3}{3}-\frac{y^2}{2}-\frac{1}{3}y^4\Big|_0^1 = \frac{5}{6}.$

③ $\int_{x=1}^{2}\int_{y=0}^{2x} 2x^2y+3xy^2\,dx\,dy$

$= \int_{x=1}^{2} x^2y^2+xy^3\Big|_{y=0}^{2x}\,dx = \int_{x=1}^{2} 4x^4+8x^4\,dx = \int_{x=1}^{2} 12x^4\,dx$

$= \frac{12x^5}{5}\Big|_1^2 = \frac{12\times31}{5} = \frac{372}{5} = 74.4$

④ $\int_{x=0}^{1}\int_{y=0}^{1-x^2} 3x^2y^2\,dx\,dy$

$= \int_{x=0}^{1} \left(x^2y^3\Big|_{y=0}^{1-x^2}\right)dx$

$= \int_{x=0}^{1} x^2(1-x^2)^3\,dx$

$= \int_{x=0}^{1} x^2-x^8-3x^4+3x^6\,dx$

$= \frac{x^3}{3}-\frac{x^9}{9}-\frac{3x^5}{5}+\frac{3x^7}{7}\Big|_0^1$

$= \frac{1}{3}-\frac{1}{9}-\frac{3}{5}+\frac{3}{7} = \frac{16}{315} \approx 0.051$

Exercise 6.14

① $\iint_D e^{r^2} r\,dr\,d\theta$ $\qquad r: [0,2] \Rightarrow u=[0,4]$

$= \int_0^{2\pi}\left[\int_0^2 e^{r^2} r\,dr\right]d\theta$ $\qquad u=r^2$, $du = 2r\,dr \Rightarrow r\,dr = \frac{du}{2}$

$= \int_0^{2\pi}\left[\frac{1}{2}\int_0^4 e^u\,du\right]d\theta = \int_0^{2\pi} \frac{1}{2}e^u\Big|_0^4 d\theta = \int_0^{2\pi}\left(\frac{1}{2}e^4-\frac{1}{2}\right)d\theta$

$= \left(\frac{1}{2}e^4 - \frac{1}{2}\right)\theta\Big|_0^{2\pi}$

$= \pi(e^4-1)$

② $\int_0^{\frac{\pi}{2}}\int_1^3 r\cos\theta \cdot r\sin\theta \; r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}}\int_1^3 r^3 \cos\theta\sin\theta\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_1^3 \cos\theta\sin\theta\,d\theta$

$= 20\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta$ $\qquad$ set $u=\sin\theta$, $\theta=[0,\frac{\pi}{2}] \Rightarrow u=[0,1]$

$= 20\int_0^1 u\,du = 20\frac{u^2}{2}\Big|_0^1 = 10$

③ $\int_0^{2\pi}\int_0^1 r\cdot\sin(r^2)\,dr\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[-\cos r^2\Big|_0^1\right]d\theta$ $d\theta$

$= \frac{1}{2}\int_0^{2\pi}[1-\cos 1]\,d\theta$ $\qquad = \frac{1}{2}\left[\theta - \theta\cos 1\right]_0^{2\pi}$

$= \frac{1}{2}\left[2\pi - 2\pi\cos 1\right] = \pi(1-\cos 1)$

≈ 1.44