Chapter 13.

Exercise 13.1

$$L = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{(-\lambda \sum_{i=1}^{n} x_i)}$$

$$\ln L = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i = n(\ln \lambda - \lambda \bar{x}), \text{ where } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

set $\frac{d \ln L}{d\lambda} = 0$, we have

$$\frac{d}{d\lambda}(n(\ln\lambda - \lambda\bar{x})) = \frac{n}{\lambda} - n\bar{x} = 0$$

$$\frac{n}{\lambda} - n \cdot \frac{1}{n} \sum_{i=1}^{n} x_i = 0$$

$$\lambda = \frac{n}{\sum_{i=1}^{n} x_i}$$

(1) $P(X=1 \mid \lambda=1) = e^{-1} \doteq 0.37$

$P(X=2 \mid \lambda=1) = e^{-2} \doteq 0.14$

$P(X=1, X=2 \mid \lambda=1) = 0.37 \times 0.14 \doteq 0.0518$

(2) $P(X=1 \mid \lambda=2) = 2e^{-2} \doteq 0.27$

$P(X=2 \mid \lambda=2) = 2e^{-4} \doteq 0.037$

$P(\cancel{X=1, X=2} \mid \lambda=2) = 0.27 \times 0.037 \doteq 0.01$

(3) $P(X=1 \mid \lambda=\frac{2}{3}) = \frac{2}{3} e^{-\frac{2}{3}} \doteq 0.34$

$P(X=2 \mid \lambda=\frac{2}{3}) = \frac{2}{3} e^{-\frac{4}{3}} \doteq 0.18$

$P(X_1=1, X_2=2 \mid \lambda=\frac{2}{3}) = 0.34 \times 0.18 \doteq 0.0612$

Exercise 13.2

Substitute $x=1$ into $y=-4+3x+\varepsilon$, we have

$$y=-4+3\times1+\epsilon=-1+\epsilon$$

1) $E(y|x=1)=-1$

2) $\operatorname{std}(y|x=1)=\operatorname{std}(-1+\epsilon)=\operatorname{std}(\varepsilon)=0.1$

Exercise 13.3

(1) $X^T=\begin{pmatrix}1&1&1\\1&3&4\end{pmatrix}$, $X^TX=\begin{pmatrix}1&1&1\\1&3&4\end{pmatrix}\begin{pmatrix}1&1\\1&3\\1&4\end{pmatrix}=\begin{pmatrix}3&8\\8&26\end{pmatrix}$

$y=\begin{pmatrix}7\\4\\1\end{pmatrix}$ $\quad(X^TX)^{-1}=\dfrac{1}{14}\begin{pmatrix}26&-8\\-8&3\end{pmatrix}$

$$\hat{a}=\frac{1}{14}\begin{pmatrix}26&-8\\-8&3\end{pmatrix}\begin{pmatrix}1&1&1\\1&3&4\end{pmatrix}\begin{pmatrix}7\\4\\1\end{pmatrix}\approx\begin{pmatrix}9.14\\-1.93\end{pmatrix}$$

$\hat{y}=9.14-1.93x+\epsilon$ where $\epsilon\sim N(0,0.64)$

$$\hat{\sigma}^2=\frac{1}{1}\left[\begin{pmatrix}7\\4\\1\end{pmatrix}-\begin{pmatrix}1&1\\1&3\\1&4\end{pmatrix}\begin{pmatrix}9.14\\-1.93\end{pmatrix}\right]^T\left[\begin{pmatrix}7\\4\\1\end{pmatrix}-\begin{pmatrix}1&1\\1&3\\1&4\end{pmatrix}\begin{pmatrix}9.14\\-1.93\end{pmatrix}\right]\approx0.64$$

(2) $y=\begin{pmatrix}7\\5\\0\end{pmatrix}$ $\quad(X^TX)^{-1}=\dfrac{1}{14}\begin{pmatrix}26&-8\\-8&3\end{pmatrix}$ again

$$\hat{a}=\frac{1}{14}\begin{pmatrix}26&-8\\-8&3\end{pmatrix}\begin{pmatrix}1&1&1\\1&3&4\end{pmatrix}\begin{pmatrix}7\\5\\0\end{pmatrix}=\begin{pmatrix}9.71\\-2.14\end{pmatrix}$$

$\hat{y}=9.71-2.14x+\epsilon$, where $\epsilon\sim N(0,4.57)$

$$\hat{\sigma}^2=\frac{1}{1}\left[\begin{pmatrix}7\\5\\0\end{pmatrix}-\begin{pmatrix}1&1\\1&3\\1&4\end{pmatrix}\begin{pmatrix}9.71\\-2.14\end{pmatrix}\right]^T\left[\begin{pmatrix}7\\5\\0\end{pmatrix}-\begin{pmatrix}1&1\\1&3\\1&4\end{pmatrix}\begin{pmatrix}9.71\\-2.14\end{pmatrix}\right]\doteq4.57$$

Exercise 13.4

$$X^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$$

$$y = \begin{pmatrix} 0.6 \\ 1 \\ 1.6 \\ 2 \end{pmatrix} \qquad (X^T X)^{-1} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$

$$\hat{a} = \frac{1}{20} \begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0.6 \\ 1 \\ 1.6 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.48 \end{pmatrix}$$

$$\tilde{\sigma}^2 = \frac{1}{2} \left[\begin{pmatrix} 0.6 \\ 1 \\ 1.6 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.48 \end{pmatrix} \right]^T \left[\begin{pmatrix} 0.6 \\ 1 \\ 1.6 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.48 \end{pmatrix} \right]$$

$$= 0.004$$

$$\tilde{y} = 0.1 + 0.48x + \epsilon, \quad \text{where } \epsilon \sim N(0, 0.004)$$

Exercise 13.5

$$y = \begin{bmatrix} 2.5 \\ 0.5 \\ 0 \end{bmatrix}, \quad X^T = \begin{bmatrix} 1 & 1 & 1 \\ -0.5 & 1 & 2 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 \\ -0.5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2.5 \\ 2.5 & 5.25 \end{bmatrix}$$

$$(X^T X)^{-1} \approx \begin{bmatrix} 0.55 & -0.26 \\ -0.26 & 0.32 \end{bmatrix}$$

$$\tilde{a} = \begin{bmatrix} 0.55 & -0.26 \\ -0.26 & 0.32 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -0.5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2.5 \\ 0.5 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 1.86 \\ -1.03 \end{bmatrix}$$

$$\tilde{\sigma}^2 = \frac{1}{3-2} \left(\begin{bmatrix} 2.5 \\ 0.5 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & -0.5 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1.86 \\ -1.03 \end{bmatrix} \right)^T \left(\begin{bmatrix} 2.5 \\ 0.5 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & -0.5 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1.86 \\ -1.03 \end{bmatrix} \right)$$

$$\doteq 0.16$$

$$\tilde{y} = 1.86 - 1.03x + \epsilon, \quad \text{where } \epsilon \sim N(0, 0.16)$$

Exercise 13.5 continue.

Let us construct the 95% CI for a.

The elements of the main diagonal of $(X^TX)^{-1} = \begin{bmatrix} 0.55 \\ 0.32 \end{bmatrix}$.

$$\tilde{a}: \begin{bmatrix} 1.86 \\ -1.03 \end{bmatrix} \pm 12.706\sqrt{0.16 \times \begin{bmatrix} 0.55 \\ 0.32 \end{bmatrix}} = \begin{bmatrix} -1.92, & 5.64 \\ -3.89, & 1.83 \end{bmatrix}$$

$$\tilde{\sigma}^2: \quad \chi^2_{0.025,1} = 5.024, \quad \chi^2_{0.975,1} = 0.001$$

$$\left[\frac{(3-2)\times 0.16}{5.024}, \quad \frac{(3-2)\times 0.16}{0.001}\right] = [0.032, \ 160]$$

$$\tilde{y}_{new} = 1.86 - 1.03 \times 1.05 = 0.315$$

95% CI for $\tilde{y}_{new}$ is

$$0.315 \pm 12.71\sqrt{0.16[1, \ 1.5]\begin{bmatrix} 0.55 & -0.26 \\ -0.26 & 0.32 \end{bmatrix}\begin{bmatrix} 1 \\ 1.5 \end{bmatrix}} \approx [-3.18, 3.81]$$

Exercise 13.6: $\quad a = \begin{pmatrix} 0.1 \\ 0.48 \end{pmatrix}, \ \tilde{\sigma}^2 = 0.04, \ N=4, \ d=1, \ N-(d+1)=2$

diagonal $(X^TX)^{-1} = \begin{pmatrix} 1.5 \\ 2 \end{pmatrix} \quad d.f. = 2, \quad t_{0.025} = 4.303$

For a. $\begin{pmatrix} 0.1 \\ 0.48 \end{pmatrix} \pm 4.303\sqrt{0.004 \times \begin{pmatrix} 1.5 \\ 0.2 \end{pmatrix}} = \begin{pmatrix} 0.1 \\ 0.48 \end{pmatrix} \pm \begin{pmatrix} 0.33 \\ 0.12 \end{pmatrix} \therefore \ a_0: [-0.23, 0.43] \quad a_1: [0.36, 0.60]$

For σ^2: $\quad \chi^2_{0.025} = 7.38, \quad \chi^2_{0.975} = 0.051, \quad d.f. = 2$

$$\left(\frac{2\times 0.004}{7.38}, \quad \frac{2\times 0.004}{0.051}\right) = (0.001, \ 0.16)$$

$x = 2.5$, then $y = 1.3$

95% CI for $y = 1.3$ is: $1.3 \pm 4.303\sqrt{0.004 \times (1, 2.5)\begin{pmatrix} 1.5, & -0.5 \\ -0.5 & 0.2 \end{pmatrix}\begin{pmatrix} 1 \\ 2.5 \end{pmatrix}} = [1.16, 1.44]$

Exercise 13.7

2nd iteration.

$$a_0^{new} = 0.473 - 0.1 \times \left((0.42-0) + (0.49-0) + (0.62-1) + (0.65-1)\right) \times 1$$
$$\doteq 0.455$$

$$a_1^{new} = 0.535 - 0.1\left[(0.42-0)\times(-1.5) + (0.49-0)\times(-1) + (0.62-1)\times 0 + (0.65-1)\times 0.3\right]$$

$$\doteq 0.658$$

Now check: $z_1 = 0.455 + 0.658 \times (-1.5) = -0.53$

$z_2 = 0.455 + 0.658 \times (-1) = -0.20$

$z_3 = 0.455 + 0.658 \times (0) = 0.455$

$z_4 = 0.455 + 0.658 \times 0.3 = 0.65$

$P(Y_1=1|x_1) \doteq 0.37$

$P(Y_2=1|x_2) \doteq 0.45$

$P(Y_3=1|x_3) \doteq 0.61$

$P(Y_4=1|x_4) \doteq 0.66$

Exercise 13.8

$z_1 = -0.3 + 0.2(-1) = -0.5$ $\quad P(Y_1=1|x_1) \doteq 0.38$

$z_2 = -0.3 + 0.2 \times 0 = -0.3$ $\quad P(Y_2=1|x_2) \doteq 0.43$

$z_3 = -0.3 + 0.2 \times 1 = -0.1$ $\quad P(Y_3=1|x_3) = 0.48$

$z_4 = -0.3 + 0.2 \times 3 = 0.3$ $\quad P(Y_4=1|x_4) = 0.57$

$$a_0^{new} = -0.3 - 0.05\left((0.38-0)\times 1 + (0.43-0)\times 1 + (0.48-1)\times 1 + (0.57-1)\times 1\right)$$
$$= -0.29$$

$$a_1^{new} = 0.2 - 0.05\left(0.38\times(-1) + 0.43\times 0 + (0.48-1)\times 1 + (0.57-1)\times 3\right) = 0.31$$

Now check. $z_1 = -0.29 + 0.31(-1) = -0.6$, $P(Y_1=1|x_1) = 0.35$

$z_2 = -0.29 + 0.31 \times 0 = -0.29$, $P(Y_2=1|x_2) = 0.43$

$z_3 = -0.29 + 0.31 \times 1 = 0.02$, $P(Y_3=1|x_3) = 0.505$

$z_4 = -0.29 + 0.31 \times 3 = 0.64$, $P(Y_4=1|x_4) = 0.65$