Problems of Chapter 10

10.1

a) The number of different "words" from Wales is $5! = 120$.
b) The number of different "words" from Scotland is $8! = 40320$.

10.2

The number of different photos they can take is $4! = 24$.

10.3

The number of passwords that can be made is $\frac{10!}{6!} = 5040$.

10.4

The number of five digits that can be made is 600.

10.5

The number of fruit salads that can be made is $\frac{7!}{4!3!} = 35$.

10.6

The number of triangles that can be made is $\frac{12!}{3!9!} = 220$.

10.7

The probability of getting two heads is 0.375.

10.8

The probability of getting four different numbers is $\frac{6\times5\times4\times3}{6^4}$ or 0.2778.
The probability of getting five different numbers is $\frac{6\times5\times4\times3\times2}{6^5}$ or 0.0926.
The probability of getting six different numbers is $\frac{6\times5\times4\times3\times2\times1}{6^6}$ or 0.0154.

10.9

The probability that all six digits are different is $\frac{10\times9\times8\times7\times6\times5}{10^6}$ or 0.1512.

10.10

See Table S.1.

Table S.1 Answer to Exercise 10.10: the probability distribution of X, where X is the total value after throwing two fair dice

x	1	2	3	4	5	6	7	8	9	10	11	12	13
$f_X(x_k) = P(X = x_k)$	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0
$F_X(x) = \sum_{x_k \le x} f_X(x_k)$	0	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1	1

10.11

See Table S.2.

10.12

$a = \frac{3}{4}$.

10.10.

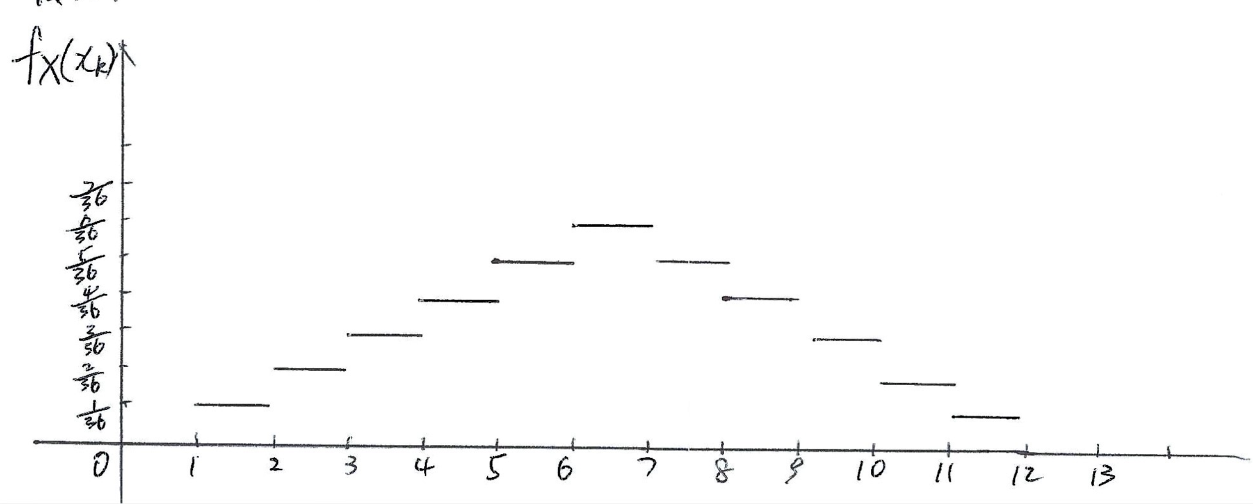
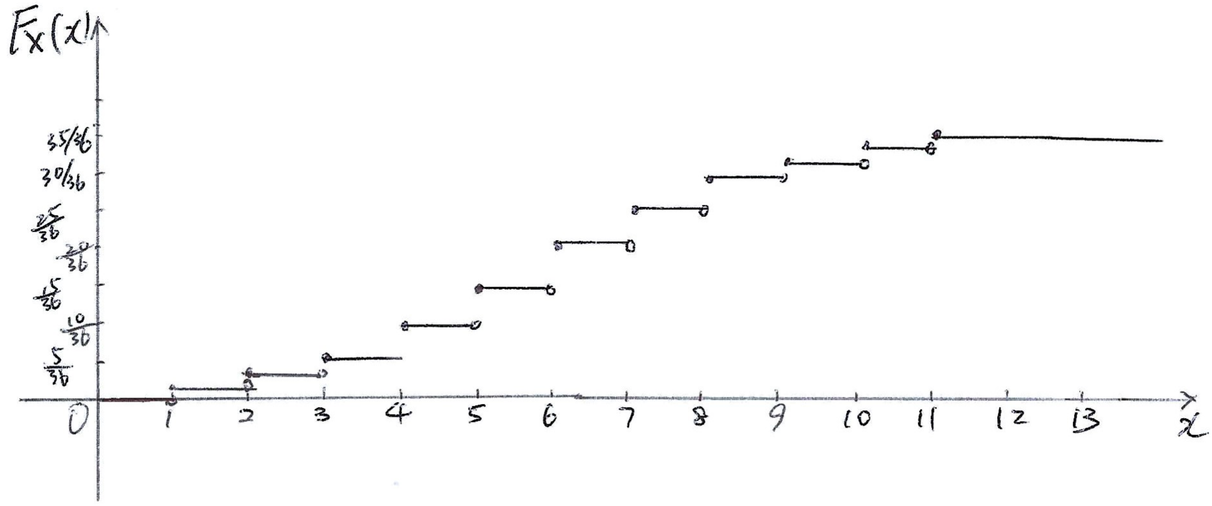


Table S.2 Answer to Exercise 10.11: the probability distribution of X, where X is the number of heads when tossing four fair coins

x	-1	0	1	2	3	4	5
$f_X(x_k) = P(X = x_k)$	0	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	0
$F_X(x) = \sum_{x_k \le x} f_X(x_k)$	0	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1	1

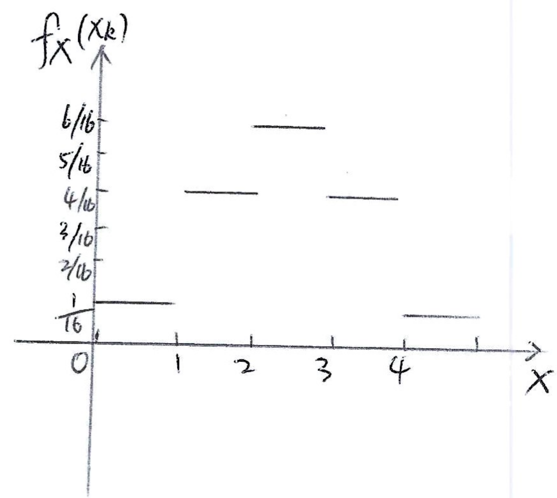
10.10.

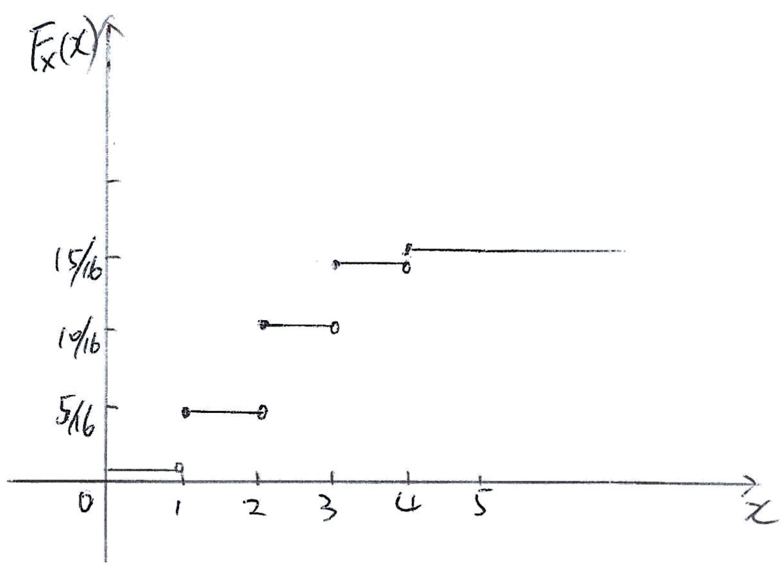


10.11

TTTT HHHH
TTTH HHHT
THTT HHTH
HTTT HTHH
TTHT THHH
TTHH HHTT
THHT HTHT
THTH HTTH

0 heads: 1/16
1 head: 4/16
2 heads: 6/16
3 heads: 4/16
4 heads: 1/16





Exercise 10.12

(1) $\int f(x)\,dx = 1$

$$\int_0^2 a(2x-x^2)\,dx$$

$$= a\cdot\left(\frac{2}{2}x^2-\frac{x^3}{3}\right)\Big|_0^2$$

$$= ax^2-\frac{a}{3}x^3\Big|_0^2$$

$$=1$$

Substitute the upper Limit 2 & the lower Limit 0, we have

$$4a-\frac{8}{3}a=1$$

$$a=\frac{3}{4}$$

(2) $F_X(x)=\int_0^x \frac{3}{4}(2t-t^2)\,dt=\frac{3}{4}x^2-\frac{x^3}{4}$ if $0<x<2$

$F_X(x)=0$ if $x\le 0$

$F_X(x)=1$ if $x\geq 2$

(3) $P(X\le 1)=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$

(4) $P(X\le\frac{1}{2})=\frac{3}{4}\times\frac{1}{4}-\frac{1}{4}\times\frac{1}{8}=\frac{5}{32}$

(5) $P(X>\frac{1}{2})=1-\frac{5}{32}=\frac{27}{32}$

Exercise 10.13

① $x^8\Big|_0^m = x^8\Big|_m^1 \Rightarrow m^8 = 1 - m^8$

$m^8 = \frac{1}{2}$

$m = \sqrt[8]{0.5}$

② $\int 8x^7\,dx = \frac{8x^8}{8} = x^8$

$x^8\Big|_n^1 = 0.05$

$1 - n^8 = 0.05$

$n = \sqrt[8]{0.95}$

Exercise 10.14

(1) $E(Z) = (-1)\frac{1}{6} + 0\times\frac{5}{12} + \frac{1}{2}\times\frac{1}{12} + 1\times\frac{1}{6} + 2\times\frac{1}{6} = \frac{3}{8} = 0.375$

(2) $E(-Z+2) = -E(Z) + 2 = -\frac{3}{8} + 2 = \frac{13}{8} = 1.625$

(3) $E(Z^2) = (-1)^2\times\frac{1}{6} + 0\times\frac{5}{12} + \left(\frac{1}{2}\right)^2\times\frac{1}{12} + 1\times\frac{1}{6} + 2^2\times\frac{1}{6}$

$= \frac{49}{48} \doteq 1.02083$

Exercise 10.15

(1) $E(Z + X_1 + X_3) = E(Z) + E(X_1) + E(X_3) = \frac{3}{8} + \frac{5}{3} + \frac{11}{6} = \frac{93}{24}$

$= 3.875$

(2) $E(ZX_4) = E(Z)E(X_4) = \frac{3}{8}\times 3 = \frac{9}{8} = 1.125$

(3) $E(3Z-5) = 3E(Z) - 5 = 3\times\frac{3}{8} - 5 = -3.875$

(4) $E(X_1 Z) = E(X_1)E(Z) = \frac{5}{3}\times\frac{3}{8} = \frac{5}{8} = 0.625$

(5) $E(X_2 + X_4 + Z) = E(X_2) + E(X_4) + E(Z) = \frac{3}{2} + 3 + \frac{3}{8} = 4.875$

Exercise 10.16

$$C_4^2 = \frac{4!}{2!\,2!} = 6$$

Let x be the absolute difference

X	1	2	3
P	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

$$E(X) = 1\times\frac{3}{6} + 2\times\frac{2}{6} + 3\times\frac{1}{6} = \frac{10}{6} = \frac{5}{3}$$

Exercise 10.17

(1) $E(X) = \int_0^1 x f(x)dx + \int_{-\infty}^0 x\cdot 0\,dx + \int_1^\infty x\cdot 0\,dx$

$$= 8\int_0^1 x^8\,dx = \frac{8}{9}x^9\Big|_0^1 = \frac{8}{9}$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx = 8\frac{x^{10}}{10}\Big|_0^1 = \frac{4}{5}$$

Exercise 10.18

(1) $E(X^2) = \int_0^{\frac{1}{2}} 2x^2\,dX = 2\frac{x^3}{3}\Big|_0^{\frac{1}{2}} = \frac{2}{24} = \frac{1}{12}$

$$E(x^4) = \int_0^{\frac{1}{2}} 2X^4\,dX = 2\cdot\frac{x^5}{5}\Big|_0^{\frac{1}{2}} = \frac{1}{80}$$

(2) $E(2X^2) = 2\int_{-\infty}^{\infty} 2X^2\,dX = 2\int_0^{\frac{1}{2}} 2X^2\,dX = 2\times\frac{2}{3}\times X^3\Big|_0^{\frac{1}{2}} = 2\times\frac{2}{3}\times\frac{1}{8}$

$$= \frac{1}{6}$$

$$Var(2X^2) = E\left[(2X^2)^2\right] - \left[E(2X^2)\right]^2$$

$$= E(4x^4) - \left(\frac{1}{6}\right)^2$$

$$E(4X^4) = 4E(X^4) = 4\int_0^{\frac{1}{2}} 2X^4\,dX = 4\times\frac{2}{5}\times X^5\Big|_0^{\frac{1}{2}} = 4\times\frac{2}{5}\times\frac{1}{32}$$

$$= \frac{1}{20}$$

$$Var(2X^2) = \frac{1}{20} - \frac{1}{36} = \frac{1}{45}$$

(3) $Var(2X^2+5) = Var(2X^2) = \frac{1}{45}$

Exercise 10.19

$$\mu_X = \frac{0+9}{2} = 4.5$$

Exercise 20.

(1) $P(X=7)=\frac{10!}{7!\times(10-7)!}0.85^7(1-0.85)^3 \doteq 0.13$

(2) $P(X=8)=\frac{10!}{8!(10-8)!}0.85^8(1-0.85)^2 \doteq 0.28$

(3) $P(X=9)=\frac{10!}{9!(10-9)!}0.85^9(1-0.85)^1 \doteq 0.35$

(4) $P(X=10)=\frac{10!}{10!\ 0!}0.85^{10}(1-0.85)^0 \doteq 0.20$

Exercise 21.

(1) $P(X=5)=e^{-6}\frac{6^5}{5!} \doteq 0.1606$

(2) $P(X=6)=e^{-6}\frac{6^6}{6!} \doteq 0.1606$

(3) $P(X=7)=e^{-6}\frac{6^7}{7!} \doteq 0.1377$

(4) $P(X=8)=e^{-6}\frac{6^8}{8!} = 0.1033$

Exercise 22.

(1) $P(X=8)=e^{-4}\frac{4^8}{8!} = 0.0298$

(2) $P(X>5)=1-e^{-4}\frac{4^0}{0!}-e^{-4}\frac{4^1}{1!}-e^{-4}\frac{4^2}{2!}-e^{-4}\frac{4^3}{3!}-e^{-4}\frac{4^4}{4!}-e^{-4}\frac{4^5}{5!}$

$= 0.215$

Exercise 10.23

$$E=\{24<x<27\}+\{33<x<36\}+\{42<x<45\}$$

$$P(E)=\int_{24}^{27}\frac{dz}{45-20}+\int_{33}^{36}\frac{dz}{45-20}+\int_{42}^{45}\frac{dz}{45-20}$$

$$=\frac{1}{25}(3+3+3)=0.36$$

Exercise 10.24

$$P=\frac{1-68\%}{2}=16\%$$

Exercise 10.25

$$z=\frac{50-54}{4}=-1$$

$$1000\times\left(\frac{1-68\%}{2}\right)=160$$

Exercise 10.26

(1) $P(x<8)=\Phi\left(\frac{8-2}{3}\right)=\Phi(2)=0.97725$

$P(X<0.5)=\Phi\left(\frac{0.5-2}{3}\right)=\Phi(-0.5)=0.30854$

$P(0.5<X<8)=0.97725-0.30854=0.66871$

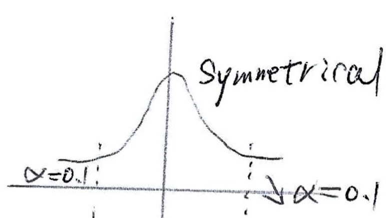
(2) $P(-1<X<5)=\Phi\left(\frac{5-2}{3}\right)-\Phi\left(\frac{-1-2}{3}\right)=\Phi(1)-\Phi(-1)=68.26\%$

(3) $P(X>C)=1-P(X\le C)$

$$\Rightarrow P(X\le C)=0.5 \xrightarrow{\text{checking a table}} z=0$$

$$z=\frac{C-2}{3}=0\Rightarrow C=2$$

Exercise 10.27



$$\Phi\left(\frac{20-16}{\sigma}\right)-\Phi\left(\frac{12-16}{\sigma}\right)=0.8$$

$$\Phi\left(\frac{4}{\sigma}\right)-\Phi\left(\frac{-4}{\sigma}\right)=0.8$$

$$\Phi\left(\frac{4}{\sigma}\right)=0.9\Rightarrow 1.28\le\frac{4}{\sigma}\le1.29\Rightarrow\sigma_{max}=\frac{4}{1.28}=3.125$$

11.1

(1) $E(Y_{15}) = 10 \times 15 = 150$

$Var(Y_{15}) = 15 \times 25 \Rightarrow std(Y_{15}) = 5\sqrt{15}$

$P(Y \leq 150) = P\left(\frac{Y-150}{5\sqrt{15}} \leq \frac{150-150}{5\sqrt{15}}\right) = P(z \leq 0) = 50\%.$

(2) $E(Y_{11}) = 11 \times 15 = 165$

$Var(Y_{11}) = 11 \times 100 \Rightarrow std(Y_{11}) = 10\sqrt{11}$

$P(Y \leq 150) = P\left(\frac{Y-165}{10\sqrt{11}} \leq \frac{150-165}{10\sqrt{11}}\right) = P(z \leq -0.4523) = 0.3264$

(3) $E(Y_{11}) = 11 \times 15 = 165$

$Var(Y_{11}) = 11 \times 25 \Rightarrow std(Y_{11}) = 5\sqrt{11}$

$P(Y \leq 150) = P\left(\frac{Y-165}{5\sqrt{11}} \leq \frac{150-165}{5\sqrt{11}}\right) = P(z \leq -0.9045) = 0.1841$

11.2 $\eta_n \sim B(100, 0.5)$ $np = 50$, $\sqrt{np(1-p)} = 5$

$1 - P(\eta_n \leq 60)$

$= 1 - P\left(\frac{\eta_n - 50}{5} \leq \frac{60-50}{5}\right)$

$= 1 - \Phi(2.0) = 1 - 0.9772 = 0.0228$

11.3

	$X=0$	$X=1$	$P(Y=y_i)$
$Y=0$	$\frac{9}{26}$	$\frac{9}{26}$	$\frac{9}{13}$
$Y=1$	$\frac{2}{13}$	$\frac{2}{13}$	$\frac{4}{13}$
$P(X=x_i)$	$\frac{13}{26}$	$\frac{13}{26}$	—

11.4

	$X=0$	$X=1$	$P(Y=y_i)$
$Y=0$	$\frac{2}{5}\times\frac{1}{4}$	$\frac{3}{5}\times\frac{1}{2}$	$\frac{2}{5}$
$Y=1$	$\frac{2}{5}\times\frac{3}{4}$	$\frac{3}{5}\times\frac{1}{2}$	$\frac{3}{5}$
$P(X=x_i)$	$\frac{2}{5}$	$\frac{3}{5}$	-

11.5

	$X=0$	$X=1$	$X=2$	$P(Y=y_i)$
$Y=0$	$\frac{2}{10}\times\frac{2}{10}$	$\frac{3}{10}\times\frac{2}{10}$	$\frac{5}{10}\times\frac{2}{10}$	$\frac{1}{5}$
$Y=1$	$\frac{2}{10}\times\frac{3}{10}$	$\frac{3}{10}\times\frac{3}{10}$	$\frac{5}{10}\times\frac{3}{10}$	$\frac{3}{10}$
$Y=2$	$\frac{2}{10}\times\frac{5}{10}$	$\frac{3}{10}\times\frac{5}{10}$	$\frac{5}{10}\times\frac{5}{10}$	$\frac{1}{2}$
$P(X=x_i)$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{2}$	-

11.6

(1) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = A\int_0^4\int_0^4 (x+y)\,dx\,dy$$
$$= A\int_0^4 \left(\frac{x^2}{2}+xy\right)\Big|_0^4 dy$$
$$= A\int_0^4 (8+4y)\,dy$$
$$= 64A = 1$$

$$\therefore A = \frac{1}{64}$$

(2) $$\int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{64}(\eta+\xi)\,d\eta\,d\xi$$
$$= \frac{1}{64}\int_0^y \left(\frac{\eta^2}{2}+\eta\xi\right)\Big|_0^x d\xi$$
$$= \frac{1}{64}\int_0^y \left(\frac{x^2}{2}+x\xi\right) d\xi$$
$$= \frac{1}{64}\left[\frac{x^2}{2}\xi + x\frac{\xi^2}{2}\right]\Big|_0^y = \frac{1}{128}x^2y + \frac{1}{128}xy^2$$

Exercise 11.6

(3) $P(0 \leq X < 2,\quad 0 \leq Y < 2)$

$$= \frac{1}{64}\int_0^2 \left(\frac{x^2}{2} + xy\Big|_0^2\right)dy$$

$$= \frac{1}{64}\int_0^2 (2+2y)\,dy$$

$$= \frac{1}{64}(2y+y^2)\Big|_0^2$$

$$= \frac{1}{64}(4+4)$$

$$= \frac{1}{8}$$

(4)

$P(X+Y \leq 4)$

$$= \int_0^4 \int_0^{4-y} \frac{1}{64}(x+y)\,dx\,dy$$

$$= \int_0^4 \left[\frac{1}{64}\left(\frac{x^2}{2}+xy\right)\Big|_0^{4-y}\right]dy$$

$$= \frac{1}{64}\int_0^4 \left(8-\frac{y^2}{2}\right)dy$$

$$= \frac{1}{64}\left(8\times 4 - \frac{64}{6}\right)$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$= \frac{1}{3}$$

11.7

$$f_X(x) = \int_0^{\infty} \frac{1}{x} e^{\frac{-y}{x}} e^{-x}\, dy$$

$$= \frac{1}{x} e^{-x} \int_0^{\infty} e^{-\frac{y}{x}} dy$$

$$= \frac{1}{x} e^{-x} \cdot \left. \frac{e^{-\frac{y}{x}}}{-\frac{1}{x}} \right|_0^{\infty} = -e^{-x}(0-1) = e^{-x}$$

11.8

(1) $$A \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(x-y)\, dx\, dy$$

$$= A \int_0^{\frac{\pi}{2}} \left[\int_0^{\frac{\pi}{2}} \cos(x-y)\, dx \right] dy$$

$$= A \int_0^{\frac{\pi}{2}} \sin(x-y) \Big|_0^{\frac{\pi}{2}} dy$$

$$= A \int_0^{\frac{\pi}{2}} \left(\sin(\tfrac{\pi}{2} - y) - \sin(0-y) \right) dy$$

$$= A \int_0^{\frac{\pi}{2}} \left[\sin\tfrac{\pi}{2}\cos y - \cos\tfrac{\pi}{2}\sin y - \sin 0 \cos y + \cos 0 \sin y \right] dy$$

$$= A \int_0^{\frac{\pi}{2}} (\cos y + \sin y)\, dy = A \left[\sin y \Big|_0^{\frac{\pi}{2}} - \cos y \Big|_0^{\frac{\pi}{2}} \right] = 2A = 1$$

$$\Rightarrow A = \frac{1}{2}$$

(2) $$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{\infty}^{y} \frac{1}{2} \cos(\eta - \xi)\, d\eta\, d\xi$$

$$= \frac{1}{2} \int_0^{y} \left[\sin(\eta - \xi) \Big|_0^{x} \right] d\xi$$

$$= \frac{1}{2} \int_0^{y} \left[(\sin\eta \cdot \cos\xi - \cos\eta \cdot \sin\xi) \Big|_0^{x} \right] d\xi$$

$$= \frac{1}{2} \int_0^{y} \sin x \cos\xi - \cos x \cdot \sin\xi + \sin\xi\, d\xi$$

$$= \frac{1}{2} \left[\sin x \sin\xi \Big|_0^{y} + \cos x \cos\xi \Big|_0^{y} - \cos\xi \Big|_0^{y} \right]$$

$$= \frac{1}{2} \sin x \sin y + \frac{1}{2} \cos x \cos y - \frac{1}{2} \cos x - \frac{1}{2} \cos y + \frac{1}{2}$$

11.8(3) $P(0 \le X < \frac{\pi}{4},\ 0 \le Y < \frac{\pi}{4})$

$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{1}{2} \cos(x-y)\, dx\, dy$

$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[\sin(x-y) \Big|_0^{\frac{\pi}{4}} \right] dy$

$= \frac{1}{2} \int_0^{\frac{\pi}{4}} [\sin x \cos y - \cos x \sin y]_0^{\frac{\pi}{4}}\, dy$

$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left(\frac{\sqrt{2}}{2} \cos y - \frac{\sqrt{2}}{2} \sin y + \sin y \right) dy$

$= \frac{1}{2} \left[\frac{\sqrt{2}}{2} \sin y + \frac{\sqrt{2}}{2} \cos y - \cos y \right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2} \times 1$

$= 1 - \frac{\sqrt{2}}{2} \approx 0.293$

11.8
(4) $f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2} \cos(x-y)\, dy$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(x-y)\, dy$

$= \frac{1}{2} \times (-1) \sin(x-y) \Big|_0^{\frac{\pi}{2}}$

$= -\frac{1}{2} \sin(x - \frac{\pi}{2}) + \frac{1}{2} \sin x$

$= \frac{1}{2} \cos x + \frac{1}{2} \sin x$

11.8
(5) $f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{2} \cos(x-y)\, dx$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(x-y)\, dx$

$= \frac{1}{2} \sin(x-y) \Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2} \sin(\frac{\pi}{2} - y) + \frac{1}{2} \sin y$

$= \frac{1}{2} \cos y + \frac{1}{2} \sin y$

Exercise 11.9

$$f_X(x) = 6\int_0^{\infty} e^{-2x} \cdot e^{3y}\, dy = 6e^{-2x}\left[\frac{e^{-3y}}{-3}\right]_0^{\infty} = 2e^{-2x}$$

$$f_Y(y) = 6e^{-3y}\int e^{-2x}\, dx = 6e^{-3y}\left[e^{-2x}\right]_0^{\infty} = 3e^{-3y}$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = 2\int_0^{\infty} xe^{-2x}\,dx = \frac{1}{2}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = 3\int_0^{\infty} ye^{-3y}\,dy$$

$$= 3\left[\frac{ye^{-3y}}{-3}\right]_0^{\infty} - \int -\frac{e^{-3y}}{3}\cdot 3\,dy$$

$$= \frac{1}{-3}e^{-3y}\Big|_0^{\infty} = \frac{1}{3}$$

$$E(XY) = 6\iint xye^{-(2x+3y)}\,dx\,dy$$

$$= 6\int xe^{-2x}\left(\int ye^{-3y}\,dy\right)dx$$

$$= \frac{2}{3}\int x \cdot e^{-2x}\,dx$$

$$= \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$$

$$\mathrm{cov}(X,Y) = \frac{1}{6} - \frac{1}{2} \times \frac{1}{3} = 0$$

Exercise 11.10

$$f_X(x) = \frac{1}{64}\int_0^4 (x+y)\,dy = \frac{1}{64}\left[xy + \frac{y^2}{2}\right]_0^4 = \frac{1}{64}(4x+8)$$

$$= \frac{1}{16}(x+2)$$

$$f_Y(y) = \frac{1}{64}\int_0^4 (x+y)\,dx = \frac{1}{64}\left[\frac{x^2}{2} + xy\right]_0^4 = \frac{1}{64}(8+4y)$$

$$= \frac{1}{16}(2+y)$$

$$E(X) = \int_0^4 x\, f_X(x)\,dx$$

$$= \frac{1}{16}\int_0^4 x(x+2)\,dx = \frac{1}{16}\int_0^4 (x^2+2x)\,dx = \frac{1}{16}\left[\frac{x^3}{3} + x^2\right]_0^4$$

$$= \frac{7}{3}$$

$$E(Y) = \int_0^4 y\, f_Y(y)\,dy = \frac{1}{16}\int_0^4 y(2+y)\,dy = \frac{1}{16}\left[y^2 + \frac{y^3}{3}\right]_0^4$$

$$= \frac{7}{3}$$

$$E(XY) = \frac{1}{64}\int_0^4\int_0^4 xy(x+y)\,dx\,dy$$

$$= \frac{1}{64}\int_0^4\int_0^4 (x^2y + xy^2)\,dx\,dy$$

$$= \frac{1}{64}\int_0^4 \left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_0^4 dx$$

$$= \frac{1}{64}\int_0^4 \left(8x^2 + \frac{64}{3}x\right) dx$$

$$= \frac{1}{64}\left[\frac{8}{3}x^3 + \frac{64}{6}x^2\right]_0^4 = \frac{16}{3}$$

$$\mathrm{cov}(X,Y) = \frac{16}{3} - \frac{7}{3}\cdot\frac{7}{3} = -\frac{1}{9}$$

Exercise 11.11

$P_1 = \frac{1}{5},\quad P_2 = \frac{3}{10},\quad P_3 = \frac{2}{5},\quad P_4 = \frac{1}{10}$

$P(x_1 = 2, x_2 = 1, x_3 = 0, x_4 = 3)$

$$= \frac{6!}{2!\,1!\,0!\,3!}\left(\frac{1}{5}\right)^2\left(\frac{3}{10}\right)^1\left(\frac{2}{5}\right)^0\left(\frac{1}{10}\right)^3$$

$$= 0.00072$$

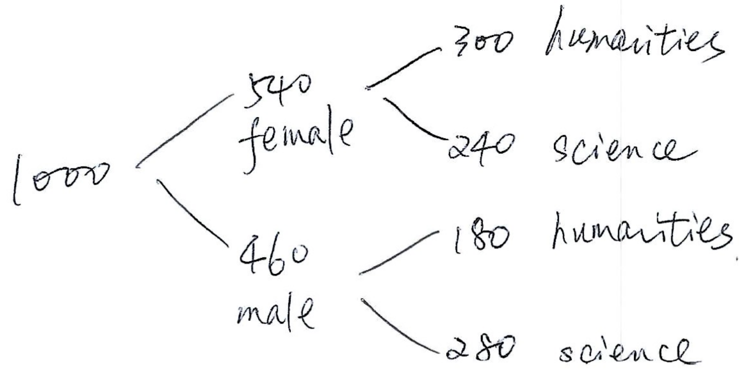
Exercise 11.12

W: woman ; M: man.

$$P(\text{depression} \mid W) = \frac{P(\text{depression} \cap W)}{P(W)} = \frac{16.67\%}{50\%} = 33.34\%$$

$$P(\text{depression} \mid M) = \frac{P(\text{depression} \cap M)}{P(M)} = \frac{10\%}{50\%} = 20\%.$$

Exercise 11.13



$$P(\text{science} \mid F) = \frac{240/1000}{\frac{540}{1000}} = 0.24/0.54 = 44.4\%$$

$$P(F \mid \text{science}) = \frac{\frac{240}{1000}}{\frac{280+240}{1000}} = 46.15\%$$

$$P(\text{science} \mid M) = \frac{\frac{280}{1000}}{\frac{460}{1000}} = 60.87\%.$$

11.14 (HHH, HHT, HTH, HTT, TTT, TTH, THT, THH)

$$P(H=2 \mid F=H) = \frac{\frac{2}{8}}{\frac{4}{8}} = 50\%.$$

11.15 DD, DS, SD, SS.

$$P(S \mid D) = \frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3}$$

11.16

$$f_X(x) = \int_0^x \tfrac{1}{2}\,dy = \tfrac{1}{2}x$$

$$f_{Y|X}(y|x) = \frac{\frac{1}{2}}{\frac{1}{2}x} = \frac{1}{x} \quad y \le x < 2,\ 0 < x < 2$$

11.17

① $A\int_0^{\frac{\pi}{2}} \sin x \left(\int_0^{\frac{\pi}{2}} \cos y\, dy\right) dx$

$= A\int_0^{\frac{\pi}{2}} \sin x \ \sin y\Big|_0^{\frac{\pi}{2}}\, dx$

$= A\int_0^{\frac{\pi}{2}} \sin x\, dx$

$= -A\cos x\Big|_0^{\frac{\pi}{2}} = -A(0-1) = 1 \Rightarrow A = 1$

② $f_Y(y) = \int_0^{\frac{\pi}{2}} \sin x \cos y\, dx = -\cos y \cos x\Big|_0^{\frac{\pi}{2}} = \cos y$

$$f_{X|Y}(x|y) = \frac{\sin x \cos y}{\cos y} = \sin x$$

③ $f_X(x) = \int_0^{\frac{\pi}{2}} \sin x \cos y\, dy = \sin x \sin y\Big|_0^{\frac{\pi}{2}} = \sin x$

$$f_{Y|X}(y|x) = \frac{\sin x \cos y}{\sin x} = \cos y$$

Exercise 11.18

$$f_X(x) = \int_0^1 12x^3y^2\,dy = 12x^3\frac{y^3}{3}\Big|_0^1 = 4x^3$$

$$E(Y|x) = \frac{\int_0^1 y\,12x^3y^2\,dy}{f_X(x)} = 3\int_0^1 y^3\,dy = \frac{3}{4}y^4\Big|_0^1 = \frac{3}{4}$$

$$E(Y^2|x) = \frac{\int_0^1 y^2\,12x^3y^2\,dy}{f_X(x)} = 3\int_0^1 y^4\,dy = \frac{3}{5}y^5\Big|_0^1 = \frac{3}{5}$$

$$Var(Y|x) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

Exercise 11.19

$$P(A) = P(B) = \frac{1}{2}, \quad P(C) = \frac{9}{36} = \frac{1}{4}$$

$$P(A\cap B) = \frac{9}{36} = \frac{1}{4} = P(A)\cdot P(B) = \frac{1}{4}$$

$$P(A\cap C) = \frac{9}{36} = \frac{1}{4} \neq P(A)\cdot P(C) = \frac{1}{8}$$

$$P(B\cap C) = \frac{9}{36} = \frac{1}{4} \neq P(B)\cdot P(C) = \frac{1}{8}$$

$\therefore$ A and B are independent.

A and C, and B and C are not independent.

Exercise 11.20. $\Omega = \{1, 2, 3, 4\}$ uniformly distribution.

$A = \{2, 4\}$ $P(A) = \frac{1}{2}$

$P(A\cap B) = P(A)\cdot P(B)$

If $B = \{1, 2, 3, 4\} \Rightarrow P(B) = 1$

$P(A\cap B) = P(\{2,4\}) = \frac{1}{2} = \frac{1}{2}\times 1$

If $B = \{2, 3\} \Rightarrow P(B) = \frac{1}{2}$

$P(A\cap B) = P(\{2\}) = \frac{1}{4} = P(A)\cdot P(B) = \frac{1}{2}\times\frac{1}{2}$

Similarly, for $B = \{1, 2\}$, $B = \{3, 4\}$, $B = \{1, 4\}$.

Exercise 11.21

1st < 6r, 4b

2nd < 7r, 8b

3rd < 6r, 6b

4th < 16r, 4b

$$\frac{6}{10}\times\frac{1}{4}+\frac{7}{15}\times\frac{1}{4}+\frac{6}{12}\times\frac{1}{4}+\frac{16}{20}\times\frac{1}{4}$$

$$=\frac{71}{120}\approx 0.592$$

If all balls in one bag:

$$\frac{6+7+6+16}{6+4+7+8+6+6+16+4}=\frac{35}{57}\doteq 0.614$$

Exercise 11.22

If the 1st is white: $\frac{2}{5}\times\frac{1}{4}$

If the 1st is red: $\frac{3}{5}\times\frac{1}{2}$

$$\frac{2}{5}\times\frac{1}{4}+\frac{3}{5}\times\frac{1}{2}=\frac{2}{5}$$

Exercise 11.23

$$P=\frac{\frac{1}{12}\times\frac{19}{20}}{\frac{1}{12}\times\frac{19}{20}+\frac{11}{12}\times\frac{1}{3}}=\frac{57}{277}\approx 20.6\%$$

Exercise 11.24

$$P=\frac{\frac{5}{10}\times\frac{9}{10}}{\frac{5}{10}\times\frac{9}{10}+\frac{3}{10}\times\frac{14}{15}+\frac{2}{10}\times\frac{19}{20}}$$

$$=\frac{270}{552}\approx 48.9\%$$

Exercise 11.25.

$$P = \frac{0.003 \times 0.95}{0.003 \times 0.95 + 0.997 \times (1 - 0.98)}$$

$$= \frac{285}{2279} \approx 12.51\%$$

Exercise 11.26

$$\underset{50}{X} \begin{cases} \frac{1}{10} & R \\ \frac{9}{10} & \bar{R} \end{cases} \quad \underset{60}{Y} \begin{cases} \frac{1}{20} & R \\ \frac{19}{20} & \bar{R} \end{cases}, \quad \underset{60}{Z} \begin{cases} \frac{1}{30} & R \\ \frac{29}{30} & \bar{R} \end{cases}, \quad \underset{30}{W} \begin{cases} \frac{1}{5} & R \\ \frac{4}{5} & \bar{R} \end{cases}$$

$$50 + 60 + 60 + 30 = 200$$

$$P(X) = \frac{50}{200}, \quad P(Y) = \frac{60}{200}, \quad P(Z) = \frac{60}{200}, \quad P(W) = \frac{30}{200}$$

$$P(Z|\bar{R}) = \frac{\frac{60}{200} \times \frac{29}{30}}{\frac{50}{200} \times \frac{9}{10} + \frac{60}{200} \times \frac{19}{20} + \frac{60}{200} \times \frac{29}{30} + \frac{30}{200} \times \frac{4}{5}}$$

$$= 31.52\%$$