Probability and Combinatorics Worksheet 4

Useful facts

- **Expected value**. For a discrete random variable with probability $p(x)$ this is

$$\langle g(X)\rangle = \sum_x p(x)g(x) \tag{1}$$

  For a continuous random variable with density $f(x)$ this is

$$\langle g(X)\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx \tag{2}$$

- **Mean and variance**. The mean is $\mu = \langle X\rangle$ and the variance is $\sigma^2 = \langle (X-\mu)^2\rangle = \langle X^2\rangle - \mu^2$.

- **Binomial distibution**. For n independent trials each with p change of success and $q = 1-p$ of failure, the probability of r successes is

$$p(r) = \left(\begin{array}{c} n \\ r \end{array}\right) p^r q^{n-r} \tag{3}$$

  and $\mu = pn$, $\sigma^2 = pqn$.

Questions

These are the questions you should make sure you work on in the workshop.

1. The illusionist Derren Brown famously flipped a coin on camera so that it landed heads ten times in a row; he claimed that this was because of his mind powers, in fact it was because of his patience, he simply kept trying the trick again and again until it worked. It took him nine hours. What is the probability of a coin landing heads ten times in a row? If you flip a coin ten times what is the probability of getting five heads and five tails?

2. You are a con person with a crooked coin, it has a 0.25 chance of head and a 0.75 change of a harp. You make a bet with someone; if the coin shows a head you pay them two points, if it shows a harp they give you one. What is the mean return from a bet; what is the variance.

3. For the binomial distribution on n trials, prove $\sigma^2 = npq$; this can be done by differenciating Z twice, an easier approach is to differenciate the expression for $\mu = np$:

$$np = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) rp^r q^{n-r} \tag{4}$$

4. Like the binomial distribution the geometric probability distribution is related to a series of independent trials where each trial has probability p of success and $q = 1-p$ of failure.

The geometric probability $p(r)$ is the probability that the rth trial is the first success. It is

$$p(r) = q^{r-1}p \tag{5}$$

It can be shown that

$$\sum_{r=1}^{\infty} p(r) = 1 \tag{6}$$

as it must be. You can assume that here. What is the mean of the geometric probability?

Extra questions

Do these in the workshop if you have time.

1. Oranmore, the village I grew up in had more people with the surname Furey than any other village or town in the world. Since I left the village has expanded ten-fold and has gone from being a small village to a commuter town for Galway. However, when I was young one in ten people in the village had the surname Furey. Imagine there are 35 children in a class at school, what is the probability that five of them were Fureys?

2. The **Fano factor** is sometimes used to describe distributions, it is

$$F = \frac{\sigma^2}{\mu} \tag{7}$$

   What is the Fano factor for the binomial distribution?

3. What is the variance of the geometric distribution?

4. The negative binomial distribution models a binomial process where, instead of n trials, the experiment continues until there are r failures. The probability of k successes is

$$p_K(k) = \left(\begin{array}{c} k+r-1 \\ k \end{array} \right) (1-p)^r p^k \tag{8}$$

   Obviously if

$$Z = \sum_{k=0}^{\infty} p_K(k) \tag{9}$$

   then $Z = 1$; this relies on the negative binomial expansion and you don't need to do this calculation. By differenciating calculate the mean of the negative binomial distribution.