

Probability and Combinatorics Worksheet 6

Useful facts

- Integrating a polynomial

$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad (1)$$

so the definite integral is

$$\int_a^b x^n dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \quad (2)$$

- Integrating an exponential

$$\int_{x_1}^{x_2} e^{ax} dx = \frac{1}{a} (e^{x_2} - e^{x_1}) \quad (3)$$

- The Gaussian distribution:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (4)$$

- Working out probabilities for the Gaussian:

$$\text{Prob}(x_1 < x < x_2) = \frac{1}{2} [\text{erf}(z_2) - \text{erf}(z_1)] \quad (5)$$

where

$$z = \frac{x - \mu}{\sqrt{2}\sigma} \quad (6)$$

The error function

$$\text{erf}(z) = \frac{1}{\sqrt{\pi}} \int_{-z}^z e^{-y^2} dy = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy \quad (7)$$

goes from -1 to one, so $\text{erf}(-\infty) = -1$ and $\text{erf}(\infty) = 1$.

Questions

These are the questions you should make sure you work on in the workshop.

1. The distribution of tree heights in a christmas tree forest is

$$p(h) = \begin{cases} 0.3 & 0 \leq h < 2 \\ 0.2 & 2 \leq h < 4 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

What is the mean height of trees in the forest?

2. Work out the mean and variance for the distribution

$$p(x) = \begin{cases} 2/a & x \in [-a, a] \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

3. Starting from the expression for the mean

$$\mu = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \quad (10)$$

and differentiating with respect to μ show the Gaussian distribution has variance σ^2 . This is a tricky question.

4. The size of a standard croquet ball is $3 \frac{5}{8}$ inches¹. The height of a croquet hoop is $3 \frac{3}{4}$ inches. If a not very good croquet-ball making machine makes croquet balls whose mean matches the standard and with standard deviation $\frac{1}{8}$ inch, what is the chance it will make a ball too large to fit through the hoop? You can write the solution in terms of the error function.

Extra questions

Do these in the workshop if you have time.

1. This will look like a long question but it is almost all background and the question is not too bad when you actually read through it. In particle physics when a collider is being used to find a new particle like the Higgs boson or the top squark scientists don't detect the sought after particle directly since it usually decays almost straight away, instead they detect the more common particles that particle will decay into, for example, a Higgs boson can decay in to two photons and these can be detected. Roughly speaking scientists count these events. However, the whole situation is very messy and there will always be some events even if the particle doesn't exist at the energy being examined. The amount of these background events will fluctuate from experiment to experiment, typically like a Gaussian. The scientific team is allowed to claim they have discovered the particle if the number of events they measure is more than five standard deviations above what would be expected if the particle didn't exist. What is the probability of this 'discovery' happening by chance?
2. Another useful distribution is the exponential distribution:

$$p(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability $\text{Prob}(x_1 < x < x_2)$ where x_1 and x_2 are both positive.

3. What is the mean of the exponential distribution?
4. Australorp hens weigh on average 4kg with a standard deviation 0.25kg; in one farm australorps who weigh less than 3.5kg are fed *patent chicken spicer*, a mixture of chalk, corn and pepper. What fraction of these hens are fed patent chicken spicer?

¹Everything in croquet is measured in old timey units