Worksheet 2 - outline solutions

Questions

1. How many distinct anagrams has the word 'BOOKKEEPER'?

   Solution: So ten slots to split into a group of three, for the 'E's, two groups of two for the 'O's and the 'K's and three groups of one for the 'B', 'P' and 'R':

$$n = \binom{10}{3,2,2,1,1,1} - 1 = \frac{10!}{3!2!2!} - 1 = 151199 \tag{1}$$

2. Two events A and B have probabilities $P(A) = 0.2$, $P(B) = 0.3$ and $P(A \cup B) = 0.4$. Find

   a) Find $P(A \cap B)$.

   b) Find $P(\bar{A} \cap \bar{B})$.

   c) Find $P(A|B)$.

   Solution: So

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{2}$$

   so

$$P(A \cap B) = 0.2 + 0.3 - 0.4 = 0.1 \tag{3}$$

   If $P(A \cap B) = 0.1$ then $P(\bar{A} \cap B) = P(B) - P(A \cap B) = 0.2$. One the other hand $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 0.6$. Finally

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.3} = \frac{1}{3} \tag{4}$$

3. How many five move games of tic-tac-toe are there?

   Solution: So to win X must have a line, there are eight lines, so the choice of line gives 8 and the order X filled it gives 3!, O has two places picked one by one from six free slots, so that's 6×5 and

$$n = 8 \times 3! \times 30 = 1440 \tag{5}$$

4. You roll a dice twice, what is the probability the second roll has a lower value than the first? You take the ace, two, three, four, five and six of hearts to make a mini-pack of cards. You draw two cards, what is the probability the second will be lower than the first, with the ace counting as a one?

   Solution: Dice first: imagine a grid of outcomes, so you tick entry x_{ij} if the first roll is i and the second is j; all entries are equally likely so we just want to count the number of elements such that $i > j$, there are 15 such elements in a 36 element grid

so the probability is $15/36 = 5/12$. Now the cards, this doesn't seem so easy since the first card is removed but if you think of the same grid, this just means that there are no diagonal elements and hence we have $15/30 = 1/2$. If this seems implausible you can calculate it the longer, but more intuitive way:

$$P = \frac{1}{6} + \frac{1}{6}\frac{4}{5} + \frac{1}{6}\frac{3}{5} + \frac{1}{6}\frac{2}{5} + \frac{1}{6}\frac{1}{5} \tag{6}$$

where the first entry corresponds to the first pick being a six, the second to the first pick being a five and so on. Doing this sum gives a half.

Extra questions

1. How many paths are there from (0,0) to (4,5)? If you pick a random path from (0,0) to (4,5) what is the probability it goes through $(2, 2)$?

   In all you make $4 + 5 = 9$ moves and you need to pick four of them to be across, so that

   $$n = \binom{9}{4} = 126 \tag{7}$$

   paths. If the path goes through $(2, 2)$ then we first count the paths from $(0, 0)$ to $(2, 2)$

   $$n_1 = \binom{4}{2} = 6 \tag{8}$$

   and then from $(2, 2)$ to $(4, 5)$ which is like going from $(0, 0)$ to $(2, 3)$

   $$n_2 = \binom{5}{2} = 10 \tag{9}$$

   so the number of paths going through $(2, 3)$ is $n_1 \times n_2 = 60$ and the probability of a random path going through $(2, 2)$ is

   $$P = \frac{60}{126} = \frac{30}{63} \tag{10}$$

2. How many six move games of tic-tac-toe are there?

   Solution: So as before there are eight ways for O to win and 3! orders for them to have occupied those three slots in the win and $6 \times 5 \times 4$ plays for X;

   $$n_1 = 8 \times 3! = 5760 \tag{11}$$

   however, X can't have won at round five for this to count as a six move game. We need to count the six move games where X won at round five and then O took one more move and produced at round six a game we have counted as part of n_1 but which shouldn't be included when calculating the total number of six round games. In a game like this there

are six choices for the location of the X line, since diagonals don't work, if a diagonal is occupied, no other line is possible. Now when X has a row there are two choices for a line of O's, the two other rows, the same is true of columns. Finally each of X and O have 3! orders they could've places their marks giving

$$n_2 = 6 \times 2 \times (3!)^2 = 432 \tag{12}$$

so

$$n = n_1 - n_2 = 5760 - 432 = 5328 \tag{13}$$

3. If p is a probability mass function and we define a function on events:

$$P(E) = \sum_{x \in E} p(x) \tag{14}$$

show P is a probability.

Solution: So the properies of a probability are that it is positive, which follows from the positivity of the mass function, that $P(X) = 1$ which follows from

$$\sum_{x \in X} p(x) = 1 \tag{15}$$

and $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$ follows from

$$\sum_{x \in A \cup B} p(x) = \sum_{x \in A} p(x) + \sum_{x \in B} p(x) \tag{16}$$

if $A \cap B = \emptyset$.

4. Ten different objects are put into three boxes. How many different configuations are there? They are sorted so that there are five in the first box, three in the second and two in the third. How many ways configurations are there?

Solutions: For the first 3^{10}, for the second

$$n = \binom{10}{5, 3, 2} = \frac{10!}{5!3!2!} = 2520 \tag{17}$$