Lecture 2: Combinatorics

COMS10014 Mathematics for Computer Science A

`cs-uob.github.io/COMS10014/ and github.com/coms10011/2020_21`

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Don't stop counting

Recall that $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$. In a discrete sample space an event can be written as the union of all the outcomes it contains, if

$$A = \{a_1, a_2, \ldots, a_k\}$$

then

$$A = \{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_k\}$$

so

$$P(A) = P(\{a_1\}) + P(\{a_2\}) + \ldots + P(\{a_k\})$$

Don't stop counting

If all the probabilities are equal, say q then, since $P(X) = 1$

$$q = \frac{1}{\#(X)}$$

and

$$P(A) = \frac{\#(A)}{\#(X)}$$

so to work out the probability we just need to calculate the number of points in A.

Simple example

A coin is flipped six times, what is the probability of getting all flips giving the same results?

Simple example

If a coin is flipped six times the set of outcomes looks like

$$X = \{HHHHHH, HHHHHT, HHHHTH, \ldots, TTTTTT\}$$

The elements look like

$$\{ABCDEF\}$$

where each element can be a H or a T

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$$\{A\color{blue}{H}\color{black}{CDEF}\}$$

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$$\#(X) = 2^6 = 64$$

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If S is the event that all the outcomes are the same, it has just two elements:

$$S = \{HHHHHH, TTTTTT\}$$

Hence

$$P(S) = \frac{\#(S)}{\#(X)} = \frac{1}{32}$$

Combinatorics

The mathematics of counting things is called combinatorics; *combinatorics is a rich area of mathematics with interesting links to number theory and many applications in computer science.*

Paul Erdös and Béla Bollabás

The power set

$$A = \{a, b, c\}$$

then the **power set** is

$$\mathcal{P}(A) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

so

$$\#(\mathcal{P}(A)) = 8$$

The power set

In fact

$$\#(\mathcal{P}(A)) = 2^{\#(A)}$$

Consider a map from the power set to binary numbers where for a given subset you use ones for the element it contains and zeros for the ones it doesn't:

$$\begin{array}{rcl}
\{\} & \leftrightarrow & 000 \\
\{a\} & \leftrightarrow & 100 \\
\{b\} & \leftrightarrow & 010 \\
 & \dots & \\
\{b, c\} & \leftrightarrow & 011 \\
\{a, b, c\} & \leftrightarrow & 111
\end{array}$$

Hence the binary numbers of length $\#(A)$ count the elements of $\mathcal{P}(A)$.

The factorial

The factorial

$$n! = n \times (n-1) \times \ldots \times 2 \times 1$$

counts the number of different orders for n objects. For example for $\{a, b, c\}$ there is

$$abc, bca, cab, acb, cba, bac$$

The factorial

There are n choices of the first element:

$$\{a_1, a_2, a_3, \dots, a_{n-1}, a_n\}$$

so

$$n! = n \times \dots$$

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The factorial

Then are $n - 1$ choices of the second element:

$$\{a_1, a_2, a_3, \ldots, a_{n-1}, a_n\}$$

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$$n! = n \times (n - 1) \times \ldots$$

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$$n! = n \times (n-1) \times \ldots$$

and so on.

Subsets of size r

$$A = \{a, b, c, d\}$$

then the set of subsets of size two is

$$[A]^2 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$$

and

$$\#([A]^2) = 6$$

where we are using $[A]^r$ for the set of subsets of A of size r.

Subsets of size r

This works much the same as the factorial. If

$$\#(A) = n$$

there are n choices for the first element, then $n-1$ for the second. This time though there are only r elements to pick, giving

$$n \times (n-1) \times \ldots (n-r+1)$$

We can write this in terms of factorials

$$\frac{n \times (n-1) \times \ldots \times (n-r+1) \times (n-r) \times \ldots \times 1}{(n-r) \times \ldots \times 1} = \frac{n!}{(n-r)!}$$

However we are overcounting, subsets don't care what order you pick them out in, so we need to divide by $r!$:

$$\#([A]^r) = \frac{n!}{(n-r)!r!}$$

Subsets of size r

$$\#([A]^r) = \frac{n!}{(n-r)!r!}$$

This is just the binomial coefficient:

$$\#([A]^r) = \frac{n!}{(n-r)!r!} = \binom{n}{r} = {}_nC_r$$

The binomial

The binomial coefficient appears in the binomial expansion:

$$(x+y)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) x^r y^{n-r}$$

We can use this to check our formulas make sense, clearly

$$\mathcal{P}(A) = \emptyset \cup [A]^1 \cup [A]^2 \cup \ldots \cup [A]^n$$

so we would expect

$$2^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right)$$

and, in fact, that follows from the binomial examples with $x = y = 1$.