COMS10013 - exam questions - 25 points

All sections are worth five points; outline solutions are just outlines, you should give more workings!

Part 1

This question is about the definition and properties of differenciation. In one sentence describe in words what df/dt means [1 points]. Give the formal definition of the limit [1 point] and of df/dt [1 point] and use this to argue [2 points] that

$$\frac{d(f+g)}{dt} = \frac{df}{dt} + \frac{dg}{dt}$$

Outline Solution

So df/dt is the rate of change of f, it quantifies how much f changes as t changes, at a single point in t. The limit

$$\lim_{t\to a} f(t) = b$$

if and only if for every ϵ there exists a δ such that $|t-a| \leq \delta$ implies $|f(t) - b| \leq \epsilon$; with this machinery we have

$$\frac{df}{dt} = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

and the linearity of the limit means

$$\frac{d(f+g)}{dt} = \lim_{h\to 0} \frac{f(t+h) + g(t+h) - f(t) - g(t)}{h} = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{g(t+h) - g(t)}{h}$$

and the second term clearly gives the answer.

Part 2

This question is about actually taking derivatives. Using the quotient rule work out [1 point]

$$\frac{d \tan x}{dx}$$

Using the chain rule work out [2 points]

$$\frac{d \tan \sin x}{dx}$$

Given the Taylor expansion of the exponential

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

show that [2 points]

$$\frac{de^x}{dx} = e^x$$

Outline Solution

So you use $\tan x = \sin x / \cos x$ so

$$\frac{d\tan x}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1/\cos^2 x$$

Next

$$\frac{d\tan\sin x}{dx} = \frac{\cos x}{\cos^2 \sin x}$$

Finally,

$$\frac{1}{n!}\frac{dx^n}{dx} = \frac{1}{(n-1)!}x^{n-1}$$

so differentiating just shuffles all the terms down one.

Part 3

This question is about partial derivatives. What is the gradient [1 point] of

$$z = xy\cos xy$$

Find the Hessian [2 point], show $(0,0)$ is an extremum [1 point] and indicate whether it is a maximum, minimum or saddle point [1 point].

0.0.1 Outline Solution

So the gradient is

$$\nabla z = \left(y\cos xy - xy^2\sin xy, x\cos xy - x^2 y\sin xy\right)$$

The Hessian is

$$H = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix}$$

where

$$z_{xx} = -2y^2\sin xy - xy^3\cos xy$$

and

$$z_{xy} = \cos xy - 3xy\sin xy - x^2y^2\cos xy = z_{yx}$$

and finally

$$z_{yy} = -2x^2\sin(xy) - x^3 y\cos(xy)$$

So the gradient is zero at $x = y = 0$ showing it is an extemum and

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which has determinant -1 and trace zero, so the two eigenvalues have opposite signs and this is a saddle point.

Part 4

This question is about complex numbers. What are the solutions of [1 point]

$$z^2 + 2z + 2 = 0$$

What are the solutions of [1 point]

$$z^2 + 2z + 5 = 0$$

In each case what is the relationship between the two solutions as complex numbers? [2 points] How many solutions do you expect for [1 point]

$$z^6 - 1 = 0$$

What are these solutions? [1 point]

Outline Solution

So the first equation has solutions $z = -1 \pm i$ and the second $z = -1 \pm 2i$; in each case the solutions are conjugates of each other. $z^6 = 1$ has six solutions, they are

$$z = \exp n\pi i/3 \tag{1}$$

for n between zero and five, inclusive.

Part 5

This question is about differential equations. Solve the following [a-c 1 point each, d 2 points] where the dot means the derivative with respect to time.

(a) $\dot{y}(t) - y(t) = 0$ with initial condition $y(0) = 1$.

(b) $\dot{y}(t) + 3y(t) = 0$ with initial condition $y(3) = 3$.

(c) $\dot{y}y(t) = 0$ with initial condition $y(5) = 2$.

(d) $\dot{y} = ry(1 - y)$ with $y(0) = y_0$.

Solution

These previously appeared in worksheet 4 and they have solutions there, except d) for which we have

$$\frac{1}{y(1-y)}\dot{y} = r \tag{2}$$

and then using the partial fraction expansion

$$\left(\frac{1}{y} + \frac{1}{1-y}\right)\dot{y} = r \tag{3}$$

and so

$$\int \frac{dy}{y} + \int \frac{dy}{1-y} = rt + C \tag{4}$$

and hence

$$\log y - \log 1 - y = rt + C \tag{5}$$

or

$$\log \frac{y}{1-y} = rt + c \tag{6}$$

and hence

$$\frac{y}{1-y} = Ae^{rt} \tag{7}$$

and solve for y to get

$$y = \frac{Ae^{rt}}{1 + Ae^{rt}} \tag{8}$$