

## Limits

The nineteenth century was a glorious one for analysis, mathematicians like Hamilton, Legendre, Laplace and greatest of all, Gauss, made deep and powerful advances that in dynamics and calculus, a sort of culmination of the programme begun by Newton and Leibniz. There were, however, growing signs that the sloppiness in the definition of the differential were causing problems and leading to theorems whose proofs were insecure and even false. At the time, with the important role mathematics played in philosophy this had profound implications in the rise of a sort of anti-realist line of philosophical thought; in mathematics though, it led to an effort to fix the problem and through the work of Weierstrass and others, this was achieved and that achievement led in turn to a new more flexible approach to calculus which continues to mathematics well, and, in particular, gave the mathematical framework that made it possible to make describe quantum mechanics.

The new, more rigorous, approach to calculus depends on the defining the *limit*. Consider the function

$$f(x) = \frac{2x(x-1)}{x-1} \quad (1)$$

It is clear to use that this is equivalent to  $x$  and that  $f(1) = 2$ ; however, if you plug  $x = 0$  into the equation for  $f(x)$  you will get zero over zero and we know that doesn't have a value, it is undefined. In fact, the logic that tells us that  $f(x)$  is  $x$  involves cancelling the  $x-1$  above and below the line, and at  $x = 1$  that cancelling is equivalent to dividing by zero. So what gives? Well, most of time, away from the special cases of interest to mathematicians, we regard the equals as good as long as it holds almost everywhere, everywhere except at specific points, so we would say without hesitation that

$$f(x) = \frac{2x(x-1)}{x-1} = 2x \quad (2)$$

A mathematician who had need to worry about these things might at a "a.e." beside that equation to mean it works almost everywhere, that is everywhere except at  $x = 1$ . We don't need to do that!

Our interest however is not in the almost everywhere business, here I want to look at this as an example of a limit. The function

$$f(x) = \frac{x(x-1)}{x-1} \quad (3)$$

might not make sense at  $x = 1$  and so we might say

$$f(x) = \begin{cases} x(x-1)/(x-1) & x \neq 1 \\ \text{undefined} & x = 1 \end{cases} \quad (4)$$

but we'd still sort of "know" the value at  $x = 1$  as the value that you'd guess if you looked at the values for  $x$  very close to one. This is the idea of the limit, we would say

$$\lim_{x \rightarrow 1} f(x) = 2 \quad (5)$$

to mean, if we look at the values of  $f(x)$  as close as you please to  $x = 1$  you'd see  $f(x)$  was clearly going to two, notwithstanding the fact that  $f(x)$  is actually undefined at  $x = 1$ . This is the sort of situation we need to deal with in defining the derivative

$$\frac{f(x+h) - f(x)}{h} \quad (6)$$

doesn't make sense at  $h = 0$  since it is zero over zero at there, but we'd like to know what the value is at  $h = 0$  based on what it is doing when  $h$  is very close to zero. In short, we want to define

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (7)$$

All we are lacking is a rigorous definition of the limit, but that we can now provide.

We don't want to make too much of a fuss of the so called epsilon-delta box definition of the limit; in the use we make of analysis it doesn't lead to anything<sup>1</sup>, it is just reassuring to know that there is a rigorous definition. The definition is this:

$$\lim_{x \rightarrow a} f(x) = b \quad (8)$$

if and only if for any  $\epsilon > 0$  there exists an  $\delta > 0$  such that if  $|x - a| < \delta$  then  $|f(x) - b| < \epsilon$ ; or, written in the language so dear to mathematicians in the early twentieth century:

$$\lim_{x \rightarrow a} f(x) = b \iff \forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - b| < \epsilon \quad (9)$$

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<sup>1</sup>This isn't completely true, cryptography and some part of algorithms do make use of limits in a way that requires a good understanding of epsilon-delta boxes, but for us, here, we'll look at it briefly and then move on.

where  $\forall$  is for all,  $\exists$  is exists, s.t. means such that and  $\epsilon$  is the Greek letter epsilon, another letter often used for small things. Anyway, the gist of all this is a sort of test, I say that the limit of  $f(x)$  as  $x$  goes to  $a$  is  $b$  and you say, well what about  $\epsilon$ , can you make  $f(x)$  within  $\epsilon$  of  $b$  and I say, why yes, if  $x$  is within  $\delta$  of  $a$  then  $f(x)$  will indeed be within  $\epsilon$  of  $b$ . If this works for any  $\epsilon$  then we have a limit.

This allows us to define a *continuous* function, a continuous function is one that always equals its limit, that is  $f(x)$  is continuous if and only if

$$f(a) = \lim_{x \rightarrow a} f(x) \quad (10)$$

for all  $x$  in its domain. As mentioned, the limit gives us a definition of the derivative that avoids the hazy notion of the infinitesimal

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (11)$$

Of course, the challenge now would be to rederive all the laws of calculus, the product rule and so on, using this definition and this is what mathematicians have done. Finally we note that a *smooth* function is one whose derivative is continuous.

## Extrema

One common practical use of the derivative is finding maxima and minima; in real use this is often done using a numerical scheme, this still involves the derivative and we'll return to this topic later. For now though we look at cases where we can find the extrema directly, the main idea is pretty simple, if a function is at a maximum or a minimum it isn't changing and so its derivative is zero.

Here is an example:

$$f(x) = \exp(-x^2) \quad (12)$$

This is like a Gaussian curve, future versions of these notes will have figures, but for now you can google what the curve looks like. Anyway

$$\frac{df}{dx} = -2x \exp(-x^2) \quad (13)$$

and the exponential part is only zero as  $x$  goes to plus or minus infinity, so the only way this can be zero is if  $x = 0$ , telling us that if  $f(x)$  has an

maximum or a minimum it has to be at  $x = 0$ ; in fact it has a maximum at  $x = 0$ .

It is important to notice that having a zero derivative is a necessary but not a sufficient condition for having a maximum or minimum; consider

$$f(x) = x^3 \quad (14)$$

This has derivative

$$\frac{df}{dx} = 3x^2 \quad (15)$$

which is zero at  $x = 0$  but if you sketch the curve it doesn't have an extremum at  $x = 0$ ; the rate of change is zero, the curve is sort of getting flatter and flatter until at  $x = 0$  it sort of instantaneously stops going up before deciding to continue going up. We call a point where  $df/dx = 0$  a *critical point*; this can be an extremum, a max or min, but it doesn't have to be: an extremum is always a critical point but not visa versa.

Now, lets think about what happens at a maximum, as you approach a maximum from the left the rate of change is positive, you are going up, then at the maximum the rate of change is zero, it stops going up and then you start going down, the rate of change is negative. That implies that at a maximum the rate of change is zero but in the context of decreasing, in other words the rate of change of the rate of change is negative. In fact, if  $x$  is critical point,  $df/dx = 0$ , and

$$\frac{d^2f}{dx} < 0 \quad (16)$$

then  $x$  is a maximum, similarly if  $x$  is a critical point and

$$\frac{d^2f}{dx} > 0 \quad (17)$$

then it is a minimum. If it is zero then we are sure, it could be a critical point that isn't an extremum, for example, if  $f(x) = x^3$  then

$$\frac{d^2f}{dx} = 6x \quad (18)$$

and this is zero at  $x = 0$ , or it could still be an extremum, if  $f(x) = x^4$  then

$$\frac{d^2f}{dx} = 12x^2 \quad (19)$$

which is also zero at the critical point,  $x = 0$ ; so here it is a maximum, just a sort of flat one. In other words, typically you find the extrema by finding the places with zero derivative and you then check by calculating the second derivative; for some special critical points the second derivative is zero, so you need to do some more messing.

As a final example lets go back to

$$f(x) = \exp(-x^2) \quad (20)$$

This has, using the chain and product rules

$$\frac{d^2 f}{dx^2} = -2 \exp(-x^2) + 4x^2 \exp(-x^2) \quad (21)$$

so

$$\left. \frac{d^2 f}{dx^2} \right|_{x=0} = -2 < 0 \quad (22)$$

telling us the critical point at  $x = 0$  is a maximum.

## Summary

Here we look at the definition of the limit:

$$\lim_{x \rightarrow a} f(x) = b \iff \forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - b| < \epsilon \quad (23)$$

and use that to define the derivative

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (24)$$

We define a continuous function as one that is equal to its limit at every point and a smooth one as one with a continuous derivative. We define a critical point as a point where the derivative is zero and note that all extrema are critical points; if the second derivative at a critical point is negative the critical point is a maximum, if positive a minimum. If it is zero it could be either, or it might not be an extremum at all.