COMS10013 - Analysis - WS3 - outline solutions

These are outline solutions to the main questions in worksheet 2, solutions to the other questions will also appear.

Questions

These are the questions you should make sure you work on in the workshop.

1. **Taylor series** Calculate the Taylor expansion, three or four terms, at $x = 0$ for
   (a) $f(x) = 1/(1 + x)$: $f'(x) = -1/(1 + x)^2$ and $f''(x) = 2/(1 + x)^3$ and $f'''(x) = -6/(1 + x)^4$ and you get the idea, so the factor in front cancels the $1/n!$ in the formula for the Taylor expansion and
   $$f(x) = 1 - x + x^2 - x^3 \ldots$$
   This satisfying fact is actually very useful.
   (b) $f(x) = \log(1 + x)$: So here everything happens just a little later, so $f'(x) = -1/(1+x)$ and then everything proceeds as before, differentiation-wise and
   $$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$$
   Again, this turns out to very useful, for example, in approximations in the variational inference spirit to objective functions in deep learning.
   (c) $f(x) = \exp(x)$: Ok look by now we know that differentiating the exponential does nothing to it so we get
   $$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ldots$$
2. **Complex numbers**: calculate the following complex numbers in the form $(a + bi)$:
   (a) $(2 + 3i) + (5 - 2i)$: just add $7 + i$.
   (b) $(-1 + i)(-1 - i)$: multiply it out and get 2.
   (c) $(1 - i)^3$: just multiple it out to get $-2(1 + i)$.
   (d) $(1 + i)/(1 - i)$, multiply above and below by the conjugate of the denominator, that is by $1 + i$, on the bottom you have $(1 + i)(1 - i) = 2$ and on the top $(1 + i)^2 = 2i$, squaring doesn't always give a purely imaginary number, as it has in the last two examples, that's just a coincidence, or rather the effect of me picking complex numbers of the form $1 \pm i$ out of laziness. With way, the answer is i.
3. **More complex numbers**: Compute the real part, imaginary part, norm, and conjugate of the following numbers:
   (a) i: real part is zero and the imaginary part i, the conjugate is $-i$ and the norm is one.
   (b) $3 - 2i$: real part 3, imaginary part $-2i$, the conjugate is $3 + 2i$ and the norm is square root of the number multiplied by its conjugate, so $\sqrt{13}$.

4. **Polar form**. Convert between rectangular $(a + ib)$ and polar $re^{i\theta}$ form:
   (a) i: gives $e^{i\pi/2}$.
   (b) $2 - i$: the norm is $sqrt5$ and the angle is some annoying angle whose tan is $1/2$.
   (c) $3e^{i\pi/2}$ is $3i$.
   (d) e^{1+2i}, this is also annoying, we have

$$e^{1+2i} = e \times e^{2i} = e[\cos 2 + i \sin 2]$$

   which I guess you could work out with a calculator.