COMS10013 - Analysis - WS4 - outline solutions

Outline solutions for the main questions.

Questions

These are the questions you should make sure you work on in the workshop.

1. **First order inhomogeneous equations**.
   (a) $f'(t) + 5f(t) = 1$ with initial condition $f(0) = 2$: well staring it at a bit $f(t) = 1/5$ is a particular solution and so
   $$y(t) = Ae^{-5t} + \frac{1}{5}$$
   is the generali solution. Putting $f(0) = 2$ gives $A = 9/5$.
   (b) $f'(t) = t - f(t)$ with initial condition $f(1) = 3e^{-1}$: again, rather than using integrating factors you can just stare at this one a bit, or substitute in $f(t) = at + b$ so $f'(t) = a$ and hence $a = t - at - b$ so $a = 1$ and $b = -a$ or $f(t) = t - 1$ is a particular solution and so the general solution is
   $$f(t) = Ae^{-t} + t - 1$$
   The so called initial condition gives
   $$3e^{-1} = Ae{-1}$$
   so $A = 3$.
   (c) $f'(t) + 2f(t) = \sin(t)$ with initial condition $f(0) = 9/5$: again, you could use an integrating factor here, multiplying both sides by $\exp 2t$ so
   $$\frac{d}{dt}e^{2t}f(t) = e^{2t}\sin(t)$$
   but, of course, this gives
   $$e^{2t}f(t) = \int e^{2t}\sin(t)dt$$
   and, while you could do that by parts, does anyone want to? Let try to use an ansatz:
   $$f(t) = a\sin t + b\cos t$$
   and hence
   $$a\cos t - b\sin t + 2a\sin t + 2b\cos t = \sin t$$
   so the cosine coefficients give $a + 2b = 0$ and the sine coefficients give $-b + 2a = 1$, or $-b - 4b = 1$ so $b = -1/5$ and $a = 2/5$ giving particular solution:
   $$f(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t$$
   and general solution
   $$f(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t + Ae{-2t}$$

If we put in $t = 0$ we get
$$\frac{9}{5} = -\frac{1}{5} + A$$
or $A = 2$.

(d) $f'(t) - 2f(t) + t^2 = 0$ with initial condition $f(2) = 13/4 + 6e^4$. In a now familiar type of laziness lets use an ansatz $f = at^2 + bt + c$ to get
$$2at + b - 2at^2 - 2bt - 2c + t^2 = 0$$
or $a = 1/2$, $b = a$ and $b - 2c = 0$ so $b = 1/2$ and $c = 1/4$. The general solution is therefore
$$f(t) = Ae^{2t} + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{4}$$
and substituting gives
$$\frac{13}{4} + 6e^4 = Ae^4 + \frac{13}{4}$$
and hence $A = 6$.

2. **Second order equations** Solve the following equations for the given initial conditions:

(a) $\ddot{y}(t) = -y(t)$ with initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$: if you substitute $y = \exp rt$ r will be imaginary so try $y = A\sin t + B\cos t$ and it works, the initial conditions give $B = 1$ and $A = 0$.

(b) $\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = 0$ with initial conditions $y(0) = 0$ and $\dot{y}(0) = -2$: now $y = \exp(rt)$ gives
$$r^2 + 4r + 3 = 0$$
or $(r + 1)(r + 3) = 0$ hence
$$y = Ae^{-t} + Be^{-3t}$$
and $A + B = 0$ while $-A - 3B = -2$, so $A = -B$ or $B = 1$ and $A = -1$.

(c) $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 0$ with initial conditions $y(0) = 2$ and $y(1) = 3/e$: whoops this is hard, $r^2 + 2r + 1 = 0$, or $(r + 1)^2 = 0$, this is a bit of an unfair question and I will move it to the 'extra question part' next year, one solution is $A\exp(-t)$ but the other is $Bt\exp -t$, this is what happens with the r-equation, called the auxiliary equation has a repeated root.

(d) $\ddot{y}(t) - 4\dot{y}(t) + 13y(t) = 0$ with initial conditions $y(0) = 2$ and $y(\pi/6) = e^{\pi/3}$: $r^2 - 4r + 13 = 0$ gives
$$r = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$$
Again this is a hard question, the solution is
$$y = e^{2t}[A\sin 3t + B\cos 3t]$$
but this should've gone in the extra questions section.