Differential equations

A differential equation is an equation for an unknown function, or functions, where the equation involves derivatives of the function as well as the function itself. Differential equations are everywhere, obviously in physics, in dynamics Newtons law, $F = ma$ is a differential equation since a, the acceleration, is the second derivative of position; in quantum mechanics, Schrödinger's equation, or even more complicated equations like the Dirac equation, are differential equations. In biology, the Hodgkin-Huxley equation links the rate of change of the voltage inside a neuron to its inputs, in finance, the Black-Scholes equation links the rate of change value of an option to the value of the underlying commodity, climate modelling is just a giant set of differential equations.

We can solve some equations exactly, others we can't solve, in fact, most non-linear equations cannot be solved exactly, but we can often solve them approximately using computers and knowing about the solutions to linear equations can give us a good understanding of how differential equations work.

Before we discuss terminology and methods, we'll look at a couple of examples, here is a very simple example, say

$$\frac{df}{dt} = t^2 + 2 \tag{1}$$

and we know the initial value of $f(t)$, it is $f(0) = 5$. We can solve this by **direct integration**, we write

$$df = (t^2 + 2)dt \tag{2}$$

and integrate both sides

$$\int df = \int (t^2 + 2)dt \tag{3}$$

or

$$f = \frac{1}{3}t^3 + 2t + C \tag{4}$$

where C is the arbitrary constant of integration. Now substitute in $t = 0$ to get

$$5 = C \tag{5}$$

and the solution is

$$f = \frac{1}{3}t^3 + 2t + 5 \tag{6}$$

Now you could object that there is nothing in the proofs and definitions we looked at to say you can treat the df/dt like a fraction and multiply across by dt, in fact, you most explicitly cannot treat df/dt as a fraction in most circumstances, but there are theorems that say that it works in cases like the one we just looked at where we are going to integrate. In other words, don't think the fact df/dt looks like a fraction allows you to treat it like a fraction in all circumstances, but, in fact, there are theorems we won't look at that says that you can in this situation. This theorem is called **the fundamental theorem of calculus** and it basically says that the inverse of differentiation, the anti-derivative, is the indefinite integral. There is a short table of integrals at the end of these notes in case you need reminding.

Here is another simple example:

$$\frac{df}{dt} = 2f \tag{7}$$

This means

$$\int \frac{df}{f} = 2\int dt \tag{8}$$

and using $\int dx/x = \log x + C$ this gives

$$\log f = 2t + C \tag{9}$$

or

$$f = e^{2t+C} \tag{10}$$

or

$$f = Ae^{2t} \tag{11}$$

Here we have taken the $\exp C$ and, since C is an arbitrary constant, we can just rewrite it as another arbitrary constant $A = \exp C$. It might appear that this is cheating since $\exp C$ is always positive, but with complex numbers that isn't always the case and, in fact, there is a bit of messing going on with the sign when you do the integration, so it actually all works out. You can check the putative solution is the solution by differentiating:

$$\frac{d}{dt}Ae^{2t} = 2Ae^{2t} = 2f \tag{12}$$

Types of differential equation

We are going to look at differential equations with just one unknown function and one variable; this is an **ordinary** differential equation, for example

$$\frac{d\theta}{dt} = w + \sin\theta - t \tag{13}$$

for constant w is an ordinary differential equation, albeit a hard one, whereas

$$\frac{\partial\phi}{\partial t} + \frac{\partial^3\phi}{\partial x^3} - 6\phi\frac{\partial\phi}{\partial x} = 0 \tag{14}$$

for a function $\phi(x,t)$ that depends on x and t is a **partial** differential equation and not an ordinary differential equation. We won't be looking at partial differential equations here. If you are curious, this very difficult partial differential equation is called the Kortweig de Vries equation and it describes waves in a shallow channel; it was written down to describe a odd wave the engineer John Scott Russell spotted in a canal near Edinburgh.

An ordinary differential equation is **linear** if, roughly speaking, if f is the unknown variables all the f terms only have f and not powers of f or functions of f. Thus

$$\frac{df}{dt} = tf + t^2 \tag{15}$$

is a linear equation but

$$\frac{df}{dt} = f^2 \tag{16}$$

is not, nor is

$$\frac{df}{dt} = \sin f \tag{17}$$

nor is

$$f\frac{df}{dt} = t \tag{18}$$

Generally speaking, we are very good at solving linear equations but often struggle with nonlinear ones. Some nonlinear equations are linear equations in disguise, for example

$$f\frac{df}{dt} = f^2 \tag{19}$$

is actually the linear equation

$$\frac{1}{2}\frac{dg}{dt} = g \tag{20}$$

where $g = f^2$, but genuinely nonlinear equations can be hard, we will look at a few, fairly easy nonlinear equations in the worksheet.

An linear ordinary differential equation can be **homogenous** or **inhomogeneous**; if f is the unknown function and t the variable then homogeneous equation is one where all the terms have an f or a derivative of f in, so

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + f = \sin t \tag{21}$$

is inhomogeneous, whereas

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + f = 0 \tag{22}$$

is homogeneous. The order of a ordinary differential equation is the order of the highest derivative, so the example above is second order because of the d^2f/dx^2 term whereas

$$\frac{df}{dt} = \sin f \tag{23}$$

is first order. Finally, the easiest examples have constant coefficients like

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + f = 0 \tag{24}$$

whereas more difficult, but for linear cases, often still solvable examples have coefficients that depend on the variable, like

$$\frac{d^2f}{dt^2} + 2t\frac{df}{dt} + f = 0 \tag{25}$$

Solving first order linear ordinary differential equations

Here we'll look at solving first order linear ordinary differential equations with constant coefficients, the collection of methods we look at will include the method called integrating factors, which often also works for equations with variable coefficients, but we won't look at that here, integrating factors are for the next lecture. In this section we will look at homogenous equations, inhomogeneous equations will be looked at in the next lecture.

The first method is direct integration, here is another example:

$$\frac{df}{dt} = \frac{1}{t} \tag{26}$$

with $f(2) = 3$ then

$$\int df = \int \frac{dt}{t} \tag{27}$$

so

$$f = \log t + C \tag{28}$$

where C is a constant of integration. A differential equation tells you how a function is changing in terms of its current value; this means that to know the solution completely, without an arbitrary constant, you usually need to know the function at at least one place: if I tell you the speed something is moving at, you can work out how far it will move but you can't work out where it will be unless I also tell you where it started. Often this extra information is an initial condition, $f(0)$ but in this problem we have been given the value at $t = 2$:

$$3 = f(2) = \log 2 + C \tag{29}$$

so $C = 3 - \log 2$ and the solution is

$$f = \log \frac{t}{2} + 3 \tag{30}$$

where we have used the law of logs that says $\log a - \log b = \log a/b$.

A second method of solving differential equation is the **ansatz**. The word ansatz is German for guess and the idea here is that you just guess the solution, a good guess often looks like

$$f(t) = Ae^{rt} \tag{31}$$

with constants r and A and then see if that works out, so basically you are guessing the solution has a particular form, in this case an exponential and you substitute in and check if it works out, if the solution doesn't have that form it won't work, if it does, you've solved the equation. The easiest example would be equations that look like

$$\frac{df}{dt} = 4f \tag{32}$$

Substitute in the ansatz and you get

$$Are^{rt} = 4Ae^{rt} \tag{33}$$

Cancel the A and the $\exp rt$ and you get

$$r = 4 \tag{34}$$

and so the ansatz works provided $r = 4$ and so the solution is

$$f = Ae4t \tag{35}$$

If you had used a bad guess, say $f = At^r$ for constant r it just won't work:

$$Art^{r-1} = 4At^r \tag{36}$$

and then dividing across by At^{r-1} we get

$$r = 4t \tag{37}$$

which isn't a constant.

Recalling integration

Recall that integration is the opposite of differentiation, so, for example

$$\frac{dt^n}{dt} = nt^{n-1} \tag{38}$$

so

$$\int t^{n-1}dt = \frac{1}{n}t^n + C \tag{39}$$

C is the integration constant, an arbitrary number. The idea is that

$$\frac{d}{dt}t^n = nt^{n-1} \tag{40}$$

but so is

$$\frac{d}{dt}\left(t^n + C\right) = nt^{n-1} \tag{41}$$

for any constant C so if you think of the indefinite integral as being the backwards of differentiation what you get isn't fully defined because you can add an arbitrary constant without changing anything.

This leads us to a list of indefinite integrals

- $\int t^n dt = t^{n+1}/(n+1) + C$.
- $\int \exp rt dt = \exp rt/r + C$ where r is constant.
- $\int 1/t dt = \log t + C$
- $\int \sin t dt = -\cos t$
- $\int \cos t dt = \sin t$

In addition you can do a **substitution**, for example if you wanted

$$\int xe^{x^2}dx \tag{42}$$

you might want to do a change of variables to $u = x^2$ since you know how to integrate $\exp u$ when you dn't know how to integrate $\exp x^2$. However, you also need to change the dx, there is a rule for this

$$dx = \frac{1}{du/dx}du \tag{43}$$

which in this case would tell you that

$$dx = \frac{du}{2x} \tag{44}$$

or $du = 2xdx$ so

$$\int xe^{x^2}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C \tag{45}$$

Now you'll see this only worked because there was an x knocking around to allow us to change the variable and this suggests some things are hard, even impossible, to integrate. We haven't looked at all the tricks, indeed we haven't looked at **integrating by parts**, one of the most important tricks, but it is indeed nonetheless true that some integrals are hard and many are impossible.