Second order equations

Here we will consider second-order homogenous differential equations; there is a rich literature on second-order differential equations of all sorts, they are the bread-and-butter of physical systems but there is already a lot of interesting ideas if you consider only the simplest second-order homogenous example. First, lets solve a second order differential equation:

$$\ddot{y} - y = 0 \tag{1}$$

and, since we don't have a method, lets try an ansatz, guessing $y = A \exp rt$:

$$Ar^2 \exp rt - A \exp rt = 0 \tag{2}$$

Cancel what can be cancelled and you get

$$r^2 = 1 \tag{3}$$

or $r = \pm 1$ so the solutions are

$$y = A_1 e^t + A_2 e^{-t} \tag{4}$$

It is easy to check this works:

$$\ddot{y} = A_1 e^t + A_2 e^{-t} = y \tag{5}$$

The first thing to notice is how easy this was; here's another example

$$\ddot{y} + \dot{y} - 6y = 0 \tag{6}$$

Substitute for $y = A \exp rt$ and

$$r^2 + r - 6 = 0 \tag{7}$$

which factorizes to $(r + 3)(r - 2) = 0$ and the solution is

$$y = A_1 e^{-3t} + A_2 e^{2t} \tag{8}$$

Of course, it won't always be that easy, these two examples have equations for r that are easy to factorize into real numbers, we will look at a more complicated example soon. There are other even more complicated examples where the equation for r has a repeated root, these will still have two solutions even if there is only one value of r, this involves a different ansatz; it isn't difficult but for reasons of time we won't look at those cases here.

Before we look at the more complicated examples, the other obvious thing is that we have a two-dimensional space of solutions, with two arbitrary constants, A_1 and A_2. This makes sense, imagine throwing a ball into the air, to map the position of the ball through time you would need not only to know where you threw it from, but how fast. In the same way that the arbitrary constant for first-order differential equations is often fixed by an initial conditions, for second order the two conditions are often fixed by an intial condition for $y(0)$ and $\dot{y}(0)$

As an example, say

$$\ddot{y} + 3\dot{y} + 2y = 0 \tag{9}$$

with $y(0) = 0$ and $\dot{y}(0) = 1$ then

$$y = A_1 e^{-t} + A_2 e^{-2t} \tag{10}$$

and $y(0) = A_1 + A_2 = 0$ and $\dot{y}(0) = -A_1 - 2A_2 = 1$, substituting $A_1 = -A_2$ from the $y(0)$ equation gives $1 = -A_2$ and so

$$y = e^{-t} - e^{-2t} \tag{11}$$

For spatical problems, the two conditions are often boundary conditions, values for y for two different values of its arguement, we will do an example like that shortly.

Now lets examine

$$\ddot{y} + y = 0 \tag{12}$$

This looks like the first one we did, but there is a change of sign, the usual ansatz gives

$$r^2 = -1 \tag{13}$$

or $r = \pm i$ and hence

$$y = A_1 e^{it} + A_2 e^{-it} \tag{14}$$

This is an inconvenient way to write it, so we can expand the exponentials

$$y = A_1(\cos t + i \sin t) + A_2(\cos t - i \sin t) \tag{15}$$

or renaming the arbitrary constant so $C_1 = A_1 + A_2$ and $C_2 = iA_1 - iA_2$ this gives

$$y = C_1 \cos t + C_2 \sin t \tag{16}$$

which is a function that oscillates up and down with period 2π. It might look like we cheated a bit to hide the is, in fact, of course, the C_1 and C_2 might be complex, but if initial or boundary conditions are real they will be real, a real solution to a real equation will stay real! Lets looks that this as a spatial problem, say

$$\frac{d^2y}{dx^2} + 4y = 0 \tag{17}$$

for $y(x)$, a function of x and with boundard conditions $y(0) = 0$ and $y(\pi/4) = 1$, then we get, but the usual process

$$y = C_1 \cos 2x + C_2 \sin 2x \tag{18}$$

and the first boundary condition gives $y(0) = C_1 = 0$ and so the second is now $y(\pi/2) = C_2 \sin \pi/2 = C_2 = 1$ so the solution is

$$y(x) = \sin 2x \tag{19}$$

You will notice that we have only chosen the simplest examples in the complex case, there are more complicated examples where the r solutions have the form $r = a \pm bi$ and these will give solutions that oscillate but with an oscillation amplitude that changes. These examples aren't really any more difficult, but we won't look at them here.

What we will look at briefly is the relationship between second order differential equations and first order. It is possible to think of second order differential equations as just two first order. This is both profound, it is at the heart of the powerful approaches to dynamics developed in the nineteenth century and then applied to quantum mechanics in the twentieth, and useful: we already know how to solve first order differential equations numerically on a computer. The key to this method is to think about dynamics were we consider position and speed different quantities, so say we have the differential equation

$$\ddot{u} + a\dot{u} + bu = 0 \tag{20}$$

and we let

$$\dot{u} = v \tag{21}$$

that is, we introduce a new function v, now $\ddot{u} = \dot{v}$ and so the differential equation for u becomes

$$\dot{v} + av + bu = 0 \tag{22}$$

so we know have two first order differential equations, $\dot{u} = v$ and $\dot{v} + av + bu = 0$; these are coupled, there are us in the v equation and vs in the u equation, but uncoupling them is actually just a process of linear algebra. Again, we won't go any further with this, but hopefully you can see how this reduces the second order problem to a first order one, albeit a first order differential equation with vectors (u, v) and matrices; this does unlock computer approaches to numerical solution.

Summary

Here we look at second order homogeneous, linear ordinary differential equations, the ansatz $y = A \exp rt$ usually works, giving a quadratic equation for r and hence, in general two solutions, or a two-dimensional family of solutions

$$y = A_1 e^{r_1 t} + A_2 e^{r_2 t} \tag{23}$$

There are cases where the quadratic equation only has one solution, it is still possible to get a two-dimensional family of solutions but we don't look at that here. Often the constants A_1 and A_2 are fixed by initial or boundary conditions. If r is imaginary the solution can be written in terms of sines and cosines, this is a periodic solution. A simple trick allows us to chanage a second-order equation into two first order equations.