

COMS10013 - exam questions - 25 points

All sections are worth five points; outline solutions are just outlines, you should give more workings!

Part 1

This question is about the definition and properties of differentiation. In one sentence describe in words what df/dt means [1 points]. Give the formal definition of the limit [1 point] and of df/dt [1 point] and use this to argue [2 points] that

$$\frac{d(f+g)}{dt} = \frac{df}{dt} + \frac{dg}{dt}$$

Outline Solution

So df/dt is the rate of change of f , it quantifies how much f changes as t changes, at a single point in t . The limit

$$\lim_{t \rightarrow a} f(t) = b$$

if and only if for every ϵ there exists a δ such that $|t - a| \leq \delta$ implies $|f(t) - b| \leq \epsilon$; with this machinery we have

$$\frac{df}{dt} = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$

and the linearity of the limit means

$$\frac{d(f+g)}{dt} = \lim_{h \rightarrow 0} \frac{f(t+h) + g(t+h) - f(t) - g(t)}{h} = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h}$$

and the second term clearly gives the answer.

Part 2

This question is about actually taking derivatives. Using the quotient rule work out [1 point]

$$\frac{d \tan x}{dx}$$

Using the chain rule work out [2 points]

$$\frac{d \tan \sin x}{dx}$$

Given the Taylor expansion of the exponential

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

show that [2 points]

$$\frac{de^x}{dx} = e^x$$

Outline Solution

So you use $\tan x = \sin x / \cos x$ so

$$\frac{d \tan x}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 / \cos^2 x$$

Next

$$\frac{d \tan \sin x}{dx} = \frac{\cos x}{\cos^2 \sin x}$$

Finally,

$$\frac{1}{n!} \frac{dx^n}{dx} = \frac{1}{(n-1)!} x^{n-1}$$

so differentiating just shuffles all the terms down one.

Part 3

This question is about partial derivatives. What is the gradient [1 point] of

$$z = xy \sin xy$$

Find the Hessian [2 point], show $(0, 0)$ is an extremum [1 point] and indicate whether it is a maximum, minimum or saddle point [1 point].

0.0.1 Outline Solution

So the gradient is

$$\nabla z = (y \sin xy + xy^2 \cos xy, x \sin xy + x^2 y \cos xy)$$

The Hessian is

$$H = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix}$$

where

$$z_{xx} = y^2 \cos(xy) + y \cos(xy) - x^2 y^3 \sin(xy) + 2xy^2 \cos(xy)$$

and

$$z_{xy} = \cos(xy) - xy^2 \sin(xy) + y \cos(xy) = z_{yx}$$

and finally

$$z_{yy} = x^2 \cos(xy) + x \cos(xy) - x^3 y \sin(xy) + 2x^2 y \cos(xy)$$

So the gradient is zero at $x = y = 0$ showing it is an extremum and

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which has determinant -1 and trace zero, so the two eigenvalues have opposite signs and this is a saddle point.

Part 4

This question is about complex numbers. What are the solutions of [1 point]

$$z^2 + 2z + 2 = 0$$

What are the solutions of [1 point]

$$z^2 + 2z + 5 = 0$$

In each case what is the relationship between the two solutions as complex numbers? [2 points]
How many solutions do you expect for [1 point]

$$z^6 - 1 = 0$$

What are these solutions? [1 point]

Outline Solution

So the first equation has solutions $z = 1 \pm i$ and the second $z = 1 \pm 2i$; in each case the solutions are conjugates of each other. $z^6 = 1$ has six solutions, they are

$$z = \exp n\pi i/3 \tag{1}$$

for n between zero and five, inclusive.

Part 5

This question is about differential equations. Solve the following [a-c 1 point each, d 2 points] where the dot means the derivative with respect to time.

- (a) $\dot{y}(t) - y(t) = 0$ with initial condition $y(0) = 1$.
- (b) $\dot{y}(t) + 3y(t) = 0$ with initial condition $y(3) = 3$.
- (c) $\dot{y}y(t) = 0$ with initial condition $y(5) = 2$.
- (d) $\dot{y} = ry(1 - y)$ with $y(0) = y_0$.

Solution

These previously appeared in worksheet 4.