Complex numbers

It probably does the imaginary number $i = \sqrt{-1}$ a disservice to call it an imaginary number; numbers are all to a certain extent imaginary. It is easy to think there is something concrete about the idea of say 'five', but it is a concept not a thing, it is the number of elements in sets that have five elements, or some such piece of semi-philosophical legerdemain. Negative numbers, or real numbers such as $\sqrt{2}$ are even less obviously 'real' despite the advertizing given by calling real numbers by that name. However, there is a long history of adding new types of numbers because they are demanded by the algebraic or arithmetical rules that have been discovered; so, if you have subtraction and are able to do $7-5=2$ you immediately wonder what $5-7$ is and hence invent negative numbers; if you have division and know $6/3=2$ you wonder what $5/2$ is and invent rationals, when you know about Pythagoras's theorem and can work out $\sqrt{25}=5$ you get worried about $\sqrt{2}$ and invent irrational numbers. Similarly we have known how to solve quadratic equations $ax^2+bx+c=0$ using

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{1}$$

since the dawn of cities in Babylon, and that immediately raises the question of what $\sqrt{-1}$ is and leads to the so called imaginary number and complex numbers. In fact, complex numbers turn out to be a very powerful and useful mathematical construction, extremely helpful in, to give a computer science example, signal processing.

So a complex number has a real part and an imaginary part:

$$z = x + iy \tag{2}$$

examples would be $1+2i$ or $-3i$ or whatever. You can add them:

$$x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + (y_1 + y_2)i \tag{3}$$

so $3+2i$ added to $-2+5i$ is $1+7i$. You can multiply them, rather than getting tangled up in symbols, lets just do a specific example:

$$(1+3i)(2-5i) = 2+6i-5i-15i^2 = 17+i \tag{4}$$

where we have used that $i^2=-1$; this is, after all, sort of the point of i.

There is some complex number specific algebraic manipulations, the **conjugate** of a complex number is the complex number you get by switching the sign of the imaginary part, so if $z = x + iy$ then the conjugate is

$$z^* = x - iy \tag{5}$$

There are actually two notations often used for the conjugate, z^* and $\bar{z}$; you see both used, sometimes by the same person; while we are talking notation, you should note that electronic engineers sometimes use j for the complex number instead of i; so they use $j = \sqrt{-1}$; this is because the use i for current. The absolute value of a complex number is

$$|z| = \sqrt{zz^*} \tag{6}$$

This is a real number, if $z = x + iy$ then, if you expand out the bracket you can see

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 \tag{7}$$

One perhaps surprising thing is that you can divide two complex numbers; a complex number has the form $x + iy$ but dividing $z_1 = x_1 + iy_1$ by $z_2 = x_2 + iy_2$ seems to give something that doesn't have this form

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \tag{8}$$

However, you can get rid of the complexness of the denominator by multiplying by z_2^*/z_2^*; you can do this because it is actually just one. Hence

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2}\frac{x_1 - iy_1}{x_2 - iy_2} = \frac{(x_1 + iy_1)(x_2 + iy_2)}{x_2^2 + y_2^2} \tag{9}$$

and if you multiply out the numerator, this does indeed have the form $x+iy$. Lets do an example@

$$z = \frac{1 + i}{3 - 2i} \tag{10}$$

Now the conjugate of the denominator is $3 + 2i$ so

$$z = \frac{1 + i}{3 - 2i}\frac{3 + 2i}{3 + 2i} = \frac{(1 + i)(3 + 2i)}{13} = \frac{1}{13} + \frac{5}{13}i \tag{11}$$

Now, this ability to divide complex numbers is interesting. Complex numbers are somewhat akin to two dimensional vectors, you can map from one to the other:

$$z = x + iy \leftrightarrow \mathbf{z} = x\mathbf{i} + y\mathbf{j} \tag{12}$$

However, while you can add two dimensional vectors, you can't divide them, the complex structure is an additional structure beyond the geometrical structural of two-dimensional space. In fact, the ability to add a structure that allows division is only possible in certain numbers of dimensions, in two-dimensions there are complex numbers, in four there are another type of number called quoternions and in eight dimensions a very difficult structure called and octonion algebra.

Apart from this musing about division and geometry, thinking of complex numbers as points in two-dimensional space leads to an important idea: the polar representation. Polar coordinates are an alternative coordinate system for two dimensions. Instead of writing the position as (x, y) where x is the distance in the x direction and y the distance in the y direction you can write the position in polar coordinates as (r, θ) where r is the distance from the origin and θ is the angle the line to the position makes with the x axis. It is easy to translate between the two, a little bit of trigonometry tells us that $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$ and, conversely, $x = r\cos\theta$ and $y = r\sin\theta$.

The same thing can be done with complex number, this is called the **polar representation** and relies on the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{13}$$

It might seem that almost everything is named after Euler! There are lots of ways to derive this formula, including using the Taylor series; but we will just accept it here. This means there are two ways to write a complex number:

$$z = x + iy = re^{i\theta} \tag{14}$$

where $r = \sqrt{zz^*} = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. As an example,

$$1 + i = \sqrt{2}e^{i\pi/4} \tag{15}$$

One advantage of the polar representation is that it allows you to find powers of complex numbers, if

$$z = re^{i\theta} \tag{16}$$

then

$$z^n = r^n e^{in\theta} \tag{17}$$

This has a slightly surprising result when applied to roots. Recall the way there are two solutions to $x^2 = a$, you have $x = \sqrt{a}$ obviously, but also $x = -\sqrt{a}$. When you include complex numbers this is only the first in a whole series of similar examples, so, consider the equation:

$$z^n = a \tag{18}$$

in polar form this give

$$\left(re^{i\theta}\right)^n = a \tag{19}$$

or

$$r^n e^{in\theta} = a \tag{20}$$

so, first off $r = \sqrt[n]{a}$, so the interesting bit is the **n-th root of unity**:

$$e^{in\theta} = 1 \tag{21}$$

Now, obviously, $\theta = 0$ is a solution, but so is $\theta = 2\pi/n$ since

$$e^{in2\pi/n} = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 \tag{22}$$

In fact there are n solution: 0, $2\pi/n$, $4\pi/n$ and so on until you get to 2π, that isn't a new solution, it is equivalent to $\theta = 0$; for example

$$e^{3i\theta} = 1 \tag{23}$$

has solutions $\theta = 0$, $\theta = 2\pi/3$ and $\theta = 4\pi/3$, or

$$e^{4i\theta} = 1 \tag{24}$$

has solutions $\theta = 0$, $\theta = \pi/2$, $\theta = \pi$ and $\theta = 3\pi/2$. For

$$e^{2i\theta} = 1 \tag{25}$$

the two solutions are $\theta = 0$ and $\theta = \pi$ and since

$$e^{\pi i} = \cos \pi + i \sin \pi = -1 \tag{26}$$

this is the $x^2 = 1$ means $x = 1$ or $x = -1$ we mentioned earlier.

Summary

Complex numbers have the form $z = x + iy$; the conjugate is

$$z^* = x - iy \tag{27}$$

while the absolute value is

$$|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2} \tag{28}$$

To divide two complex numbers you multiple above and below by the conjugate of the denominator, this will get rid of the is below the bar:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} \tag{29}$$

You can rewrite a complex number in polar form

$$z = re^{i\theta} \tag{30}$$

using the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{31}$$

This is particularly useful when calculating powers of complex numbers. When taking roots of complex numbers, remember there are n n-roots of unity.