COMS10013 - Analysis - WS6 - more outline solutions

Extra questions

1. **Divergence and Laplacian**. The divergence of a vector is a sort of complement to the gradient of a function. If
$$\mathbf{v}(x, y) = [v_1(x, y), v_2(x, y)] \tag{1}$$
is a vector function in two dimensions then the divergence is
$$\text{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{2}$$
This can be written as
$$\text{div}\mathbf{v} = \nabla \cdot \mathbf{v} \tag{3}$$
Now if you have a function $f(x, y)$ we define the Laplacian as
$$\Delta f = \nabla \cdot \nabla(f) = \text{div}\,\text{grad}\, f \tag{4}$$
write this down in terms of partial derivatives. How is it related to the Hessian? So this is easy enough:
$$\Delta f = \nabla \cdot [f_x, f_y] = f_{xx} + f_{yy} \tag{5}$$
and this is the trace of the Hessian.

2. **Curl**. After gradient and divergence, the curl fills out the list of vector differential operators, weirdly it only exists in three dimensions. If
$$\mathbf{v} = [v_1, v_2, v_3] \tag{6}$$
is a vector function then
$$\text{curl}\,\mathbf{v} = \left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right] \tag{7}$$
if $f(x, y, z)$ is a function and $\mathbf{v}(x, y, z)$ a vector function, both in three dimensions what are $\text{curl}\,\text{grad}\, f$ and $\text{div}\,\text{curl}\,\mathbf{v}$? Again, this is easy that it sounds, I think.
$$\text{curl}\,\text{grad}\, f = \text{curl}\,[f_x, f_y, f_z] \tag{8}$$
and hence
$$\text{curl}\,\text{grad}\, f = [f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}] = 0 \tag{9}$$
For the other one
$$\text{div}\,\text{curl}\,\mathbf{v} = \mathbf{div}(v_{3y} - v_{2z}, v_{1z} - v_{3x}, v_{2x} - v_{1y}) = v_{3yx} - v_{2zx} + v_{1zy} - v_{3xy} + v_{2xz} - v_{1yz} \tag{10}$$
and, since the order of the differentiation doesn't matter, this all cancels. These formula play a surprisingly important role in differential topology, which studies the relationships between differential operators, like these, and the shapes of spaces.

3. **Quaternions**. The quaternions are a type of generalization of complex numbers; they have some deep mathematical properties but in practice they are usually used to help describe rotations in three-dimensional space. Instead of just i there are three imaginary numbers i, j and k, and these all square to minus one: $i^2 = j^2 = k^2 = -1$. In addition $ijk = -1$ and the numbers are *anti-commutative*: $ij = -ji$, $jk = -kj$ and so on. Lots of other relationships can be derived from these rules, for example if you multiply $ijk = -1$ you get $jk = i$, or if you switch it $jik = 1$ and multiply by j you get $ik = 1$. Anyway, if

$$q = w + xi + yj + zk \tag{11}$$

and

$$q^* = w - xi - yj - zk \tag{12}$$

prove

$$qq^* = w^2 + x^2 + y^2 + z^2 \tag{13}$$

This is just a big calculation and everything cancels!