COMS10013 - Analysis - WS2 outline solutions

These are outline solutions to the main questions in worksheet 1, solutions to the other questions will also appear.

1. **Gradients and Hessians** Let $z(x,y) = x^2y + 3xy^2 + xy$.
   (a) Find the gradient of $z(x,y)$.
   $$\nabla(z) = \begin{pmatrix} \partial z/\partial x \\ \partial z/\partial y \end{pmatrix} = \begin{pmatrix} 2xy + 3y^2 + y \\ x^2 + 6xy + x \end{pmatrix}. \tag{1}$$
   (b) Find the derivative of $z(x,y)$ along the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
   $$\nabla z \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2xy + 3y^2 + y \\ x^2 + 6xy + x \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = x^2 + 9y^2 + 12xy + x + 3y. \tag{2}$$
   (c) Compute $\nabla_{\left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right)} z\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)$: plug in $(x,y) = (2,0)$ to get $2^2 + 0 + 0 + 2 + 0 = 6$.
   (d) What is the Hessian of $z(x,y)$? So we have $z_x = 2xy + 3y^2 + y$ and $z_y = x^2 + 6xy + x$, differentiating again gives $z_{xx} = 2y$ and $z_{xy} = 2x + 6y + 1$ and $z_{yy} = 6x$ and $z_{yx} = 2x + 6y + 1$ and we note that $z_{xy} = z_{yx}$ as it will be for any normal function. Hence, the Hessian is
   $$H = \begin{pmatrix} 2y & 2x + 6y + 1 \\ 2x + 6y + 1 & 6x \end{pmatrix} \tag{3}$$

2. **Extremal points in two dimensions**; this question is pretty hard!
   (a) Find the local extrema, and determine their types, for
   $$z(x,y) = x^3 + y^3 - \frac{1}{2}(15x^2 + 9y^2) + 18x + 6y + 1.$$
   (b) Find the local extrema, and determine their types, for
   $$z(x,y) = 3xy^2 - 30y^2 + 30xy - 300y + 2x^3 - 15x^2 + 111x + 7.$$

   Solutions:
   (a) Compute the gradiant first:
   $$\nabla z = \begin{pmatrix} 3x^2 - 15x + 18 \\ 3y^2 - 9y + 6 \end{pmatrix}.$$
   The top entry is zero if and only if $x = 2$ or 3 and the bottom entry is zero if and only if $y = 1$ or 2. So there are 4 possible extremal points:
   $$(2,1),\ (2,2),\ (3,1),\ (3,2).$$
   To determine their types, we compute the Hessian
   $$H = \begin{pmatrix} 6x - 15 & 0 \\ 0 & 6y - 9 \end{pmatrix},$$
   which has determinant $\det(H) = (6x - 15)(6y - 9) = 9(2x - 5)(2y - 3)$. At

- $(2, 1)$, $\det(H) = 9 \cdot -1 \cdot -1 > 0$ and $6x - 15 = -3 < 0$, so $(2, 1)$ is a local maximum.
- $(2, 2)$, $\det(H) = 9 \cdot -1 \cdot 1 < 0$ so $(2, 2)$ is a saddle point.
- $(3, 1)$, $\det(H) = 9 \cdot 1 \cdot -1 < 0$ so $(3, 1)$ is a saddle point.
- $(3, 2)$, $\det(H) = 9 \cdot 1 \cdot 1 > 0$ and $6x - 15 = 3 > 0$, so $(3, 2)$ is a local minimum.

(b) The gradient is

$$\nabla z = \begin{pmatrix} 3y^2 + 30y + 6x^2 - 30x + 111 \\ 6xy - 60y + 30x - 300 \end{pmatrix}.$$

Look first at the second entry $6(xy - 10y + 5x - 50)$. We want to understand when this is $= 0$, which then gives us an equation we can rearrange and solve:

$$y(x - 10) + 5x - 50 = 0 \Rightarrow y = 5\frac{10 - x}{x - 10} = -5,$$

for $x \neq 10$. When $x = 10$, the second entry gives us $60y - 60y + 300 - 300 = 0$, so the second entry is 0 when $x = 10$ or $y = -5$. Plugging in $y = -5$ to the first entry and setting it to zero gives us an equation for x:

$$0 = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6),$$

which has roots at $x = 2$ and 3. Plugging in $x = 10$ to the first entry and setting it to zero gives us an equation for y:

$$0 = 3y^2 + 30y + 411 = 3(y^2 + 10y + 137),$$

which has no real solutions. So the only possible extrema are

$$(2, -5) \text{ and } (3, -5).$$

To check their types, we look at the Hessian

$$H = \begin{pmatrix} 12x - 30 & 6y + 30 \\ 6y + 30 & 6x - 60 \end{pmatrix} = 6 \begin{pmatrix} 2x - 5 & y + 5 \\ y + 5 & x - 10 \end{pmatrix},$$

which has determinant

$$\det(H) = 36 \cdot ((2x - 5)(x - 10) - (y + 5)^2).$$

Then, at

- $(x, y) = (2, -5)$ we get $\det(H) = 36 \cdot ((-1 \cdot -8) - 0) > 0$ and $2x - 5 = -1 < 0$ so $(2, -5)$ is a local maximum.
- $(x, y) = (3, -5)$ we get $\det(H) = 36 \cdot ((1 \cdot -7) - 0) < 0$ so $(3, -5)$ is a saddle point.

3. **Taylor series**

(a) Compute the Taylor series of e^x at $x = 2$: well $de^x/dx = e^x$ so we get

$$e^{2+\delta} = e^2 \left[1 + \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \right] \tag{4}$$

(b) Compute the Taylor series of $1/(1-x)^2$ at $x=0$: so this one is cool,

$$\frac{d}{dx}\frac{1}{(1-x)^2}=\frac{2}{(1-x)^3} \tag{5}$$

and

$$\frac{d^2}{dx^2}\frac{1}{(1-x)^2}=\frac{6}{(1-x)^4} \tag{6}$$

if you do a few more you might guess

$$\frac{d^n}{dx^n}\frac{1}{(1-x)^2}=\frac{(n+1)!}{(1-x)^{n+2}} \tag{7}$$

and you can prove this is true by induction. Now

$$\left.\frac{d^n}{dx^n}\frac{1}{(1-x)^2}\right|_{x=0}=(n+1)! \tag{8}$$

so

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n. \tag{9}$$

(c) Compute the Taylor series of $1/x$ at $x=2$. Again if you differentiate a few times you'll see

$$\frac{d^n}{dx^n}\frac{1}{x}=\frac{(-1)^n n!}{x^{n+1}} \tag{10}$$

and substituting that in to the Taylor series gives

$$f(2+\delta)=2\sum_{n=0}^{\infty}(-1)^n\left(\frac{\delta}{2}\right)^n \tag{11}$$