COMS10013 - Analysis - WS3

These worksheets are partly, well mostly, taken from worksheets prepared by Chloe Martindale.

Useful facts

- the **Taylor series** is

$$f(a+x) = f(a) + f'(a)x + \frac{1}{2}f''(a)x^2 + \frac{1}{6}f'''(a)x^3 + \dots$$

  or

$$f(a+x) = f(a) + \sum_{n=1}^{\infty} \frac{1}{n!}x^n \left.\frac{d^n f}{dx^n}\right|_{x=a}$$

- if $z = x + iy$ the conjugate $z^* = x - iy$ and the absolute value is $|z|^2 = zz^* = x^2 + y^2$.
- to work out w/z where w and z are complex, multiply above and below by z^*.
- in polar form we write

$$x + iy = re^{i\theta} \tag{1}$$

  To convert from polar form to Cartesian form we use

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2}$$

  To convert the other way $r = |z|$ and $\theta = \tan^{-1} y/x$.

Questions

These are the questions you should make sure you work on in the workshop.

1. **Taylor series** Calculate the Taylor expansion, three or four terms, at $x = 0$ for
   (a) $f(x) = 1/(1+x)$
   (b) $f(x) = \log(1+x)$
   (c) $f(x) = \exp(x)$
2. **Complex numbers**: calculate the following complex numbers in the form $(a + bi)$:
   (a) $(2 + 3i) + (5 - 2i)$
   (b) $(-1 + i)(-1 - i)$
   (c) $(1 - i)^3$
   (d) $(1 + i)/(1 - i)$
3. **More complex numbers**: Compute the real part, imaginary part, norm, and conjugate of the following numbers:
   (a) i
   (b) $3 - 2i$
4. **Polar form**. Convert between rectangular $(a + ib)$ and polar $re^{i\theta}$ form:

(a) i

(b) $2-i$

(c) $3e^{i\pi/2}$

(d) e^{1+2i}

Discussion questions

These are questions you could discuss with your group:

1. What does conjugating a complex number mean as a geometric operation on a point in the complex plane?
2. What is the formula for conjugating a complex number given in polar form?
3. What is the formula for the norm of a complex number given in polar form?
4. What is the formula for the inverse of a complex number in polar form (e.g. $1/re^{i\theta}$, give the solution in polar form again) and what does this mean geometrically?
5. If you write a complex number $(a+bi)$ as a vector (a,b), how would you express the function which rotates the number around the origin by an angle θ? (Hint: think about matrices).
6. How are complex numbers different from two-dimensional vectors? (Hint: think about division)
7. What about other dimensions? Can they be thought of as being like complex numbers? This is a very difficult problem!

Extra questions

These are extra questions you might attempt in the workshop or at a later time; in fact these questions are tricky so you might want to come back to them later when you've had some more lectures.

1. **L'Hôpital's rule**: this says that if you interested in the limit of a function

$$\lim_{x\to a}\frac{f(x)}{g(x)} \tag{3}$$

and $f(a)=g(a)=0$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{4}$$

where I am using $f'(x)=df/dx$. Use this to work out

$$\lim_{x\to 0}\frac{\sin(x)}{x} \tag{5}$$

2. **Taylor series**: calculate the Taylor series for

$$f(x) = \begin{cases} \exp(-1/x) & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{6}$$

at $x = 0$. This is a bit of a joke, so don't spend too much time on it; weird though, isn't it!

3. **Equations with complex solutions**. Solve the following equations over the complex numbers
   (a) $x^2 - 2x + 5 = 0$
   (b) $x^2 - 2x + 8 = 0$
   (c) $x^2 - ix - 1 = 0$
   (d) $x^5 - 1 = 0$. How many solutions do you expect?

4. **Some hard complex number questions**.
   (a) Over the complex numbers, what are the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}?$$

   (b) The square roots of a number x are all numbers y with $y^2 = x$. How would you find the square root of any given complex number?
   (c) Compute $(1+i)^{1-i}$, either as an exact/symbolic result or to 3 decimal places.