COMS10013 - Analysis - WS6 - outline solutions

Questions

1. **Differentiation**. Differentiate this functions:
   (a) $f(t) = \sin t^2$; chain rule, $\dot{f} = 2t \sin t^2$.
   (b) $f(t) = \tan t$: quotient rule with $u = \sin t$ and $v = \cos t$ giving $1/\cos^2 t$ since $\sin^2 t + \cos^2 t = 1$.
   (c) $f(t) = 1/(1+t^2)$: chain rule again, $-2t/(1+t^2)^2$.
   (d) $f(t) = \log\exp(t)$: trick question since this is just t.

2. **Integration**. What is the indefinite integral $\int t \sin(t) dt$? What about $\int \sin(t)\cos(t) dt$? Everyone hates integrating by parts but here we go, for the first one let $u = t$ and $dv = d \sin t dt$ so $du = dt$ and $v = -\cos t$ so the answer is $-t\cos t + \int costdt = -t\cos t + \sin t + C$; the second one is more difficult, well it is easy if you just use $2\sin t \cos t = \sin 2t$ but you were asked to do it by parts, let:
$$I = \int \sin(t)\cos(t) dt \tag{1}$$
Now $u = \sin(t)$ and $dv = \cos(t)dt$ and they both sort of swap
$$I = \sin^2(t) - I \tag{2}$$
giving $2I = \sin^2 t$; so yeah everyone hates integrating by parts but that one is sort of fun.

3. **Integration**. There is no indefinite integral $\int \exp(t^2) dt$; have a go at failing to find one: no solution here, which is kind of the point.

4. **Partial fractions**. Now try
$$F = \frac{1}{(t-3)(3t+1)} \tag{3}$$
Well same craic:
$$\frac{1}{(t-3)(3t+1)} = \frac{A}{t-3} + \frac{B}{3t+1} \tag{4}$$
and multiply across
$$1 = A(3t+1) + B(t-3) \tag{5}$$
and $t = 3$ gives $A = 1/10$ and $t = -1/3$ gives $B = -3/10$ so
$$\frac{1}{(t-3)(3t+1)} = \frac{1}{10(t-3)} - \frac{3}{10(3t+1)} \tag{6}$$
Now use that to work out the indefinite integral
$$I = \int \frac{dt}{(t-3)(3t+1)} \tag{7}$$
Well the idea here is that
$$I = \frac{1}{10}\int \frac{dt}{t-3} - \frac{3}{10}\int \frac{dt}{3t+1} \tag{8}$$

and these integrals are easy, by sustitution using $u = t - 3$ and so $dt = du$ for the first one and $u = 3t + 1$ so $dt = du/3$ in the second, hence

$$I = \frac{1}{10}\log(t-3) - \frac{1}{10}\log(3t+1) \tag{9}$$

5. **The Laplace transform**. The transform maps a function $f(t)$ to another function $F(s)$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{10}$$

What is the Laplace transform of $f(t) = \exp at$? Well

$$F(s) = \int_0^\infty e^{at}e^{-st}dt \tag{11}$$

and $\exp(at)\exp(-st) = \exp[(a-s)t]$ and hence

$$F(s) = \frac{1}{s-a} \tag{12}$$

Now consider the differential equation

$$\frac{df}{dt} = 3f \tag{13}$$

with $f(0) = 2$. Now from the above

$$sF(s) - 2 = 3F(s) \tag{14}$$

so $(s-3)F(s) = 2$ or

$$F(s) = \frac{2}{s-3} \tag{15}$$

and hence $f(t) = 2e^{3t}$, which you can easily check is correct. This works for more complicated cases too, but you will generally need a few more rows in your Laplace transform table.