COMS10013 - Analysis - WS4 - outline solutions

This is for the main questions, solutions for the other questions will come later.

Questions

These are the questions you should make sure you work on in the workshop.

1. **A linear accelerated motion question**. A train is travelling from Bristol to London Paddinton at the maximum speed of 55.9 m/s, 125 mph, when the driver activates the emergency break. This causes the train to decelerate uniformly at 1.2 m/ss^2. How far will the train travel until it stops and how long will this take, in seconds. Do this using differential equations, for example:

$$\frac{dv}{dt} = -1.2 \tag{1}$$

   not by looking up formulas.
   Solution:
   Let

   - $y(t)$ be the position of the train at time t,
   - $v(t) = \frac{dy}{dt}$ be the velocity of the train at time t.
   - $a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2}$ be the acceleration of the train at time t.

   It is given that $y(0) = 0$, $v(0) = 55.9$, and that for $t \geq 0$ the acceleration $a(t) = -1.2$ is constant. Then

$$v(t) = \int a(t)dt = \int -1.2\,dt = -1.2t + v(0) = -1.2t + 55.9.$$

   So the train reaches velocity $v = 0$ at time $t = 55.9/1.2 \approx 46.6$, that is, after 46.6 seconds. Also the position

$$y(t) = \int v(t)dt = \int -1.2t + 55.9\,dt = -0.6t^2 + 55.9t + y(0) = -0.6t^2 + 55.9t.$$

   So the position of y at the moment the velocity v reaches zero, which we just computed occurs at time $t = 55.9/1.2$, is

$$y(55.9/1.2) = -0.6 \cdot (55.9/1.2)^2 + 55.9 \cdot (55.9/1.2) \approx 1302.0,$$

   i.e., 1302 metres further on from the point when the brake was activated.

2. **Types of differential equations** Write down (but don't solve) an example of a differential equation that is:
   (a) First-order, linear but not homogeneous, with constant coefficients.
   (b) First-order, linear, homogeneous but without constant coefficients.
   (c) Second-order, linear, homogeneous, with constant coefficients.
   (d) Second-order, linear, not homogeneous, without constant coefficients.

Solution:

(a) Anything of the form
$$ay'(x) + by(x) = c,$$
where a, b, c are independent of x and y, and a and c are non-zero.

(b) Anything of the form
$$a(x)y'(x) + b(x)y(x) = 0,$$
where at least one of $a(x)$ and $b(x)$ is non-constant, and a is non-zero.

(c) Anything of the form
$$ay''(x) + by'(x) + cy(x) = 0,$$
where a, b, c are independent of x and y and a is non-zero.

(d) Anything of the form
$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = d(x),$$
where at least one of $a(x), b(x), c(x), d(x)$ is non-constant, and $a(x)$ and $d(x)$ are non-zero.

3. **Differential equations** Solve the following, linear, homogeneous, first-order, constant coefficients, differential equations once using separation of variables and once with the *ansatz*.

(a) $\dot{y}(t) - y(t) = 0$ with initial condition $y(0) = 2$: this is the classic: $y = A\exp t$ and the initial condition gives $A = 2$.

(b) $\dot{y}(t) + 3y(t) = 0$ with initial condition $y(3) = 3$: this one isn't much different, $y = A\exp -3t$ and the intial conditionisn't an initial condition, which is sneaky, but $A\exp -9 = 3$ so $A = 3\exp 9$.

(c) $\dot{y}y(t) = 0$ with initial condition $y(5) = 2$: oh dear, this was typo, but lets pretend it wasn't.
$$\dot{y}y = \frac{1}{2}\frac{dy^2}{dt}$$
so integrating give $y^2 = C$ or, with some sloppiness renaming the arbitrary constant, $y = C$ and then $y(5) = 2$ tells us that $C = 2$.

(d) $\dot{y}(t) + 5y(t) = 0$ with initial condition $y(1) = 1$. Not sure what the point here is, it's the same as the others really, $y = A\exp(-5t)$ and the condition tells us that $A = \exp 5$.