EMAT10001 Lecture 13.

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Preface

These are outline notes for lecture 13. As usual there is a bounty of between 20p and £2 for errors, you can tell me at the end of a lecture or email me at `conor.houghton@bristol.ac.uk`.

Introduction

This lecture is about the exponential and about simple differential equations, well mostly one differential equation, the growth equation, which models the dynamics of growth and decay.

The limit of continuous compounding

Imagine you have money in a bank, say 1000 euro at 10% interest per annum; this example is set quite a few years ago. After one year you get your interest and you'll have 1100 euro. However, this isn't quite fair, what would happen if the bank gave you your interest every six months; 5% per semiannum. In this system after six months you would get 50 euro giving you 1050 euro, and after another six months you would get 5% of this, giving you 1102.50 euro, an extra 2.50 euro. How about doing the interest every three months; then you would have 1025 euro after three months, 1050.625 euro after six months, about 1076.90 after nine months and 1103.82 after the full year, an extra 3.82 euro.

Now lets think about the limit, where you continually add the interest. If the interest rate is r, written as a fraction, and P is the amount of money you are adding interest to, then the interest is rP and so after the interest is added you have $P + rP = (1+r)P$; in other words, adding the interest is like multiplying by $(1+r)$. Writing the amount after interest is added as P_1, one since it added only once at the end of the year, we have

$$P_1 = (1+r)P \tag{1}$$

If it is added every six months, the interest is $r/2$ and

$$P_2 = \left(1 + \frac{r}{2}\right)\left(1 + \frac{r}{2}\right)P = \left(1 + \frac{r}{2}\right)^2 P \tag{2}$$

Now, by the same argument adding the interest n times gives

$$P_n = \left(1 + \frac{r}{n}\right)^n P \tag{3}$$

It turns out you can take the limit as n goes to infinity, often called the *limit of continual compounding*. The name given to this limit is the exponential function:

$$\exp r = \lim n \to \infty \left(1 + \frac{r}{n}\right)^n \tag{4}$$

and so after one year you should get

$$P_\infty = \exp r P \tag{5}$$

or, for our case, 1105.17 euro.

Now, what about the properties of this exponential function? First off, lets differenciate it

$$\frac{d}{dx}\exp(x) = \lim_{n\to\infty}\frac{d}{dx}\left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty}\frac{1}{n}n\left(1 + \frac{x}{n}\right)^{n-1} \tag{6}$$

where we have used the chain rule

$$\frac{d}{dx}f(g(x)) = \frac{d}{dx}g(x)\frac{d}{dg}f(g) \tag{7}$$

In this case that means we want df/dx where

$$f(x) = \left(1 + \frac{x}{n}\right)^n \tag{8}$$

or

$$f(x) = g(x)^n \tag{9}$$

where

$$g(x) = \left(1 + \frac{x}{n}\right) \tag{10}$$

so

$$\frac{df}{dg} = ng^{n-1} \tag{11}$$

and

$$\frac{dg}{dx} = \frac{1}{n} \tag{12}$$

Now, going back to where we started,

$$\frac{d}{dx}\exp(x) = \lim_{n\to\infty}\frac{1}{n}n\left(1 + \frac{x}{n}\right)^{n-1} \tag{13}$$

and, since there is no difference between n and $n-1$ when n is going to infinity, we have

$$\frac{d}{dx}\exp(x) = \exp(x) \tag{14}$$

This is the signature property of the exponential, differentiating it gives you back the same function.

We can use this to work out how to calculate the exponential function. Say

$$\exp(x) = \sum_{i=0}^{\infty} a_i x^i \tag{15}$$

for some constants a_i, now

$$\frac{d}{dx}\exp(x) = \frac{d}{dx}\sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} i a_i x^{i-1} \tag{16}$$

so the equation $d\exp(x)/dx = \exp(x)$ tells us that

$$\sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} i a_i x^{i-1} \tag{17}$$

We want to solve this for the a_i, to do this we have to change the index in the second sum, so we write $j = i - 1$ and put it back in

$$\sum_{i=0}^{\infty} a_i x^i = \sum_{j=-1}^{\infty} (j+1)a_{j+1} x^j \tag{18}$$

then, just like it doesn't matter what you call a local variable in a for loop, it doesn't matter what you call an index in a sum, so we can write

$$\sum_{i=0}^{\infty} a_i x^i = \sum_{i=-1}^{\infty} (i+1)a_{i+1} x^i \tag{19}$$

Now, when $i = -1$ we have $i + 1 = 0$ so the $i = -1$ term in the right hand side sum can be ignored, giving

$$\sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} (i+1)a_{i+1} x^i \tag{20}$$

and, finally,

$$\sum_{i=0}^{\infty} [a_i - (i+1)a_{i+1}]x^i = 0 \tag{21}$$

so

$$a_{i+1} = \frac{1}{i+1}a_i \tag{22}$$

Finally, from

$$\exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \tag{23}$$

we can see $\exp(0) = 1$ and hence $a_0 = 1$, then from the formula

$$a_1 = a_0 \tag{24}$$

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and

$$a_2 = \frac{1}{2}a_1 = \frac{1}{2} \tag{25}$$

and

$$a_3 = \frac{1}{3}a_2 = \frac{1}{3!} \tag{26}$$

and so on. Hence

$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \ldots \tag{27}$$

One thing that isn't obvious from all of this, but which can be proven by messing around with the power series above, the exponential has the form of a power

$$\exp(x) = e^x \tag{28}$$

where e is a special number

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.7182818283\ldots \tag{29}$$

Like π this is not a rational number, it isn't of the form n/m for any n and m; and, again like π, it is *transendental*; it isn't the solution of any polynomial equation with rational coefficients.

Now, consider the following common type of situation; you have a quantity of bacteria, say 0.25 kg and each cell has 0.1 chance of dividing into two every hour; the challenge is to work out the quantity of bacteria after a 24 hours. Now, call the quantity of bacteria P and the rate of change of the quantity is

$$\frac{dP}{dt} = rP \tag{30}$$

where $r = 0.1$ and we are measuring t in hours; notice that it is rP, the rate of change is the probability of a given bacteria dividing multiplied by the total quantity of bacteria. To do this we need to solve the equation. There are lots of ways to solve differential equations like this but, perhaps, the best way is to guess based on our knowledge of exponentials; broadly speaking this is called the method of ansatze. Hence, we guess the solution might be

$$P(t) = Ae^{st} \tag{31}$$

where A and s are constants. Now

$$\frac{d}{dt}P(t) = A\frac{d}{dt}e^{st} \tag{32}$$

and using the chain rule again, that give
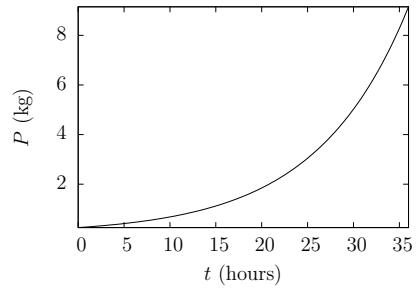
$$\frac{d}{dt}P(t) = Ase^{st} \tag{33}$$

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Figure 1: Plot of $0.25 \exp(0.1t)$.

so substituting into the differential equation we have

$$Ase^{st} = rAe^{st} \tag{34}$$

so that's a solution provided $s = r$. We also have to consider the initial condition $P(0) = 0.25$, substituting $t = 0$ into our solution we have

$$P(0) = A = 0.25 \tag{35}$$

and

$$P(t) = 0.25e^{0.1t} \tag{36}$$

so $P(24) = 0.25 \exp(2.4) \approx 2.756$ and the growth is plotted in Fig. 1.