EMAT10001 Lecture 17.

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Preface

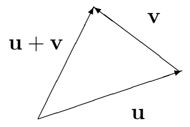
These are outline notes for lecture 16. As usual there is a bounty of between 20p and £2 for errors, you can tell me at the end of a lecture or email me at `conor.houghton@bristol.ac.uk`.

Introduction

This is a lecture about the Fourier series and the Fourier transform.

Vector spaces and inner product spaces

We are familiar with vectors, things like $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are two vectors we can add them $\mathbf{u}+\mathbf{v}$:



We can multiply vectors by scalars, ordinary numbers, so if we have $\mathbf{u}$ and a then $a\mathbf{u}$ is the vector which points in the same direction as $\mathbf{u}$ but is a times longer. We also have the *dot product*,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \tag{1}$$

Just like symmetry is abstracted away to give groups, there is a mathematical structure called a *vector space* for a set of objects that have addition and scalar multiplication and with some property to demand that the addition and scalar multiplication behave nicely, we won't go through these now but they are things like $(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$. Some vector spaces are inner product spaces if there is an *inner product*, that is a map that sends pairs of vectors to a number, just like the *dot product* does. Again, there are niceness properties, like

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w} \tag{2}$$

and

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} \tag{3}$$

Anyway, the key point is that the dot product is an example of such a thing. Furthermore, if you have a inner product, you have a *norm*, a way of measuring length

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} \tag{4}$$

The reason we mention this is that being an inner product space allows you to convert between two ways of thinking about vectors, the original way, as something with a length and a direction, and the component form, things like $\mathbf{u} = (u_1, u_2, u_3)$. The story goes like this, it is possible to find independent unit length orthogonal vectors, in three dimensions these are often taken to be the three vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ where

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1 \tag{5}$$

and

$$\mathbf{i}\cdot\mathbf{j} = 0 \tag{6}$$

and so on. Now, if we are in three dimensions and we have three of these vectors, we are able to write any other vector in terms of them

$$\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k} \tag{7}$$

Now try dot producting both sides with $\mathbf{i}$

$$\mathbf{u}\cdot\mathbf{i} = u_1\mathbf{i}\cdot\mathbf{i}+u_2\mathbf{j}\cdot\mathbf{i}+u_3\mathbf{k}\cdot\mathbf{i} = u_1 \tag{8}$$

In other words

$$u_1 = \mathbf{u}\cdot\mathbf{i} \tag{9}$$

Similarly $u_2 = \mathbf{u}\cdot\mathbf{j}$ and $u_3 = \mathbf{u}\cdot\mathbf{k}$. Hence, if we are in an inner product space we can work out components (u_1, u_2, u_3) where u_1 is the projection in the $\mathbf{i}$-direction, and so on.

Fourier series

The amazing thing is that the space of functions is also an inner product space. Lets first of all think about periodic function and for simplicity we are going to imagine the period is 2π. Hence, we are thinking about functions $f(t)$ such that

$$f(t+2\pi) = f(t) \tag{10}$$

Now this is a vector space, if $f(t)$ and $g(t)$ are both functions with period 2π so is $h(t) = f(t)+g(t)$:

$$h(t+2\pi) = f(t+2\pi)+g(t+2\pi) = f(t)+g(t) = h(t) \tag{11}$$

and, if $af(t)$ is a periodic function with period 2π if $f(t)$ is.

It is also an inner product space, though this might be less obvious, but given two periodic functions $f(t)$ and $g(t)$ an inner product is given by

$$(f,g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt \tag{12}$$

Don't worry about the $1/\pi$, it is just there to make some formulas later look neater, the basic point is that there is a way of going from two of these functions to a scalar and, though we haven't listed them, it does turn out that this satisfies all the required niceness properties.

So, the space of periodic functions is an inner product space. This is nice to know since it means we can now start seeing if stuff that works with vectors works with periodic functions. The obvious thing to try is the splitting up into components that worked so well with vectors. Thus, the question is, is there a set of basic functions like the basis vectors **i**, **j** and **k**? The answer is yes, they are sine and cosines. In fact

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nt \sin mt dt &= \delta_{nm} \\ \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nt \cos mt dt &= 0 \\ \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt \cos mt dt &= \delta_{nm} \end{aligned} \tag{13}$$

where δ_{nm} is the Kronecker δ-function which is one if n and m are equal, and zero if they aren't

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

So, it looks a little confusing since we have two different types of basis functions, but sine and cosines have the same sort of orthogonality relationship that the **i**, **j** and **k** had. In fact, to make it even more confusing for there to be enough basis vectors we have to include the possibility that $n = 0$ for the cosines, giving just a constant, and the normalization is different for the constant. However, the net effect is we can decompose periodic functions just like we can ordinary vectors.

So, in short, we can write

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \tag{15}$$

We haven't, in a sense, proven that we have enough basis vectors for this to work, and that's actually very tricky to prove, but assuming there are we can work out the a_n and b_n the same way we did for u_1 and so on. For example, for a_n with $n \neq 0$ multiply both side by $\cos mt$ and integrate

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt f(t) dt &= \frac{1}{2}a_0 \frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt dt + \sum_{n=1}^{\infty} a_n \frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt \cos nt dt \\ &+ \sum_{n=1}^{\infty} b_n \frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt \sin nt dt \end{aligned} \tag{16}$$

then, since almost everything is zero, we end up with

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt f(t) dt \tag{17}$$

so this is like the projection of $f(t)$ onto $\cos mt$. Doing the same thing with sine and with just straight integration of both sides we get

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) dt \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt dt \end{aligned} \tag{18}$$

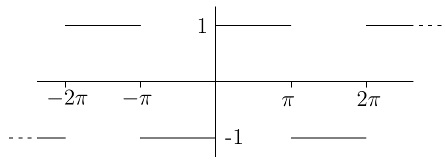
This is the Fourier series.

Example

Consider the block wave with period 2π

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases} \tag{19}$$

with $f(t + 2\pi) = f(t)$.



So

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt dt \tag{20}$$

Since $f(t)$ is *odd*, that is $f(t) = -f(-t)$, this can be shown to be zero by a standard trick. Let $t' = -t$ and do a change of variable in the integral: $dt' = -dt$ but when $t = \pi$, $t' = -\pi$ and visa versa; flipping the limits around gives an extra minus, so

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(-t') \cos(-nt') dt' = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t') \cos(nt') dt' = -a_n \tag{21}$$

and, if $a_n = -a_n$ then it must be zero. A similar trick works for b_n is the function is even, that is, if $f(t) = f(-t)$. In this way a_n is thought of as dealing with the even part of the function and b_n with the odd part.

As for the b_n, you need to do this by integration

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt f(t)\sin nt = \frac{2}{\pi}\int_0^{\pi} dt \sin nt = -\left.\frac{2\cos nt}{n\pi}\right|_0^{\pi} = \frac{2}{\pi n}[1-(-1)^n] \tag{22}$$

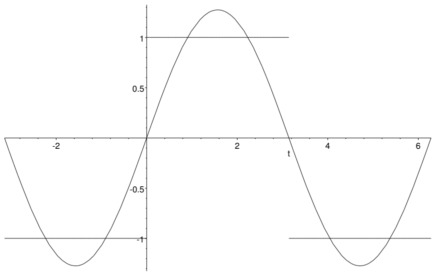
where we have used $\cos n\pi = (-1)^n$. Hence

$$f(t) = \frac{4}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\sin nt \tag{23}$$

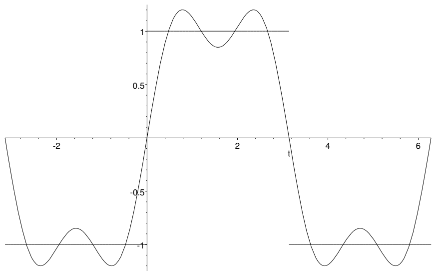
If we write

$$f_N(t) = \frac{4}{\pi}\sum_{n\text{ odd, }n\le N}\frac{1}{n}\sin nt \tag{24}$$

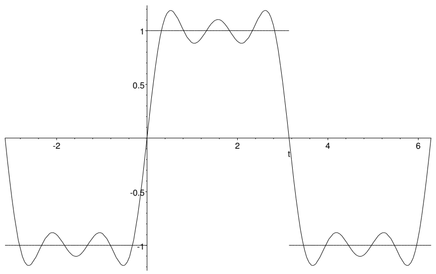
then



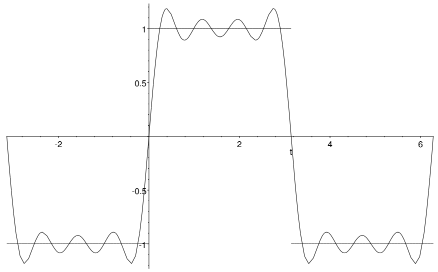
$f_1(t)$



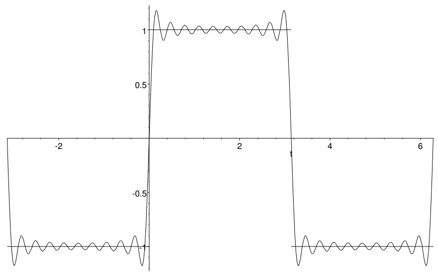
$f_3(t)$



$f_5(t)$



$f_7(t)$



$f_{19}(t)$

In fact, this is a very artificial example, not at all like the examples that arise in image processing. It is commonly looked at only because the integration is relatively easy to do. In real examples the integration can't be done explicitely, but there is an extremely powerful numerical technique called the Fast Fourier Transform.

Application

This might all seem a bit odd, but, in fact, the Fourier series has many applications. Basically what is happening in a Fourier series is that we are replacing a picture of the function that looks at the value of the function at different values of t with a picture that looks at how much of the function can be explained by different sines and cosines. Since the sines and cosines get wavier at higher and higher n then the size of a_n and b_n describes how much of the detail of the function is found at a length scale of $2\pi/n$.

One common computer science or electronic engineering application of Fourier series is cleaning up or compressing a signal. For example, for sound, if you convert a sound wave into its Fourier series and then truncate it so that you leave out values of n that correspond to frequencies humans can't hear, the sound will be described in a more compressed way, but without any loss in how the sound is heard. Common video compression methods, like mpeg, work like this; they are cleverer in that they not only drop the high frequencies that we can't hear, the also take into account masking, an auditory effect were if some a_n and b_ns are large you can set the smaller ones to zero without affecting sound quality. Other compression methods use a similar approach, but replace the sines and cosines with other basis sets.

Another application relates to sampling. Typically a signal is known only from a discrete sampling, a sound wave may be measured at discrete points in time, for example. It is useful to reconstruct the signal, but this can't be done at a level of detail that excedes the sampling rate, basically the reconstructed signal is like the truncated Fourier series, leaving out a_n and b_ns for the ns where the detail probed by the corresponding $\sin nt$ and $\cos nt$ is smaller than the gap between the sample points. This is dealt with in a theorem called the Shannon-Nyquist sampling theorem.

In Fourier series also has many applications in mathematical physics, there are equations, like the heat equation, that can only be solved using Fourier methods and quantum mechanics is usually formulated in a way that hops between the original $f(t)$-style perspective of a function and a perspective based on the Fourier approach.

We have been looking specifically at functions with period 2π; this can easily be generalized to other periods. Another change is to replace the sines and cosines with $\exp(int) = \cos nt + i \sin nt$, this looks more annoying at first since it includes complex numbers, but it turns out to be more elegant. A bigger change is to consider non-periodic functions, a similar approach works provided they decay rapidly to zero for large $|t|$, however in this case just looking at $\cos nt$ and $\sin nt$ for integer n does not give enough basis functions, you need to use a continuous set of basis functions, this leads to the Fourier transform

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikt} \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{-ikt} \end{aligned} \tag{25}$$

where a real number k have replaced the n, an integral has replaced the sum and $\widetilde{f(k)}$ has replaced the a_n and b_n. In fact, in practice, we use a mixture of Fourier series, Fourier transform, and windowed Fourier transforms, where you look at the little bit of the original signal at a time and do a Fourier transform on that.