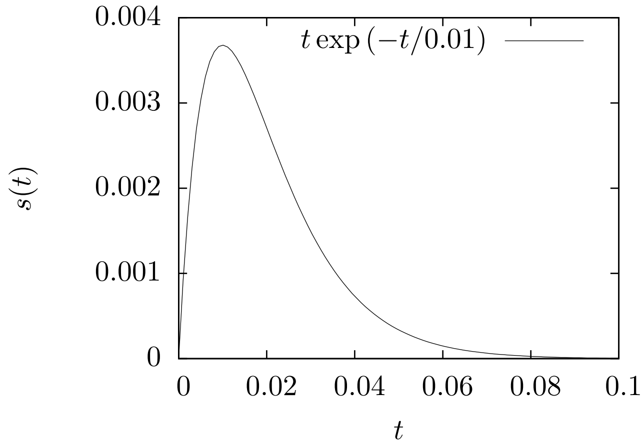

Figure 1: The α-function profile often used to model synaptic conductances, shown here with $\tau_m = 10$ ms.

Modelling the synaptic conductance

This is a quick note on modelling the synaptic conductance. It has been claimed in the leture notes that the synaptic conductance is well described by an α function so after a spike arrives $g_s \propto s(t)$ where

$$s(t) = te^{-t/\tau_s} \tag{1}$$

where τ_s is a time scale, see Fig. 1. Mostly this is used because it has roughly the right shape. Here we will outline how synaptic conductances might be modelled.

The basic idea is that a closed ligand gated channel has a probability of opening which depends on the concentration of the neurotransmitter. Lets call this opening rate o; since the level of neurotransmitter varies, $o(t)$. Conversely when a channel is open there is a chance the neurotransmitter will unbind because of thermal fluctuations, lets call this closing rate c, c is a constant. Now if x is the fraction of gates that are open

$$\frac{dx}{dt} = (1-x)o - xc \tag{2}$$

since $(1-x)$ is the number of open gates and o is the opening rate.

Now we need to have some description of how o varies with time. One idea is that o might be an exponential: perhaps the more neurotransmitter there is the faster it gets transported away by the neurotransmitter reuptake pumps. Let

$$o(t) = \bar{o}e^{-t/\tau_o} \tag{3}$$

in response to a spike at time $t = 0$. Hence

$$\frac{dx}{dt} + (\bar{o}e^{-t/\tau_o} + c)x = \bar{o}e^{-t/\tau_o} \tag{4}$$

Clearly this is going to be a complete pain to solve, we can make it easier with a few approximations, say $c >> \bar{o}$, that is, because of the decay in the exponential, the coefficient of x is quickly dominated by c

$$\frac{dx}{dt} + cx = \bar{o}e^{-t/\tau_o} \tag{5}$$

This can be solved, it is an inhomogenous first order differential equation and using an integrating factor the solution with $x(0) = 0$ is

$$x(t) = \tau_o \bar{o} \left(e^{-ct} - e^{-t/tau_o} \right) \tag{6}$$

This is quite complicated, it has three timescales in it, τ_o, the decay time for the neurotransmitter, the maximum opening rate $1/\bar{o}$ and the closing rate $1/c$. Often this is simplified by assuming $1/c = \tau_o$, there is no particular reason for doing this except that, maybe, the precise shape, as opposed to amplitude or overall duration, of the synapse conductance does not have much consequence and so it is reasonable to simplify. If $1/c = \tau_o$ the equation becomes

$$\frac{dx}{dt} + cx = \bar{o}e^{-ct} \tag{7}$$

and the previous solution is uniformly zero. In the language of ordinary differential equations this is called the critically damped case. It can also be solved by integating factor, multiplying both sides by $\exp(ct)$ gives

$$e^{ct}\frac{dx}{dt} + ce^{ct}x = \bar{o} \tag{8}$$

or

$$\frac{d}{dt}\left[e^{ct}x\right] = \bar{o} \tag{9}$$

Integrating gives

$$x(t) = \bar{o}te^{-ct} + Ae^{-ct} \tag{10}$$

where A is an integration constant that gets set to zero if $x(0) = 0$. Thus, there is a derivation for the α function, just not a paritcularly good one!

Another model describes the period of time that the neurotransmitter is in the cleft as a square pulse, this might not be all that realistic either, but it is a simplification that makes it easier to deal with how the conductance changes in response to more than one spike. Say there is a spike at $t = 0$:

$$o(t) = \begin{cases} \bar{o} & 0 < t < \tau_o \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Imagine $\bar{o}$ is much bigger than c so that in $0 < t < \tau_o$

$$\frac{dx}{dt} = (1 - x)\bar{o} = -\bar{o}x - \bar{o} \tag{12}$$

We can solve this:

$$x(t) = \bar{o} + [x(0) - \bar{o}]e^{-\bar{o}t} \tag{13}$$

For $t > \tau_o$

$$\frac{dx}{dt} = -cx \tag{14}$$

so

$$x(t) = x(\tau_o)e^{-ct} \tag{15}$$

Now, a useful approximation here is to ignore the short time o is non-zero, in other words, to assume that the dynamics during the period the cleft is full of neurotransmitter is uninteresting, we have in any case approximated it rather crudely. Now between the spike arriving and the end of this short period $x(t)$ has gone from $x(0)$ to

$$x(\tau_o) = \bar{o} + [x(0) - \bar{o}]e^{-\bar{o}t} \tag{16}$$

which can be rearranged to give

$$x(\tau_o) = x(0) + [\bar{o} - x(0)]\left(1 - e^{-\bar{o}t}\right) \tag{17}$$

If there hasn't be a spike in a long time $x(0) = 0$ and in that case

$$x(\tau_o) = \bar{o}\left(1 - e^{-\bar{o}t}\right) \tag{18}$$

This is actually the largest increase in the conductance that is possible, call this value x_m and substitute back into the general equation for $x(\tau_o)$:

$$x(\tau_o) = x(0) + \frac{[\bar{o} - x(0)]x_m}{\bar{o}} \tag{19}$$

This gives another popular model for synapses where

$$\frac{dx}{dt} = -cx \tag{20}$$

but every time there is a spike

$$x(t) \to x(t) + \frac{[\bar{o} - x(t)]x_m}{\bar{o}} \tag{21}$$

This becomes a bit easier to interpret if we write $x(t) = \bar{o}p(t)$ and $x_m = \bar{o}p(t)$, then

$$\frac{dp}{dt} = -cp \tag{22}$$

but every time there is a spike

$$p(t) \to p(t) + [1 - p(t)]p_m \tag{23}$$

and we can see that p_m is the fraction of available gates that open with each spike.