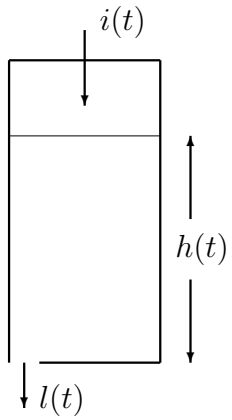
Equation for leaky buckets

These notes are about a simple but very useful model of spiking neurons: the integrate and fire model. First, though, we consider the dynamics of a leaky bucket of water, in turns out these dynamics are very similar to those seen in the integrate and fire model.

Buckets of water

In the simplest model of neurons their voltage dynamics is similar to the dynamics of a bucket with a leak and the class of equations that apply in this case will also be applied to synapses, for example.



Consider a bucket with straight sides which is filled to a height h with water. Imagine the water leaks out of a hole in the bottom. The rate the water leaks out depends on h; the larger h is the larger the pressure at the bottom is and hence the faster the water pours out. In other words

$$l(t) \propto h(t) \tag{1}$$

or

$$l(t) = Gh(t) \tag{2}$$

where G is a constant which will depend on the size of the hole and complicated things like the viscosity of water. Of course, we are also ignoring lots of complicated stuff, like turbulence and so forth, but since we are interested in the equation rather than the amount of water in a bucket, this is fine. Imagine water also pours in the top at a rate $i(t)$. This means the total rate of change of the amount of water is $i(t) - Gh(t)$.

Now, $h(t)$ is the height of the water not the volume: the volume is $Ch(t)$ where C is the cross-sectional area of the bucket. The rate of change of the volume is therefore

$$\frac{dCh(t)}{dt} = i(t) - Gh(t) \tag{3}$$

or

$$\frac{dh}{dt} = \frac{1}{C}(i - Gh) \tag{4}$$

This is an equation we know how to solve; we saw it at the start of the course when learning about differential equations. This same equation will appear when we look at the dynamics of neuron and for similar reasons, something, water inthe case of the bucket, charge in the case of a neuron, is being added to a leaky container.

Constant input

This equation can be solved analytically if the current flowing in is constant, but we won't try that calculation here since it only works for this specific special case, normally we have to solve numerically, using a computer. The solution is

$$h(t) = [h(0) - i/G]e^{-t/\tau} + i/G \tag{5}$$

where $\tau = C/G$. This makes sense, when $h = i/G$ then the equation says $dh/dt = 0$ so this is an equilibrium point, a value where everything stops changing. The dynamics describe the systems as decaying exponentially to the equilibrium.

These dynamics make good intuitive sense; the more water there is in the bucket, the higher the pressure will be at the leak and the quicker the water will pour out. If there is just the right amount of water the rate the water pours out the leak will precisely match the rate it pours in, this is the equilibrium. If there is more water than required for equilibrium it will pour out faster than the flow coming in, if there is less, it will pour out slower.

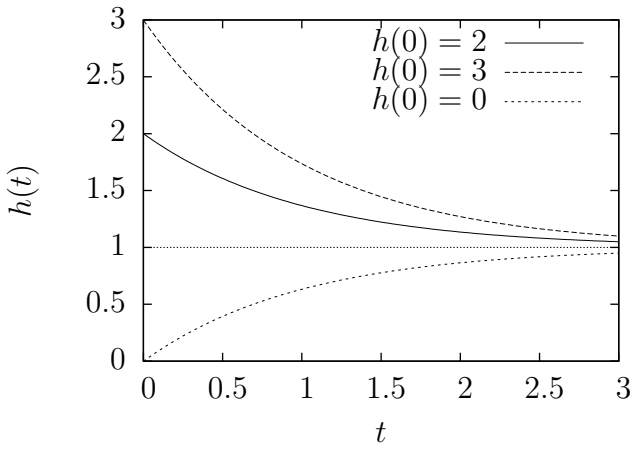


Figure 1: Exponential relaxation. The dynamics described by the 'bucket equation' is very common. Here $h(t)$ is plotted with $i = G$, $\tau = 1$ and three different values of $h(0)$. $h(t)$ relaxes towards the equilibrium value, the closer it gets, the slower it approaches.

Either way, as time passes the height of the water will reach the equilibrium. The plot in Fig. 1 illustrates this.

Variable input

We have only discussed constant inputs; the variable input case where i depends on time is harder and although it can sometimes be solved it is often easier just to compute it numerically, we will look briefly at how to do this, but first note that the effect of variable inputis that the solution kind of chases the input with a timescale set by τ, that is for very small τ is chases it quickly, so it is close to the input, but for large τ it lags behind it and smooths it out. This is sometimes described by saying that it *filters* the inout. There is an illustration in Fig. 2.

Figure 2: Variable input. Here the input is a sine wave $i(t) = \sin t$ and the equation is evolved with $h(0) = 0$ and three different τ values. For $\tau = 0.25$ we see $h(t)$ closely matches the input whereas for larger τ it is smoother and lags behind.

Numerical solutions

Consider the problem of solving a general first order differential equation

$$\frac{df(t)}{dt} = F(f, t) \tag{6}$$

This includes our bucket case if we replace f with h and $F(f, t)$ with $(i - Gh)/C$. Now consider the situation where we want to solve this but can't do it analytically, as will be the case for many values of the time dependent input $i(t)$ or, indeed, if $i(t)$ itself is only known numerically.

The simplest numerical approach is Euler's method. Say you know the value of $f(t)$ and want to approximate $f(t + \delta t)$ where δt is some small increment in time. Now, it is known that

$$f(t + \delta t) = f(t) + \delta t \frac{df(t)}{dt} + O(\delta t^2) \tag{7}$$

That is, it changes by its rate of change multiplied by how long it is changing for. The $O(\delta t^2)$ says there are δt^2 sized errors in this approximation, this comes about because dh/dt is itself changing but our approximation assumes it is fixed still. Now we know what $df(t)/dt$ is, it is $F(f, t)$ from the equation, so

$$f(t + \delta t) = f(t) + \delta t F(f, t) + O(\delta t^2) \tag{8}$$

so approximating $f(t + \delta t)$ with $f(t) + \delta t F(f, t)$ is actual to first order in the time step δt. This is the Euler approximation. More formally if we have an initial condition $f(t_0)$ and write

$$f_n = f(t_0 + n\delta t) \tag{9}$$

$$t_n = t_0 + n\delta t \tag{10}$$

then the Euler approximation is

$$f_{n+1} = f_n + \delta t F(f_n, t_n) \tag{11}$$

Thus, in the case we are interested in here

$$h_{n+1} = \frac{[i(t_n) - Gh_n]\delta t}{C} \tag{12}$$

Of course, this is just the easiest way to do the integration, there are more sophisticated approaches like Runge-Kutta or backwards Euler which are more complex and computational costly for each step, but which are also much more accurate.