A note on the derivation of the Runge Kutta formula[1]

The Runge-Kutta method uses the Taylor expansion in a clever way to find a better approximation than the Euler method. It is a bit convoluted, so there is a lot of notation, but it does give a very useful numerical algorithm.

As before, we want to solve

$$\frac{df}{dt} = G(f) \tag{1}$$

with a time discretization of δt, f_n is the approximate value the algorithm calculates for $f(n\delta t)$ and $f_0 = f(0)$, the initial condition. Now say we are at f_n and let

$$k_1 = G(f_n) \tag{2}$$

so the Euler approximation would be $f_{n+1} = f_n + k_1\delta t$. Next, let

$$k_2 = G\left(f_n + \frac{\delta t k_1}{2}\right) \tag{3}$$

Now, using the Taylor expansion

$$k_2 = G(f_n + \delta t k_1/2) = G(f_n) + \left.\frac{dG}{df}\right|_{f=f_n} \delta t \frac{k_1}{2} + \ldots \tag{4}$$

Substituting back for k_1 this gives

$$k_2 = G(f_n) + \frac{1}{2}\left.\frac{dG}{df}\right|_{f=f_n} \left.\frac{df}{dt}\right|_{t=t_n} \delta t + \ldots \tag{5}$$

Using the chain rule

$$\frac{d^2 f}{dt^2} = \frac{dG}{dt} = \frac{dG}{df}\frac{df}{dt} \tag{6}$$

so

$$k_2 = G(f_n) + \frac{1}{2}\frac{d^2 f}{dt^2}\delta t + \ldots \tag{7}$$

Now, recall

$$f(n\delta t + \delta t) = f_n + \left.\frac{df}{dt}\right|_{t=t_n} \delta t + \frac{1}{2}\left.\frac{d^2 f}{dt^2}\right|_{t=t_n} \delta t^2 + \ldots \tag{8}$$

[1] this is a supplementary note, it isn't examinable

and from the formula for k_1 and k_2 we see that this can be written as

$$f_{n+1} = f_n + k_2\delta t \tag{9}$$

This means that the calculation of f_{n+1} takes into account more of the Taylor expansion than the Euler method, it includes the δt^2 part and so the errors will come in at δt^3. This is the *second order Runge Kutta method*, it is called second order because it includes the first and second order terms in the Taylor expansion, the Euler method is like a first order Runge Kutta method.

The second order Runge Kutta method isn't usually used; it is the fourth order Runge Kutta that is considered the standard way of doing numerical integration. The idea is just the same as the one we saw above, by combining different terms more of the Taylor expansion is accounted for, in fact, as the name suggests, the fourth order Runge Kutta gets everything up to the fourth order, the errors are like δt^5.

Here I will give the fourth order Runge Kutta and will include the possibility that the right hand side of the differential equation also includes a dependence on t so, writing $t_n = n\delta t$

$$\frac{df}{dt} = G(t, f) \tag{10}$$

Now

$$\begin{aligned}
k_1 &= G(t_n, f_n) \\
k_2 &= G\left(t_n + \frac{1}{2}\delta t, f_n + \frac{1}{2}\delta t k_1\right) \\
k_3 &= G\left(t_n + \frac{1}{2}\delta t, f_n + \frac{1}{2}\delta t k_2\right) \\
k_4 &= G\left(t_n + \delta t, f_n + \delta t k_3\right)
\end{aligned} \tag{11}$$

and

$$f_{n+1} = f_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\delta t \tag{12}$$