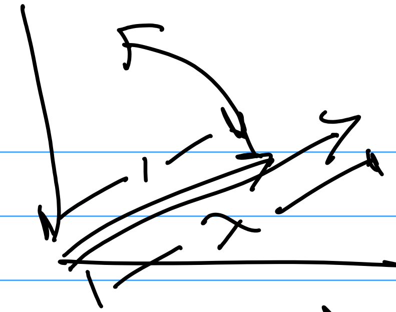
③ $\Sigma v = \lambda v$



a) eigenvalues of $\Sigma + \sigma^2 I$

$$\Sigma = \begin{bmatrix} \sigma_1^2, & \sigma_{12} \\ \sigma_{21}, & \sigma_2^2 \end{bmatrix}$$

$E(XX^T)$

$$\sigma^2 I + \Sigma = \begin{bmatrix} \sigma_1^2, & \sigma_{12} \\ \sigma_{21}, & \sigma_2^2 \end{bmatrix} + \begin{bmatrix} \sigma^2, & \sigma \\ \sigma, & \sigma^2 \end{bmatrix}$$

original covariance

$$\sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

$$\Sigma + \sigma^2 I = \begin{bmatrix} \sigma_1^2 + \sigma^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 + \sigma_2 \end{bmatrix}$$

$$\boxed{(\Sigma + \sigma^2 I) v' = \lambda' v'}$$

$Iv = v$

$$\Sigma v' + \sigma^2 I v' = \lambda' v'$$

$$\Sigma v' = \lambda' v' - \sigma^2 I v'$$

$\boxed{\Sigma v = \lambda v}$

$$\Sigma v' = \lambda' v' - \sigma^2 v'$$

$$\boxed{\Sigma v' = (\lambda' - \sigma^2) v'}$$

v' eigenvector of Σ

$(\lambda' - \sigma^2)$ eigenvalue of Σ

$\lambda' - \sigma^2 = \lambda$

$$\boxed{\lambda' = \lambda + \sigma^2}$$

(old matrix) v = (old eigenval) v

(new matrix) v = (new eigenval) v

rearrange

(old matrix) v = [old eigen in terms of new eigen] v

equival

new eigval

$(\Sigma + \blacksquare) v = \square\, v$ $\qquad \Sigma v = \lambda v$

$\Sigma v = (\underbrace{\square \cancel{+} \blacksquare}_{\lambda}) v$

new evec

v_2

$(\Sigma + \sigma^2 I)^{-1} v' = \lambda' v' \qquad \times(\Sigma + \sigma^2 I)$

$v' = \lambda' (\Sigma + \sigma^2 I) v'$

$$\frac{1}{\lambda'} v' = (\Sigma + \sigma^2 I) v'$$

$$= \Sigma v' + \sigma^2 v'$$

$$\Sigma v' = \frac{1}{\lambda'} v' - \sigma^2 v'$$

$$\Sigma v' = \underbrace{\left(\frac{1}{\lambda'} - \sigma^2\right)}_{\lambda} v'$$

$$\lambda = \frac{1}{\lambda'} - \sigma^2$$

$$\lambda + \sigma^2 = \frac{1}{\lambda'}$$

$$\boxed{\lambda' = \frac{1}{\lambda + \sigma^2}}$$

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad R_{12} = R_{21} \qquad R = R^T$$

$$R^{-1} \qquad \frac{1}{5} 5 = 1$$

$$RR = \Sigma + \sigma^2 I \qquad R^{-1} R = I$$

$$RR^{-1} = I$$

$$\Sigma v_1 = \lambda_1 v_1$$

$$Y = \underbrace{\underset{1 \times D}{v_1^T} \underset{(D \times D)}{R^{-1}} \underset{D \times 1}{X}}_{1 \times 1}$$

$$E(X) = 0$$

$$\mu_Y = E(Y) = E(\underbrace{v_1^T R^{-1}}_{\text{const.}} X)$$

$$= v_1^T R^{-1} \underbrace{E(X)}_{0}$$

$$= 0$$

$$\mathrm{var}(Y) = E\left[(Y - \cancel{\mu})^2\right] = E\left[Y^2\right]$$

$$= \cancel{E[X]^2}$$

$(2^{-1})^2$

$$= E\left[(v_1^T R^{-1} X)(v_1^T R^{-1} X)^T\right]$$

2^{-2}

$$= E\left[v_1^T R^{-1} X X^T R^{-T} v_1\right]$$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to [v_1\ v_2]$

$$= v_1^T R^{-1} E(XX^T) R^{-T} v_1$$

$\Sigma = E(XX^T)$

$$= v_1^T \underbrace{R^{-1} \Sigma R^{-T}}_{} v_1 \leftarrow$$

$$RR = \Sigma + \sigma^2 I \qquad R = R^T \longrightarrow R^{-1} = (R^{-1})^T$$

$$\underbrace{R^{-1}R}_{I}R = R^{-1}\Sigma + \sigma^2 R^{-1} I$$

$$R = R^{-1}\Sigma + \sigma^2 R^{-1} \qquad \times (R^{-1})^T = R^{-1}$$

$$\underbrace{RR^{-1}}_{} = R^{-1}\Sigma R^{-T} + \sigma^2 R^{-1} R^{-1}$$

$$I = R^{-1}\Sigma R^{-T} + \sigma^2 R^{-1} R^{-1}$$

$$R^{-1}\Sigma R^{-T} = I - \sigma^2 R^{-1} R^{-1}$$

$$\mathrm{var}(Y) = v_1^T (I - \sigma^2 R^{-1} R^{-1}) v_1$$

$$= \underbrace{v_1^T v_1}_{} - \sigma^2 v_1^T R^{-1} R^{-1} v_1$$

$$= \underbrace{1 - \sigma^2 v_1^T R^{-1} R^{-1} v_1}_{}$$

$$\Sigma v_1 = \lambda_1 v_1 \qquad RR = \Sigma + \sigma^2 I$$

$$\cancel{RRv_1 = RR\lambda_1 v_1} \qquad \underbrace{\Sigma = RR - \sigma^2 I}$$

$$(RR - \sigma^2 I)\, v_1 = \lambda_1 v_1$$

$$RRv_1 - \sigma^2 v_1 = \lambda_1 v_1$$

$$\boxed{RRv_1 = (\lambda_1 + \sigma^2)\, v_1}$$

multipy both sides by v_1^T

$$v_1^T RRv_1 = (\lambda_1 + \sigma^2)\, \underbrace{v_1^T v_1}_{1}$$

$$v_1^T RRv_1 = (\lambda_1 + \sigma^2)$$

multiply both sides v

multiply both sides by R^{-1}

$$\underbrace{R^{-1}R}_{I}Rv_1 = R^{-1}(\lambda_1 + \sigma^2)\, v_1$$

$$Rv_1 = (\lambda_1 + \sigma^2)\, R^{-1}v_1$$

$$\boxed{R^{-1}v_1 = \frac{Rv_1}{\lambda_1 + \sigma^2}}$$

transpose

$$(R^{-1}v_1)^T = v_1^T R^{-T} = v_1^T R^{-1} = \frac{v_1^T R^T}{\lambda_1 + \sigma^2}$$

$$\boxed{v_1^T R^{-1} = \frac{v_1^T R}{\lambda_1 + \sigma^2}}$$

$$\mathrm{var}(Y) = 1 - \sigma^2 \underbrace{v_1^T R^{-1}} \underbrace{R^{-1} v_1}$$

$$= 1 - \sigma^2 \left(\frac{v_1^T R}{\lambda_1 + \sigma^2}\right)\left(\frac{R v_1}{\lambda_1 + \sigma^2}\right)$$

$$= 1 - \frac{\sigma^2}{(\lambda_1 + \sigma^2)^2} v_1^T (RR) v_1$$

$RR = \Sigma + \sigma^2 I$

$$= 1 - \frac{\sigma^2}{(\lambda_1 + \sigma^2)^2} v_1^T (\Sigma + \sigma^2 I) v_1$$

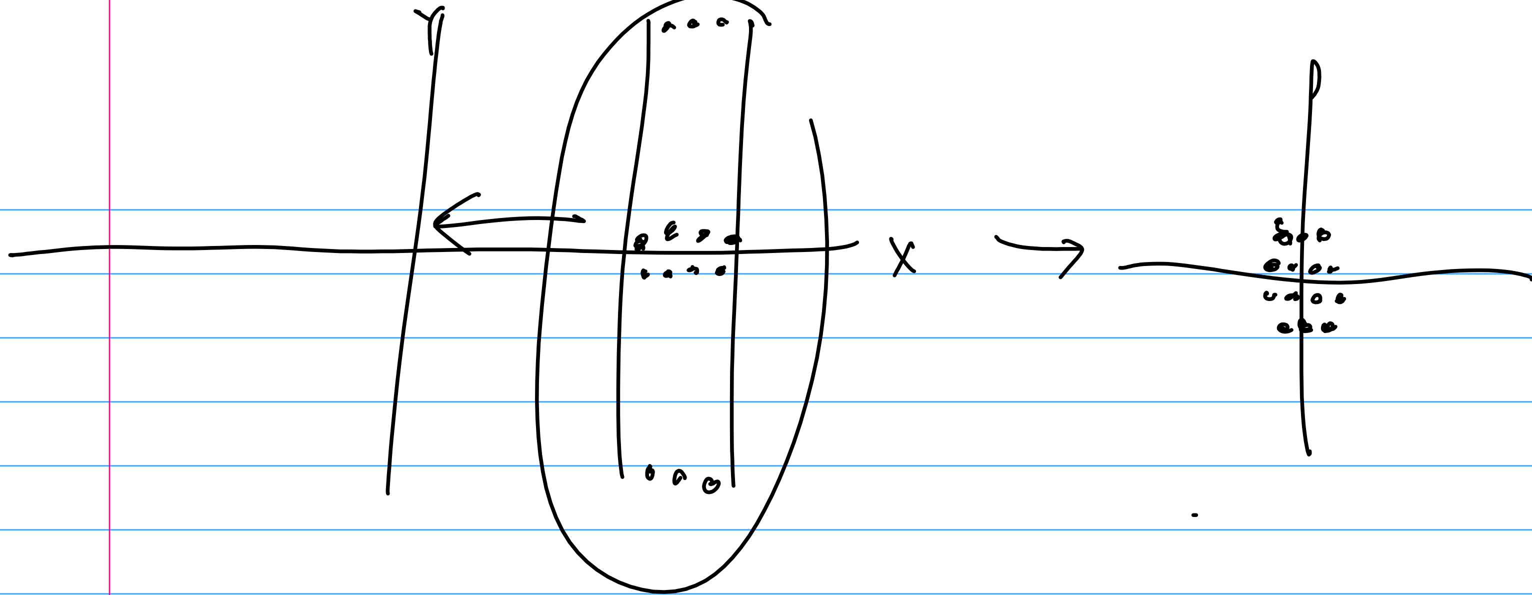
$$= 1 - \frac{\sigma^2}{(\lambda_1 + \sigma^2)^2} \left[\underbrace{v_1^T \Sigma v_1}_{\lambda} + \sigma^2 \underbrace{v_1^T v_1}_{1}\right]$$

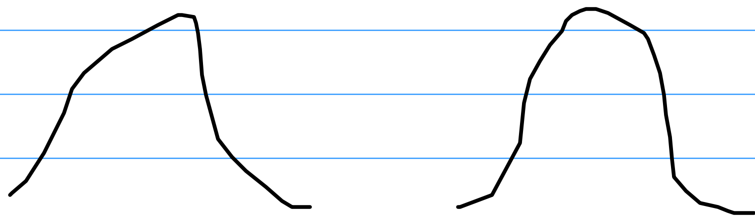
$\Sigma v_1 = \lambda_1 v_1$

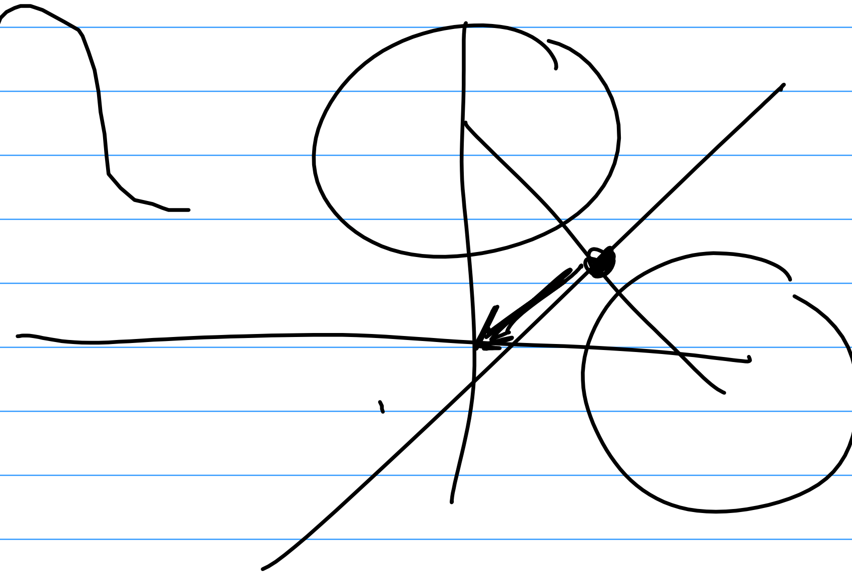
$v_1^T \Sigma v_1 = \lambda_1 v_1^T v_1 = \lambda$

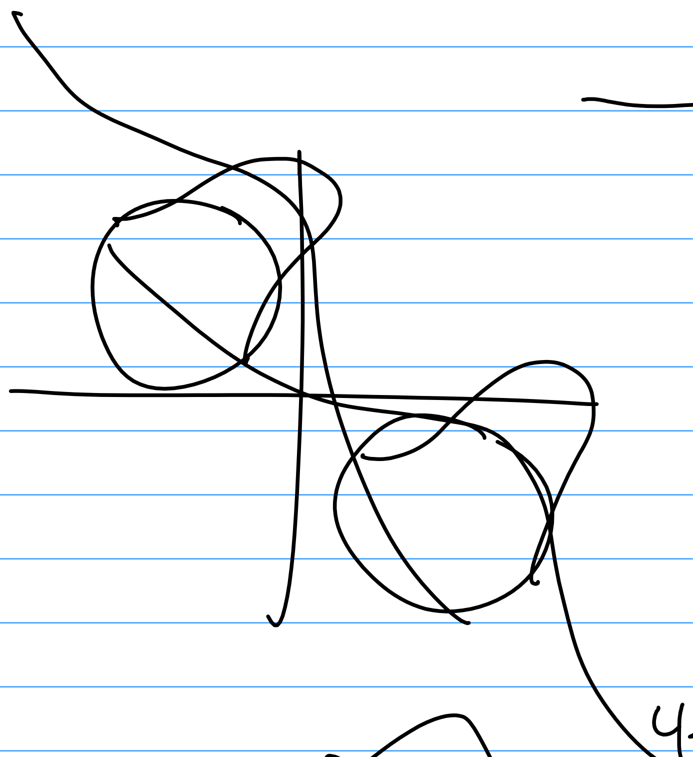
$$= 1 - \frac{\sigma^2}{(\lambda_1 + \sigma^2)^2} (\lambda_1 + \sigma^2)$$

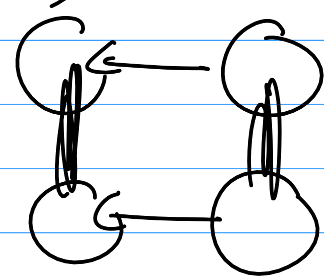
$$= 1 - \frac{\sigma^2}{\lambda_1 + \sigma^2}$$





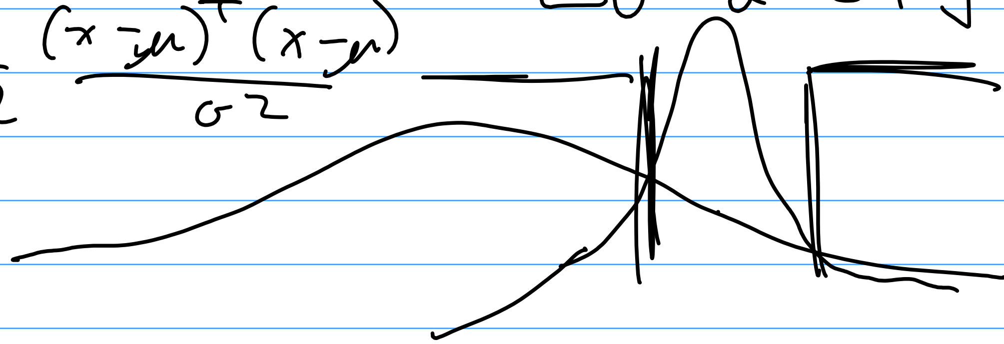


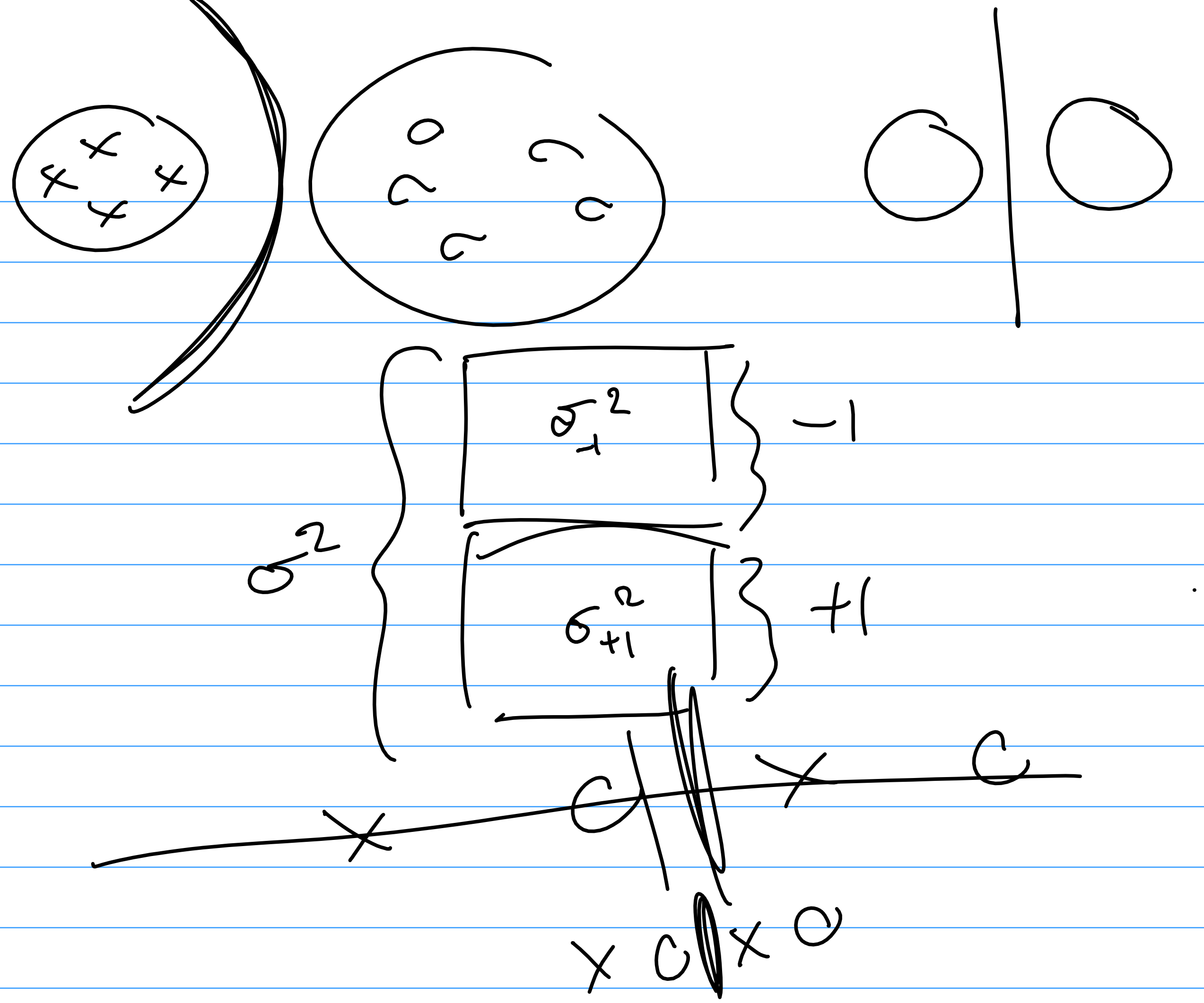




$$4 - \begin{bmatrix} 16 & -2 & -8 \\ 6 & -2 & -4 \end{bmatrix} - 4$$

$$e^{-\frac{1}{2}\frac{(x-\mu)^T(x-\mu)}{\sigma^2}}$$





{ label : { 'mu' : ___ , 'cov' : ___ }

$$P(x \mid e_1)\,\pi_1 > P(x \mid e_1)\,\pi_0$$

$\Sigma = E(xx^T)$

$$\cancel{\frac{1}{\sqrt{(2\pi)^D|\Sigma|}}} \exp\left[-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right]\pi_1$$

$$> \cancel{\frac{1}{\sqrt{(2\pi)^D|\Sigma|}}} \exp\left[-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\right]\pi_0$$

$$-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1) + \log\pi_1 > -\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0) + \log\pi_0$$

$\times 2$

$$(x-\mu_0)^T\Sigma^{-1}(x-\mu_0) - (x-\mu_1)^T\Sigma^{-1}(x-\mu_1) > 2(\log\pi_0 - \log\pi_1)$$

$$x^T\Sigma^{-1}x - x^T\Sigma^{-1}\mu_0 - \mu_0^T\Sigma^{-1}x + \mu_0^T\Sigma^{-1}\mu_0$$

$$- \left[x^T\Sigma^{-1}x - \underbrace{x^T\Sigma^{-1}\mu_1 - \mu_1^T\Sigma^{-1}x}_{\text{equal b/c } \Sigma \text{ is symmetric}} + \mu_1^T\Sigma^{-1}\mu_1\right] > 2\log\frac{\pi_0}{\pi_1}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$2x^T\Sigma^{-1}\mu_1 - 2x^T\Sigma^{-1}\mu_0 + \underbrace{\mu_0^T\Sigma^{-1}\mu_0 - \mu_1^T\Sigma^{-1}\mu_1} > 2\log\frac{\pi_0}{\pi_1}$$

$$\underset{1\times D}{x^T}\underset{D\times 1}{\left(\Sigma^{-1}(\mu_1-\mu_0)\right)} + \underbrace{\frac{1}{2}(\mu_0+\mu_1)^T\Sigma^{-1}(\mu_0-\mu_1) - \log\frac{\pi_1}{\pi_0}} > 0$$

$$w^Tx + b > 0$$

$$w = \Sigma^{-1}(\mu_1-\mu_0)$$

$$b = \underline{\qquad\qquad}$$

4f) don't have Σ

instead Σ_0, Σ_1

$$x^TAx + w^Tx + b > 0$$

$$A = \frac{1}{2}(\Sigma_0^{-1} - \Sigma_1^{-1})$$

$$w = \Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0$$

$$b = \frac{1}{2}\mu_0^T\Sigma_0^{-1}\mu_0 - \frac{1}{2}\mu_1^T\Sigma_1^{-1}\mu_1 + \log\frac{\pi_1}{\pi_0}$$