Bayes rule

Consider X and Y, two random variables with X taking values in $\mathcal{X}$ and Y in $\mathcal{Y}$. The *joint probability* is

$$p_{X,Y}(x,y) = \text{probability that } X = x \text{ and } Y = y \tag{1}$$

Here is an example with $\mathcal{X} = \{0,1\}$ and $\mathcal{Y} = \{0,1,2\}$

$$\begin{array}{c|ll} & 0 & 1 \\ \hline 0 & 1/4 & 1/8 \\ 1 & 1/16 & 3/8 \\ 2 & 1/16 & 1/8 \end{array} \tag{2}$$

So the probability of $(X,Y) = (0,1)$ is $1/16$.

From the joint probability we can define the *marginal distributions*

$$\begin{aligned} p_X(x) &= \text{probability that } X = x \text{ irrespective of what } Y \text{ is} \\ p_Y(y) &= \text{probability that } Y = y \text{ irrespective of what } X \text{ is} \end{aligned} \tag{3}$$

and, it follows that

$$\begin{aligned} p_X(x) &= \sum_{y\in\mathcal{Y}} p_{X,Y}(x,y) \\ p_Y(y) &= \sum_{x\in\mathcal{X}} p_{X,Y}(x,y) \end{aligned} \tag{4}$$

For the example above

$$\begin{array}{c|ll} X & 0 & 1 \\ \hline & 3/8 & 5/8 \end{array} \tag{5}$$

and

$$\begin{array}{c|lll} Y & 0 & 1 & 2 \\ \hline & 3/8 & 7/16 & 3/16 \end{array} \tag{6}$$

We can also define the *conditional probabilities*

$$\begin{aligned} p_{X|Y}(x|y) &= \text{probability that } X = x \text{ if } Y = y \\ p_{Y|X}(y|x) &= \text{probability that } Y = y \text{ if } X = x \end{aligned} \tag{7}$$

These are calculated using Bayes rule, basically this says that the probability of $X = x$ and $Y = y$ is the probability of $X = x$ multiplied by the probability of $Y = y$ given that $X = x$:

$$p_{X,Y}(x, y) = p_X(x)p_{Y|X}(y|x) \tag{8}$$

and, similarily

$$p_{X,Y}(x, y) = p_Y(y)p_{X|Y}(x|y) \tag{9}$$

and, of course, this means

$$\begin{aligned} p_{Y|X}(y|x) &= \frac{p_{X,Y}(x,y)}{p_X(x)} \\ p_{X|Y}(x|y) &= \frac{p_{X,Y}(x,y)}{p_Y(y)} \end{aligned} \tag{10}$$

so in the example above For the example above

$$\begin{array}{c|cc} X|Y=0 & 0 & 1 \\ \hline & 2/3 & 1/3 \end{array} \tag{11}$$

$$\begin{array}{c|cc} X|Y=1 & 0 & 1 \\ \hline & 1/7 & 6/7 \end{array} \tag{12}$$

$$\begin{array}{c|cc} X|Y=2 & 0 & 1 \\ \hline & 1/3 & 2/3 \end{array} \tag{13}$$

and

$$\begin{array}{c|ccc} Y|X=0 & 0 & 1 & 2 \\ \hline & 4/6 & 1/6 & 1/6 \end{array} \tag{14}$$

$$\begin{array}{c|ccc} Y|X=1 & 0 & 1 & 2 \\ \hline & 1/5 & 3/5 & 1/5 \end{array} \tag{15}$$

Finally, since $p_{X,Y}(x, y) = p_{Y,X}(y, x)$ we can write Bayes rule in the more familiar form

$$p_{Y|X}(y|x) = \frac{p_Y(y)p_{X|Y}(x|y)}{p_X(x)} \tag{16}$$