Gambling

COMSM0075 Information Processing and Brain

comsm0075.github.io

October 2020

Gambling

Gambling presents a set of useful 'toy examples' for considering the role of information in decision making and strategy.

Gambling

Do not gamble - you will lose all your money!

A sulky race

Image from wikipedia

A race

n outcomes corresponding to n different potential winners. Each outcome has some probably unknown probability p_a.

Odds

The bookie will offer **odds**; we will use r_a the amount paid out for each euro bet, so if you bet A on horse a and it won, you would get Ar_a back.

Odds

Odds of $5/2$ means if you get bive and win you get your five back and two extra giving you seven in total, so that's equivalent to $r = 7/5$.

Gambling strategy

Should you bet everything on the horse you think mostly likely to win or should you spread your bets around.

Gambling strategy

> *As a simplified problem, imagine we are intending to bet repeatedly on the same race and we are intending to go* all in *every time.*

Your gambling strategy is how you spread you bets across the horses. Let b_a be the fraction of your total cash you bet on the ath horse. The vector $\mathbf{b}$ is your **gambling strategy** with $\sum_a b_a = 1$.

Earnings - one race

Your total amount of cash is your **float**; in this simplied version you always bet all your float.

Recall b_a be the fraction of your float you bet on the ath horse. If A was your float and the horse w wins you will have

$$S(w)A = b_w r_w A$$

after the race.

Earnings - one race

So ratio of your float afterwards to your float before is

$$S(w) = b_w r_w$$

after the race if w wins. We can think of this as a random variable Swhich is a function of the random variable W, the winner: $S(W)$.

Earnings - many races

If there are N races then at the end of the day you will have

$$S_N = \prod_{i=1}^{N} S(w(i)) = \prod_{i=1}^{N} b_{w(i)} r_{w(i)}$$

where $w(i)$ is the winner of the ith race and we are setting the initial float to one.

The doubling rate

The **doubling rate** is defined as

$$R(\mathrm{p}, \mathrm{b}) = \langle \log_2 S(W) \rangle = \sum_a p_a \log_2 b_a r_a$$

Why the 'doubling rate'?

Assume the results $\{W_1, W_2, \ldots, W_N\}$ are independent and have the same distribution, then if we bet with strategy b we have

$$\frac{1}{N}\log_2 S_N = \frac{1}{N}\sum_i \log_2 b_{w(i)} r_{w(i)} \to \langle \log_2 b_{w(i)} r_{w(i)} \rangle_W = R(\mathrm{p}, \mathrm{b})$$

so

$$S_N \approx 2^{NR(\mathrm{p},\mathrm{b})}$$

Best strategy

We want to maximize $R(\mathrm{p}, \mathrm{b})$ over all choices of $\mathrm{b_i}$ subject to the constraint that $b_a \geq 0$ and $\sum_a b_a = 1$. For a constrained optimization we use a Lagrange multiplier:

$$J = \sum \sum_a p_a \log_2 b_a r_a + \lambda \sum_a b_a$$

Now

$$\frac{dJ}{db_a} = \frac{p_a}{b_a} + \lambda$$

and, for a stationary point $dJ/db = 0$ so

$$p_a = -\lambda b_a$$

Finally sum over a:

$$\sum_a p_a = -\lambda \sum_a b_a$$

giving $\lambda = -1$ and

$$b_a = p_a$$

Proportional betting is best

$$b_a = p_a$$

In fact

$$R(\mathrm{p},\mathrm{b}) = \sum_a p_a \log_2 b_a r_a = \sum_a p_a \log_2 \frac{b_a}{p_a} p_a r_a$$

then splitting the log

$$R(\mathrm{p},\mathrm{b}) = \sum_a p_a \log_2 \frac{b_a}{p_a} + \sum_a p_a \log_2 p_a r_a$$

and splitting the second term again this give

$$R(\mathrm{p},\mathrm{b}) = \sum_a p_a \log_2 r_a - H(\mathrm{p}) - D(\mathbf{p}\|\mathbf{b})$$

and since the first two terms don't depend on r_a this shows that proportional betting is the best strategy.

The cost of ignorance

$$R(\mathrm{p}, \mathrm{b}) = \sum_a p_a \log_2 r_a - H(p) - D(\mathbf{p} \| \mathbf{b})$$

If the odds are fair

Say

$$\sum_a \frac{1}{r_a} = 1$$

then

$$R(\mathrm{p}, \mathrm{b}) = D(\mathbf{p} \| \mathbf{r}) - D(\mathbf{p} \| \mathbf{b})$$

and betting is a competition of knowledge between you and the bookie.

The Kelly criterion - a question

The **Kelly criterion** is an investment strategy which can be derived in its simplest form using the methods we have discussed. In a simple example you have the opportunity to bet on event, that two dice roll a seven for example. If your bet is a success you get r times your stake, if it fails you get nothing. The probability of success is p. What fraction of your float should you bet each time?