# Mutual information: lecture 4

COMSM0075 Information Processing and Brain

`comsm0075.github.io`

September 2020

# The chain rule for entropy

$$H(X, Y) = H(X) + H(Y|X)$$

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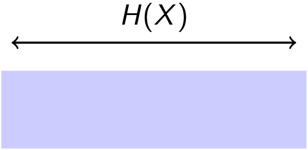

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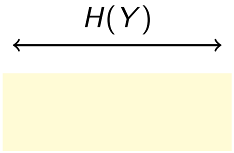

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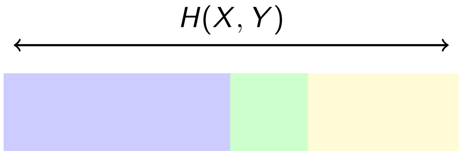

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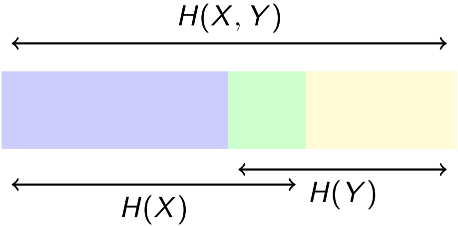

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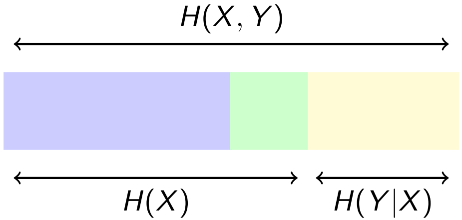
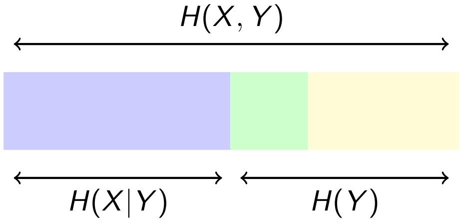
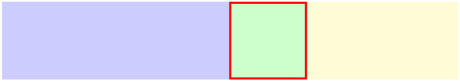


$$H(X, Y) = H(Y) + H(X|Y)$$



# Mutual information



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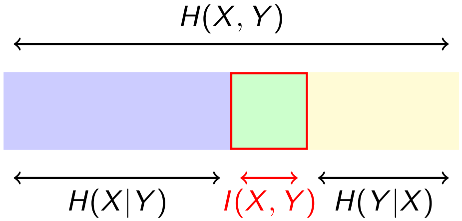

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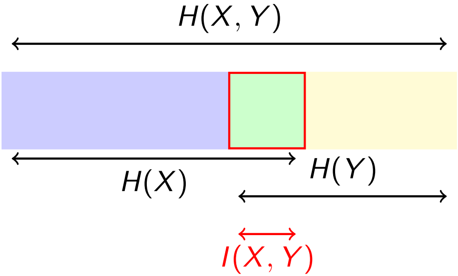

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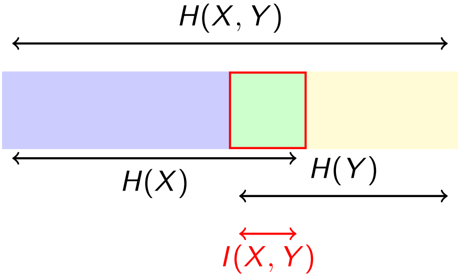
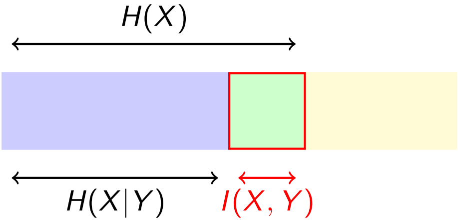



$$I(X, Y) = H(X) + H(Y) - H(X, Y)$$

# Mutual information

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then substitute $H(X,Y) = H(Y) + H(X|Y)$ to get

$$I(X,Y) = H(X) - H(X|Y)$$

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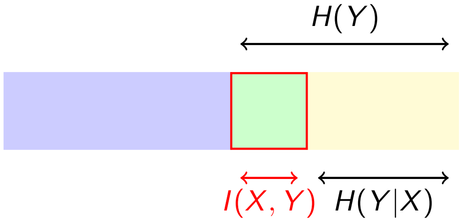


$$I(X, Y) = H(X) + H(Y) - H(X, Y)$$

then substitute $H(X, Y) = H(Y) + H(X|Y)$ to get

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# All the mutual informations

- $I(X,Y) = H(X) + H(Y) - H(X,Y)$
- $I(X,Y) = H(X) - H(X|Y)$
- $I(X,Y) = H(Y) - H(Y|X)$
- $I(X,Y) = H(X,Y) - H(X|Y) - H(Y|X)$

# Mutual information - direct formula

By substituting the formulas for $H(X)$, $H(Y)$ and $H(X,Y)$ we get

$$I(X,Y) = \sum_{i,j} p_{X,Y}(x_i, y_j) \log_2 \frac{p_{X,Y}(x_i, y_j)}{p_X(x_j) p_Y(y_j)}$$

# Example

| | $x_0$ | $x_1$ |
|---|---|---|
| $y_0$ | $1/4$ | $1/4$ |
| $y_1$ | $1/2$ | $0$ |

has $H(X, Y) = 3/2$, $H(X) \approx 0.81$ and $H(Y) = 1$ so

$$I(X, Y) \approx 0.31$$

# Mutual information - independent variables

$$I(X, Y) = \sum_{i,j} p_{X,Y}(x_i, y_j) \log_2 \frac{p_{X,Y}(x_i, y_j)}{p_X(x_j) p_Y(y_j)}$$

but if $X$ and $Y$ are independent $p_{X,Y}(x_i, y_j) = p_X(x_j) p_Y(y_j)$ and hence

$$I(X, Y) = 0$$

# Mutual information - independent variables

In fact

$$I(X, Y) \geq 0$$

with equality if and if $X$ and $Y$ are independent.

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In fact

$$I(X, Y) \geq 0$$

with equality if and if $X$ and $Y$ are independent; note that this is equivalent to

$$H(X) \geq H(X|Y)$$

with equality if and if $X$ and $Y$ are independent; as claimed earlier.

# Correlation

$$C(X, Y) = \frac{\langle (X - \mu_X)(Y - \mu_Y) \rangle}{\sigma_X \sigma_Y}$$

where $\mu_X$ is the average of $X$ and $\sigma_X$ is its standard deviation with similar notation for $Y$.

# Correlation

Consider

| | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $1$ | $1/4$ | $0$ | $1/4$ |
| $0$ | $0$ | $1/2$ | $0$ |

then

$$C(X, Y) = 0$$

whereas $I(X, Y) = 1$.