Gambling

COMSM0075 Information Processing and Brain

`comsm0075.github.io`

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Gambling

Gambling presents a set of useful 'toy examples' for considering the role of information in decision making and strategy.

Gambling

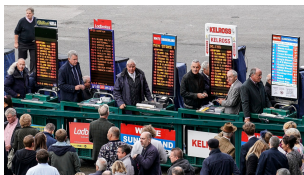Do not gamble - you will lose all your money!

A sulky race

A race

n outcomes corresponding to n different potential winners. Each outcome has some probably unknown probability p_a.

Odds

The bookie will offer **odds**; we will use r_a the amount paid out for each euro bet, so if you bet A on horse a and it won, you would get Ar_a back.

Odds

Odds of $5/2$ means if you get bive and win you get your five back and two extra giving you seven in total, so that's equivalent to $r = 7/5$.

Gambling strategy

Should you bet everything on the horse you think mostly likely to win or should you spread your bets around.

Gambling strategy

As a simplified problem, imagine we are intending to bet repeatedly on the same race and we are intending to go all in every time.

Your gambling strategy is how you spread you bets across the horses. Let b_a be the fraction of your total cash you bet on the ath horse. The vector b is your **gambling strategy** with $\sum_a b_a = 1$.

Earnings - one race

Your total amount of cash is your **float**; in this simplied version you always bet all your float.

Recall b_a be the fraction of your float you bet on the ath horse. If A was your float and the horse w wins you will have

$$S(w) = Ab_w r_w$$

after the race.

Earnings - many races

If there are N races then at the end of the day you will have

$$S = \prod_{i=1}^{N} S(w(i)) = \prod_{i=1}^{N} b_{w(i)} r_{w(i)}$$

where $w(i)$ is the winner of the ith race and we are setting the initial float to one.

The doubling rate

The **doubling rate** is defined as

$$R(\mathrm{p}, \mathrm{b}) = \langle \log_2 S(W) \rangle = \sum_a p_a \log_2 b_a r_a$$

Why the 'doubling rate'?

Assume the results $\{W_1, W_2, \ldots, W_N\}$ are independent and have the same distribution, then if we bet with strategy b we have

$$\frac{1}{n} \log_2 S = \frac{1}{n} \sum_i \log_2 b_{w(i)} r_{w(i)} \to \langle \log_2 b_{w(i)} r_{w(i)} \rangle_W = R(\mathsf{p}, \mathsf{b})$$

so

$$S \approx 2^{nR(\mathsf{p},\mathsf{b})}$$

Best strategy

We want to maximize $R(p, b)$ over all choices of b_i subject to the constraint that $b_a \geq 0$ and $\sum_a b_a = 1$. For a constrained optimization we use a Lagrange multiplier:

$$J = \sum \sum_a p_a \log_2 b_a r_a + \lambda \sum_a b_a$$

Now

$$\frac{dJ}{db_a} = \frac{p_a}{b_a} + \lambda$$

and, for a stationary point $dJ/db = 0$ so

$$p_a = -\lambda b_a$$

Finally sum over a:

$$\sum_a p_a = -\lambda \sum_a b_a$$

giving $\lambda = -1$ and

$$b_a = p_a$$

Proportional betting is best

$$b_a = p_a$$

In fact

$$R(\mathrm{p}, \mathrm{b}) = \sum_a p_a \log_2 b_a r_a = \sum_a p_a \log_2 \frac{b_a}{p_a} p_a r_a$$

then splitting the log

$$R(\mathrm{p}, \mathrm{b}) = \sum_a p_a \log_2 \frac{b_a}{p_a} + \sum_a p_a \log_2 p_a r_a$$

and splitting the second term again this give

$$R(\mathrm{p}, \mathrm{b}) = \sum_a p_a \log_2 r_a - H(\mathrm{p}) - D(\mathbf{p} \| \mathbf{b})$$

and since the first two terms don't depend on r_a this shows that proportional betting is the best strategy.

The cost of ignorance

$$R(\mathrm{p}, \mathrm{b}) = \sum_a p_a \log_2 r_a - H(p) - D(\mathbf{p} \| \mathbf{b})$$

If the odds are fair

Say

$$\sum_a \frac{1}{r_a} = 1$$

then

$$R(\mathrm{p}, \mathrm{b}) = D(\mathbf{p}\|\mathbf{r}) - D(\mathbf{p}\|\mathbf{b})$$

and betting is a competition of knowledge between you and the bookie.

The Kelly criterion - a question

The **Kelly criterion** is an investment strategy which can be derived in its simplest form using the methods we have discussed. In a simple example you have the opportunity to bet on event, that two dice roll a seven for example. If your bet is a success you get r times your stake, if it fails you get nothing. The probability of success is p. What fraction of your float should you bet each time?