# Leaky buckets

These notes describe the modelling of a leaky bucket of water, something that is useful to think about as an analogue of how neurons behave; both because it is a useful way to think about the equations and because the actual principles involves are surprisingly similar to the principles involved in neurons.

Buckets of water

In the simplest model of neurons their voltage dynamics is similar to the dynamics of a bucket with a leak and the class of equations that apply in this case will also be applied to synapses, for example.
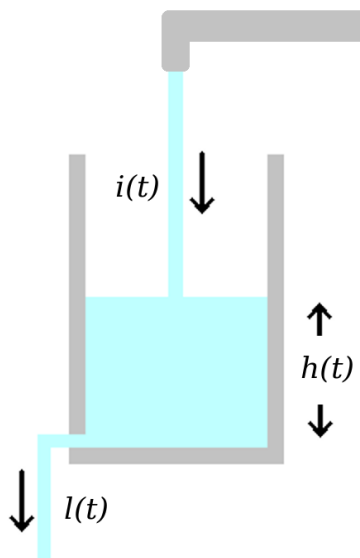
Consider a bucket with straight sides which is filled to a height h with water. Imagine the water leaks out of a hole in the bottom. The rate the water leaks out depends on h; the larger h is the larger the pressure at the bottom is and hence the faster the water pours out. In other words

$$l(t) \propto h(t) \tag{1}$$

or

$$l(t) = Gh(t) \tag{2}$$

where G is a constant which will depend on the size of the hole and complicated things like the viscosity of water. Of course, we are also ignoring lots of complicated stuff, like turbulence and so forth, but since we are interested in the equation rather than the amount of water in a bucket, this is fine: the main point is that in our model there is a linear relationship between leak and height.

Imagine water also pours in the top at a rate $i(t)$. This means the total rate of change of the amount of water is $i(t) - Gh(t)$.

Now, $h(t)$ is the height of the water not the volume: the volume is $Ch(t)$ where C is the cross-sectional area of the bucket. The rate of change of the volume is therefore

$$\frac{dCh(t)}{dt} = i(t) - Gh(t) \tag{3}$$

or, in other words, the rate the volume of water changes is the difference between the water flowing in and the water flowing out. Divinging across by G this gives

$$\frac{G}{C}\frac{dh}{dt} = \frac{i}{C} - h \tag{4}$$

or, if we define $\tau = G/C$ and $y = i/C$

$$\tau\frac{dh}{dt} = y - h \tag{5}$$

It is useful to note the dimensions at this point, that is dimensions in the sense of units: h is a distance, measured in meters or feet or cubits or whatever, dh/dt in contrast, is distance per time: this means that τ has to be timescale for the dimensions to all match. The unit for τ will be a unit of time, like seconds or whatever.

Let's solve this equation for constant y before going on to look at neurons: to remind ourselves that we are looking at the constant input case we'll write $y = \bar{y}$ for the constant values. There are lots of ways to do this, including Laplace transforms and integrating factors. Here we will use an ansatz; ansatz just means guess and what we do is make a guess as to the solution, the guess will usually have a few constant in it and we show, by substituing in that we can pick those constants to make the solution work. Here we use the ansatz:

$$h(t) = A + Be^{-rt} \tag{6}$$

where A, B and r are the constants. There is nothing deep happening here, I have seen equations like this before so I think I will be able to find values of the constants that make this a solution, if I am wrong that won't be possible, though, of course, I won't be wrong.

So differenciating:

$$\frac{dh}{dt} = -rBe^{-rt} \tag{7}$$

using the rules for differenting exponentials. Substituting that into the equation I get:

$$-\tau rBe^{-rt} = \bar{y} - A - Be^{-rt} \tag{8}$$

For the ansatz to be a solution of the differential equation this equation needs to hold for all values of t so, basically, it most hold for the bits multiplying the exponential and for the bits that don't. Thus
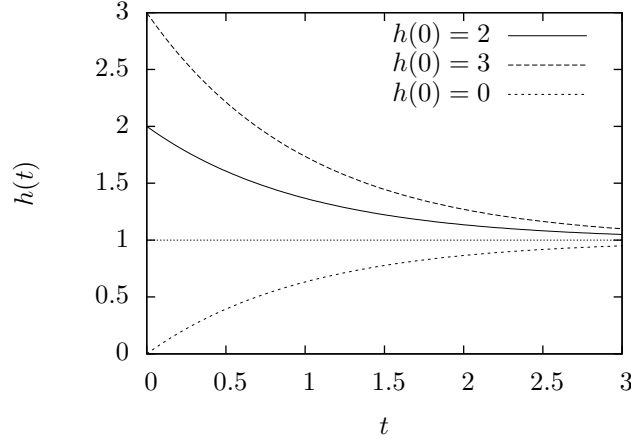
$$A = \bar{y} \tag{9}$$

Figure 1: Exponential relaxation. The dynamics described by the 'bucket equation' are very common. Here $h = [h(0) - \bar{y}]\exp(-t/\tau) + \bar{y}$ is plotted with $\tilde{\imath} = 1$, $\tau = 1$ and three different values of $h(0)$. $h(t)$ relaxes towards the equilibrium value $\bar{y} = 1$, the closer it gets, the slower it approaches.

and

$$r = 1/\tau \tag{10}$$

Hence, the ansatz did work and tells us that the equation is solved by

$$h(t) = \bar{y} + Be^{-t/\tau} \tag{11}$$

for any value of B. In fact, we can fix this constant using an initial condition: lets substitute in for $t = 0$, the exponential of zero is one so

$$h(0) = \bar{y} + B \tag{12}$$

and hence

$$h(t) = \bar{y} + [h(0) - \bar{y}]e^{-t/\tau} \tag{13}$$

These dynamics make good intuitive sense; the more water there is in the bucket, the higher the pressure will be at the leak and the quicker the water will pour out. If there is just the right amount of water the rate the water pours out the leak will precisely match the rate it pours in, this is the equilibrium. If there is more water than required for equilibrium it will pour out faster than the flow coming in, if there is less, it will pour out slower. Either way, as time passes the height of the water will reach the equilibrium. The plot in Fig. 1 illustrates this.
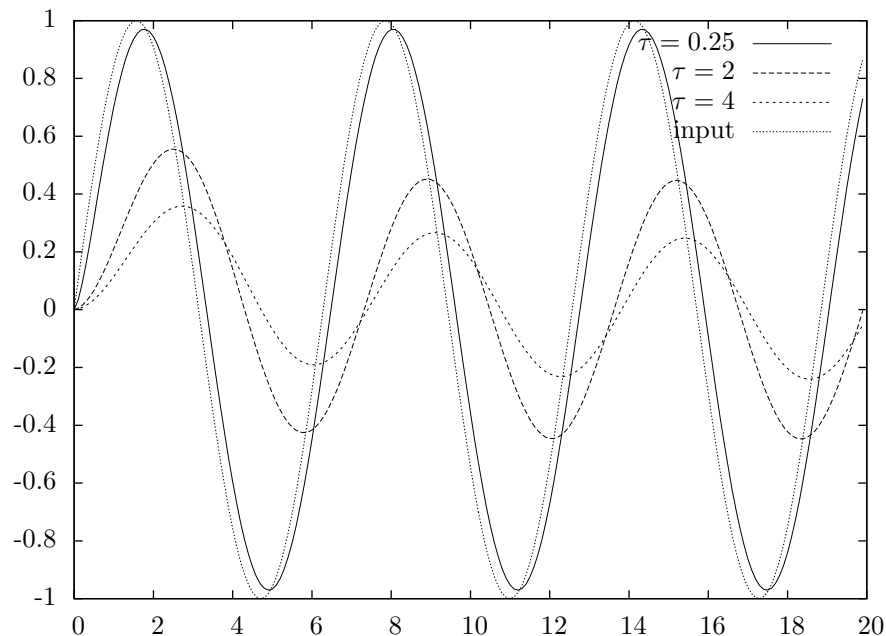
Figure 2: Variable input. Here the input is a sine wave $\bar{y} = \sin t$ and the equation is evolved with $h(0)$ and three different τ values. For $\tau = 0.25$ we see $h(t)$ closely matches the input whereas for larger τ it is smoother and lags behind.

Variable input

We have only discussed constant inputs; the variable input case where y depends on time is harder and although it can sometimes be solved it is often easier just to compute it numerically, we will look briefly at how to do this, but first note that the effect of variable input is that the solution kind of chases the input with a timescale set by τ. For very small τ is chases it quickly, so it is close to the input, but for large τ it lags behind it and smooths it out. This is sometimes described by saying that it *filters* the input. There is an illustration in Fig. 2.
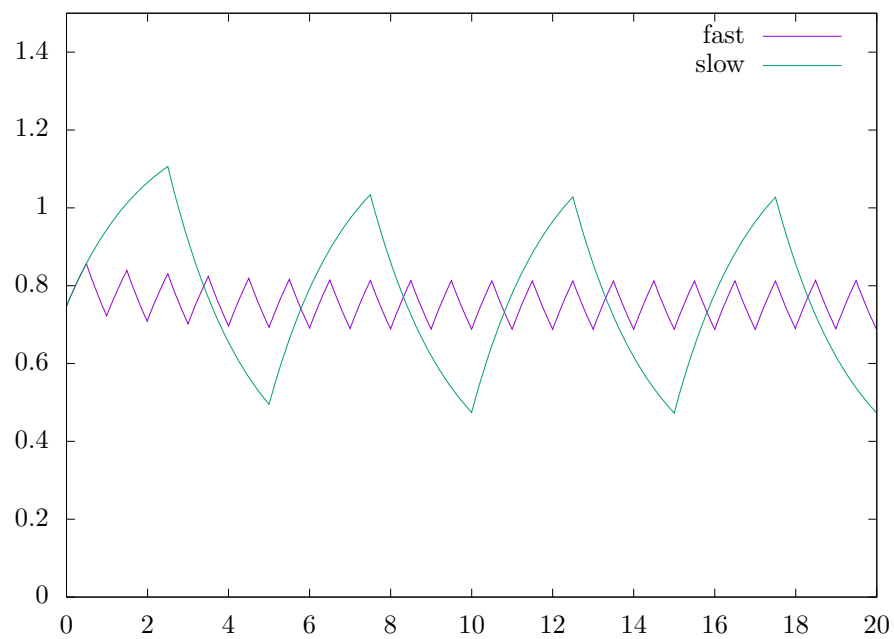
Figure 3: Piecewise constant input. Here the input is a function that is either $y = 0.5$ or $y = 1.5$; for **fast** this flips every 0.5 so the period is one, for **slow** it flips ever 2.5 so the persion is 5; in both cases $\tau = 2$ and the initial v is 0.75.