Assignment 1
MATH271

Connor Braun

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Problemset 1

a) For all integers x, there is an integer y so that $3|x+y$

—The statement is True.—

Proof. Suppose x is an integer. Then, by the quotient-remainder theorem, we have that

$$x = 3q + r$$

where $3 \neq 0$ and q and r are integers with r is on the interval $[0, 3)$. Subtracting r from both sides yields

$$x - r = 3q.$$

Next, we set $-r = y$. Clearly, y is an integer since r is an integer. With this substitution the equation becomes

$$x + y = 3q$$

which, by the definition of divisibility, implies that $3|x+y$ since $x+y$, 3, q are integers. Therefore, for any integer x, there exists an integer y such that $3|x+y$.

b) For all integers x, there is an integer y so that $3|x+y$ and $3|x-y$

—The statement is False.—

Negation: There exists an integer x such that for any integer y, either $3 \nmid x+y$ or $3 \nmid x-y$.

Proof (by contradiction)**.** Consider $x = 1$ where 1 is an integer and suppose y is some integer. For the purposes of deriving a contradiction, suppose that both $3|x+y$ and $3|x-y$. From the definition of divisibility, $3|x+y$ implies that

$$x + y = y + 1 = 3k$$

for some integer k. Subtracting 1 from both sides reveals that

$$y = 3k - 1.$$

Similarly, by the definition of divisibility, $3|x-y$ implies that

$$x - y = 1 - y = 3t$$

for some integer t. This further implies that

$$y = 1 - 3t.$$

Now, since both $3k-1$ and $1-3t$ are equal to y, we can set them to be equal to one another to find that

$$3k - 1 = 1 - 3t.$$

Adding $3t + 1$ to both sides yields

$$3k + 3t = 2.$$

Finally, factoring a 3 from the left hand side and dividing both sides by 3 yields

$$k + t = \frac{2}{3}$$

which shows that $k + t$ cannot be an integer since $\frac{2}{3}$ is not an integer. Since the integers are closed under addition, this equality would require that either k or t not be integers, violating the established supposition that k and t be integers in order for both $3|x + y$ and $3|x - y$. Hence, either $3 \nmid x + y$ or $3 \nmid x - y$. Furthermore, since the definition of divisibility by 3 is not met for at least one of $x + y$ or $x - y$, we conclude the negation of the original statement to be true by contradiction.

c) For all integers x and y, if $3|x + y$ then $3|x$ or $3|y$

—The statement is False.—

Negation: There exists two integers x and y such that $3|x + y$ but $3 \nmid x$ and $3 \nmid y$.

Proof. Consider $x = 1$ and $y = 2$, where both x and y are integers. Clearly, it is the case that $3|x + y$, since $x + y$ can be written in the form

$$x + y = 1 + 2 = 3(1)$$

where 3 and 1 on the right hand side are both integers with $3 \neq 0$. Despite this, $x = 1$ is not divisible by 3, since, by the quotient-remainder theorem, the unique integers 0 and 1 (where 1 is on the interval $[0, 3)$) allow us to write 1 in the form

$$1 = 3(0) + 1$$

Since the quotient-remainder theorem guarantees uniqueness of integers 0 and 1, there is no way to write 1 in the form $1 = 3(k)$ with k being an integer. Therefore, $3 \nmid 1$ which by substitution implies that $3 \nmid x$.

Similarly, 2 is not divisible by 3, since, by the quotient-remainder theorem, the unique integers 0 and 2 (where 2 is on the interval $[0, 3)$) allow us to write 2 in the form

$$2 = 3(0) + 2.$$

Just as before, since the quotient-remainder theorem guarantees uniqueness of integers 0 and 2, there is no way to write 2 in the form $2 = 3(t)$ with t being an integer. Therefore, $3 \nmid 2$ which by substitution implies that $3 \nmid y$. Moreover, in the case of $x = 1$ and $y = 2$, we have shown that $3|x + y$ but $3 \nmid x$ and $3 \nmid y$.

d) For all integers x and y, if $3|xy$ then $3|x$ or $3|y$.

—The statement is True.—

Proof (by contradiction)**.** Suppose that x and y are integers and that $3|xy$. Suppose also for the purpose of deriving a contradiction that $3 \nmid x$ and $3 \nmid y$. By the definition of divisibility, $3|xy$ implies that

$$xy = 3k$$

for some integer k, where $3 \neq 0$. By the quotient-remainder theorem, $3 \nmid x$ implies that

$$x = 3q_1 + r_1$$

where q_1 and r_1 are integers, and r_1 is on the interval $[1, 3)$. As a brief aside, normally the quotient-remainder theorem would specify that integer r_1 be on the interval $[0, 3)$. However, $r_1 = 0$ would imply that $3|x$, since then we'd have that $x = 3q_1$ where q_1 is an integer. Since we supposed that $3 \nmid x$, we need to restrict r_1 to interval $[1, 3)$. Similarly, by the quotient-remainder theorem, $3 \nmid y$ implies that

$$y = 3q_2 + r_2$$

where q_2 and r_2 are integers, and r_2 is on the interval $[1, 3)$ instead of $[0, 3)$ to satisfy our supposition that $3 \nmid y$ (by the same logic as argued for r_1 being on the interval $[1, 3)$). Substituting these definitions for x and y into $xy = 3k$, we see that

$$(3q_1 + r_1)(3q_2 + r_2) = 3k$$

Expanding the left hand side yields

$$9q_1q_2 + 3q_1r_2 + 3q_2r_1 + r_1r_2 = 3k$$

Factoring out a 3 from the left hand side and then dividing both sides by 3 yields

$$3q_1q_2 + q_1r_2 + q_2r_1 + \frac{1}{3}r_1r_2 = k$$

In order for this equality to hold, $3q_1q_2 + q_1r_2 + q_2r_1 + \frac{1}{3}r_1r_2$ must be an integer since k is an integer. Clearly, $3q_1q_2 + q_1r_2 + q_2r_1$ is an integer since the integers are closed under both multiplication and addition, but it is not necessarily the case that $\frac{1}{3}r_1r_2$ is since $\frac{1}{3}$ is not an integer.

In order for $\frac{1}{3}r_1r_2$ to be an integer, $3|r_1r_2$ since then $r_1r_2 = 3t$ for some integer t. When this is the case, $\frac{1}{3}r_1r_2$ can be rewritten as

$$\frac{3t}{3} = t \in \mathbb{Z}$$

There are 4 possible cases, considering the fact that both $r_1 \in [1, 3)$ and $r_2 \in [1, 3)$:

Case 1: $r_1 = 1$ and $r_2 = 1$

Then $r_1r_2 = 1$. By the quotient-remainder theorem, for dividend 1 and divisor $3 \neq 0$, the unique integers $q = 0$ and $r = 1 \in [0, 3)$ satisfy the equation

$$1 = 3q + r = 3(0) + 1$$

Since $r \neq 0$ and these integers are unique for dividend 1 and divisor 3, there is no way to write $1 = 3(h)$ with h being an integer. Therefore, when $r_1 = 1$ and $r_2 = 1$, $3 \nmid r_1r_2$.

Case 2: $r_1 = 2$ and $r_2 = 1$

Then $r_1r_2 = 2$. By the quotient-remainder theorem, for dividend 2 and divisor $3 \neq 0$, the unique integers $q = 0$ and $r = 2 \in [0, 3)$ satisfy the equation

$$2 = 3q + r = 3(0) + 2$$

Since $r \neq 0$ and these integers are unique for dividend 2 and divisor 3, there is no way to write $2 = 3(h)$ with h being an integer. Therefore, when $r_1 = 2$ and $r_2 = 1$, $3 \nmid r_1r_2$.

Case 3: $r_1 = 1$ and $r_2 = 2$

Then $r_1r_2 = 2$. This product, and hence the result, is identical to that of case 2. Therefore, when $r_1 = 1$ and $r_2 = 2$, $3 \nmid r_1r_2$.

Case 4: $r_1 = 2$ and $r_2 = 2$

Then $r_1r_2 = 4$. By the quotient-remainder theorem, for dividend 4 and divisor $3 \neq 0$, the unique integers $q = 1$ and $r = 1 \in [0, 3)$ satisfy the equation

$$4 = 3q + r = 3(1) + 1$$

Since $r \neq 0$ and these integers are unique for dividend 4 and divisor 3, there is no way to write $4 = 3(h)$ with h being an integer. Therefore, when $r_1 = 2$ and $r_2 = 2$, $3 \nmid r_1r_2$.

No matter the case, $3 \nmid r_1r_2$ and hence there is no integer t for which $r_1r_2 = 3t$. Furthermore, it is impossible for $\frac{1}{3}r_1r_2$ to be an integer, which means it is impossible for $3q_1q_2 + q_1r_2 + q_2r_1 + \frac{1}{3}r_1r_2$ to be an integer. Therefore, we have that

$$3q_1q_2 + q_1r_2 + q_2r_1 + \frac{1}{3}r_1r_2 \neq k$$

which, working backwards, implies that

$$(3q_1 + r_1)(3q_2 + r_2) \neq 3k$$

which, working backwards once more, gives rise to the contradiction

$$xy \neq 3k$$

Which implies that $3 \nmid xy$. This contradicts with our supposition that $3|xy$. Therefore, for all integers x and y, if $3|xy$ then either $3|x$ or $3|y$.

Problemset 2

a) If x is an irrational number, then for all integers m, n where $n \neq 0$, $m + nx$ is irrational

—The statement is True.—

Proof (by contradiction). Suppose that x is an irrational number and that m, n are integers with $n \neq 0$. For the purpose of deriving a contradiction, suppose also that $m + nx$ is a rational number.By the definition of rationality, $m + nx$ can be represented by coprime integers a, b where $b \neq 0$ such that

$$m + nx = \frac{a}{b}.$$

Multiplying both sides of the equation by b yields

$$bm + bnx = a.$$

Next, subtracting bm from both sides yields

$$bnx = a - bm$$

Since b, n, a, m are all integers and $b \neq 0$ and $n \neq 0$, both $a - bm$ and bn are integers, with $bn \neq 0$. To show that this equality does not hold for irrational number x, consider the following Lemma.

Lemma 1. For any irrational number y and any integer $h \neq 0$, hy is irrational.

Proof (by contradiction). Suppose y is an irrational number and h is an integer where $h \neq 0$. For the purpose of deriving a contradiction, suppose also that the product hy is rational. Then by the definition of rationality, hy can be represented by coprime integers c, d where $d \neq 0$ such that

$$hy = \frac{c}{d}.$$

Dividing both sides by h yields

$$y = \frac{c}{hd}$$

However, since it was specified that $c, h, d \in \mathbb{Z}$ and both $h \neq 0$ and $d \neq 0$, both the numerator and denominator of $\frac{c}{hd}$ are integers. Furthermore, $hd \neq 0$ since neither h nor d are 0. Hence, $\frac{c}{hd}$ meets the definition of a rational number, while y was supposed to be irrational and we have that

$$y \neq \frac{c}{hd}$$

and

$$hy \neq \frac{c}{d}$$

which contradicts the supposition that hy was rational. Therefore, for any irrational number y, no matter what integer $h \neq 0$ one selects, the product hy is irrational.

Lemma 1 shows that bnx must be irrational since $bn \neq 0$ and is an integer while x is irrational. However, it was already shown that $a - bm$ is an integer. This results in the inequality

$$bnx \neq a - bm$$

Which, working backwards, reveals that

$$bm + bnx \neq a$$

and

$$m + nx \neq \frac{a}{b}$$

Which contradicts the supposition that $m+nx$ is rational. Therefore, $m+nx$ must be irrational for an irrational number x and integers m, n with $n \neq 0$.

b) For all real numbers x, there is a real number y so that $x + y$ is irrational.

—The statement is True.—

Proof. Suppose x is a real number and consider the real number $y = \sqrt{2} - x$. Then the sum of x and y yields

$$x + y = x + (\sqrt{2} - x) = \sqrt{2}$$

Where $\sqrt{2}$ is irrational. Therefore, for any real number x, there exists a real number y so that their sum $x + y$ is irrational.

c) For all real numbers x and y, if $x + y$ is rational then x or y is rational.

—The statement is False.—

Negation: There exists real numbers x and y such that $x + y$ is rational and both x and y are irrational.

Proof. Consider the example of $x = \sqrt{2}$ and $y = -\sqrt{2}$. By Lemma 1, $y = -\sqrt{2}$ is irrational since -1 is an integer, $-1 \neq 0$, and $\sqrt{2}$ is an irrational number. This also means that $x = \sqrt{2}$ is an irrational number, and both y and x are real numbers. Taking the sum of x and y yields

$$x + y = \sqrt{2} - \sqrt{2} = 0$$

where 0 is a rational number since it can be written in the form

$$0 = \frac{0}{a}$$

For any integer $a \neq 0$. Therefore, it is not the case that the sum of any two real numbers x and y being rational guarantees that either x or y be rational.

d) For all real numbers x and y, if xy is irrational then x or y is irrational.

—*The statement is True.*—

Proof (by contradiction)**.** Suppose that x and y are both real numbers and that their product xy is irrational. For the purpose of deriving a contradiction, suppose that x and y are both rational. This means that there exists integers a, b, c, d with both $b \neq 0$ and $d \neq 0$. Then, by the definition of a rational number, we have

$$x = \frac{a}{b}$$

and

$$y = \frac{c}{d}$$

Taking the product xy reveals that

$$xy = \frac{ac}{bd}$$

The products ac and bd are both integers, and $bd \neq 0$ since neither $b = 0$ nor $d = 0$. This means that the fraction $\frac{ac}{bd}$ meets the definition of rationality, which contradicts the supposition that xy is irrational. Thus, the supposition that x and y are both rational cannot be the case, and furthermore that for any real numbers x and y, if their product is irrational, then either x or y must be irrational.

Problemset 3

a) Use the Euclidean Algorithm to compute $\gcd(2021, 271)$ and use that to find integers x and y so that $\gcd(2021, 271) = 2021x + 271y$.

—*Part one: compute* $\gcd(2021, 271)$—

Solution (by Euclidean Algorithm)**.** The quotient-remainder theorem guarantees that for any integer dividend a and integer divisor $d > 0$, there exits two unique integers q and r (called the quotient and remainder respectively) with r on the interval $[0, d)$ such that

$$a = dq + r.$$

Which allows one to make use of the fact that if $a, d, q, r \in \mathbb{Z}$ with not both $a \neq 0$ and $d \neq 0$ (satisfied by the requirements of the quotient-remainder theorem so long as $a \neq 0$) then we have that

$$\gcd(a, d) = \gcd(d, r).$$

Now, since 2021 and 271 are integers with neither equal to 0, 2021 can be written in the form

$$2021 = 7(271) + 124, \qquad \text{such that } \gcd(2021, 271) = \gcd(271, 124).$$

Reapplying the same fact that 271 and 124 are integers with neither equal to 0, 271 can be written in the form

$$271 = 2(124) + 23, \qquad \text{such that } \gcd(271, 124) = \gcd(124, 23).$$

We reapply the exact same step repeatedly until one of integers s, t in $\gcd(s, t)$ is 0.

$$124 = 5(23) + 9, \qquad \text{such that } \gcd(124, 23) = \gcd(23, 9).$$

$$\begin{aligned} 23 &= 2(9) + 5, && \text{such that } \gcd(23, 9) = \gcd(9, 5). \\ 9 &= 1(5) + 4, && \text{such that } \gcd(9, 5) = \gcd(5, 4). \\ 5 &= 1(4) + 1, && \text{such that } \gcd(5, 4) = \gcd(4, 1). \\ 4 &= 4(1) + 0, && \text{such that } \gcd(4, 1) = \gcd(1, 0). \end{aligned}$$

From which we conclude that $\gcd(1, 0) = \gcd(2021, 271) = 1$.

—Part two: use previous result to compute integers x and y such that $\gcd(2021, 271) = 2021x + 271y$*—*

Solution. Using the so-called table method, we solve equation $2021n + 271m = r_i$, where n, m, r_i are integers and r_i is the remainder in each equation from part one. To start, we solve for the trivial case of 2021 and 271

$$\textbf{Row 1: } 2021 = (1)2021 + (0)271$$

and

$$\textbf{Row 2: } 271 = (0)2021 + (1)271.$$

Next we solve for 124 using equation $2021 = 7(271) + 124$ from part one

$$\textbf{Row 3: } 124 = (1)2021 + (-7)271 \qquad \textbf{(Row 1 - 7(Row 2))}$$

Which is equivalent to a row operation which scales the linear equation for 271 by -7 before adding it to the equation for 2021. We repeat this process until we have found integers x, y which satisfy equation $\gcd(2021, 271) = 2021x + 271y = 1$:

$$\textbf{Row 4: } 23 = (-2)2021 + (15)271 \qquad \textbf{(Row 2 - 2(Row 3))}$$

$$\textbf{Row 5: } 9 = (11)2021 + (-82)271 \qquad \textbf{(Row 3 - 5(Row 4))}$$

$$\textbf{Row 6: } 5 = (-24)2021 + (179)271 \qquad \textbf{(Row 4 - 2(Row 5))}$$

$$\textbf{Row 7: } 4 = (35)2021 + (-261)271 \qquad \textbf{(Row 5 - 1(Row 6))}$$

$$\textbf{Row 8: } 1 = (-59)2021 + (440)271 \qquad \textbf{(Row 6 - 1(Row 7))}$$

Where Row 8 reveals that integers $x = -59$ and $y = 440$ satisfy equation $\gcd(2021, 271) = 1 = 2021x + 271y$.

b) Is it true that for all positive integers a and b, $\gcd(a, b) \leq \gcd(a + b, a - b)$?

—The statement is True.—

Proof. Suppose that a and b are positive integers. Suppose also that $\gcd(a, b) = g_1$ which is an integer. Since this means that g_1 divides both a and b, we have that

$$a = kg_1 \text{ and } b = tg_1$$

For some integers k and t. By the definition of greatest common factor, we have the fact that for any integer x which divides both a and b

$$\gcd(a, b) \geq x.$$

So, if both $g_1|a + b$ and $g_1|a - b$, then we will have that $\gcd(a, b) \leq \gcd(a + b, a - b)$. By substitution, we have that

$$a + b = kg_1 + tg_1.$$

Factoring out g_1 from the right hand side yields

$$a + b = g_1(k + t)$$

Where $k + t$ is an integer. Therefore, $g_1|a + b$. Similarly, substituting in for $a - b$ yields

$$a - b = g_1k - g_1t.$$

Factoring out g_1 from the right hand side yields

$$a - b = g_1(k - t)$$

where $k - t$ is an integer. Therefore, we also have that $g_1|a - b$. Furthermore, since $g_1 = \gcd(a, b)$ divides both $a + b$ and $a - b$, it must be the case that $\gcd(a, b) \leq \gcd(a + b, a - b)$.

c) Is it true that for all positive integers a and b, $\gcd(a+b, a-b) \leq \gcd(a,b)$?

—*The statement is False.*—

Negation: There exists positive integers a and b such that $\gcd(a+b, a-b) > \gcd(a,b)$.

Proof. Consider integers $a = 3$ and $b = 1$. Now, using the Euclidean algorithm and quotient-remainder theorem as described in Problemset 3-c, we find that

$$a = 3 = 3(1) + 0, \qquad \text{such that } \gcd(3,1) = \gcd(1,0).$$

From which it is shown that $\gcd(3,1) = \gcd(1,0) = 1$. Similarly, to determine $\gcd(a+b, a-b)$, we begin with

$$a + b = k(a-b) + r$$

for some integers k, r. Substituting in $a = 3$ and $b = 1$ we get

$$4 = 2(2) + 0, \qquad \text{such that } \gcd(4,2) = \gcd(2,0).$$

From which it is shown that $\gcd(a+b, a-b) = \gcd(4,2) = \gcd(2,0) = 2$. Hence we have that

$$\gcd(a+b, a-b) > \gcd(a,b).$$

Therefore, for positive integers $a = 3$ and $b = 1$ we have that $\gcd(a+b, a-b) > \gcd(a,b)$.