Assignment 3
MATH271

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Problem 1

Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(x) = 2x - 3$ for all $x \in \mathbb{Z}$.

a) Prove or disprove the statement: f is injective.

Solution: the statement is true.

Proof. Let a and b be integers and suppose that $f(a) = f(b)$. Then we have that

$$\begin{aligned}
a &= \frac{(2a-3)+3}{2} \\
&= \frac{f(a)+3}{2} && \text{Since } 2a - 3 = f(a) \\
&= \frac{f(b)+3}{2} && \text{Since we've supposed that } f(a) = f(b) \\
&= \frac{(2b-3)+3}{2} && \text{Since } f(b) = 2b - 3 \\
&= b
\end{aligned}$$

Therefore, for all integers a and b in the domain of f, whenever $f(a) = f(b)$ it is always the case that $a = b$. Furthermore, f is injective.

b) Prove or disprove the statement: f is surjective.

Solution: The statement is false.

Negation: f is not surjective $\iff \exists_y \in \mathbb{Z}$ such that $\forall_x \in \mathbb{Z}$, $y \neq f(x)$.

Proof. Suppose that $y = 2$ where $2 \in \mathbb{Z}$ be an element of the codomain of f and let $x \in \mathbb{Z}$ be some element of the domain of f. Assume for the purpose of deriving a contradiction that we have $f(x) = y$. Then we see that

$$\begin{aligned}
y &= f(x) \\
2 &= 2x - 3 && \text{Since we've chosen } y = 2 \text{ and by the definition of } f \\
5 &= 2x && \text{Adding 3 to both sides} \\
\frac{5}{2} &= x && \text{Dividing both sides by 2}
\end{aligned}$$

However, since 5 and 2 are coprime, we have that $\frac{5}{2} \notin \mathbb{Z}$ and furthermore $x \notin \mathbb{Z}$. This violates our assumption that $x \in \mathbb{Z}$. To resolve the contradiction, it must be the case that $f(x) \neq y$. Therefore, there is no x in the domain of f which satisfies $f(x) = 2$. However, since $2 \in \mathbb{Z}$, 2 is in the codomain of f. Hence f is not surjective.

c) Prove or disprove the statement: for all functions g and h from $\mathbb{Z} \to \mathbb{Z}$, if $f \circ g = f \circ h$ then $g = h$.

Solution: The statement is true.

Proof. Let $g : \mathbb{Z} \to \mathbb{Z}$ and $h : \mathbb{Z} \to \mathbb{Z}$ be functions and suppose that $f \circ g = f \circ h$. In order to show that $g = h$, we must have that g and h share the same domain and that for all elements x of the domain, $g(x) = h(x)$. We've already supposed g and h to have the same domain, so to show that $\forall_x \in \mathbb{Z}$, $g(x) = h(x)$, let $x \in \mathbb{Z}$. Now, for the purpose of deriving a contradiction, suppose that $g(x) \neq h(x)$. Then we have that

$$\begin{aligned} g(x) &\neq h(x) & & \\ f(g(x)) &\neq f(h(x)) & &\text{Since, as shown in 1.a, } f \text{ is injective.} \\ f \circ g(x) &\neq f \circ h(x) & &\text{By the definition of function composition.} \end{aligned}$$

However, this violates our assumption that $f \circ g = f \circ h$ since we have shown that there exists some x in the domain of both $f \circ g$ and $f \circ h$ such that $f \circ g(x) \neq f \circ h(x)$. To resolve the contradiction, we must have that $g(x) = h(x)$. Therefore, we have that for any element x of $\mathbb{Z}$, $g(x) = h(x)$. Therefore, we've shown that $g = h$.

d) Prove or disprove the statement: for all functions g and h from $\mathbb{Z} \to \mathbb{Z}$, if $g \circ f = h \circ f$ then $g = h$.

Solution: The statement is false.

Negation: There exists functions g and h from $\mathbb{Z} \to \mathbb{Z}$ such that $g \circ f = h \circ f$ but $g \neq h$.

Proof. Let $g : \mathbb{Z} \to \mathbb{Z}$ and $h : \mathbb{Z} \to \mathbb{Z}$ be the functions defined by:

$$g(a) = 2a \qquad h(b) = \begin{cases} 2b & \text{if b is odd} \\ b & \text{if b is even} \end{cases}$$

for any integers a and b in the domain of g and h respectively. We clearly have that $g \neq h$ since 2 is an element of both of their domains, but $g(2) = 2(2) = 4$ while $h(2) = 2$ since 2 is even. Furthermore, $4 \neq 2$ implies that $g(2) \neq h(2)$ and hence $g \neq h$. Now, by composing f with each of these, we see that

$$g \circ f(c) = 2(2c - 3) \qquad h \circ f(d) = \begin{cases} 2(2d - 3) & \text{if d is odd} \\ 2d - 3 & \text{if d is even} \end{cases}$$

for any two integers c and d by the definitions of f, g, h and the composition of functions. Furthermore, we have that for any odd integer x

$$g(x) = 2x = h(x)$$

Therefore, if we can show that $\forall_z \in \mathbb{Z}$, $f(z)$ is an odd integer, then we'll have that

$$g \circ f(z) = g(f(z)) = 2f(z) = h(f(z)) = h \circ f(z)$$

as required. Consider the following lemma.

Lemma 1. Let z be an integer in the domain of f. Then we have that

$$\begin{aligned} f(z) &= 2z - 3 \\ &= 2z - 4 + 1 \\ &= 2(z - 2) + 1 \end{aligned}$$

Where since $z - 2$ is an integer, we have that for any integer z in the domain of f, $f(z)$ is odd.

By lemma 1, we now have that for any integer z in the domain of $g \circ f$ and $h \circ f$

$$g \circ f(z) = g(f(z)) = 2(2z-3) = h(f(z)) = h \circ f(z)$$

since $f(z)$ is an odd integer. This in conjunction with the fact that $g \circ f$ and $h \circ f$ share the same domain ($\mathbb{Z}$) means we have that $g \circ f = g \circ h$. Therefore, we've shown the existence of functions $g : \mathbb{Z} \to \mathbb{Z}$ and $h : \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f = h \circ f$ but $g \neq h$.

Problem 2

Let A={1,2,3,4}. Let $f : A \to A$ be the function defined by $f = \{(1,2),(2,3),(3,2),(4,4)\}$.

a) How many functions $g : A \to A$ so that $f \circ g(1) = 2$?

Solution: there are $1 \times 4^3 + 1 \times 4^3 = 2(4^3) = 128$ *such functions* g.

We define 2 different recipes, where using either will construct a unique definition of a function $g : A \to A$ guaranteeing that $f \circ g(1) = 2$. Furthermore, using either of the two recipes allow one to define every function $g : A \to A$ guaranteeing that $f \circ g(1) = 2$ in exactly one way.

Recipe 1.
Setting $g(1) = 1$, where $1 \in A$, we have that $f \circ g(1) = f(g(1)) = f(1) = 2$ as required. One can construct all possible functions g where $(1,1) \in g$ by the recipe:

(I) Let the image of $1 \in A$ under g be 1.
(II) Choose an element of A to be the image of $2 \in A$ under g.
(III) Choose an element of A to be the image of $3 \in A$ under g.
(IV) Choose an element of A to be the image of $4 \in A$ under g.

Hence we have that in the case where we require $(1,1) \in g$, there are $1 \times 4 \times 4 \times 4 = 1 \times 4^3$ possible ways to define $g : A \to A$.

Recipe 2.
Setting $g(1) = 3$, where $1, 3 \in A$, we have that $f \circ g(1) = f(g(1)) = f(3) = 2$ as required. One can construct all possible functions g where $(1,3) \in g$ by the recipe:

(I) Let the image of $1 \in A$ under g be 3.
(II) Choose an element of A to be the image of $2 \in A$ under g.
(III) Choose an element of A to be the image of $3 \in A$ under g.
(IV) Choose an element of A to be the image of $4 \in A$ under g.

Hence we have that in the case where we require $(1,3) \in g$, there are $1 \times 4 \times 4 \times 4 = 1 \times 4^3$ possible ways to define $g : A \to A$.

b) How many injective functions $g : A \to A$ so that $f \circ g(1) = 2$?

Solution: there are $1 \times 3 \times 2 \times 1 + 1 \times 3 \times 2 \times 1 = 6 + 6 = 12$ *such functions* g.

We define 2 different recipes, where using either will construct a unique definition of a function $g : A \to A$ guaranteeing that $f \circ g(1) = 2$. Furthermore, using either of the two recipes allow one to define every function $g : A \to A$ guaranteeing that $f \circ g(1) = 2$ in exactly one way.

Recipe 3.
Setting $g(1) = 1$, where $1 \in A$, we have that $f \circ g(1) = f(g(1)) = f(1) = 2$ as required. One can construct all possible injective functions g where $(1,1) \in g$ by the recipe:

(I) Let the image of $1 \in A$ under g be 1.

(II) Choose an element a of A to be the image of $2 \in A$ under g such that for all $2 \neq x \in A$, $(x, a) \notin g$ yet.

(II) Choose an element b of A to be the image of $3 \in A$ under g such that for all $3 \neq y \in A$, $(y, b) \notin g$ yet.

(II) Choose an element c of A to be the image of $4 \in A$ under g such that for all $4 \neq z \in A$, $(z, c) \notin g$ yet.

Hence we have that in the case where we require $(1, 1) \in g$, there are $1 \times 3 \times 2 \times 1 = 6$ possible ways to define $g : A \to A$.

Recipe 4.
Setting $g(1) = 3$, where $1, 3 \in A$, we have that $f \circ g(1) = f(g(1)) = f(3) = 2$ as required. One can construct all possible injective functions g where $(1, 3) \in g$ by the recipe:

(I) Let the image of $1 \in A$ under g be 3.

(II) Choose an element a of A to be the image of $2 \in A$ under g such that for all $2 \neq x \in A$, $(x, a) \notin g$ yet.

(II) Choose an element b of A to be the image of $3 \in A$ under g such that for all $3 \neq y \in A$, $(y, b) \notin g$ yet.

(II) Choose an element c of A to be the image of $4 \in A$ under g such that for all $4 \neq z \in A$, $(z, c) \notin g$ yet.

Hence we have that in the case where we require $(1, 3) \in g$, there are $1 \times 3 \times 2 \times 1 = 6$ possible ways to define $g : A \to A$.

c) How many functions $h : A \to A$ so that $f \circ h \circ f(2) = 3$?

Solution: there are $1 \times 4^3 = 4^3 = 64$ *such functions* h.

We define one recipe, which when followed will construct a unique definition of a function $h : A \to A$ guaranteeing that $f \circ h \circ f(2) = 3$. This recipe will be capable of defining all possible functions h that have this property in exactly one way.

Recipe 5.
First, we notice that $f \circ h \circ f(2) = f(h \circ f(2))$. Now, in order for $h : A \to A$ to have the property that $f \circ h \circ f(2) = 3$ we require that $h \circ f(2) = 2$ since by the definition of f, $\forall_x \in A$, $f(x) = 3$ only when $x = 2$. Furthermore, we have that

$$\begin{aligned} 2 &= h \circ f(2) && \text{As required for the desired property that } f \circ h \circ f(2) = 3. \\ &= h(f(2)) && \text{By the definition of function composition} \\ &= h(3) && \text{By the definition of } f. \end{aligned}$$

Therefore, so long as $(3, 2) \in h$, we will have that $f \circ h \circ f(2) = 3$ as required. Now, one can construct any function $h : A \to A$ such that $(3, 2) \in h$ by the steps:

(I) Let the image of $3 \in A$ under h be 2.

(II) Choose an element of A to be the image of $1 \in A$ under h.

(III) Choose an element of A to be the image of $2 \in A$ under h.

(IV) Choose an element of A to be the image of $4 \in A$ under h.

Hence we have that in the case where we require $(3, 2) \in h$, there are $1 \times 4 \times 4 \times 4 = 1 \times 4^3 = 64$ possible ways to define $h : A \to A$.

d) How many injective functions $h : A \to A$ so that $f \circ h \circ f(2) = 3$?

Solution: there are $1 \times 3 \times 2 \times 1 = 6$ *such functions* h.

We define one recipe, which when followed will construct a unique definition of a function $h : A \to A$ guaranteeing that $f \circ h \circ f(2) = 3$. This recipe will be capable of defining all possible functions h that have this property in exactly one way.

Recipe 6.
First, we notice that $f \circ h \circ f(2) = f(h \circ f(2))$. Now, in order for $h : A \to A$ to have the property that $f \circ h \circ f(2) = 3$ we require that $h \circ f(2) = 2$ since by the definition of f, $\forall_x \in A$, $f(x) = 3$ only when $x = 2$. Furthermore, we have that

$$\begin{aligned} 2 &= h \circ f(2) && \text{As required for the desired property that } f \circ h \circ f(2) = 3. \\ &= h(f(2)) && \text{By the definition of function composition} \\ &= h(3) && \text{By the definition of } f. \end{aligned}$$

Therefore, so long as $(3, 2) \in h$. we will have that $f \circ h \circ f(2) = 3$ as required. In order to ensure injectivity, we further require that no two distinct elements of the domain A of h can have the same image in the codomain A of h. Now, one can construct any injective function $h : A \to A$ with $(3, 2) \in h$ by the steps:

(I) Let the image of $3 \in A$ under h be 2.

(II) Choose an element a of A to be the image of $1 \in A$ under h such that for all $1 \neq x \in A$, $(x, a) \notin h$ yet.

(II) Choose an element b of A to be the image of $2 \in A$ under h such that for all $2 \neq y \in A$, $(y, b) \notin h$ yet.

(II) Choose an element c of A to be the image of $4 \in A$ under h such that for all $4 \neq z \in A$, $(z, c) \notin h$ yet.

Hence we have that in the case where we require $(3, 2) \in h$, there are $1 \times 3 \times 2 \times 1 = 6$ possible ways to define $h : A \to A$ where h is injective.

Problem 3

Let $\mathbb{Z}^+$ be the set of all positive integers. Let R be the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined by: For all (a, b), $(c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(a.b)R(c, d)$ if and only if $a + b \leq c + d$.

a) Prove whether or not R is reflexive, symmetric, antisymmetric, transitive.

Solution I: The relation R is reflexive.

Proof. Let $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ be an ordered pair of positive integers. Then we have that

$$\begin{gathered} x + y = x + y \\ x + y \leq x + y \end{gathered}$$

Which, by the defintion of R, means that we have $(x, y)R(x, y)$. Therefore, for any ordered pair of positive integers $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(x, y)R(x, y)$ and R is reflexive.

Solution II: The relation R is not symmetric.

Negation: There exists some $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(u, v) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $(x, y)R(u, v)$ but $(u, v)\not{R}(x, y)$.

Proof. Let $x = 1$, $y = 2$, $u = 3$, $v = 4$ such that we have $(x, y) = (1, 2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(u, v) = (3, 4) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Then we can see that

$$\begin{gathered} 3 \leq 7 \\ 1 + 2 \leq 3 + 4 \\ x + y \leq u + v \end{gathered}$$

Which implies that $(x, y)R(u, v)$ by the definition of the relation R. However, we also have that

$$\begin{gathered} 7 \not\leq 3 \\ 3 + 4 \not\leq 1 + 2 \\ u + v \not\leq x + y \end{gathered}$$

Which implies that $(u, v)\not{R}(x, y)$ by the definition of the relation R. Therefore, the relation R is not symmetric.

Solution III: The relation R is not antisymmetric.

Negation: There exists some $(x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(u,v) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $(x,y)R(u,v)$ and $(u,v)R(x,y)$ but $(x,y) \neq (u,v)$.

Proof. Let $x = 1$, $y = 2$, $u = 2$, $v = 1$ such that we have $(x,y) = (1,2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(u,v) = (2,1) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Then we have that

$$\begin{gathered}(1,2) \neq (2,1)\\(x,y) \neq (u,v).\end{gathered}$$

Furthermore, we can see that

$$\begin{gathered}3 \leq 3\\1+2 \leq 2+1\\x+y \leq u+v\end{gathered}$$

Which means we have that $(x,y)R(u,v)$ by the definition of the relation R. Similarly, we have that

$$\begin{gathered}3 \leq 3\\2+1 \leq 1+2\\u+v \leq x+y\end{gathered}$$

Which means we also have that $(u,v)R(x,y)$ by the definition of the relation R. Hence we've shown the existence of two ordered pairs $(x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(u,v) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $(x,y)R(u,v)$ and $(u,v)R(x,y)$ but $(x,y) \neq (u,v)$. Therefore, the relation R is not antisymmetric.

Solution IV: The relation R is transitive.

Proof. Let $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(e,f) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Next, suppose that both $(a,b)R(c,d)$ and $(c,d)R(e,f)$. Then we have that both

$$a+b \leq c+d \qquad \text{and} \qquad c+d \leq e+f$$

However, these two facts together imply that

$$a+b \leq e+f$$

Which means that we have that $(a,b)R(e,f)$ by the definition of the relation R. Therefore, R is transitive.

b) List all pairs $(x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $(x,y)R(2,2)$.

$$\begin{aligned}\text{I: }&(1,1)\\\text{II: }&(1,2)\\\text{III: }&(2,1)\\\text{IV: }&(2,2)\\\text{V: }&(3,1)\\\text{VI: }&(1,3)\end{aligned}$$

c) Let $n \in \mathbb{Z}^+$. How many $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $(x, y)R(n, n)$?

Solution: there are $\sum_{i=0}^{2n-1} i$ ordered pairs $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $(x, y)R(n, n)$ for some $n \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$ be some positive integer and let $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ be an ordered pair such that $(x, y)R(n, n)$. The number of such ordered pairs depends on the selected value of n. In fact, by the definition of relation R we can see that

$$\begin{aligned}(x, y)R(n, n) &\iff x + y \leq n + n \\ (x, y)R(n, n) &\iff x + y \leq 2n\end{aligned}$$

Now, since $\forall_z \in \mathbb{Z}^+$, $1 \leq z$, and both $x \in \mathbb{Z}^+$ and $y \in \mathbb{Z}^+$, we have that

$$\begin{aligned}2 &= 1 + 1 \\ &\leq x + y\end{aligned}$$

Taking together the upper and lower bound of $x + y$, we see that

$$2 \leq x + y \leq 2n$$

Which means that for a chosen value of $n \in \mathbb{Z}^+$, there are $2n - 1$ possible values of $x + y$ such that $(x, y)R(n, n)$ by relation R. Now, let $s = x + y$ be one of the sums on interval $2 \leq s \leq 2n$. For each s, we define a recipe that, when followed, will generate any ordered pair $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $a + b = s$ (guaranteeing that $(a, b)R(n, n)$ under relation R) in exactly one way. The number of such ordered pairs that can be generated by each recipe is dependent on the corresponding value of s only. Finally, by adding together the number of possible ordered pairs generated by each recipe, we arrive at the total number of ordered pairs $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $(a, b)R(n, n)$ under relation R.

Recipe 7. Let $n \in \mathbb{Z}^+$ and $2 \leq s \leq 2n$. Then we can generate any ordered pair $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $(a, b)R(n, n)$ under relation R in exactly one way by the steps:

(I) Choose $p \in \mathbb{Z}$ with $1 \leq p \leq s - 1$.
(II) Set $a = p$.
(III) Set $b = s - p$.

Then we have that $a + b = p + s - p = s$. Furthermore, since $s \leq 2n$, we have that $a + b \leq n + n$ implying that $(a, b)R(n, n)$ by the definition of relation R as required. Importantly, the number of ordered pairs (a, b) that can be generated by this recipe is $s - 1$ since that is how many choices of p there are. Since a recipe can be defined for every s where $2 \leq s \leq 2n$, and since each recipe generates $s - 1$ unique ordered pairs (a, b) with $(a, b)R(n, n)$, the total number of ordered pairs (a, b) where $(a, b)R(n, n)$ is $1 + 2 + ... + 2n - 1 = \sum_{i=1}^{2n-1} i$.