Assignment 2
STAT321

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Problem 1

a) Compute $E[X]$. Let $x_1 = -10$, $x_2 = 0$ and $x_3 = 5$. Then let $1 \leq i \leq 3$ where $i \in \mathbb{Z}$. Then

$$E[X] = \sum_{i=1}^{3} x_i P(X = x_i) = -10(\frac{3}{10}) + 0(\frac{1}{2}) + 5(\frac{1}{5}) = -3 + 1 = -2$$

Solution: $E[X] = -2$.

b) Compute $VAR[X]$.

$$VAR[X] = E[X^2] - E[X]^2 = 100(\frac{3}{10}) + 0(\frac{1}{2}) + 25(\frac{1}{5}) - (-2)^2 = 31$$

Solution: $VAR[X] = 31$.

c) Compute $E[3X]$

$$E[3X] = 3E[X] = 3(-2) = -6$$

Solution: $E[3X] = -6$.

Problem 2

a) Construct a probability distribution table for X.

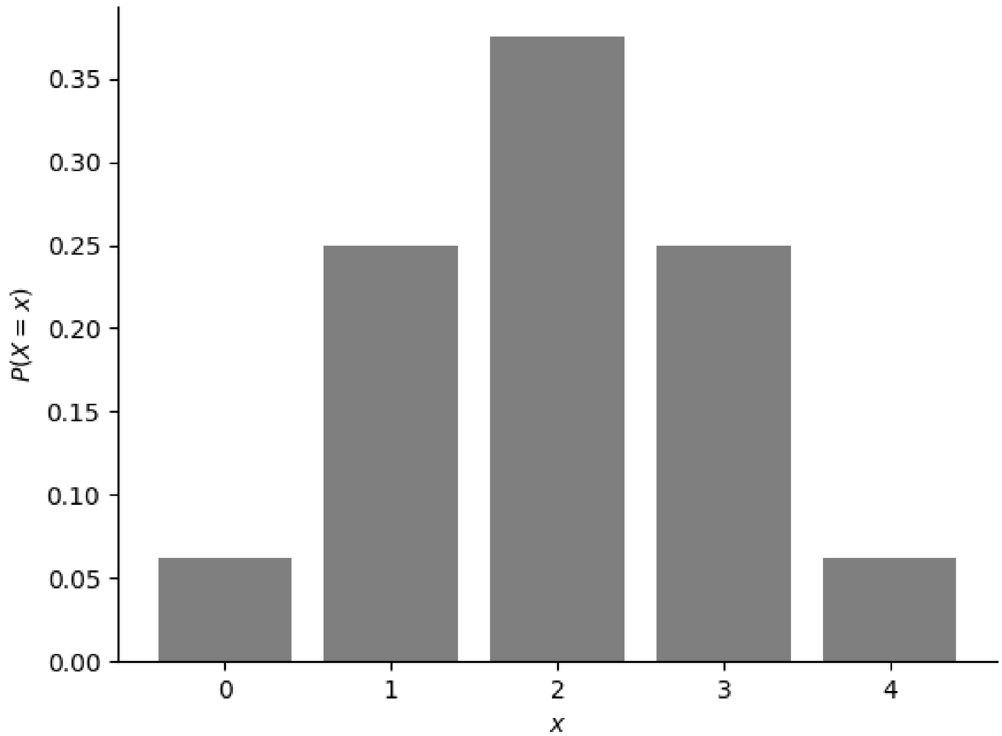
$X \sim binomial(4, \frac{1}{2})$. Therefore, we have that

$$P(X = 0) = \binom{4}{0}(\frac{1}{2})^0(1 - \frac{1}{2})^4 = \frac{1}{16}$$
$$P(X = 1) = \binom{4}{1}(\frac{1}{2})^1(1 - \frac{1}{2})^3 = \frac{1}{4}$$
$$P(X = 2) = \binom{4}{2}(\frac{1}{2})^2(1 - \frac{1}{2})^2 = \frac{3}{8}$$
$$P(X = 3) = \binom{4}{3}(\frac{1}{2})^3(1 - \frac{1}{2})^1 = \frac{1}{4}$$
$$P(X = 4) = \binom{4}{4}(\frac{1}{2})^4(1 - \frac{1}{2})^0 = \frac{1}{16}$$

So a probability distribution table for the described experiment could be

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

b) Create a probability distribution graph for X.



The probability distribution graph of random variable X is symmetric.

c) Compute $P(X \geq 2)$.

$$P(X \geq 2) = P(X=2) + P(X=3) + P(X=4) = \frac{3}{8} + \frac{1}{4} + \frac{1}{16} = \frac{11}{16}$$

Solution: $P(X \geq 2) = \frac{11}{16}$.

Problem 3

Let X be the random variable for how many defective screws are found in a box. We have that $X \sim binomial(20, \frac{1}{100})$.

a) Compute $P(X=0)$.

$$P(X=0) = \binom{20}{0}\left(\frac{1}{100}\right)^0\left(1-\frac{1}{100}\right)^{20} = (99/100)^20 = 0.81791$$

Solution: $P(X=0) = 0.81791$.

b) Compute $P(X = 1)$.

$$P(X = 1) = \binom{20}{1}\left(\frac{1}{100}\right)\left(\frac{99}{100}\right)^{19} = 0.16523$$

Solution $P(X = 1) = 0.16523$.

c) Compute $P(X > 1)$.

$P(X > 1) = 1 - P(X > 1)^C = 1 - P(X \leq 1) = 1 - (P(X = 0) + P(X = 1))$.

$$\begin{aligned} P(X > 1) &= 1 - 0.81791 - 0.16523 \qquad \text{(from part 3.a and 3.b).} \\ &= 0.01686 \end{aligned}$$

Solution: the probability that any given box of screws will be eligible for a refund is 0.0169, or 1.69%.

Problem 4

Let H_1, H_2 be the events that the first and second coin (respectively) lands heads up. Then let $P(T_1) = 1 - P(H_1) = P(H_1)^C$ and $P(T_2) = 1 - P(H_2) = P(H_2)^C$ be the probability that the first and second coin (respectively) land tails. Then $P(H_1) = 0.7$, $P(T_1) = 0.3$, $P(H_2) = 0.8$ and $P(T_2) = 0.2$. Furthermore,

$$\begin{aligned} P(X = 2) &= P(H_1 \cap H_2) = (0.7)(0.8) = 0.56 \\ P(X = 1) &= P(H_1 \cap T_2) + P(T_1 \cap H_2) = (0.7)(0.2) + (0.3)(0.8) = 0.38 \\ P(X = 0) &= P(T_1 \cap T_2) = (0.3)(0.2) = 0.06 \end{aligned}$$

a) Compute $E[X]$.

$$\begin{aligned} E[X] &= 2P(X = 2) + P(X = 1) + 0P(X = 0) \\ &= 2(0.56) + 1(0.38) + 0 \\ &= 1.5 \end{aligned}$$

Solution: $E[X] = 1.5$.

b) Compute $SD[X]$.

$$\begin{aligned} SD[X] &= \sqrt{VAR[X]} \\ &= \sqrt{E[X^2] - E[X]^2} \\ &= \sqrt{4(0.56) + 1(0.38) - \frac{9}{4}} \\ &= 0.60828 \end{aligned}$$

Solution: $SD[X] = 0.60828$.

Problem 5

Let X be the number of trials for Bob to win r rounds. Suppose that the probability of winning is $\frac{1}{6}$.

a) Let $r = 1$. Compute $P(X = 4)$.

Here we have that $X \sim neg.binomial(1, \frac{1}{6})$, so

$$P(X = 4) = \binom{3}{0}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = 0.09645$$

Solution: The probability that Bob's first win is on the fourth round is 0.09645.

b) Let $r = 2$. Compute $P(X = 3)$.

Now we have that $X \sim neg.binomial(2, \frac{1}{6})$, so

$$P(X = 3) = \binom{2}{1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = 0.04629$$

Solution: The probability that Bob's second win is on the third round is 0.04629.

Problem 6

Note that $X \sim binomial(2, \frac{2}{9})$.

a) Create a probability distribution table for X.

$$P(X = 0) = \binom{2}{0}\left(\frac{2}{9}\right)^0\left(\frac{7}{9}\right)^2 = 0.60494$$
$$P(X = 1) = \binom{2}{1}\left(\frac{2}{9}\right)^1\left(\frac{7}{9}\right)^1 = 0.34568$$
$$P(X = 2) = \binom{2}{2}\left(\frac{2}{9}\right)^2\left(\frac{7}{9}\right)^0 = 0.04938$$

x	0	1	2
$P(X = x)$	0.60494	0.34568	0.04938

b) Compute $E[X]$.

For any discrete random variable Y where $Y \sim binomial(n, p)$, we have that $E[Y] = np$. Hence

$$E[X] = 2\left(\frac{2}{9}\right) = \frac{4}{9}.$$

Solution: $E[X] = \frac{4}{9}$.

Problem 7

a) Compute $E[4Y - 2]$.

$$\begin{aligned} E[4Y - 2] &= E[4Y] + E[-2] \\ &= 4E[Y] - 2 \end{aligned}$$

$$= 4\left(120 \times \frac{3}{10}\right) - 2$$
$$= 142$$

Solution: $E[4Y - 2] = 142$.

b) Compute $VAR[4Y - 2]$**.**

$$\begin{aligned} VAR[4Y-2] &= VAR[4Y] + VAR[-2] \\ &= 16VAR[Y] \\ &= 16\left(120 \times \left(\frac{3}{10}\right)\left(\frac{7}{10}\right)\right) \\ &= 403.2 \end{aligned}$$

Solution: $VAR[X] = 403.2$.

Problem 8

Let Y be the random variable for the number of yellow M&M's in a pack of 23, $p = 0.24 = \frac{6}{25}$. Then $Y \sim binomial(23, \frac{6}{25})$.

a) Compute $P(Y > 10)$.

$$P(Y > 10) = \sum_{n=11}^{23} P(Y = n) = \sum_{n=11}^{23} \binom{23}{n} p^n (1-p)^{23-n}$$

We will compute this with the following code in R:

```
x<-11:23
p<-sum(dbinom(x, 23, 0.24))
```

Where here the variable $p = P(Y > 10)$ takes the value 0.01087513.
Solution: $P(Y > 10) = 0.01087513$.

b) Compute $P(5 \leq Y \leq 10)$.

$$P(5 \leq Y \leq 10) = \sum_{n=5}^{10} P(Y = n) = \sum_{n=5}^{10} \binom{23}{n} p^n (1-p)^{23-n}$$

We will compute this with the following code in R:

```
x<-5:10
p<-sum(dbinom(x, 23, 0.24))
```

Where here the variable $p = P(5 \leq Y \leq 10)$ takes the value 0.6674023.
Solution: $P(5 \leq Y \leq 10) = 0.6674023$.

c) Compute $P(Y = 7|5 \leq Y \leq 10)$.

By the law of total probability, we have that

$$P(Y = 7) = P(Y = 7 \cap (5 \leq Y \leq 10)) + P(Y = 7 \cap (5 \leq Y \leq 10)^C)$$

However, the events that $Y = 7$ and $(5 \leq Y \leq 10)^C$ are mutually exclusive, since Y cannot simultaneously be 7 and not in the interval $[5, 10]$. Hence, $P(Y = 7 \cap 5 \leq Y \leq 10) = 0$, so we have

$$P(Y = 7) = P(Y = y \cap 5 \leq Y \leq 10)$$

Then we can compute $P(Y = 7|5 \leq Y \leq 10)$ directly:

$$\begin{aligned}
P(Y = 7|5 \leq Y \leq 10) &= \frac{P(Y = 7 \cap 5 \leq Y \leq 10)}{P(5 \leq Y \leq 10)} \\
&= \frac{P(Y = 7)}{P(5 \leq Y \leq 10)} \\
&= \frac{\binom{23}{7} p^7 (1-p)^{16}}{\sum_{n=5}^{10} \binom{23}{n} p^n (1-p)^{23-n}} \\
&= \frac{0.1393}{0.6674023} \qquad \text{(taking the denominator from 8.b).} \\
&= 0.20872
\end{aligned}$$

Solution: $P(Y = 7|5 \leq Y \leq 10) = 0.20872$.

Problem 9

a) Let Y be the number of hit targets out of 3 trials. The probability that Bob hits the target is 0.7, so $Y \sim binomial(3, 0.7)$. Compute $P(Y = 3)$.

$$P(Y = 3) = \binom{3}{3} (0.7)^3 (0.3)^0 = 0.343$$

Solution: $P(Y = 3) = 0.343$.

b) Let $X = Y + 1$ be the random variable for the number of the trial on which Bob misses the target. Compute $E[Y]$.

First, we have that $X \sim geometric(\frac{3}{10})$. Then, we can show that:

$$\begin{aligned}
P(X \geq 4) &= 1 - P(X \geq 4)^C \\
&= 1 - P(X < 4) \\
&= 1 - P(X = 1) - P(X = 2) - P(X = 3) \\
&= 1 - \left(\left(\frac{7}{10}\right)^0 \left(\frac{3}{10}\right)^1 + \left(\frac{7}{10}\right)^1 \left(\frac{3}{10}\right)^1 + \left(\frac{7}{10}\right)^2 \left(\frac{3}{10}\right)^1 \right) \\
&= 1 - \left(\frac{3}{10}\right) - \left(\frac{21}{100}\right) - \left(\frac{147}{1000}\right) \\
&= 0.3430
\end{aligned}$$

Then, we have that

$$P(X=3) = \left(\frac{147}{1000}\right) \qquad \text{(as found above in 9.b).}$$
$$P(X=2) = \left(\frac{21}{100}\right) \qquad \text{(as found above in 9.b).}$$
$$P(X=1) = \left(\frac{3}{10}\right) \qquad \text{(as found above in 9.b).}$$

Note that $P(X \geq 4)$ is the event that Bob's first miss would occur after his third hit, i.e., after he had already hit all three shots. Then we can construct the following probability distribution table. And proceed to compute

y	0	1	2	3
x	1	2	3	≥ 4
$P(X=x)$	$\frac{3}{10}$	$\frac{21}{100}$	$\frac{147}{1000}$	0.3430

$E[Y]$ by the conventional formula:

$$E[Y] = 0P(Y=0) + 1P(Y=1) + 2P(Y=2) + 3P(Y=3) = \frac{21}{100} + \frac{294}{1000} + 1.0290 = 1.5330$$

Solution: We'd expect Bob to hit an average of 1.5330 shots in the long run.