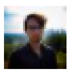
Final Reflection: Proof Writing
MATH311

Connor Braun

November 15, 2021

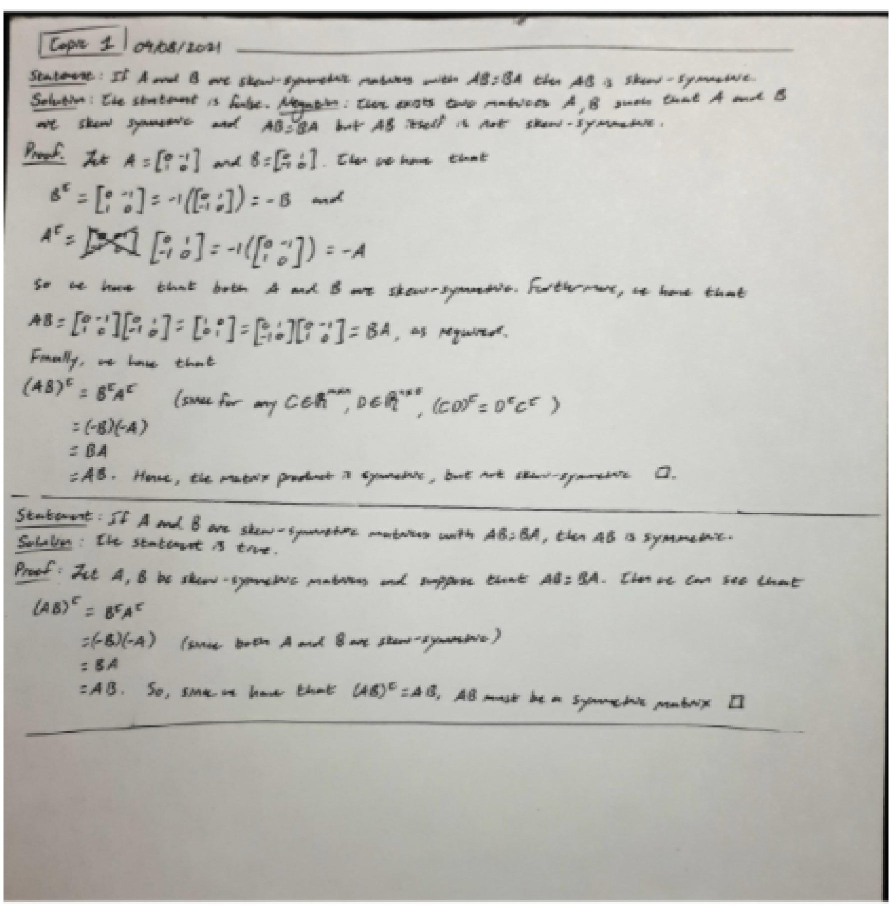
Topic 1



Connor Braun - Topic 1 Proof

Connor Braun posted Sep 8, 2021 9:56 PM • 32 Words ★ Subscribed

Scroll down to see my proof attempt as an embedded image. I've also attached a pdf document of the same image in case the embedded file is not visible for some reason.



Topic 1 09/08/2021

Statement: If A and B are skew-symmetric matrices with $AB=BA$ then AB is skew-symmetric.
Solution: The statement is false. Negation: there exists two matrices A, B such that A and B are skew symmetric and $AB=BA$ but AB itself is not skew-symmetric.

Proof. Let $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$. Then we have that

$$B^t=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=-1\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)=-B \quad \text{and}$$

$$A^t=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=-1\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)=-A$$

So we have that both A and B are skew-symmetric. Furthermore, we have that

$$AB=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=BA, \text{ as required.}$$

Finally, we have that

$$\begin{aligned}(AB)^t &= B^tA^t \quad (\text{since for any } C\in\mathbb{R}^{m\times n}, D\in\mathbb{R}^{n\times p}, (CD)^t=D^tC^t)\\ &=(-B)(-A)\\ &=BA\\ &=AB.\end{aligned}$$

Hence, the matrix product is symmetric, but not skew-symmetric □.

Statement: If A and B are skew-symmetric matrices with $AB=BA$, then AB is symmetric.
Solution: The statement is true.

Proof: Let A, B be skew-symmetric matrices and suppose that $AB=BA$. Then we can see that

$$\begin{aligned}(AB)^t &= B^tA^t\\ &=(-B)(-A) \quad (\text{since both } A \text{ and } B \text{ are skew-symmetric})\\ &=BA\\ &=AB.\end{aligned}$$

So, since we have that $(AB)^t=AB$, AB must be a symmetric matrix □

Connor Braun

September 9 at 9:26 AM • **58** Words

The negation of the original statement (i.e. what you're trying to prove) requires that AB=BA. Your choices for matrices A, B do actually satisfy this, but you don't explicitly show it.

Since you showed that there exist two matrices A, B which are skew-symmetric but their product AB is not, you appear to have solved the wrong problem.

Connor Braun

September 9 at 9:36 AM • **103** Words

Your proof is completely correct (and your handwriting is frighteningly neat) as far as I can tell, but I just noticed that your negation of the original statement appears incorrect.

Since the original statement was of the form 'if c and d then e' which is equivalent to 'for all c and d, e' the negation should have been 'there exists c and d, but not e'. Instead you kept the if-then structure of the statement, which implies a universal quantifier and is not the negation.

I feel as though this would be categorized as a 'false statement' but I'm not entirely sure.

Topic 1 reflection

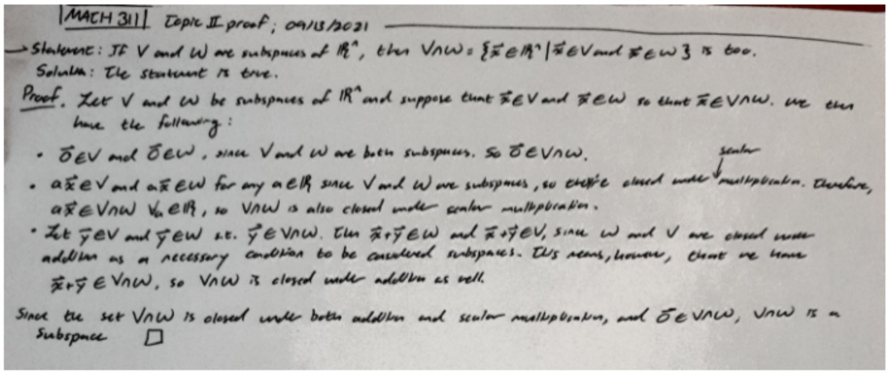
Looking back now, I'm happy with this proof and believe it to be correct, but one thing does stand out. As a matter of trying to form good proof writing habits, I try to use plain English since it's very easy to overdo the symbolic notation and muddy the readability of the proof. However, this approach benefits from 'good writing', which is a major goal of mine – even though it's a bit nebulous. Despite this, one comment from a classmate was essentially to reduce my usage of the phrase: 'we have that...', which is a good suggestion since I habitually use this phrase often enough to detract from the prose – 5 times in just this one proof! What's worse is that I *still* do this, and I haven't actually been mindful of it until just now while writing this reflection. I'll try to vary my language going forward, and also take this as an opportunity to work on forming a cohesive narrative; at least when writing longer proofs.

Topic 2

Connor Braun - Topic 2 Proof

Connor Braun posted Sep 13, 2021 7:25 PM • **11** Words ★ **Subscribed**

Addendum: the last line should read "...is a subspace of R^n".



MATH 311 Topic II proof; 09/13/2021

→ Statement: If V and W are subspaces of $\mathbb{R}^n$, then $V \cap W = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \in V \text{ and } \vec{x} \in W\}$ is too.

Solution: The statement is true.

Proof. Let V and W be subspaces of $\mathbb{R}^n$ and suppose that $\vec{x} \in V$ and $\vec{x} \in W$ so that $\vec{x} \in V \cap W$. We then have the following:

- $\vec{0} \in V$ and $\vec{0} \in W$, since V and W are both subspaces. So $\vec{0} \in V \cap W$.
- $a\vec{x} \in V$ and $a\vec{x} \in W$ for any $a \in \mathbb{R}$ since V and W are subspaces, so they're closed under scalar multiplication. Therefore, $a\vec{x} \in V \cap W$ $\forall a \in \mathbb{R}$, so $V \cap W$ is also closed under scalar multiplication.
- Let $\vec{y} \in V$ and $\vec{y} \in W$ s.t. $\vec{y} \in V \cap W$. Then $\vec{x} + \vec{y} \in W$ and $\vec{x} + \vec{y} \in V$, since W and V are closed under addition as a necessary condition to be considered subspaces. This means, however, that we have $\vec{x} + \vec{y} \in V \cap W$, so $V \cap W$ is closed under addition as well.

Since the set $V \cap W$ is closed under both addition and scalar multiplication, and $\vec{0} \in V \cap W$, $V \cap W$ is a subspace □

Connor Braun

September 18 at 6:02 PM • **76** Words

Nicely done, I for one am partial to lots of plain English over inundating a proof with mathematical notation.

I would like to point out that in your last argument you write that 'scalar multiplication is closed', however the correct way to state this would be to say that both V and W are closed under scalar multiplication. Mostly a nitpick, but its the set that's closed (under some operation) not the operation itself.

Nice one!

Connor Braun

September 18 at 6:16 PM • **122** Words

Hiya Kate,

Just a couple suggestions for this one:

1. Your point (1) should probably be supported by some argument rather than simply asserted.
2. Your point (2) is similar, in the sense that <v1 + w1, v2 + w2, ..., vn + wn> is true, but there's no argument made here as to *why* it's true.
3. Same as for (2), your assertion in (3) that the vector <kx1, kx2, ..., kxn> is in the intersection isn't supported by any mathematical argument... although it is true.

By now you've most likely seen some of the other proofs; the solution to each of my suggestions would be to use the fact that both V and W are subspaces explicitly.

That's all! Hope this was helpful.

Topic 2 reflection

I remember this proof feeling straightforward at the time of my writing it, and even now it strikes me as concise, clear and correct. One suggestion from a peer was to improve the prose by introducing variables as I use them rather than at the outset of the proof. Upon reflection I actually disagree with their comment, since then if a reader wanted to revisit the definition of some variable they would have to scan the body of the writing rather than simply direct their attention to the top. My colleague's suggestion isn't without merit; introducing variables as they're used means that the reader wouldn't have to disrupt their reading to revisit the variable's definition elsewhere. I choose to disregard their suggestion on the grounds that reading proofs usually consists of a lot of pausing and thinking, so there is little benefit to scattering my variable definitions so that the more advanced reader needn't lift their eyes from the page.

Topic 3

Connor Braun - Topic 3 Proof

Connor Braun posted Sep 22, 2021 4:31 PM • **0** Words Subscribed

MATH 311 - 09/22/2021 - Topic III

Theorem: Let U be a subspace of $\mathbb{R}^n$ of dimension m. Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$. Then S is linearly independent iff S spans U.

Proof. Let U be a subspace of $\mathbb{R}^n$ of dimension m and $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$. To complete the proof, we proceed by considering the statements: "if S is linearly independent, then S spans U" and it's converse separately.

Lemma. Suppose that S is linearly independent, and let $\vec{y} \in \text{span}(S)$. Then $\vec{y} = t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_m\vec{v}_m$ for some $t_1, t_2, \dots, t_m \in \mathbb{R}$. Noticing that $\vec{y}$ is given here as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in U$, and U (as a subspace) is closed under addition and scalar multiplication, we can conclude that $\vec{y} \in U$ as well. Next, let $\vec{t} = [t_1\ t_2 \dots t_m]^t$ and $V = [\vec{v}_1\ \vec{v}_2 \dots \vec{v}_m]$ then the above can be written as $V\vec{t} = \vec{y}$. However, since S is linearly independent, we have that $V\vec{t} = \vec{0}_n$ has only the trivial solution: $\vec{t} = \vec{0}_m$. This means that $V\vec{t} = \vec{y}$ is consistent $\forall \vec{y} \in U$. Now let $\vec{w} \in U$. Since $V\vec{t} = \vec{y}$ consistent, $\forall \vec{y} \in U$, $\exists \vec{a} \in \mathbb{R}^n$ such that $V\vec{a} = \vec{w}$, so if $\vec{a} = [a_1\ a_2 \dots a_m]^t$, then we have $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m$. Since $a_1, a_2, \dots, a_m \in \mathbb{R}$, $\vec{w} \in \text{span}(S)$. Then we have that $\text{span}(S) \subseteq U$ and $U \subseteq \text{span}(S)$, so $U = \text{span}(S)$, i.e., U is spanned by S.

Lemma. Suppose that S spans U. Then $\text{span}(S) = U$. Since U is a subspace, it can be written as the span of some set of vectors, and since U is m-dimensional, it is spanned by a basis of exactly m vectors. Furthermore, these m vectors are linearly independent by definition of a basis. Let this set of vectors be called M. Then we have

$$\text{span}(S) = U = \text{span}(M) \Rightarrow \text{span}(S) = \text{span}(M)$$

and since $|S| = m = |M|$ and M is linearly independent, it follows that S is linearly independent as well

Now, since we have that if S is linearly independent then S spans U and if S spans U then S is linearly independent, we have that S spans U iff S is linearly independent $\square$.

Connor Braun

September 27 at 10:45 PM • **142** Words

Heya Bilan,

Just a couple suggestions from me on this one.

1. To prove a biconditional the general strategy is to take the proof on in two steps. First, suppose one side of the biconditional, then show that the other side must also be true. Repeat this process by first supposing the other side of the biconditional and showing that the unsupposed side is true.
2. Near the end of your proof you assert that S is linearly dependent, but this is unsupported. You showed that T would be linearly dependent (if the new vector v is in U), but this says nothing about S.
3. At the very end, you write "S being linearly dependent means that the above equation only has the trivial solution". This would be true if S were linearly *independent*.

Hopefully this has been at least somewhat constructive.

C. Braun

Connor Braun

September 27 at 10:32 PM • **147** Words

Hiya Evan,

Just a few comments on this one.

1. Your initial suppositions were not exactly clear to me -- that is, I'm having a hard time determining when your preamble ends and your proof begins. Try using phrases like: 'suppose that' or 'let ____ be _____'. When you instead say 'we know that S spans U' it sounds like you're making an assertion rather than supposing one side of a conditional.
2. Note that S is a set of vectors in this question, so rank(S) isn't defined. You could construct a matrix A where the ith column of A is the ith vector of S and then proceed to talk about rank.
3. Note that when you have >2 vectors in a set (in R^n, n>1), you can have linear dependence where none of the vectors are parallel.

Hope that at least some of this was constructive.

C. Braun

Topic 3 reflection

This proof took me the longest to write of any of the 5 I chose to complete. Furthermore, it's actually incorrect since I assume that linear independence of the columns of a matrix make it invertible. This holds for square matrices, but the suppositions here do not preclude the possibility of a nonsquare matrix, for which this result does not hold. Perhaps more importantly, however, was the approach I took and the reason for it. This proof is a bit tricky, and my general inclination when dealing with tricky concepts in linear algebra is to try and formulate the problem as a system of linear equations, with which I'm more comfortable. The problem I've found with this approach is that it is not a good default strategy since it often does not lend any clarity to the problem at hand and can be quite verbose. I recognized this around the time of submitting this proof, and have since taken any inclination to form a system of equations as an indication that I had ought review the relevant concepts for a more direct solution.

Topic 4

Connor Braun - Topic 4 Proof

Connor Braun posted Sep 27, 2021 2:13 PM • **0** Words Subscribed

Proof. Let A and B be square matrices and suppose that $A \sim B$. Suppose also that A^2 is diagonalizable. By definition of similarity of matrices, we have

$$A = PBP^{-1}$$

for some invertible matrix P. Then we have that

$$AA = PBP^{-1}PBP^{-1} = PBIBP^{-1} = PBBP^{-1}$$

Since $AA = A^2$ is diagonalizable, we have that for diagonal matrix D and invertible matrix V

$$AA = VDV^{-1}$$

Now we can show that

$$AA = VDV^{-1} = PBBP^{-1} \Rightarrow BB = P^{-1}VDV^{-1}P.$$

Since $(P^{-1}V)^{-1} = V^{-1}P$, we can define F to be a new invertible matrix where $F = P^{-1}V$. Then finally we have that

$$BB = FDF^{-1}$$

So $BB = B^2 \sim D$, and since D is diagonal we call $BB = B^2$ diagonalizable $\square$.

Connor Braun

October 5 at 4:32 PM • **58** Words

Hiya Dien,

Proof looks great (both technically and aesthetically -- your handwriting rocks). My one comment is that the matrix D that you introduce for A^2's diagonalization needn't be raised to the second power. That is, D^1 would've been perfectly fine.

Not technically incorrect (since multiplying diagonal matrices results in a diagonal matrix) but quite unnecessary.

C. Braun

Connor Braun

October 5 at 4:43 PM • **76** Words

Hiya Marko,

Just a few comments for this one:

1. You inexplicably change the diagonalization of A^2 from PDP^-1 to PD^2P^-1 (equation 2).

2. You didn't take the square root of the RHS when finding a diagonalization for A (equation 2) so the implication doesn't hold.

3. You don't explicitly introduce matrix B anywhere.

4. You don't take the square of the RHS when showing the diagonalization of B^2 (equation 8).

Hope this helps.

Connor B.

Topic 4 reflection

I like this proof and think it's correct, but there are two salient points I've gleaned from my reflection on it. First is that while I thought some symbolic manipulations were clear enough to afford me an implication arrow, a colleague actually found that this interrupted their interpretation of my work. This makes for a good reminder on how biased the writer can be as to the readability of a proof – next time I think that an implication arrow might suffice I'll default to a one sentence explanation instead.

Secondly, I actually collaborated with a colleage in person to discuss our approach for this proof. This made for a very productive conversation, and I remember we even proved some only tangentially-related statements out of curiosity (for example, I believe we proved that all diagonal and symmetric matrices are diagonalizable). In light of the pandemic (and the apparent reclusiveness of math majors in general) I rarely get to collaborate on anything in mathematics. I loved this experience and have made more of an effort to rally classmates in all of my math classes since then (within the constraints of academic integrity, of course).

Topic 5

Connor Braun - Topic 5 Proof

Connor Braun posted Oct 6, 2021 9:56 AM • **0** Words Subscribed

Proof. Let V be a vector space and $S \subset V$ be a finite subset of V. Then $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $\text{span}(S) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n \mid a_1, \ldots, a_n \in \mathbb{R}\}$. Then $\vec{0} \in S$, since $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_n$, and $0 \in \mathbb{R}$. Now suppose that $\vec{\mu}, \vec{\omega} \in \text{span}(S)$ and $k \in \mathbb{R}$. Then span(S) can be shown to have the following properties:

I) ~~Closure~~ Closure under addition. $\cancel{\vec{\mu} = b_1}$ $\vec{\mu} = b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$ for some $b_1, \ldots, b_n \in \mathbb{R}$ and $\vec{\omega} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$ for some $c_1, \ldots, c_n \in \mathbb{R}$. Then $\vec{\mu} + \vec{\omega} = (b_1+c_1)\vec{v}_1 + (b_2+c_2)\vec{v}_2 + \ldots + (b_n+c_n)\vec{v}_n$ where since $b_i + c_i \in \mathbb{R}$, $1 \le i \le n$, $\vec{\mu} + \vec{\omega} \in \text{span}(S)$.

II) Closure under scalar multiplication. $\cancel{k\vec{\mu} = k}$ Letting $\vec{\mu} = b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$ again, for some $b_1, \ldots, b_n \in \mathbb{R}$, $k\vec{\mu} = kb_1\vec{v}_1 + kb_2\vec{v}_2 + \ldots + kb_n\vec{v}_n$ where $kb_i \in \mathbb{R}$, $1 \le i \le n$, so $k\vec{\mu} \in \text{span}(S)$.

Therefore, since $\vec{0} \in \text{span}(S)$, and span(S) is closed under addition/scalar multiplication, and $\text{span}(S) \subseteq V$, span(S) is a subspace of vector space V. $\square$

Connor Braun

October 7 at 8:58 AM • **75** Words

Hiya Jayden,

Just a couple comments for this one:

1. Note that you introduce **v** E V, where **v** should be an element of span(S). Even if you had made **v** E span(S), you cannot simply say **v***0=**0** so **0** E span(S), since you haven't established that span(S) is closed under multiplication.
2. Minor, but you do not define vectors **v1,...,vn**. They should be elements of S, but this is left unspecified.

Hope this helps,

C. Braun

Connor Braun

October 7 at 9:09 AM • **120** Words

Hiya Usama,

Couple minor comments for this one:

1. Your use of curly brackets for the linear combinations makes them out to be sets with one element. This would make assertions like {a1**v1** +...+an**vn**} E span(S) untrue, since while span(S) *is itself* a set, it does not contain any sets. Parentheses would probably be clearer.
2. It's hard to distinguish the zero vector from the real number zero in your proof. Consider using a vector arrow or something.
3. Your implication arrow that asserts that **p**+**q** E span(S) could probably use some justification, namely that since each coefficient (a2+b2) is real, we have a linear combination of vectors in S so **p**+**q** is in span(S).

That's about it. Hope this helps,

C. Braun

Topic 5 reflection

Upon reflection, I like this proof and think it's both concise and correct. One problem I recall encountering was that I actually scrapped and rewrote the thing some two or three times. The reason for this was that the prompt struck me as fairly straightforward, so instead of planning my approach on a scratchpad beforehand I simply dived into writing the final draft. Despite rewriting it a few times, there are many little mistakes which have been crossed out and corrected in the margins, which makes for a messy read. For me this is a dead giveaway that I didn't think ahead before writing (and a common feature of every math test I've ever written) and is a good reminder to not proceed so boldly – time permitting.