Assignment 1
STAT321

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Problem 1

Let $P(A) = 0.83$, $P(B|A) = 0.22$ and $P(A^c \cap B^c) = 0.05$.

a) Find $P(A \cap B)$.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow 0.22 = \frac{P(B \cap A)}{0.83} \Rightarrow P(B \cap A) = (0.22)(0.83) = 0.1826$$

Solution: $P(A \cap B) = 0.1826$.

b) Find $P(B)$.

$$\begin{aligned} P(A^c \cap B^c) = P(A \cup B)^c = 0.05 &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cup B)) \\ &= 1 - 0.83 - P(B) + 0.1826 \qquad (P(A \cap B) = 0.1826 \text{ from 1.a.}) \\ 0.05 &= 0.3526 - P(B) \\ P(B) &= 0.3026 \end{aligned}$$

Solution: $P(B) = 0.3026$.

c) Find $P(B|A^c)$.

$$\begin{aligned} P(B|A^c) &= \frac{P(B \cap A^c)}{P(A^c)} \\ &= \frac{P(B) - P(A \cap B)}{1 - P(A)} \\ &= \frac{0.3026 - 0.1826}{1 - 0.83} \qquad (P(B) = 0.3026,\ P(A \cap B) = 0.1826 \text{ from 1.a, 1.b respectively.}) \\ &= \frac{0.12}{0.17} \\ P(B|A^c) &= 0.7059 \end{aligned}$$

Solution: $P(B|A^c) = 0.7059$.

Problem 2

Let H denote the outcome of the coin flip being heads, T denoting tails. Let $1, 2, 3, 4, 5, 6$ indicate the outcome of the dice roll, so that $H3 = 3H$ indicates an outcome of 3 on the die roll with a simultaneous heads outcome on the coin flip.

a) List the sample space S for the experiment described.

Solution: $S = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\}$, $|S| = 12$.

b) Let E indicate that the number returned by the dice roll was even. Let $T\prime$ be the event that the coin flip came up tails. Find $P(T\prime|E)$.

First, we have that

$$E = \{2H, 2T, 4H, 4T, 6H, 6T\}$$

so $|E| = 6$. Hence, we have

$$P(E) = \frac{|E|}{|S|} = \frac{6}{12} = \frac{1}{2}.$$

Next, we have that

$$T\prime = \{1T, 2T, 3T, 4T, 5T, 6T\}$$

Taking the intersection of $T\prime$ and E

$$T\prime \cap E = \{2T, 4T, 6T\}$$

Since only $2T, 4T, 6T$ are in both E and $T\prime$. The probability of the intersection is then

$$P(E \cap T\prime) = \frac{|E \cap T\prime|}{|S|} = \frac{3}{12} = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(E)\frac{6}{12} = P(E)\frac{|T\prime|}{|S|} = P(E)P(T\prime).$$

Allowing us to conclude that E and $T\prime$ are independent events. Finally, we have that

$$P(T\prime|E) = \frac{P(T\prime \cap E)}{P(E)} = \frac{P(T\prime)P(E)}{P(E)} = P(T\prime) = \frac{1}{2}$$

Solution: $P(T\prime|E) = \frac{1}{2}$.

c) Let $A = \{6H, 6T\}$ be the event that the die roll yields a 6 and $B = T\prime$ from 2.b. Show that A and B are independent.

First, we have that $A \cap B = \{6T\}$ since only $6T \in A$ and $6T \in B$. Then $|A \cap B| = 1$ and it follows that

$$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{1}{12} = \frac{1}{2} \times \frac{1}{6} = \frac{|A|}{|S|}\frac{|B|}{|S|} = P(A)P(B)$$

Solution: since $P(A \cap B) = P(A)P(B)$ we have that A and B are independent events.

Problem 3

Let M indicate the event that a randomly sampled person is male and F the event that a randomly sampled person is female. Then $P(F) = 0.53$, $P(M) = P(F^c) = 1 - P(F) = 1 - 0.53 = 0.47$. Suppose C is the event that a randomly sampled person is colourblind, and that we have $P(C|M) = 0.08$ and $P(C|F) = 0.01$.

a) Find $P(C)$.

First, we compute $P(C \cap M) = P(C \cap F^c)$.

$$P(C|M) = \frac{P(C \cap M)}{P(M)} \Rightarrow 0.08 = \frac{P(C \cap M)}{0.47} \Rightarrow P(C \cap M) = P(C \cap F^c) = (0.08)(0.47) = 0.0376$$

Next, we compute $P(C \cap F)$.

$$P(C|F) = \frac{P(C \cap F)}{P(F)} \Rightarrow 0.01 = \frac{P(C \cap F)}{0.53} \Rightarrow P(C \cap F) = (0.01)(0.53) = 0.0053$$

Finally, by the law of total probability we arrive at the solution.

$$P(C) = P(C \cap F) + P(C \cap F^c) = 0.0053 + 0.0376 = 0.0429$$

Solution: $P(C) = 0.0429$.

b) Find $P(M \cup C)$.

$$\begin{aligned} P(M \cup C) &= P(M) + P(C) - P(M \cap C) \\ &= 0.47 + 0.0429 - 0.0376 \qquad \text{(from 3.a.)} \\ P(M \cup C) &= 0.4753 \end{aligned}$$

Solution: $P(M \cup C) = 0.4753$.

c) Find $P(F|C^c)$.

By the law of total probability, we have that $P(F) = P(F \cap C) + P(F \cap C^c)$ so $P(F \cap C^c) = P(F) - P(F \cap C) = 0.53 - 0.0053 = 0.5247$. Then we have that

$$\begin{aligned} P(F|C^c) &= \frac{P(F \cap C^c)}{P(C^c)} \\ &= \frac{0.5247}{1 - 0.0429} \\ &= \frac{0.5247}{0.9571} \\ P(F|C^c) &\approx 0.5482 \qquad \text{(rounding to 4 decimal places.)} \end{aligned}$$

Solution: $P(F|C^c) \approx 0.5482$.

Problem 4

Let S be the sample space containing all possible outcomes when five balls are randomly drawn from the urn without replacement.

a) Let B_5 be the event that all five balls drawn are blue. Find $P(B_5)$.

$$P(B_5) = \frac{|B_5|}{|S|} = \frac{\binom{8}{5}}{\binom{16}{5}} = \frac{8!}{5!3!} \times \frac{5!11!}{16!} = \frac{1}{78} \approx 0.0128 \qquad \text{(rounding to 4 decimal places.)}$$

Solution: $P(B_5) \approx 0.0128$.

b) Let R_5 be the event that all five balls drawn are red. Find $P(R_5)$.

$$P(R_5) = \frac{|R_5|}{|S|} = \frac{\binom{3}{5}}{\binom{16}{5}} = \frac{0 \times 5!11!}{16!} = 0$$

Solution: $P(R_5) = 0$.

c) Let $R_1G_2B_2$ be a single event where one of the five balls drawn are red, two are green and two are blue. Find $P(R_1G_2B_2)$.

$$\begin{aligned} P(R_1G_2B_2) &= \frac{|R_1G_2B_2|}{|S|} \\ &= \frac{\binom{3}{1}\binom{5}{2}\binom{8}{2}}{\binom{16}{5}} \\ &= \frac{3!5!8!}{1!2!2!(3-1)!(5-2)!(8-2)!} \times \frac{5!11!}{16!} \\ &= \frac{5}{26} \end{aligned}$$

$$P(R_1G_2B_2) \approx 0.1923 \qquad \text{(rounding to 4 decimal places.)}$$

Solution: $P(R_1G_2B_2) \approx 0.1923$.

d) Let R_2B_3 be the event that two of five balls drawn are red and three are blue. Then let R_3B_2 be the event that three of five balls drawn are red and two are blue. Find $P(R_2B_3 \cup R_3B_2)$.

Let $x \in R_2B_3$ be the outcome of an experiment. Then exactly two balls drawn were red, so it is not the case that exactly three balls drawn were red. Therefore, $x \notin R_3B_2$. Furthermore, there are no experimental outcomes which are in both event R_2B_3 and R_3B_2, so $R_2B_3 \cap R_3B_2 = \emptyset$. Then we compute $P(R_2B_3 \cup R_3B_2)$ directly.

$$\begin{aligned}
P(R_2B_3 \cup R_3B_2) &= P(R_2B_3) + P(R_3B_2) - P(R_2B_3 \cap R_3B_2) \\
&= P(R_2B_3) + P(R_3B_2) - P(\emptyset) \\
&= \frac{|R_2B_3|}{|S|} + \frac{|R_3B_2|}{|S|} - 0 \\
&= \frac{\binom{3}{2}\binom{8}{3}}{\binom{16}{5}} + \frac{\binom{3}{3}\binom{8}{2}}{\binom{16}{5}} \\
&= \frac{3!8!}{2!3!5!} \times \frac{5!11!}{16!} + \frac{3!8!}{3!2!6!} \times \frac{5!11!}{16!} \\
&= \frac{1}{4368}(168 + 28) \\
P(R_2B_3 \cup R_3B_2) &= \frac{7}{156} \approx 0.0449 \qquad \text{(rounding to 4 decimal places.)}
\end{aligned}$$

Solution: $P(R_2B_3 \cup R_3B_2) = 0.0449$.

e) Let ξ be the event that at most four of five balls drawn are blue. Find $P(\xi)$.

Noticing that ξ^c is the event that all five of five balls drawn are blue, we have that $\xi^c = B_5$ (from 4.a). Then we compute $P(\xi)$ directly.

$$P(\xi) = 1 - P(\xi^c) = 1 - P(B_5) = 1 - 0.0128 = 0.9872.$$

Solution: $P(\xi) = 0.9872$.

Problem 5

a) Let S be the sample space containing all permutations of the letters in 'ALBERTA', treating the A's as distinct. Find $|S|$.

$$|S| = P_7^7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = 7! = 5040.$$

Solution: $|S| = 5040$.

b) Now let S be the sample space containing all permutations of the letters in 'ALBERTA', treating the A's as indistinct. Find $|S|$.

By the multinomial theorem, we compute $|S|$ directly.

$$|S| = \binom{7}{2,1,1,1,1,1} = \frac{7!}{2!} = 2520.$$

Solution: $|S| = 2520$.

Problem 6

Although the four-sided die are rolled simultaneously, suppose we keep track of which is which, such that we are able to identify outcome of dice roll one by x and dice roll two by y, where $x, y \in \mathbb{Z}$ and $1 \leq x \leq 4$, $1 \leq y \leq 4$.

a) Let $z = xy$ be the product of the two dice rolls, and z_E be the event that z is even. Find $P(z_E)$.

If z is even, then $z = 2k$ for some $k \in \mathbb{Z}$, which further implies that $xy = 2k$. Therefore, either $2|x$ or $2|y$, so both x and y are either 2 or 4. Let x_E be the event that x is even, and y_E be the event that y is even. We begin be computing $P(x_E \cap y_E)$.

$$P(x_E \cap y_E) = \frac{\binom{2}{1}\binom{2}{1}}{\binom{4}{1}\binom{4}{1}} = \frac{2 \times 2}{4 \times 4} = \frac{4}{16} = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = \frac{|\{2,4\}|}{|\{1,2,3,4\}|} \times \frac{|\{2,4\}|}{|\{1,2,3,4\}|} = P(x_E)P(y_E)$$

Hence we have that x_E and y_E are independent events. As defined above, $z_E = (x_E \cup y_E)$, so we now compute $P(z_E)$ directly.

$$P(z_E) = P(x_E \cup y_E) = P(x_E) + P(y_E) - P(x_E \cap y_E) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} = 1 - \frac{1}{4} = \frac{3}{4}$$

Solution: $P(z_E) = 0.75$.

b) Let $z_{<5}$ be the event that z is less than 5 and $z_E^c = 1 - z_E = z_O$ be the event that z is odd. Find $P(z_O \cup z_{<5})$.

Let the ordered pair (x, y) be one possible the outcome of an experiment and S_z the sample space of z. Then we can list S_z as

$$S_z = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$$

with $|S_z| = 16$. For each element $a = (x, y) \in S_z$, we compute $z = xy$ as the product of the two members of the ordered pair. S_z can then be relisted as

$$S_z = \{1,2,3,4,2,4,6,8,3,6,9,12,4,8,12,16\}.$$

Furthermore, we have

$$z_O = \{1,3,3,9\}$$

with $|z_O| = 4$ and

$$z_{<5} = \{1,2,3,4,2,4,3,4\}$$

where $|z_{<5}| = 8$. Then we find that

$$z_{<5} \cap z_O = \{1,3,3\}$$

since $z = 1,3,3$ are the outcomes where z is both less than 5 and odd, so $|z_O \cap z_{<5}| = 3$. Then we compute $P(z_O \cup z_{<5})$ directly.

$$\begin{aligned} P(z_O \cup z_{<5}) &= P(z_O) + P(z_{<5}) - P(z_O \cap z_{<5}) \\ &= \frac{|z_O|}{|S_z|} + \frac{|z_{<5}|}{|S_z|} - \frac{|z_O \cap z_{<5}|}{|S_z|} \\ &= \frac{4}{16} + \frac{8}{16} - \frac{3}{16} \\ &= \frac{9}{16} \end{aligned}$$

Solution: $P(z_O \cup z_{<5}) = 0.5625$.

c) Find $P(z_E \cap z_{<5})$.

$$z_E = \{2, 4, 2, 4, 6, 8, 6, 12, 4, 8, 12, 16\}$$

so we have that

$$z_E \cap z_{<5} = \{2, 4, 2, 4, 4\}$$

since $2, 4, 2, 4, 4 \in z_E$ and $2, 4, 2, 4, 4 \in z_{<5}$. Then we compute $P(z_E \cap z_{<5})$ directly.

$$P(z_E \cap z_{<5}) = \frac{|z_E \cap z_{<5}|}{|S_z|} = \frac{5}{16}$$

Solution: $P(z_E \cap z_{<5}) = 0.3125$.

Problem 7

Let A be the event that Alice hits the target, and B be the event that Bonnie hits the target. Then $P(A) = x$ and $P(B) = y$. Also suppose that H is the event that the target is hit during an experiment.

a) Find $P(A \cap B|H)$.

First, by the law of total probability we find that

$$P(A \cap B) = P(A \cap B \cap H) + P(A \cap B \cap H^c).$$

However, $(A \cap B \cap H^c) = \emptyset$, since there is no experimental outcome where Alice and Bonnie both hit the target, but the target does not get hit. Hence,

$$\begin{aligned} P(A \cap B) &= P(A \cap B \cap H) + P(\emptyset) \\ P(A)P(B) &= P(A \cap B \cap H) \qquad \text{(since } A \text{ and } B \text{ are independent.)} \\ xy &= P(A \cap B \cap H) \end{aligned}$$

Next, we compute $P(H)$. Since H is the event that the target gets hit, $H = A \cup B$ and $P(H) = P(A \cup B)$.

$$\begin{aligned} P(H) = P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= x + y - P(A)P(B) \qquad \text{(since A and B are independent.)} \\ P(H) &= x + y - xy \end{aligned}$$

Finally, we compute $P(A \cap B|H)$ directly.

$$P(A \cap B|H) = \frac{P(A \cap B \cap H)}{P(H)} = \frac{xy}{x + y - xy}$$

Solution: $P(A \cap B|H) = \frac{xy}{x+y-xy}$.

b) Find $P(B|H)$.

First, by the law of total probability we find that

$$P(B) = P(B \cap H) + P(B \cap H^c).$$

However, $(B \cap H^c) = \emptyset$, since there is no experimental outcome where Bonnie hits the target, but the target does not get hit. Hence,

$$\begin{aligned} P(B) &= P(B \cap H) + P(\emptyset) \\ P(B) &= P(B \cap H) \end{aligned}$$

$$y = P(B \cap H)$$

Since $P(H) = x + y - xy$ from 7.a, we can compute $P(B|H)$ directly.

$$P(B|H) = \frac{P(B \cap H)}{P(H)} = \frac{y}{x + y - xy}$$

Solution: $P(B|H) = \frac{y}{x+y-xy}$.

Problem 8

Let 321_s be the event that a randomly sampled student from the STAT321 spring class was in the faculty of science. Let 321_s^c be the event that a randomly sampled student from the STAT321 spring class was not in the facult of science. Let 323 be the event that a student who took STAT321 went on to take STAT323 that summer. We have that $P(323|321_s) = 0.48$, $P(323|321_s^c) = 0.22$, $P(321_s) = 0.65$.

a) Find $P(323 \cap 321_s)$.

We compute $P(323 \cap 321_s)$ directly.

$$P(323|321_s) = \frac{P(323 \cap 321_s)}{P(321_s)} \Rightarrow P(323 \cap 321_s) = P(323|321_s)P(321_s) = (0.48)(0.65) = 0.312$$

Solution: $P(323 \cap 321_s) = 0.312$ is the probability that a randomly sampled STAT323 student was in the faculty of science.

b) Find $P(323)$.

Starting from the law of total probability, we compute $P(323)$ directly.

$$\begin{aligned} P(323) &= P(323 \cap 321_s) + P(323 \cap 321_s^c) \\ &= P(323|321_s)P(321_s) + P(323|321_s^c)P(321_s^c) \\ &= (0.48)(0.65) + (0.22)(1 - 0.65) \\ &= 0.389 \end{aligned}$$

Solution: $P(323) = 0.389$.

Problem 9

Let S be the sample space containing all possible hands of 5 cards from a deck of 52, dealt randomly.

a) Let F be the event that a poker hand is a flush. Find $P(F)$.

There are 4 suits in a deck. For each given suit, there are 13 cards of that suit in a deck of 52. Then, a flush is made by selecting one of the 4 suits, then drawing 5 cards of that suit from the 13 available in a deck. We can compute $P(F)$ directly.

$$P(F) = \frac{|F|}{|S|} = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = \frac{5148}{2598960} \approx 0.001981 \qquad \text{(rounding to 6 decimal places.)}$$

Solution: $P(F) \approx 0.001981$.

b) Let F_R be the event that a poker hand is a flush with a red suit. Find $P(F_R)$.

The reasoning is identical to that of 9.a, except now there are 2 suits to choose from instead of 4. Then we can compute $P(F_R)$ directly.

$$P(F_R) = \frac{|F_R|}{|S|} = \frac{\binom{2}{1}\binom{13}{5}}{\binom{52}{5}} = \frac{2574}{2598960} \approx 0.0009903 \qquad \text{(rounding to 7 decimal places.)}$$

Solution: $P(F_R) \approx 0.0009903$.

c) Let T be the event that a poker hand contains a three-of-a-kind. Find $P(T)$.

To construct all possible hands with three-of-a-kind, we first select 1 of 13 denominations. Then, of the 4 available cards for the selected denomination, we select 3 to make up the three-of-a-kind. There are 12 denominations remaining. We select a new denomination, then 1 of the 4 cards available of that denomination. Of the remaining 11 denominations, we choose one, then choose one of the 4 available cards to be in the hand. By this reasoning, we compute $P(T)$ directly.

$$P(T) = \frac{|T|}{|S|} = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{1}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{109824}{2598960} \approx 0.04226 \qquad \text{(rounding to 5 decimal places.)}$$

Solution: $P(T) = 0.04226$.