Assignment 1

MATH391

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Problem 1

a) Function to return coefficients of Newton form interpolating polynomial using divided differences algorithm; function specifications are as described in the assignment.

```
function [a] = newton_dd(X, F, n)
    a = F;
    b = [];
    c = F;
    for i = 1:n
        for j = 1:(numel(X) - i)
            b(j, 1) = X(i + j) - X(j);
        end
        c((i + 1):end) = diff(a(i:end));
        a((i + 1):end) = c((i + 1):end)./b;
        b = [];
    end
end
```

b) Function to compute values of a Newton form interpolating polynomial over some domain, then plot the result as a curve alongside the data points which the polynomial was originally meant to interpolate.

```
function plot_newton_poly(X, F, a, n, num_points)

    x = linspace(min(X), max(X), num_points);
    y = [];
    for i = 1:numel(x)
        y(i) = horner_newton(x(i), a, X, n);
    end

    hold on
    scatter(X, F, 'r', 'filled');
    plot(x, y, 'k', linewidth=0.8);
    xlabel('x'), ylabel('y');
    axis padded
    hold off
end
```

where the function *horner_newton*$(\dots)$ is defined as given in the assignment as:

```
function y = horner_newton(c, a, x, n)
    y = a(n + 1);
    for i = 1:n
        y = y*(c - x(n + 1 - i)) + a(n + 1 - i);
    end
end
```

Problem 2

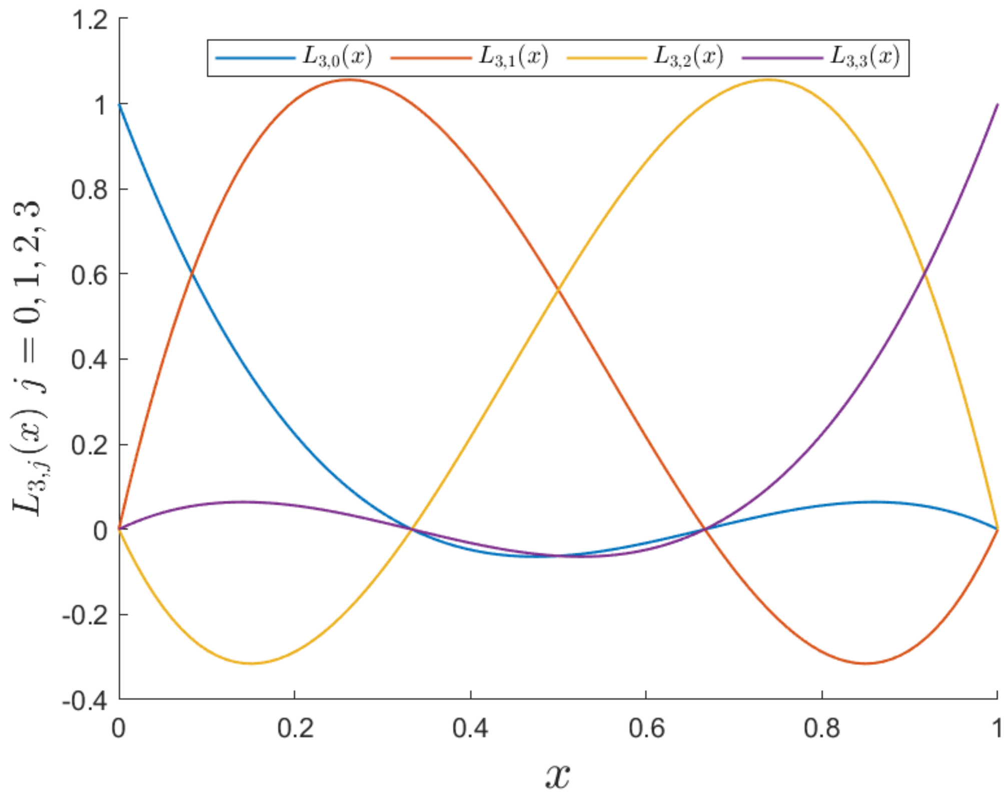

Fig 1. Degree 3 Lagrange polynomials, denoted $L_{3,j}(x)$ associated with four equidistant nodes $\{x_j\}_{1\leq j\leq 4}$ with $x_0 < x_1 < x_2 < x_3$. In this example, $x_0 = 0$ and $x_3 = 1$.

Problem 3

Let n be a natural number and define

$$T_n(x) = \cos(n\arccos(x)), \qquad -1 \leq x \leq 1$$

to be the degree n Chebeyshev polynomial.

a) Plot the graphs of $T_i(x)$, $i = 1, 2, 3, 4, 5$.

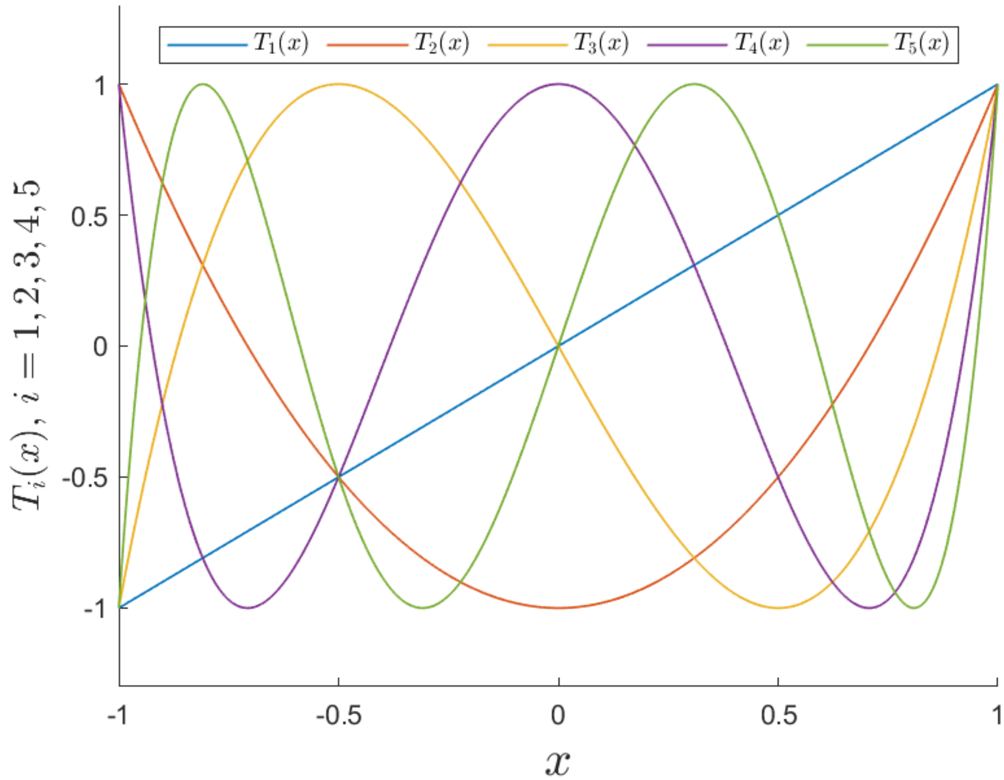

Fig 2. Degree i Chebeyshev polynomials denoted $T_i(x)$ for $i = 1, 2, 3, 4, 5$ over their domain $-1 \leq x \leq 1$.

b) Show that $T_n(x)$ is a polynomial of degree n.

Let n be a natural number. Then we begin with the trigonometric identity

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(\theta)\cos(n\theta).$$

Now, let $\theta = \arccos(x)$. Then the above can be rewritten as

$$\begin{aligned} \cos((n+1)\arccos(x)) + \cos((n-1)\arccos(x)) &= 2\cos(\arccos(x))\cos(n\arccos(x)) \\ T_{n+1}(x) + T_{n-1}(x) &= 2T_1(x)T_n(x). \end{aligned}$$

However, $T_1(x) = \cos(\arccos(x)) = x$. From this we can easily construct the recurrence relation.

$$\begin{aligned} T_{n+1}(x) + T_{n-1}(x) &= 2xT_n(x) \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x) \qquad \text{(shifting the indices.)} \end{aligned}$$

Where since $T_k(x)$ is defined for $k \in \mathbb{N}$, we require that $2 \leq n$ for the recurrence relation to hold. We proceed to use the recurrence relation to prove that $T_n(x)$ is a polynomial of degree n.

Proof (by induction)**.** Let $n \in \mathbb{N}$ with $n \geq 2$ and $x \in \mathbb{R}$ with $-1 \leq x \leq 1$. Then let $T_k(x)$ where $k \in \mathbb{N}$ be the degree k Chebeyshev polynomial defined by

$$T_k(x) = \cos(k\arccos(x))$$

and $T_n(x)$ defined by the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Basis($k = 0$, $k = 1$). Suppose that $k = 0$. Then we have that

$$T_k(x) = T_0(x) = \cos(0 \arccos(x)) = \cos(0) = 1$$

Where $T_0(x) = 1$ is a polynomial of degree $0 = k$. Next, suppose that $k = 1$. Then we have that

$$T_k(x) = T_1(x) = \cos(\arccos(x)) = x$$

Where $T_1(x) = x$ is a polynomial of degree $1 = k$. Hence, in the base cases of $k = 0$ and $k = 1$, $T_k(x)$ is a polynomial of degree k.

Inductive step. Let $n \geq 2$ and suppose that for all $k \in \mathbb{N}$ where $0 \leq k < n$ we have that $T_k(x)$ is a polynomial of degree k (inductive hypothesis). We want to show that $T_n(x)$ is a polynomial of degree n. Let deg() be the function which maps any polynomial to its degree. We begin with the recurrence relation.

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

By the inductive hypothesis, we have that $T_{n-1}(x)$ is a polynomial of degree $n-1$ since $0 \leq n-1 < n$. Furthermore, we have that $T_{n-2}(x)$ is a polynomial of degree $n-2$ since $0 \leq n-2 < n$. Now we see that $2xT_{n-1}(x)$ is a polynomial of degree n, since $\deg(2x) = 1$. However, for any two polynomials $P_a(x)$ and $P_b(x)$ where $\deg(P_a(x)) = a$ and $\deg(P_b(x)) = b$ with $a > b$, we have that $P_a(x) - P_b(x)$ is a polynomial and $\deg(P_a(x) - P_b(x)) = a$. Therefore since $\deg(2xT_{n-1}(x)) = n$ and $\deg(T_{n-2}(x)) = n-2$, we have that $\deg(2xT_{n-1}(x) - T_{n-2}(x)) = n$, so $T_n(x)$ is a polynomial and $\deg(T_n(x)) = n$.

Now, by the base cases of $k = 0$ and $k = 1$, the inductive step and the principle of mathematical induction (strong form) we have that $T_n(x)$ is a polynomial of degree n for all $n \in \mathbb{N}$. $\square$

c) Find the roots $\{\xi_i\}_{1 \leq i \leq n}$ of $T_n(x)$.

To begin, consider the definition of $T_n(x)$.

$$T_n(x) = \cos(n \arccos(x))$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $-1 \leq x \leq 1$. To find the roots, first let $\theta = n \arccos(x)$ so that we have $T_n(x) = \cos(\theta)$. Importantly, we have that $0 \leq \theta \leq n\pi$. Then we rewrite $T_n(x)$.

$$T_n(x) = \cos(\theta)$$

Letting θ_c be the possible values of θ for which $\cos(\theta) = 0$, we have

$$\cos(\theta_c) = 0 \Rightarrow \theta_c = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots, \pm\frac{(2k-1)\pi}{2}, \qquad k \in \mathbb{Z}.$$

However, since θ is bounded, we need to enforce the appropriate bounds on θ_c.

$$0 \leq \theta_c \leq n\pi \Rightarrow 0 \leq \frac{(2k-1)\pi}{2} \leq n\pi \Rightarrow \frac{1}{2} \leq k \leq n + \frac{1}{2} \Rightarrow k = 1, 2, 3, ..., n \qquad (\text{since } k \in \mathbb{Z}).$$

Finally, we can determine the n roots of $T_n(x)$, the i^{th} of which we will denote ξ_i, $1 \leq i \leq n$.

$$\begin{aligned} n \arccos(\xi_i) &= \theta_c \\ n \arccos(\xi_i) &= \frac{(2i-1)\pi}{2} \qquad (\text{where } i = 1, 2, 3, ..., n) \\ \arccos(\xi_i &= \frac{(2i-1)\pi}{2n} \\ \xi_i &= \cos\left(\frac{(2i-1)\pi}{2n}\right) \end{aligned}$$

Where for $T_n(x)$ there are n distinct values of i, so there are i roots of the form

$$\xi_i = \cos\left(\frac{(2i-1)\pi}{2n}\right).$$

d) Compute $\max_{-1\le x\le 1}|(x-\xi_1)(x-\xi_2)\dots(x-\xi_n)|$, where $\{\xi_i\}_{1\le i\le n}$ are the n roots of $T_n(x)$.

First, suppose that $-1 \le x \le 1$. Clearly, $(x-\xi_1)(x-\xi_2)\dots(x-\xi_n)$ is a degree n polynomial with the same roots as $T_n(x)$. This means that $(x-\xi_1)(x-\xi_2)\dots(x-\xi_n) = CT_n(x)$ for some constant C so long as $n \ne 0$. Furthermore, we have that

$$\begin{aligned}
&\max_{-1\le x\le 1}|(x-\xi_1)(x-\xi_2)\dots(x-\xi_n)| = \max_{-1\le x\le 1}|CT_n(x)|\\
&= C(1) \qquad \text{(since } T_n(x) = \cos(n\arccos(x))\text{, so } -1 \le T_n(x) \le 1 \text{ and } |T_n(x)| \le 1).
\end{aligned}$$

Importantly, the leading coefficient of $(x-\xi_1)(x-\xi_2)\dots(x-\xi_n)$ is 1, so if we suppose that B is the leading coefficient of $T_n(x)$, then $C = B^{-1}$. We proceed by finding B. $T_{n=2}(x)$ is given by

$$T_{n=2}(x) = 2x(x) - 1 = 2x^2 - 1$$

which has leading coefficient $2 = 2^1 = 2^{n-1}$. By the recurrence relation, we can compute the next few Chebeyshev polynomials.

$$\begin{aligned}
T_{n=3}(x) &= 2xT_2(x) - T_1(x) = 4x^3 - 2x - x = 4x^3 - 3x = 2^2x^3 - 3x = 2^{n-1}x^3 - 3x\\
T_{n=4}(x) &= 2xT_3(x) - T_2(x) = 8x^4 - 6x^2 - 2x^2 + 1 = 8x^4 - 8x^2 + 1 = 2^3x^4 - 8x^2 + 1 = 2^{n-1}x^4 - 8x^2 + 1\\
T_{n=5}(x) &= 2xT_4(x) - T_3(x) = 16x^5 - 16x^3 + 2x - 4x^3 + 3x = 16x^5 - 20x^3 + 5x = 2^{n-1}x^5 - 20x^3 + 5x\\
&\vdots
\end{aligned}$$

So the leading coefficient of $T_n(x)$ appears to have the form 2^{n-1}, which we will now endeavor to prove by induction. Taking the above to satisfy the base cases of $0 < n \le 5$, we proceed directly to the inductive step.

Inductive step. Let $n \ge 6$ be a natural number and suppose that $\forall_k \in \mathbb{N}$, $1 \le k < n$ we have that the leading coefficient of $T_k(x)$ is 2^{k-1} (inductive hypothesis). We want to show that the leading coefficient of $T_n(x)$ is 2^{n-1}. By the recurrence relation we have

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Now, since $1 \le n-1 < n$, the leading coefficient of $T_{n-1}(x)$ is 2^{n-2} by the inductive hypothesis. Note that this term is also the highest order among those in $T_{n-1}(x) - T_{n-2}(x)$. Let $2^{n-2}P_{n-1}(x) = T_{n-1}(x)$, where now $P_{n-1}(x)$ is a degree $n-1$ polynomial with leading coefficient 1. Then we have

$$2xT_{n-1}(x) = 2x(2^{n-2}P_{n-1}(x)) = 2^{n-1}xP_{n-1}(x)$$

so $2xT_{n-1}(x)$ is a degree n polynomial with leading coefficient 2^{n-1}. Since this term is the highest order term on the RHS of the recurrence relation, it is also the highest order term in $T_n(x)$, so the leading coefficient of $T_n(x)$ is 2^{n-1}.

By the base case of $0 < n \le 5$, the inductive step and the principle of mathematical induction, we have that the leading coefficient of the degree n Chebeyshev polynomial (where $n > 0$) is 2^{n-1}. □

As found above, letting B be the leading coefficient of $T_n(x)$, we have that $B = 2^{n-1}$. But $C = B^{-1}$, so $C = 2^{1-n} = \frac{1}{2^{n-1}}$. Then we have that

$$\begin{aligned}
\max_{-1\le x\le 1}|(x-\xi_1)(x-\xi_2)\dots(x-\xi_n)| &= C\\
&= \frac{1}{2^{n-1}}.
\end{aligned}$$

Problem 4

a) Plot the graphs of $f(x)$, $P_2(x)$, $P_4(x)$, $P_6(x)$, $P_8(x)$.

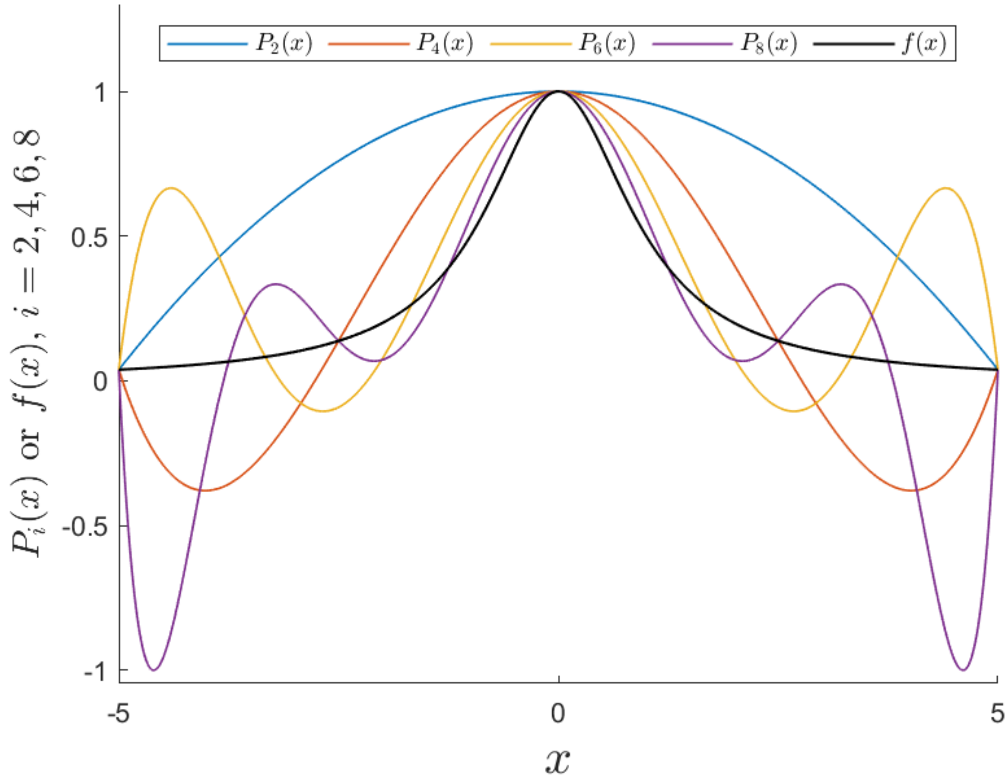

Fig 3. Function $f(x) = \frac{1}{1+x^2}$ plotted alongside degree n interpolating polynomials $P_n(x)$ for $n = 2, 4, 6, 8$. Here, $P_n(x)$ interpolates $n + 1$ evenly spaced points defined by $\{(x_i, f(x_i))\}_{0 \le i \le n}$ where $x_i = -5 + \frac{10i}{n}$.

b) Comment on the closeness of the interpolating polynomials to $f(x)$ around the middle and endpoints of $[-5, 5]$.

Let $f(x) = \frac{1}{1+x^2}$. Then $f(x)$ has at least 10 continuous derivatives on the interval $[-5, 5]$, which contains the interpolatory points. Let $P_n(x)$ be the polynomial of degree n interpolating $n + 1$ linearly spaced nodes on $[-5, 5]$. Then we have that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)(x - x_1)\dots(x - x_n)$$

for some $\theta(x) \in [-5, 5]$ for some $x \in [-5, 5]$. In part **a)** we considered the cases of $n = 2, 4, 6, 8$, so we first consider the graphs of $f^{(n+1)}(x)$ on the interval $[-5, 5]$ for these cases.

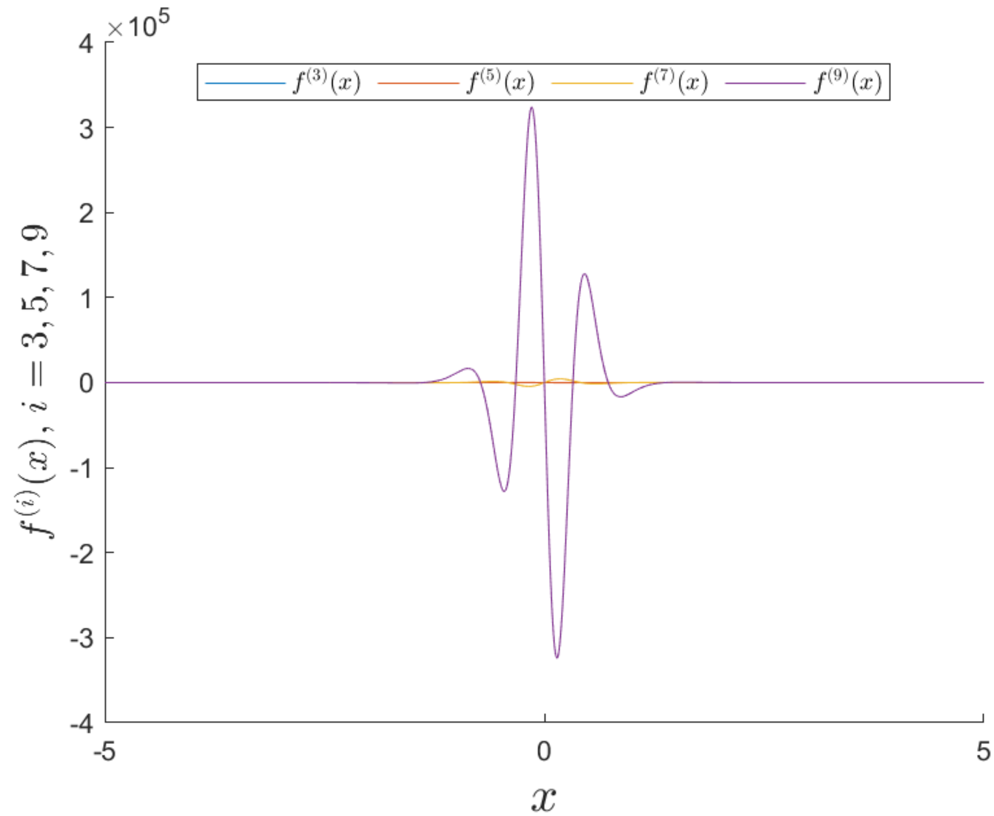

Fig 4. Graphs of $f^{(3)}(x)$, $f^{(5)}(x)$, $f^{(7)}(x)$, $f^{(9)}(x)$ for $x \in [-5, 5]$.

Where it is clear that $||f^{(9)}(x)||_\infty$ for $x \in [-5, 5]$ dwarfs the other derivatives plotted, so $f(x) - P_n(x)$ has the potential to be much higher for $n = 9$ than $n < 9$. This does not tell us where on the interval the error will be greatest since $\theta(x)$ is not known. Let $\omega_n(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$. Then for x near either endpoint, with $x \neq x_0$ and $x \neq x_n$, x is distant from all nodes except the endpoints, so the maxima of $\prod_{i=1}^{n-1}(x - x_i)$ are maximized (multiplied by several numbers > 1). Likewise, near the middle of the interval the maxima of the product are not allowed to take great values, since nodes on either side of x scale the product by smaller values. To visualize this, consider the following figure depicting $\omega_n(x)$ for $n = 2, 4, 6, 8$ over interval $[-5, 5]$.

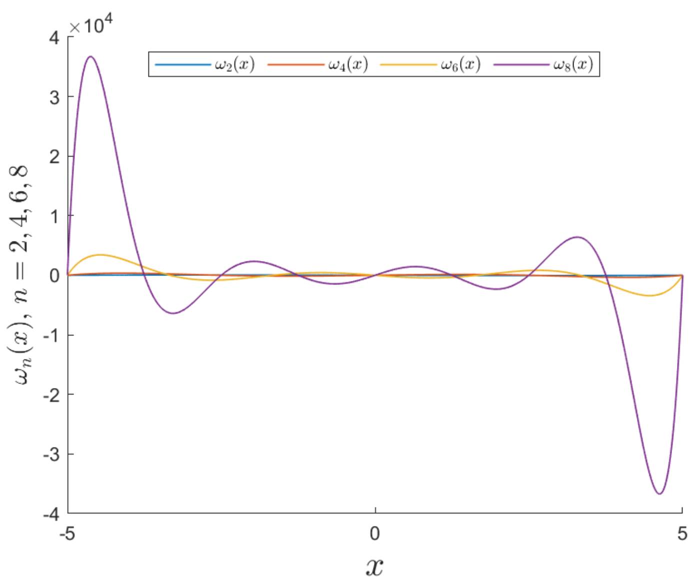

Fig 5. Graphs of $\omega_2(x)$, $\omega_4(x)$, $\omega_6(x)$, $\omega_8(x)$ plotted for x on interval $[-5, 5]$.

Here the phenomena described above is visualized, with $||\omega_n(x)||_\infty$ located near the endpoints. Furthermore, the maxima shown here are greater for increasing n. The consequence is that $f(x) - P_n(x)$ is highest near the endpoints as well, since only $\omega_n(x)$ has direct dependence on x in $f(x) - P_n(x) = \frac{f^{(n+1)}(\theta(x))}{(n+1)!}\omega_n(x)$.

Problem 5

Let $P_n(x)$ be the degree n polynomial interpolating $f(x)$ at the $n+1$ pairwise distinct points $X = \{x_0, \dots x_n\}$. Let $t \notin X$. Then let the degree $n+1$ polynomial interpolating $X \cup \{t\}$ be given by $P_{n+1}(x)$. Then by the Newton form of the interpolating polynomials, we have that

$$
\begin{aligned}
&P_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) \\
&P_{n+1}(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) + b_{n+1}(x - x_0)(x - x_1)\dots(x - x_n)
\end{aligned}
$$

Where a_i, $0 \le i \le n$ are the $n+1$ real coefficients of $P_n(x)$ and similarly b_j, $0 \le j \le n+1$ are the $n+2$ real coefficients of $P_{n+1}(x)$. Then we have a theorem which allows us to assert that

$$a_i = f[x_0, \dots, x_i] = b_i$$

where $f[x_0, \dots, x_i]$ is the i^{th} divided difference for the nodes X under f. This is true for all $0 \le i \le n$. Hence we have that

$$
\begin{aligned}
&P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1)\dots(x - x_{n-1}) \\
&P_{n+1}(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1)\dots(x - x_{n-1}) \\
&+ f[x_0, x_1, \dots, x_n, t](x - x_0)(x - x_1)\dots(x - x_n)
\end{aligned}
$$

So from this we can see that

$$P_{n+1}(x) = P_n(x) + f[x_0, x_1, \dots, x_n, t](x - x_0)(x - x_1)\dots(x - x_n)$$

Now, since $P_{n+1}(x)$ interpolates $f(x)$ at the nodes $X \cup \{t\}$, $P_{n+1}(t) = f(t)$, so we have that

$$f(t) = P_{n+1}(t) = P_n(t) + f[x_0, x_1, \dots, x_n, t](t - x_0)(t - x_1)\dots(t - x_n)$$

as required.