Assignment 2

MATH391

Connor Braun

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Problem 1

a) Let $x_0 < x_1 < x_2$ and let $f(x)$ be a function that is twice differentiable in $[x_0, x_2]$. Let $p(x)$ be a cubic polynomial so that $p(x) = a + bx + cx^2 + dx^3$ for some $a, b, c, d \in \mathbb{R}$. Furthermore, suppose that $p(x_0) = f(x_0)$, $p(x_2) = f(x_2)$, $p'(x_1) = f'(x_1)$, $p''(x_1) = f''(x_1)$.

Then we have the following system of linear equations in a, b, c, d.

$$\begin{aligned}
p(x_0) &= a + bx_0 + cx_0^2 + dx_0^3 = f(x_0) \\
p(x_2) &= a + bx_2 + cx_2^2 + dx_2^3 = f(x_2) \\
p'(x_1) &= b + 2cx_1 + 3dx_1^2 = f'(x_1) \\
p''(x_1) &= 2c + 6dx_1 = f''(x_1).
\end{aligned}$$

We can rewrite the system in matrix form as follows:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_2) \\ f'(x_1) \\ f''(x_1) \end{bmatrix} = \vec{f} \in \mathbb{R}^4$$

Let U be the matrix in the left hand side of this equation. In order for this system to have a unique solution $\forall \vec{f} \in \mathbb{R}^4$, it is sufficient to show that

$$det(U) = det\left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix}\right) \neq 0.$$

Indeed, we have that

$$\begin{aligned}
det\left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix}\right) &= det\left(\begin{bmatrix} x_2 & x_2^2 & x_2^3 \\ 1 & 2x_1 & 3x_1^2 \\ 0 & 2 & 6x_1 \end{bmatrix}\right) - det\left(\begin{bmatrix} x_0 & x_0^2 & x_0^3 \\ 1 & 2x_1 & 3x_1^2 \\ 0 & 2 & 6x_1 \end{bmatrix}\right) \\
&= x_2(6x_1^2) - (6x_1x_2^2 - 2x_2^3) - x_0(6x_1^2) + (6x_1x_0^2 - 2x_0^3) \\
det(U) &= x_1^2(6x_2 - 6x_0) + x_1(6x_0^2 - 6x_2^2) + 2(x_2^3 - x_0^3)
\end{aligned}$$

Which is a quadratic polynomial in x_1. Let Δ be the discriminant of this polynomial. Then,

$$\begin{aligned}
\Delta &= (6x_0^2 - 6x_2^2)^2 - 4(6x_2 - 6x_0)(2x_2^3 - 2x_0^3) \\
&= (36x_0^4 - 72x_0^2x_2^2 + 36x_2^4) - 4(12x_2^4 - 12x_0^3x_2 - 12x_0x_2^3 + 12x_0^4) \\
&= 36x_0^4 - 72x_0^2x_2^2 + 36x_2^4 - 48x_2^4 + 48x_0^3x_2 + 48x_0x_2^3 - 48x_0^4
\end{aligned}$$

$$
\begin{aligned}
&= -12x_0^4 + 48x_0^3x_2 - 72x_0^2x_2^2 + 48x_0x_2^3 - 12x_2^4 \\
&= -12(x_0^4 - 4x_0^3x_2 + 6x_0^2x_2^2 - 4x_0x_2^3 + x_2^4) \\
&= -12(x_0 - x_2)^4 \qquad \text{(by the binomial theorem).}
\end{aligned}
$$

Where $\forall_{x_0,x_2} \in \mathbb{R}$ we have that $(x_0 - x_2)^4 > 0$ since $x_0 < x_2$. This means that $\Delta < 0$, so $det(U)$ has no real roots and furthermore $det(U) = x_1^2(6x_2 - 6x_0) + x_1(6x_0^2 - 6x_2^2) + 2(x_2^3 - x_0^3) \neq 0 \ \forall_{x_0,x_1,x_2} \in \mathbb{R}$, provided $x_0 < x_1 < x_2$ as previously supposed.

b) Let $x_0 < x_1 < x_2$ and $f(x)$ be a function which is twice differentiable in $[x_0, x_2]$. Formulate the third degree polynomial $\xi(x)$ that meets the following interpolatory conditions:

$$
\begin{aligned}
\xi(x_0) &= f(x_0) \\
\xi(x_2) &= f(x_2) \\
\xi'(x_1) &= f'(x_1) \\
\xi''(x_1) &= f''(x_1).
\end{aligned}
$$

To construct $\xi(x)$ we first seek to meet the interpolatory conditions $\xi(x_0) = f(x_0)$ and $\xi(x_2) = f(x_2)$. We begin by treating this as a Lagrange interpolation problem. Let $r(x)$ be the linear Newton form interpolating polynomial defined by

$$
r(x) = a + b(x - x_0).
$$

For some $a, b \in \mathbb{R}$. We now impose the interpolatory conditions to find that

$$
\begin{aligned}
r(x_0) &= a = f(x_0), \qquad \text{and} \\
r(x_2) &= a + b(x_2 - x_0) = f(x_2) \\
\Rightarrow f(x_2) &= f(x_0) + b(x_2 - x_0) \\
\Rightarrow b &= \frac{f(x_2) - f(x_0)}{(x_2 - x_0)} \\
&= f[x_0, x_2].
\end{aligned}
$$

Where $f[x_0, x_2]$ is the divided difference formula between points x_0 and x_2. We next try to find a third-degree polynomial $q(x)$ such that

$$
\xi(x) = r(x) + q(x).
$$

Now, since $r(x)$ already interpolates $f(x)$ at x_0 and x_2, we find that

$$
\begin{aligned}
f(x_0) = \xi(x_0) &= r(x_0) + q(x_0) \\
&= f(x_0) + q(x_0) \qquad \text{(since } r(x_0) = f(x_0)\text{)} \\
&\Rightarrow q(x_0) = 0
\end{aligned}
$$

and similarly

$$
\begin{aligned}
f(x_2) = \xi(x_2) &= r(x_2) + q(x_2) \\
&= f(x_2) + q(x_2) \qquad \text{(since } r(x_2) = f(x_2)\text{)} \\
&\Rightarrow q(x_2) = 0
\end{aligned}
$$

so $q(x)$ is a third degree polynomial with roots at $x = x_0$ and $x = x_2$. Hence, $q(x)$ takes the form

$$
q(x) = (x - x_0)(x - x_2)(Ax + B)
$$

for some $A, B \in \mathbb{R}$. To finish constructing $\xi(x)$, we determine A and B by imposing the last two interpolatory conditions on $\xi(x)$. First, derivatives of $\xi(x)$ are

$$\begin{aligned}
\xi'(x) &= f[x_0, x_2] + (x - x_2)(Ax + B) + (x - x_0)(Ax + B) + A(x - x_0)(x - x_2) \\
\xi''(x) &= 2A(x - x_2) + 2A(x - x_0) + 2(Ax + B) \\
&= 2A(x - x_2) + 2A(x - x_0) + 2Ax + 2B \\
&= 2A(3x - x_0 - x_2) + 2B.
\end{aligned}$$

Next, imposing the interpolatory conditions yields the following system of linear equations in A and B

$$\begin{aligned}
\xi'(x_1) &= f[x_0, x_2] + (x_1 - x_2)(Ax_1 + B) + (x_1 - x_0)(Ax_1 + B) + A(x_1 - x_0)(x_1 - x_2) = f'(x_1) \\
\Rightarrow f'(x_1) - f[x_0, x_2] &= (Ax_1^2 + Bx_1 - Ax_1x_2 - Bx_2) + (Ax_1^2 + Bx_1 - Ax_0x_1 - Bx_0) \\
&\quad + (Ax_1^2 - Ax_1x_2 - Ax_0x_1 + Ax_0x_2) \\
\Rightarrow f'(x_1) - f[x_0, x_2] &= A(3x_1^2 - 2x_1x_2 - 2x_0x_1 + x_0x_2) + B(2x_1 - x_2 - x_0)
\end{aligned}$$

$$\xi''(x_1) = 2A(3x_1 - x_0 - x_2) + 2B = f''(x_1).$$

This can be rewritten in matrix form as

$$\begin{bmatrix} (3x_1^2 - 2x_1x_2 - 2x_0x_1 + x_0x_2) & (2x_1 - x_2 - x_0) \\ 2(3x_1 - x_0 - x_2) & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} f'(x_1) - f[x_0, x_2] \\ f''(x_1) \end{bmatrix}$$

Letting V be the matrix in the left hand side of the equation, we first find $det(V)$.

$$\begin{aligned}
det(V) &= det\left(\begin{bmatrix} (3x_1^2 - 2x_1x_2 - 2x_0x_1 + x_0x_2) & (2x_1 - x_2 - x_0) \\ 2(3x_1 - x_0 - x_2) & 2 \end{bmatrix}\right) \\
&= 2(3x_1^2 - 2x_1x_2 - 2x_0x_1 + x_0x_2) - 2(2x_1 - x_2 - x_0)(3x_1 - x_0 - x_2).
\end{aligned}$$

Expanding and then collecting like terms of $det(V)$ yields a quadratic polynomial in x_0

$$det(V) = -2(x_0^2 + x_0(x_2 - 3x_1) + (3x_1^2 - 3x_1x_2 + x_2^2)).$$

Next, let Δ be the discriminant of $det(V)$. We find Δ by the following

$$\begin{aligned}
\Delta &= (x_2 - 3x_1)^2 - 4(3x_1^2 - 3x_1x_2 + x_2^2) \\
&= x_2^2 - 6x_1x_2 + 9x_1^2 - 12x_1^2 + 12x_1x_2 - 4x_2^2 \\
&= -3x_2^2 + 6x_1x_2 - 3x_1^2 \\
&= -3(x_2^2 - 2x_1x_2 + x_1^2) \\
&= -3(x_2 - x_1)^2 \qquad \text{(by the binomial theorem).}
\end{aligned}$$

Provided $x_1 < x_2$, we have that $\forall_{x_1,x_2} \in \mathbb{R}$, $(x_2 - x_1)^2 > 0$, so $\Delta < 0$. This means that $det(V)$ has no real roots, and thus since $x_0 < x_1 < x_2$, we have that $\forall_{x_0,x_1,x_2} \in \mathbb{R}$, $det(V) \neq 0$. This allows us to compute A and B directly by way of Cramer's rule.

$$\begin{aligned}
A &= \frac{det\left(\begin{bmatrix} f'(x_1) - f[x_0, x_2] & (2x_1 - x_2 - x_0) \\ f''(x_1) & 2 \end{bmatrix}\right)}{det(V)} \\
A &= \frac{2(f'(x_1) - f[x_0, x_2]) - f''(x_1)(2x_1 - x_2 - x_0)}{-2(x_0^2 + x_0(x_2 - 3x_1) + (3x_1^2 - 3x_1x_2 + x_2^2))}.
\end{aligned}$$

$$B = \frac{det\left(\begin{bmatrix} (3x_1^2 - 2x_1x_2 - 2x_0x_1 + x_0x_2) & f'(x_1) - f[x_0, x_2] \\ 2(3x_1 - x_0 - x_2) & f''(x_1) \end{bmatrix}\right)}{det(V)}$$

$$B = \frac{f''(x_1)(3x_1^2 - 2x_1x_2 - 2x_0x_1 + x_0x_2) - 2(f'(x_1) - f[x_0, x_2])(3x_1 - x_0 - x_2)}{-2(x_0^2 + x_0(x_2 - 3x_1) + (3x_1^2 - 3x_1x_2 + x_2^2))}.$$

Finally, we arrive at the explicit formula for $\xi(x)$ meeting the specified interpolatory conditions on $f(x)$

$$\xi(x) = f(x_0) + f[x_0, x_2](x - x_0) + (x - x_0)(x - x_2)(\zeta)$$

Where ζ is given by

$$\begin{aligned} \zeta = & \frac{x(2(f'(x_1) - f[x_0, x_2]) - f''(x_1)(2x_1 - x_2 - x_0))}{-2(x_0^2 + x_0(x_2 - 3x_1) + (3x_1^2 - 3x_1x_2 + x_2^2))} \\ & + \frac{f''(x_1)(3x_1^2 - 2x_1x_2 - 2x_0x_1 + x_0x_2) - 2(f'(x_1) - f[x_0, x_2])(3x_1 - x_0 - x_2)}{-2(x_0^2 + x_0(x_2 - 3x_1) + (3x_1^2 - 3x_1x_2 + x_2^2))}. \end{aligned}$$

Problem 2

a) Let $f(x)$ be a function with as many derivatives as needed at and around the point x_0. Derive a difference formula approximating $f'(x_0)$ which uses the points $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ for $h \in \mathbb{R}$ small.

We begin by finding the degree 3 Lagrange form polynomial $p_3(x)$ interpolating $f(x)$ at the points $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$.

$$\begin{aligned} p_3(x) &= L_{3,0}(x)f(x_0 - h) + L_{3,1}(x)f(x_0) + L_{3,2}(x)f(x_0 + h) + L_{3,3}(x)f(x_0 + 2h) \\ &= \frac{(x - x_0)(x - x_0 - h)(x - x_0 - 2h)}{(x_0 - h - x_0)(x_0 - h - x_0 - h)(x_0 - h - x_0 - 2h)} f(x_0 - h) \\ &+ \frac{(x - x_0 + h)(x - x_0 - h)(x - x_0 - 2h)}{(x_0 - x_0 + h)(x_0 - x_0 - h)(x_0 - x_0 - 2h)} f(x_0) \\ &+ \frac{(x - x_0 + h)(x - x_0)(x - x_0 - 2h)}{(x_0 + h - x_0 + h)(x_0 + h - x_0)(x_0 + h - x_0 - 2h)} f(x_0 + h) \\ &+ \frac{(x - x_0 + h)(x - x_0)(x - x_0 - h)}{(x_0 + 2h - x_0 + h)(x_0 + 2h - x_0)(x_0 + 2h - x_0 - h)} f(x_0 + 2h) \end{aligned}$$

The derivative of which is easily computable as

$$\begin{aligned} p_3'(x) &= L_{3,0}'(x)f(x_0 - h) + L_{3,1}'(x)f(x_0) + L_{3,2}'(x)f(x_0 + h) + L_{3,3}'(x)f(x_0 + 2h) \\ &= \frac{(x - x_0 - h)(x - x_0 - 2h) + (x - x_0)(x - x_0 - 2h) + (x - x_0)(x - x_0 - h)}{(x_0 - h - x_0)(x_0 - h - x_0 - h)(x_0 - h - x_0 - 2h)} f(x_0 - h) \\ &+ \frac{(x - x_0 - h)(x - x_0 - 2h) + (x - x_0 + h)(x - x_0 - 2h) + (x - x_0 + h)(x - x_0 - h)}{(x_0 - x_0 + h)(x_0 - x_0 - h)(x_0 - x_0 - 2h)} f(x_0) \\ &+ \frac{(x - x_0)(x - x_0 - 2h) + (x - x_0 + h)(x - x_0 - 2h) + (x - x_0 + h)(x - x_0)}{(x_0 + h - x_0 + h)(x_0 + h - x_0)(x_0 + h - x_0 - 2h)} f(x_0 + h) \\ &+ \frac{(x - x_0)(x - x_0 - h) + (x - x_0 + h)(x - x_0 - h) + (x - x_0 + h)(x - x_0)}{(x_0 + 2h - x_0 + h)(x_0 + 2h - x_0)(x_0 + 2h - x_0 - h)} f(x_0 + 2h) \end{aligned}$$

Where since $f(x) = p_3(x) + E(x)$, where $E(x)$ is an error term depending on x, we have that $f(x) \approx p_3(x)$ so $f'(x) \approx p_3'(x)$. Furthermore, $f'(x_0) \approx p_3'(x_0)$, so

$$f'(x_0) \approx p_3'(x_0) = \frac{(-h)(-2h)}{(-h)(-2h)(-3h)} f(x_0 - h)$$

$$
\begin{aligned}
&+ \frac{(-h)(-2h)+(h)(-2h)+(h)(-h)}{(h)(-h)(-2h)} f(x_0) \\
&+ \frac{(h)(-2h)}{(2h)(h)(-h)} f(x_0+h) \\
&+ \frac{(h)(-h)}{(3h)(2h)(h)} f(x_0+2h) \\
&= \frac{2h^2}{-6h^3} f(x_0-h) + \frac{-h^2}{2h^3} f(x_0) + \frac{-2h^2}{-2h^3} f(x_0+h) + \frac{-h^2}{6h^3} f(x_0+2h) \\
&= \frac{-1}{3h} f(x_0-h) + \frac{-1}{2h} f(x_0) + \frac{1}{h} f(x_0+h) + \frac{-1}{6h} f(x_0+2h)
\end{aligned}
$$

So we arrive at a formula for the approximation of $f'(x_0)$ using the four points

$$
f'(x_0) \approx p_3'(x) = \frac{-2f(x_0-h) - 3f(x_0) + 6f(x_0+h) - f(x_0+2h)}{6h}.
$$

b) Determine the order of convergence of the approximation derived in 2.a.

We begin by constructing the Taylor expansion of $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$, $f(x_0+2h)$

$$
\begin{aligned}
&f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)h^2}{2!} - \frac{f'''(x_0)h^3}{3!} + \frac{f^{(4)}(\theta_1)h^4}{4!} \\
&f(x_0) = f(x_0) \\
&f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2!} + \frac{f'''(x_0)h^3}{3!} + \frac{f^{(4)}(\theta_2)h^4}{4!} \\
&f(x_0+2h) = f(x_0) + 2f'(x_0)h + 2f''(x_0)h^2 + \frac{4f'''(x_0)h^3}{3} + \frac{2f^{(4)}(\theta_3)h^4}{3}
\end{aligned}
$$

for some $\theta_1 \in I(x_0, x_0-h)$, $\theta_2 \in I(x_0, x_0+h)$, $\theta_3 \in I(x_0, x_0+2h)$. Next we can scale these by their numerator coefficients in the approximation from 2.a and add the result to find that

$$
\begin{aligned}
-2f(x_0-h) - 3f(x_0) + 6f(x_0+h) - f(x_0+2h) &= (-2-3+6-1)f(x_0) \\
&+ (2+6-2)f'(x_0)h \\
&+ (-1+3-2)f''(x_0)h^2 \\
&+ (\frac{1}{3} + 1 - \frac{4}{3})f'''(x_0)h^3 \\
&+ (-\frac{f^{(4)}(\theta_1)}{12} + \frac{3f^{(4)}(\theta_2)}{12} - \frac{8f^{(4)}(\theta_3)}{12})h^4
\end{aligned}
$$

Which, letting $E = 3f^{(4)}(\theta_2) - f^{(4)}(\theta_1) - 8f^{(4)}(\theta_3)$, reduces to

$$
\begin{aligned}
&-2f(x_0-h) - 3f(x_0) + 6f(x_0+h) - f(x_0+2h) = 6f'(x_0)h + \frac{1}{12}Eh^4 \\
&\Rightarrow -2f(x_0-h) - 3f(x_0) + 6f(x_0+h) - f(x_0+2h) - \frac{1}{12}Eh^4 = 6f'(x_0)h \\
&\Rightarrow f'(x_0) = \frac{-2f(x_0-h) - 3f(x_0) + 6f(x_0+h) - f(x_0+2h)}{6h} - \frac{1}{72}Eh^3
\end{aligned}
$$

Where now it is clear that the error term between $f'(x_0)$ and the approximation derived in 2.a is proportional to h^3, and so the approximation has order of convergence 3.

Problem 3

a)Let $f(x)$ be a function with as many derivatives as required at and around a point x_0. Find an error term for the difference formula

$$f''(x_0) \approx \frac{-f(x_0+3h)+4f(x_0+2h)-5f(x_0+h)+2f(x_0)}{h^2}$$

We begin by writing the Taylor expansions of $-f(x_0+3h)$, $4f(x_0+2h)$ and $-5f(x_0+h)$.

$$\begin{aligned}
-f(x_0+3h) &= -f(x_0) - 3f'(x_0)h - \frac{9f''(x_0)h^2}{2!} - \frac{3^3 f'''(x_0)h^3}{3!} - \frac{3^4 f^{(4)}(\theta_1)h^4}{4!} \\
4f(x_0+2h) &= 4f(x_0) + 8f'(x_0)h + \frac{16f''(x_0)h^2}{2!} + \frac{32f'''(x_0)h^3}{3!} + \frac{4(2^4)f^{(4)}(\theta_2)h^4}{4!} \\
-5f(x_0+h) &= -5f(x_0) - 5f'(x_0)h - \frac{5f''(x_0)h^2}{2!} - \frac{5f'''(x_0)h^3}{3!} - \frac{5f^{(4)}(\theta_3)h^4}{4!}.
\end{aligned}$$

For some $\theta_1 \in I(x_0, x+3h)$, $\theta_2 \in I(x_0, x_0+2h)$, $\theta_3 \in I(x_0, x_0+h)$. Next we add these and rearrange them to isolate $f''(x_0)$

$$\begin{aligned}
&f''(x_0)h^2 - \frac{1}{4!}\left(64f^{(4)}(\theta_2) - 81f^{(4)}(\theta_1) - 5f^{(4)}(\theta_3)\right)h^4 = -3f(x_0+3h)+4f(x_0+2h)-5f(x_0+h)+2f(x_0) \\
&f''(x_0) = \frac{-3f(x_0+3h)+4f(x_0+2h)-5f(x_0+h)+2f(x_0)}{h^2} + \frac{1}{4!}\left(64f^{(4)}(\theta_2) - 81f^{(4)}(\theta_1) - 5f^{(4)}(\theta_3)\right)h^2 \\
&\Rightarrow f''(x_0) - \frac{-3f(x_0+3h)+4f(x_0+2h)-5f(x_0+h)+2f(x_0)}{h^2} = \frac{1}{4!}\left(64f^{(4)}(\theta_2) - 81f^{(4)}(\theta_1) - 5f^{(4)}(\theta_3)\right)h^2
\end{aligned}$$

So we have that the given finite difference formula has an error term $E(h)$ given by

$$E(h) = \frac{1}{4!}\left(64f^{(4)}(\theta_2) - 81f^{(4)}(\theta_1) - 5f^{(4)}(\theta_3)\right)h^2$$

For some $\theta_1 \in I(x_0, x+3h)$, $\theta_2 \in I(x_0, x_0+2h)$, $\theta_3 \in I(x_0, x_0+h)$.

Problem 4

Let $f(t)$ be a function and suppose that $a, b \in \mathbb{R}$ and that the closed interval $[a, b]$ is subdivided into $n \in \mathbb{N}$ equal subintervals of length $h = \frac{b-a}{n}$. Let $Q_s^{c,n}(f)$ be the n-interval composite Simpson's rule on [a,b].

a) Find $Q_s^{c,n}(f)$.

First we divide the definite integral of $f(t)$ on $[a, b]$ into n subintervals of equal size.

$$\int_a^b f(t)dt = \int_a^{a+h} f(t)dt + \int_{a+h}^{a+2h} f(t)dt + \cdots + \int_{a+(n-1)h}^{a+nh} f(t)dt.$$

This sum can be rewritten as

$$\int_a^b f(t)dt = \sum_{i=1}^{n} \int_{a+(i-1)h}^{a+ih} f(t)dt.$$

Where now we can use Simpson's rule to approximate each integral in the sum as follows

$$\begin{aligned}
\sum_{i=1}^{n} \int_{a+(i-1)h}^{a+ih} f(t)dt \approx \sum_{i=1}^{n} \Big(&\frac{a+ih-a-(i-1)h}{6} f(a+(i-1)h) \\
&+ \frac{2(a+ih-a-(i-1)h)}{3} f(\frac{2a+(2i-1)h}{2}) \\
&+ \frac{a+ih-a-(i-1)h}{6} f(a+ih)\Big)
\end{aligned}$$

Furthermore, simplifying the right hand side we can manipulate the sum to show that

$$\begin{aligned}
&\sum_{i=1}^{n}\Big(\frac{h}{6}f(a+(i-1)h)+\frac{2h}{3}f(a+\frac{(2i-1)h}{2})+\frac{h}{6}f(a+ih)\Big)\\
&=\frac{h}{6}\sum_{i=1}^{n}f(a+(i-1)h)+\frac{2h}{3}\sum_{i=1}^{n}f(a+\frac{(2i-1)h}{2})+\frac{h}{6}\sum_{i=1}^{n}f(a+ih)\\
&=\frac{h}{6}(f(a)+f(a+h)+\cdots+f(a+(n-1)h))\\
&\quad+\frac{2h}{3}(f(a+\frac{h}{2})+f(a+\frac{3h}{2})+\cdots+f(a+\frac{(2n-1)h}{2}))\\
&\quad+\frac{h}{6}(f(a+h)+f(a+2h)+\cdots+f(a+nh))\\
&=\frac{h}{6}(f(a)+f(a+nh))\\
&\quad+\frac{h}{6}(f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h))\\
&\quad+\frac{2h}{3}(f(a+\frac{h}{2})+f(a+\frac{3h}{2})+\cdots+f(a+\frac{(2n-1)h}{2}))\\
&\quad+\frac{h}{6}(f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h))\\
&=\frac{h}{6}(f(a)+f(a+nh))\\
&\quad+\frac{h}{3}(f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h))\\
&\quad+\frac{2h}{3}(f(a+\frac{h}{2})+f(a+\frac{3h}{2})+\cdots+f(a+\frac{(2n-1)h}{2})).
\end{aligned}$$

Finally, since $h=\frac{b-a}{n}$, we have that $a+nh=b$, we arrive at the solution

$$\begin{aligned}
Q_s^{c,n}(f)=&\frac{h}{6}\big(f(a)+f(b)\big)\\
&+\frac{h}{3}\big(f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)\big)\\
&+\frac{2h}{3}\big(f(a+\frac{h}{2})+f(a+\frac{3h}{2})+\cdots+f(a+\frac{(2n-1)h}{2})\big).
\end{aligned}$$

b) Suppose that $f(t)$ has a continuous fourth derivative and let $\theta\in[a,b]$. Show that

$$\int_a^b f(t)dt=Q_S^{c,n}(f)-\frac{(b-a)^5}{2880n^4}f^{(4)}(\theta).$$

As before, we begin by dividing the definite integral of $f(t)$ on interval $[a,b]$ into n subintervals of equal size.

$$\int_a^b f(t)dt=\int_a^{a+h}f(t)dt+\int_{a+h}^{a+2h}f(t)dt+\cdots+\int_{a+(n-1)h}^{a+nh}f(t)dt.$$

Let $Q_S(g,m,k)$ be the Simpson's quadrature formula for some function g on interval $[m,k]$. Then, by the error of the Simpson's quadrature (theorem), we have that

$$\begin{aligned}
\int_a^b f(t)dt=&Q_S(f,a,a+h)-\frac{f^{(4)}(\theta_1)h^5}{2880}\\
&+Q_S(f,a+h,a+2h)-\frac{f^{(4)}(\theta_2)h^5}{2880}+\ldots
\end{aligned}$$

$$\cdots + Q_S(f, a+(n-1)h, a+nh) - \frac{f^{(4)}(\theta_n)h^5}{2880}$$

For some $\theta_1 \in [a, a+h]$, $\theta_2 \in [a+h, a+2h]$,...,$\theta_n \in [a+(m-1)h, a+nh]$. This can be rewritten as

$$\begin{aligned}\int_a^b f(t)dt &= Q_S^{c,n}(f) - \frac{h^5}{2880}(f^{(4)}(\theta_1) + f^{(4)}(\theta_2) + \cdots + f^{(4)}(\theta_n)) \\ &= Q_S^{c,n}(f) - \frac{nh^5}{2880}\left(\frac{1}{n}\sum_{j=1}^{n} f^{(4)}(\theta_j)\right)\end{aligned}$$

Where now since $f^{(4)}(\theta_j)$ is continuous for all $1 \leq j \leq n$, the intermediate value theorem guarantees $\exists_\theta \in I[\theta_1, \theta_2, \ldots, \theta_n]$ such that

$$\int_a^b f(t)dt = Q_S^{c,n}(f) - \frac{nh^5}{2880}f^{(4)}(\theta).$$

However, since $h = \frac{b-a}{n}$, we can rewrite this formula as

$$\int_a^b f(t)dt = Q_S^{c,n}(f) - \frac{(b-a)^5}{2880n^4}f^{(4)}(\theta).$$

So we have derived the desired equality.

c)Suppose that $-9 \leq f^{(4)}(t) \leq 2\ \forall_t \in [0,2]$. Find the smallest number of subintervals n required to guarantee that

$$\left|\int_0^2 f(t)dt - Q_S^{c,n}(f)\right| \leq 10^{-5}.$$

First we apply the result from 4.b

$$\left|\int_0^2 f(t)dt - Q_S^{c,n}(f)\right| = \left|\frac{2^5}{2880n^4}f^{(4)}(\theta)\right|$$

for some $\theta \in [0,2]$. Applying the constraint on the error

$$\left|\frac{2^5}{2880n^4}f^{(4)}(\theta)\right| \leq 10^{-5}$$

Now, since $n^4 > 0$ and $\forall_\theta$, $-9 \leq f^{(4)}(\theta) \leq 2$, we can solve for n directly

$$\begin{aligned}&\left|\frac{2^5}{2880n^4}f^{(4)}(\theta)\right| \leq 10^{-5} \\ &\Rightarrow 9\left|\frac{32}{2880}\right| \leq n^4 10^{-5} \\ &\Rightarrow \frac{10^5}{10} \leq n^4 \\ &\Rightarrow \sqrt[4]{10000} \leq n \\ &\Rightarrow n \geq 10\end{aligned}$$

Therefore, to bound the error between $\int_0^2 f(t)dt$ and $Q_S^{c,n}(f)$ by 10^{-5}, we require that the interval $[0,2]$ be subdivided into at least $n = 10$ equal subintervals.

Problem 5

Function for evaluating definite integrals using composite Simpson's rule. Function specifications are as described in the assignment.

```
function Q = Composite_Simpson(fname, a, b, n)

    h = (b - a)/n;
    Q = 0;
    for i = 1:n
        sub_int_a = a + (i-1)*h;
        sub_int_b = a + i*h;
        sub_int_mid = (sub_int_a + sub_int_b)/2;

        Q = Q + (h/6)*fname(sub_int_a) + ...
            (2*h/3)*fname(sub_int_mid) + ...
            (h/6)*fname(sub_int_b);
    end
end
```