Assignment 2
MATH271

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Problem 1

a) Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all integers $n \geq 0$ by induction on n.

Proof (by induction).
Basis ($n = 0$). Suppose n is an integer but specifically $n = 0$. Then we have the following

$$\begin{aligned} 5^{2n+1} + 2^{2n+1} &= 5^{2(0)+1} + 2^{2(0)+1} && \text{Substituting } n = 0 \\ &= 5^1 + 2^1 \\ &= 7 \\ &= 7(1) \end{aligned}$$

Where since both 7 and 1 are integers with $7 \neq 0$ we have that $7|5^{2n+1} + 2^{2n+1}$ in the case of $n = 0$.
Inductive step. Now, suppose that $\forall_k \in \mathbb{Z}$ where $k \geq 0$ we have that 7 divides $5^{2k+1} + 2^{2k+1}$. That is:

$$5^{2k+1} + 2^{2k+1} = 7a \qquad \text{(Inductive hypothesis)}$$

for some integer a. Let the acronym IH henceforth stand for 'Inductive Hypothesis'. We want to show that

$$\begin{aligned} 5^{2(k+1)+1} + 2^{2(k+1)+1} &= 5^{2k+3} + 2^{2k+3} \\ &= 7b \end{aligned}$$

for some integer b. Starting with $5^{2k+3} + 2^{2k+3}$ we can derive the desired equality as follows:

$$\begin{aligned} 5^{2k+3} + 2^{2k+3} &= 5^2 \cdot 5^{2k+1} + 2^2 \cdot 2^{2k+1} \\ &= 5^2(5^{2k+1} + 2^{2k+1}) - 21 \cdot 2^{2k+1} \\ &= 5^2(7a) - 21 \cdot 2^{2k+1} && \text{by the IH} \\ &= 7(25a - 3 \cdot 2^{2k+1}) && \text{factoring out 7} \\ &= 7b. && \text{setting } b = 25a - 3 \cdot 2^{2k+1} \end{aligned}$$

Now since $b = 25a - 3 \cdot 2^{2k+1}$ is an integer, we've shown that $7|5^{2k+3} + 2^{2k+3}$ given the IH. Furthermore, by the base case of $n = 0$, the inductive step and the principle of mathematical induction, we have that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all integers $n \geq 0$.

b) Prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > 2(\sqrt{n+1} - 1)$ for all integers $n \geq 1$ by induction on n.

Proof (by induction).
Basis ($n = 0$). Suppose n is an integer but specifically $n = 1$. Then we have that

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} = \sum_{i=1}^{1} \frac{1}{\sqrt{i}} \qquad \text{Substituting } n = 1$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{1}} \\
&= \frac{1}{1} \\
&= 1 \\
&> \sqrt{8} - 2
\end{aligned}
$$

It is worth pausing to explain why $1 > \sqrt{8} - 2$. To see why this strict inequality holds, consider the following

$$
\begin{aligned}
4 < 8 < 9 \\
\sqrt{4} < \sqrt{8} < \sqrt{9} \\
2 < \sqrt{8} < 3 \\
0 < \sqrt{8} - 2 < 1
\end{aligned}
$$

With this fact, we return to where we left off in establishing the base case for $n = 1$:

$$
\begin{aligned}
\sum_{i=1}^{1} \frac{1}{\sqrt{i}} &> \sqrt{8} - 2 \\
&= \sqrt{4 \cdot 2} - 2 \\
&= 2\sqrt{2} - 2 \\
&= 2(\sqrt{2} - 1) \\
&= 2(\sqrt{n+1} - 1)
\end{aligned}
$$

So we have that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > 2(\sqrt{n+1} - 1)$ holds for the base case of $n = 1$.
Inductive step. Now, suppose that $\forall_k \in \mathbb{Z}$ where $k \geq 1$ we have that

$$
\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > 2(\sqrt{k+1} - 1) \qquad \text{(Inductive hypothesis)}
$$

From this, we aim to establish that

$$
\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > 2(\sqrt{k+2} - 1).
$$

We can derive the desired inequality by starting with the trivial fact that $9 > 8$ as follows:

$$
\begin{aligned}
9 &> 8 && \\
4k^2 + 12k + 9 &> 4k^2 + 12k + 8 && \text{adding } 4k^2 + 12k \text{ to both sides} \\
4(k^2 + 3k + \frac{9}{4}) &> 4(k^2 + 3k + 2) && \text{factoring 4 from both sides} \\
4(k + \frac{3}{2})^2 &> 4(k+2)(k+1) && \text{factoring each quadratic separately} \\
(2k+3)^2 &> 4(k+2)(k+1) && \text{distributing the 4 on the left hand side} \\
2k + 3 &> 2\sqrt{k+2}\sqrt{k+1} && \text{taking the square root of both sides} \\
\frac{2k+3}{\sqrt{k+1}} &> 2\sqrt{k+2} && \text{dividing both sides by } \sqrt{k+1} \\
\frac{2k+3}{\sqrt{k+1}} - 2 &> 2\sqrt{k+2} - 2 && \text{subtracting 2 from both sides} \\
\frac{2k+2+1}{\sqrt{k+1}} - 2 &> 2(\sqrt{k+2} - 1) && \text{expanding the numerator on the left hand side}
\end{aligned}
$$

$$\frac{2(k+1)}{\sqrt{k+1}} - 2 + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1) \qquad \text{splitting the fraction on the left hand side and then factoring a 2}$$

For the next step we simply multiply the first term on the left hand side by $\frac{\sqrt{k+1}}{\sqrt{k+1}}$, yielding

$$\begin{aligned}
\frac{2(k+1)\sqrt{k+1}}{(k+1)} - 2 + \frac{1}{\sqrt{k+1}} &> 2(\sqrt{k+2} - 1) \\
2\sqrt{k+1} - 2 + \frac{1}{\sqrt{k+1}} &> 2(\sqrt{k+2} - 1) && \text{canceling } k+1 \text{ in the first term on the left hand side} \\
2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} &> 2(\sqrt{k+2} - 1) && \text{factoring a 2 from the first two terms on the left hand side} \\
\sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} &> 2(\sqrt{k+2} - 1) && \text{by the IH} \\
\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} &> 2(\sqrt{k+2} - 1) && \text{including the second term on the left hand side in the sum}
\end{aligned}$$

Now, by the base case of $n = 1$, the inductive step and the principle of mathematical induction, we have that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > 2(\sqrt{n+1} - 1)$ for all integers $n \geq 1$.

Problem 2

a) Let $b_n = \sum_{i=1}^{n} \frac{2i-1}{2^i}$. Compute b_1, b_2, b_3 and b_4.

$$\begin{aligned}
b_1 &= \sum_{i=1}^{1} \frac{2i-1}{2^i} = \frac{2(1)-1}{2^1} = \frac{1}{2} \\
b_2 &= \sum_{i=1}^{2} \frac{2i-1}{2^i} = b_1 + \frac{2(2)-1}{2^2} = b_1 + \frac{3}{4} = \frac{1}{2} + \frac{3}{4} = \frac{5}{4} \\
b_3 &= \sum_{i=1}^{3} \frac{2i-1}{2^i} = b_2 + \frac{2(3)-1}{2^3} = b_2 + \frac{5}{8} = \frac{5}{4} + \frac{5}{8} = \frac{15}{8} \\
b_4 &= \sum_{i=1}^{4} \frac{2i-1}{2^i} = b_3 + \frac{2(4)-1}{2^4} = b_3 + \frac{7}{16} = \frac{15}{8} + \frac{7}{16} = \frac{37}{16}
\end{aligned}$$

b) Guess a simple formula for b_n.

$$b_n = 3 - \frac{2n+3}{2^n}$$

c) Let $b_1, b_2, ..., b_n$ be the sequence with terms defined by $b_n = \sum_{i=1}^{n} \frac{2i-1}{2^i}$. Prove that $b_n = 3 - \frac{2n+3}{2^n}$ for all integers $n \geq 1$ by induction on n.

Proof (by induction)**.**
Basis $(n = 1)$. Suppose that n is an integer but specifically that $n = 1$. Then we have that

$$b_n = \sum_{i=1}^{n} \frac{2i-1}{2^i} = \sum_{i=1}^{1} \frac{2i-1}{2^i} = \frac{2(1)-1}{2^1}$$

$$
\begin{aligned}
&= \frac{1}{2} \\
&= \frac{6}{2} - \frac{5}{2} \\
&= 3 - \frac{2+3}{2} \\
&= 3 - \frac{2(1)+3}{2^1} \\
&= 3 - \frac{2n+3}{2^n} \qquad \text{substituting } n = 1.
\end{aligned}
$$

Hence we have the equality $b_n = \sum_{i=1}^{n} \frac{2i-1}{2^i} = 3 - \frac{2n+3}{2^n}$ in the base case of $n = 1$.
Inductive step. Now suppose that $\forall_k \in \mathbb{Z}$ where $k \geq 1$ we have that

$$b_k = \sum_{i=1}^{k} \frac{2i-1}{2^i} = 3 - \frac{2k+3}{2^k} \qquad \text{(Inductive hypothesis)}$$

We want to show that

$$\sum_{i=1}^{k+1} \frac{2i-1}{2^i} = 3 - \frac{2(k+1)+3}{2^{k+1}} = 3 - \frac{2k+5}{2^{k+1}}.$$

Beginning with $\sum_{i=1}^{k+1} \frac{2i-1}{2^i}$ we can derive the desired equality as follows

$$
\begin{aligned}
\sum_{i=1}^{k+1} \frac{2i-1}{2^i} &= \sum_{i=1}^{k} \frac{2i-1}{2^i} + \frac{2(k+1)-1}{2^{k+1}} && \text{removing the last term from the sum} \\
&= 3 - \frac{2k+3}{2^k} + \frac{2(k+1)-1}{2^{k+1}} && \text{by the IH} \\
&= 3 - \frac{4k+6}{2^{k+1}} + \frac{2(k+1)-1}{2^{k+1}} && \text{multiplying the second term by } \frac{2}{2} \\
&= 3 + \frac{2k+1-4k-6}{2^{k+1}} && \text{adding the two fractions} \\
&= 3 - \frac{2k+5}{2^{k+1}}
\end{aligned}
$$

Now, by the base case of $n = 1$, the inductive step and the principle of mathematical induction, we have the equality $b_n = \sum_{i=1}^{n} \frac{2i-1}{2^i} = 3 - \frac{2n+3}{2^n}$ for all integers $n \geq 1$.

Problem 3

a) Prove or disprove. For all sets A, B and C, if $A \cup B \subseteq A \cup C$ then $B \subseteq C$.

Solution: the statement is false.

Negation: There exists sets A, B and C such that $A \cup B \subseteq A \cup C$ but $B \nsubseteq C$.

Proof.
Suppose that A, B and C are sets with $A = \{1\}, B = \{1\}$ and $C = \varnothing$. From this, we see that

$$
\begin{aligned}
A \cup B &\subseteq A \cup C \\
\{1\} \cup \{1\} &\subseteq \{1\} \cup \varnothing && \text{substituting the sets supposed for } A, B \text{ and } C \\
\{1\} &\subseteq \{1\} \cup \varnothing && \text{since 1 is the only element which is in } \{1\} \text{ or } \{1\} \\
\{1\} &\subseteq \{1\} && \text{since 1 is the only element which is in } \{1\} \text{ or } \varnothing
\end{aligned}
$$

Which is true since all elements contained in $\{1\}$ are also in $\{1\}$. In fact, these two sets are quite obviously equivalent, implying that each is a subset of the other. Hence, for $A = \{1\}, B = \{1\}$ and $C = \varnothing$ we have that $A \cup B \subseteq A \cup C$. In addition, we have $B \nsubseteq C$ since

$$\begin{aligned} \{1\} &\nsubseteq \varnothing && \text{since 1 is an element of the left hand side, but not the right} \\ B &\nsubseteq C && \text{substituting } B = \{1\} \text{ and } C = \varnothing. \end{aligned}$$

Therefore, for the case of $A = \{1\}, B = \{1\}$ and $C = \varnothing$, we have that $A \cup B \subseteq A \cup C$ but $B \nsubseteq C$. Hence the negation of the original statement has been shown true.

b) Prove or disprove. For all sets A, B and C, if $A \cup B \subseteq A \cup C$ and $A \cap B \subseteq A \cap C$ then $B \subseteq C$.

Solution: the statement is true.

Proof.
Suppose that A, B and C are sets. Now suppose that both $A \cup B \subseteq A \cup C$ and $A \cap B \subseteq A \cap C$. Finally, to show that $B \subseteq C$, suppose that x is an element of B. We aim to show that $x \in C$. Since we have $x \in B$, we have that $x \in A$ or $x \in B$. This means that $x \in A \cup B$. Now, since we supposed that $A \cup B \subseteq A \cup C$, $x \in A \cup B$ implies that $x \in A \cup C$. Furthermore, we have two cases: either $x \in A$ or $x \in C$.

Case 1: $x \in C$
Then, having started originally with $x \in B$, we've shown that $x \in C$.

Case 2: $x \in A$
Then we have that $x \in A$ and $x \in B$, with the latter being one of the original suppositions. This means that $x \in A \cap B$. Since we also have supposed that $A \cap B \subseteq A \cap C$, we have that $x \in A \cap C$, which means that $x \in A$ and $x \in C$. Therefore, in the case that $x \in A$, we necessarily have that $x \in C$ as well.

Therefore, for any three sets A, B and C, if $A \cup B \subseteq A \cup C$ and $A \cap B \subseteq A \cap C$ then any $x \in B$ must also be an element of C – in other words, $B \subseteq C$.

c) Prove or disprove. For all sets A, B and C, if $A \setminus B = A \setminus C$, then $(A \cap B) \setminus C = \varnothing$.

Solution: the statement is true.

Proof.
Suppose that A, B and C are sets such that $A \setminus B = A \setminus C$. This means two things. First, if we have some $x \in A$ with the same $x \notin B$, then $x \in A$ and $x \notin C$. Second, if we have some $x \in A$ with the same $x \notin C$, then we have that $x \in A$ with the same $x \notin B$. We want to establish two things. First, that $\varnothing \subseteq (A \cap B) \setminus C$. This is vacuously true, since $\varnothing$ has no elements. Hence, any element of $\varnothing$ is an element of any set. This makes $\varnothing$ a subset of all sets, including $(A \cap B) \setminus C$. Second, we seek to establish is that $(A \cap B) \setminus C \subseteq \varnothing$. Importantly, the only subset of $\varnothing$ is $\varnothing$ itself, so we next try to show that $(A \cap B) \setminus C$ is the empty set. Assume for the purpose of deriving a contradiction that $(A \cap B) \setminus C$ is not empty, such that there is some $x \in (A \cap B) \setminus C$. This means we have some $x \in A \cap B$ but $x \notin C$. Furthermore, we have both $x \in A$ and $x \in B$. Since we have both $x \in A$ and $x \notin C$, we have $x \in A$ and $x \notin B$. This gives rise to the contradiction $x \in B$, $x \notin B$. To resolve the contradiction, we conclude that $(A \cap B) \setminus C$ must be empty. This means we've shown $(A \cap B) \setminus C \subseteq \varnothing$, since $\varnothing \subseteq \varnothing$. Finally, we've shown that both $\varnothing \subseteq (A \cap B) \setminus C$ and $(A \cap B) \setminus C \subseteq \varnothing$. This means we have that $(A \cap B) \setminus C = \varnothing$. Therefore, for all sets A, B and C, if $A \setminus B = A \setminus C$, then $(A \cap B) \setminus C = \varnothing$.

d) Prove or disprove. For all sets A, B and C, if $(A \cap B) \setminus C = \varnothing$, then $A \setminus B = A \setminus C$.

Solution: the statement is false.

Negation: There exists sets A, B and C such that $(A \cap B) \setminus C = \varnothing$ but $A \setminus B \neq A \setminus C$.

Proof. Suppose A, B and C are sets with $A = \{1\}, B = \varnothing$ and $C = \{1\}$. Then we have that

$$\begin{aligned}
(A \cap B) \setminus C &= (\{1\} \cap \varnothing) \setminus \{1\} \\
&= (\varnothing) \setminus \{1\} && \text{since there are no elements which are in both } \{1\} \text{ and } \varnothing \\
&= \varnothing && \text{since there are no elements which are both in } \varnothing \text{ and not in } \{1\}
\end{aligned}$$

So we have that $(A \cap B) \setminus C = \varnothing$ in this case. Next we show that $A \setminus B \neq A \setminus C$.

$$\begin{aligned}
A \setminus B &= \{1\} \setminus \varnothing \\
&= \{1\} && \text{since only the element 1 is in } \{1\} \text{ while not being in } \varnothing \\
&\neq \varnothing \\
&= \{1\} \setminus \{1\} && \text{since there are no elements in } \{1\} \text{ which aren't elements of } \{1\} \\
&= A \setminus C && \text{substituting } A = \{1\} \text{ and } C = \{1\}
\end{aligned}$$

So we also have that $A \setminus B \neq A \setminus C$ in this case. Therefore, since in the case of $A = \{1\}, B = \varnothing$ and $C = \{1\}$ we have both that $(A \cap B) \setminus C = \varnothing$ and $A \setminus B \neq A \setminus C$, the negation of the original statement has been shown true.