Assignment 3
STAT321

Connor Braun

October 30, 2021

Problem 1

a) Compute probability that no one enters the store from $12:00$ to $12:03$.

Let $\lambda = 1$ be the typical number of people to enter the store every three minutes and let X be the discrete random variable for the number of people to enter the store from $12:00$ to $12:03$. Then we have that

$$X \sim Poisson(\lambda).$$

So we can compute $P(X = 0)$ directly as

$$\begin{aligned} P(X=0) &= \frac{\lambda^0 e^{-\lambda}}{0!} \\ &= e^{-1} \\ &= 0.3678794412 \end{aligned}$$

Solution: $P(X = 0) = 0.3678794412$.

b) Compute probability that no one enters the store from $12:00$ to $12:15$

Let λ_r be the typical number of people to enter the store every fifteen minutes, and let Y be the discrete random variable for the number of people to enter the store from $12:00$ to $12:15$. Then we have that

$$\lambda_r = 5 \times \lambda = 5(1) = 5$$

and

$$Y \sim Poisson(\lambda_r = 5).$$

So we compute $P(Y = 0)$ directly as

$$\begin{aligned} P(Y=0) &= \frac{\lambda_r^0 e^{-\lambda_r}}{0!} \\ &= e^{-5} \\ &= 0.006737946999 \end{aligned}$$

Solution: $P(Y = 0) = 0.006737946999$

c)Determine the probability that ≥ 3 people enter the store from $12:00$ to $12:15$ given that > 1 person enter the store during the same interval.

Let Y and λ_r be as defined in 1.b. Then we have that

$$P(Y \geq 3 | Y > 1) = \frac{P(Y \geq 3 \cap Y > 1)}{P(Y > 1)}.$$

Now, by the law of total probability we can see that

$$P(Y \geq 3) = P(Y \geq 3 \cap Y > 1) + P(Y \geq 3 \cap Y \leq 1).$$

However, $Y \leq 1$ is mutually exclusive with $Y \geq 3$, so $P(Y \geq 3 \cap Y \leq 1) = 0$, so we have that

$$P(Y \geq 3) = P(Y \geq 3 \cap Y > 1).$$

Which now allows us to compute $P(Y \geq 3|Y > 1)$ directly.

$$\begin{aligned}
P(Y \geq 3|Y > 1) &= \frac{P(Y \geq 3 \cap Y > 1)}{P(Y > 1)} \\
&= \frac{P(Y \geq 3)}{P(Y > 1)} \\
&= \frac{1 - P(Y \leq 2)}{1 - P(\leq 1)} \\
&= \frac{1 - 0.124652}{1 - 0.04042768} \qquad \text{(Computed using the } R \text{ code found below)} \\
&= \frac{0.875348}{0.95957232} \\
&= 0.9122272306
\end{aligned}$$

With $P(Y \leq 2)$ and $P(Y \leq 1)$ computed as P_1 and P_2 respectively in the following R code:

```
P_1 <- ppois(2, 5)
P_2 <- ppois(1, 5)
```

Solution: $P(Y \geq 3|Y > 1) = 0.9122272306$.

Problem 2

a) Create a probability distribution table for X.

Let $r = 2$ be the number of red balls, $n = 2$ be the number of balls drawn, and $N = 9$ be the number of balls in the urn. Then we have that

$$X \sim hypergeometric(2, 9, 2).$$

Furthermore, we have that

$$\begin{aligned}
P(X = 0) &= \frac{\binom{2}{0}\binom{7}{2}}{\binom{9}{2}} = \frac{21}{36} = \frac{7}{12} \\
P(X = 1) &= \frac{\binom{2}{1}\binom{7}{1}}{\binom{9}{2}} = \frac{14}{36} = \frac{7}{18} \\
P(X = 2) &= \frac{\binom{2}{2}\binom{7}{0}}{\binom{9}{2}} = \frac{1}{36}
\end{aligned}$$

Solution: So a probability distribution table for X could be

x	0	1	2
$P(X = x)$	$\frac{7}{12}$	$\frac{7}{18}$	$\frac{1}{36}$

b) Compute $E_{[X]}$.

Since $X \sim hypergeometric(2, 9, 2)$, we can compute $E_{[X]}$ directly as

$$E_{[X]} = \frac{(2)(2)}{9} = \frac{4}{9}$$

Solution: one would expect to draw $0.44444\ldots$ red balls from the urn on average (under the conditions specified).

Problem 3

a)Compute $P(X \leq \frac{1}{2})$.

$$\begin{aligned} \int_0^{\frac{1}{2}} f(x)dx &= \int_0^{\frac{1}{2}} xdx \\ &= \frac{x^2}{2}\bigg|_0^{\frac{1}{2}} = \frac{1}{8}. \end{aligned}$$

Solution: $P(X \leq \frac{1}{2}) = \frac{1}{8}$.

b) Compute $P(0 \leq X \leq 2)$.

$$\begin{aligned} \int_0^2 f(x)dx &= \int_0^1 f(x)dx + \int_1^2 f(x)dx \\ &= \int_0^1 xdx + \int_1^2 \frac{1}{4}dx \\ &= \frac{x^2}{2}\bigg|_0^1 + \frac{x}{4}\bigg|_1^2 \\ &= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

Solution: $P(0 \leq X \leq 2) = \frac{3}{4}$.

Problem 4

a) Find c such that $f(x)$ is a probability density function.

We require that

$$\int_0^2 f(x)dx = 1.$$

This allows us to compute c directly by

$$\begin{aligned} 1 = \int_0^2 f(x)dx &= \int_0^2 c(2x - x^2)dx \\ &= c\left[\int_0^2 2xdx - \int_0^2 x^2dx\right] \\ &= c\left[x^2\bigg|_0^2 - \frac{x^3}{3}\bigg|_0^2\right] \end{aligned}$$

$$= c\left[4 - \frac{8}{3}\right]$$

$$\frac{4c}{3} = 1 \Rightarrow c = \frac{3}{4}.$$

Solution: For $f(x)$ to be a valid probability density function, we require that $c = \frac{3}{4}$.

Problem 5

a) Compute probability that the region will see exactly 10 hurricanes over 2 years.

Let X be the discrete random variable for the number of hurricanes the region experiences in 2 years and $\lambda = 4.04$ be the number of hurricanes the region typically experiences per year. Finally, let λ_r be the typical number of hurricanes over a 2 year period. Then we have that

$$\lambda_r = 2 \times \lambda = 8.08$$

and

$$X \sim Poisson(\lambda_r = 8.08).$$

Therefore, we compute the desired probability directly as

$$\begin{aligned}
P(X = 10) &= \frac{\lambda_r^1 0}{e}^{-\lambda_r} 10! \\
&= \frac{8.08^{10} e^{-8.08}}{10!} \\
&= 0.101216464.
\end{aligned}$$

Solution: $P(X = 10) = 0.101216464$.

b) Compute $E_{[X]}$.

With X and λ_r defined as in 5.a, we have that since $X \sim Poisson(\lambda_r)$, we have

$$E_{[X]} = \lambda_r \qquad = 8.08$$

Solution: we'd expect the region to experience 8.08 hurricanes over a 2 year period.

Problem 6

a) Show that $f(x)$ is not a valid probability density function.

$$\begin{aligned}
\int_1^2 f(x)dx &= \int_1^2 2x - 1dx \\
&= \left[x^2 - x\right]\Big|_1^2 \\
&= 2 - 0 = 2
\end{aligned}$$

Solution: since the area under $f(x)$ is $2 > 1$ over the real numbers, $f(x)$ is not a probability density function.

Problem 7

a) Compute the probability that no defective screws appear in the display bin.

Let $N = 500$ be the total number of screws in the case, $r = 11$ be the known number of defective screws and $n = 125$ be the number selected from N without replacement for display. Then let X be the random variable for the number of defective screws in the display bin. We have that

$$X \sim hypergeometric(11, 500, 125).$$

So we can compute $P(X = 0)$ as

$$\begin{aligned} P(X = 0) &= \frac{\binom{11}{0}\binom{489}{125}}{\binom{500}{125}} \\ &= \frac{2.155861 \times 10^{119}}{5.298292 \times 10^{120}} \\ &= 0.04068972683 \end{aligned}$$

Solution: $P(X = 0) = 0.04068972683$.

b) Compute $P(X \geq 2)$.

Let N, n, r and X be as defined in 7.a. Then we seek to compute directly $P(X \geq 2)$ as

$$\begin{aligned} P(X \geq 2) &= 1 - P(X \leq 1) \\ &= 1 - (P(X = 0) + P(X = 1)) \\ &= 1 - \left(0.04068972683 + \frac{\binom{11}{1}\binom{489}{124}}{\binom{500}{125}}\right) \qquad \text{(from 7.}a\text{)} \\ &= 1 - (0.04068972683 + 0.15328321751) \\ &= 1 - (0.1939729443) \\ &= 0.8060270557 \end{aligned}$$

Solution: $P(X \geq 2) = 0.8060270557$.

c) Compute $P(X = 2) + P(X = 3)$.

Let N, n, r and X be as defined in 7.a. Then we compute $P(X = 2) + P(X = 3)$ by first computing each probability separately, and then adding the results.

$$\begin{aligned} P(X = 2) &= \frac{\binom{11}{2}\binom{489}{123}}{\binom{500}{125}} \\ &= 0.25966009524 \\ P(X = 3) &= \frac{\binom{11}{3}\binom{489}{122}}{\binom{500}{125}} \\ &= 0.26107513663 \\ P(X = 2) + P(X = 3) &= 0.25966009524 + 0.26107513663 = 0.5207352319 \end{aligned}$$

Solution: $P(X = 2) + P(X = 3) = 0.5207352319$.

Problem 8

a)Show that $E_{[X]} = \lambda$ for $X \sim Poisson(\lambda)$ using the provided $M_X(t)$ for the Poisson distribution.

We have that

$$M_X(t) = e^{\lambda(e^t - 1)}$$

And we know that $E_{[X]} = \frac{d}{dt} M_X(t) \Big|_{t=0}$, so we proceed to differentiate $M_X(t)$ with respect to t.

$$\begin{aligned}
E_{[X]} &= \frac{d}{dt} M_X(t) \\
&= \frac{d}{dt} e^{\lambda(e^t - 1)} \\
&= \lambda e^t e^{\lambda(e^t - 1)} \\
&= \lambda e^{\lambda e^t - \lambda + t}
\end{aligned}$$

Then we simply evaluate the equation at $t = 0$

$$E_{[X]} = \lambda e^{\lambda e^t - \lambda + t} \Big|_{t=0} = \lambda e^{\lambda - \lambda} = \lambda e^0 = \lambda.$$

Solution: thus, by the above we have verifed that $E_{[X]} = \lambda$ for $X \sim Poisson(\lambda)$.