However division is not a binary operation on the set of reac numbers, since the quotient $x=y$ is not de ned when $y = 0$. (Under a binary operation on a set must determine an element $x \quad y$ of the set for every pair of elements x and y of that set.)

4.2 Commutative Binary Operations

De nition A binary operation on a set A is said to be *commutative* if $x \quad y = y \quad x$

De nition

4.6 Identity elements

De nition Let $(A;\ )$ be a semigroup. An element e of A is said to be an *idents element* for the binary operation if $e \ x = x \ e = x$ for all elements x of

Proof Let e denote the identity element of the monoid. Then $x \quad x^{1} =$
x

The appropriate definitions ensure that the identity $a^m \; a^n = a^{m+n}$ holds if $m = 0$ or if $n = 0$.

The result has already been verified if both m and

Example Let n be a natural number, and let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$$

$_9$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	4	6	8	1	3	5	7
3	0	3	6	0	3	6	0	3	6
4	0	4	8	3	7	2	6	1	5
5	0	5	1	6	2	7	3	8	4
6	0	6	3	0	6	3	0	6	3
7	0	7	5	3	1	8	6	4	2
8	0	8	7	6	5	4	3	2	1

We recall that a function $f\colon A \to B$ is said to be *injective* if distinct

The de nitions of addition and subtraction are straightforward. The *sum* and *di erence* of two quaternions $w + xi + yj + zk$ and *ww* *ofxiw* *w* above formula de ning multiplication of quaternions that

$$i^2 = j^2 = k^2 = \quad 1;$$

$$ij = \quad ji = k; \quad jk = \quad kj = i; \quad ki = \quad ik = j;$$

where $i^2 = i \quad i$, *1*

4.13 Quaternions and Rotations

Let us consider the e ect of a rotation through an angle about an axis in three-dimensional space passing through the origin. Let l, m and n be the cosines of the angles between the axis of the rotation and the three coordinate axes. In Cartesian coordinates, the axis of rotation is then in the direction of the vector $(l; m; n)$, where $l^2 + m$

since

$$
\begin{aligned}
q\,\overline{q} &= \cos\frac{}{2} + \sin\frac{}{2}\mathbf{u} \quad \cos\frac{}{2} \quad \sin\frac{}{2}\mathbf{u} \\
&= \cos^2\frac{}{2} + \quad \sin^2\frac{}{2} \quad \mathbf{u} : \mathbf{u} \\
&\qquad\quad \sin^2\frac{}{2} \quad \mathbf{u} \wedge \mathbf{u} \\
&= \cos^2\frac{}{2} + \sin^2\frac{}{2} \\
&= 1.
\end{aligned}
$$

Also we nd that

$$
\begin{aligned}
q^2 &= \cos\frac{}{2} + \sin\frac{}{2}\mathbf{u} \quad \cos\frac{}{2} + \sin\frac{}{2}\mathbf{u} \\
&= \cos^2\frac{}{2} \qquad \sin^2\frac{}{2} \quad \mathbf{u} : \mathbf{u} \\
&\qquad + 2\sin\frac{}{2}\cos\frac{}{2}\mathbf{u} + \quad \sin^2\frac{}{2} \quad \mathbf{u} \wedge \mathbf{u} \\
&= \cos^2\frac{}{2} \quad \sin^2\frac{}{2} + 2\sin\frac{}{2}\cos\frac{}{2}\mathbf{u} \\
&= \cos \quad + \sin \quad \mathbf{u}.
\end{aligned}
$$

Let us now calculate the quaternion products $q\mathbf{u}$

$$= \cos^2_2$$