Course 2BA1: Trinity Term 2003
Section 7: Harmonic Analysis

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7 Harmonic Analysis

7.1 Basic Trigonometric Identities

An anticlockwise rotation about the origin through an angle of θ radians sends a point (x, y) of the plane to the point (x', y'), where

$$\left\{ \begin{array}{rcl} x' &=& x\cos\theta - y\sin\theta \\ y' &=& x\sin\theta + y\cos\theta \end{array} \right. \tag{1}$$

(This follows easily from the fact that such a rotation takes the point $(1, 0)$ to the point $(\cos\theta, \sin\theta)$ and takes the point $(0, 1)$ to the point $(-\sin\theta, \cos\theta)$.) An anticlockwise rotation about the origin through an angle of ϕ radians then sends the point (x', y') of the plane to the point (x'', y''), where

$$\left\{ \begin{array}{rcl} x'' &=& x'\cos\phi - y'\sin\phi \\ y'' &=& x'\sin\phi + y'\cos\phi \end{array} \right. \tag{2}$$

Now an anticlockwise rotation about the origin through an angle of $\theta+\phi$ radians sends the point (x, y), of the plane to the point (x'', y''), and thus

$$\left\{\begin{array}{lcl} x'' & = & x\cos(\theta+\phi) - y\sin(\theta+\phi) \\ y'' & = & x\sin(\theta+\phi) + y\cos(\theta+\phi) \end{array}\right. \tag{3}$$

But if we substitute the expressions for x' and y' in terms of x, y and θ provided by equation (1) into equation (2), we find that

$$\left\{\begin{array}{lcl} x'' & = & x(\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) - y(\sin\theta\,\cos\phi + \cos\theta\,\sin\phi) \\ y'' & = & x(\sin\theta\,\cos\phi + \cos\theta\,\sin\phi) + y(\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) \end{array}\right. \tag{4}$$

On comparing equations (3) and (4) we see that

$$\cos(\theta+\phi) = \cos\theta\,\cos\phi - \sin\theta\,\sin\phi, \tag{5}$$

and

$$\sin(\theta+\phi) = \sin\theta\,\cos\phi + \cos\theta\,\sin\phi. \tag{6}$$

On replacing ϕ by $-\phi$, and noting that $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$, we find that

$$\cos(\theta-\phi) = \cos\theta\,\cos\phi + \sin\theta\,\sin\phi, \tag{7}$$

and

$$\sin(\theta-\phi) = \sin\theta\,\cos\phi - \cos\theta\,\sin\phi. \tag{8}$$

If we add equations (5) and (7) we find that

$$\cos\theta\,\cos\phi = \tfrac{1}{2}(\cos(\theta+\phi) + \cos(\theta-\phi)). \tag{9}$$

If we subtract equation (5) from equation (7) we find that

$$\sin\theta\,\sin\phi = \tfrac{1}{2}(\cos(\theta-\phi) - \cos(\theta+\phi)). \tag{10}$$

And if we add equations (6) and (8) we find that

$$\sin\theta\,\cos\phi = \tfrac{1}{2}(\sin(\theta+\phi) + \sin(\theta-\phi)). \tag{11}$$

If we substitute $\phi=\theta$ in equations (5) and (6), and use the identity $\cos^2\theta + \sin^2\theta = 1$, we find that

$$\sin 2\theta = 2\sin\theta\,\cos\theta \tag{12}$$

and

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta. \tag{13}$$

It then follows from equation (13) that

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta) \tag{14}$$

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta). \tag{15}$$

Remark Equations (1) and (2) may be written in matrix form as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Also equation (3) may be written

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

It follows from basic properties of matrix multiplication that

$$\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and therefore

$$\begin{aligned} \cos(\theta+\phi) &= \cos\theta\,\cos\phi - \sin\theta\,\sin\phi \\ \sin(\theta+\phi) &= \sin\theta\,\cos\phi + \cos\theta\,\sin\phi. \end{aligned}$$

This provides an alternative derivation of equations (5) and (6).

7.2 Basic Trigonometric Integrals

On differentiating the sine and cosine function, we find that

$$\frac{d}{dx}\sin kx = k\cos kx \tag{16}$$
$$\frac{d}{dx}\cos kx = -k\sin kx. \tag{17}$$

for all real numbers k.

It follows that

$$\int \sin kx = -\frac{1}{k}\cos kx + C \tag{18}$$
$$\int \cos kx = \frac{1}{k}\sin kx + C, \tag{19}$$

for all non-zero real numbers k, where C is a constant of integration.

Theorem 7.1 *Let m and n be positive integers. Then*

$$
\begin{aligned}
\int_{-\pi}^{\pi} \cos nx\,dx &= 0, && (20)\\
\int_{-\pi}^{\pi} \sin nx\,dx &= 0, && (21)\\
\int_{-\pi}^{\pi} \cos mx\,\cos nx\,dx &= \begin{cases} \pi & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} && (22)\\
\int_{-\pi}^{\pi} \sin mx\,\sin nx\,dx &= \begin{cases} \pi & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} && (23)\\
\int_{-\pi}^{\pi} \sin mx\,\cos nx\,dx &= 0. && (24)
\end{aligned}
$$

Proof First we note that

$$\int_{-\pi}^{\pi} \cos nx\,dx = \left[\frac{1}{n}\sin nx\right]_{-\pi}^{\pi} = \frac{1}{n}\left(\sin n\pi - \sin(-n\pi)\right) = 0$$

and

$$\int_{-\pi}^{\pi} \sin nx\,dx = \left[-\frac{1}{n}\cos nx\right]_{-\pi}^{\pi} = -\frac{1}{n}\left(\cos n\pi - \cos(-n\pi)\right) = 0$$

for all non-zero integers n, since $\cos n\pi = \cos(-n\pi) = (-1)^n$ and $\sin n\pi = \sin(-n\pi) = 0$ for all integers n.

Let m and n be positive integers. It follows from equations (9) and (10) that

$$\int_{-\pi}^{\pi} \cos mx\,\cos nx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi} \left(\cos((m-n)x) + \cos((m+n)x)\right)dx.$$

and

$$\int_{-\pi}^{\pi} \sin mx\,\sin nx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi} \left(\cos((m-n)x) - \cos((m+n)x)\right)dx$$

But

$$\int_{-\pi}^{\pi} \cos((m+n)x)\,dx = 0$$

(since $m+n$ is a positive integer, and is thus non-zero). Also

$$\int_{-\pi}^{\pi} \cos((m-n)x)\,dx = 0 \text{ if } m \neq n,$$

and

$$\int_{-\pi}^{\pi} \cos((m-n)x)\,dx = 2\pi \text{ if } m = n$$

(since $\cos((m-n)x) = 1$ when $m = n$). It follows that

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx\, \cos nx\,dx &= \int_{-\pi}^{\pi} \sin mx\, \sin nx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x)\,dx \\
&= \begin{cases} \pi & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}
\end{aligned}$$

Using equation (11), we see also that

$$\int_{-\pi}^{\pi} \sin mx\, \cos nx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi} (\sin((m+n)x) + \sin((m-n)x))\,dx = 0$$

for all positive integers m and n. (Note that $\sin((m-n)x) = 0$ in the case when $m = n$). ∎

7.3 Fourier Series

Definition A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *periodic* if there exists some positive real number l such that $f(x+l) = f(x)$ for all real numbers x. The smallest real number l with this property is the *period* of the periodic function f.

A periodic function f with period l satisfies $f(x+ml) = f(x)$ for all real numbers x and integers m.

The period l of a periodic function f is said to *divide* some positive real number K if K/l is an integer. If the period of the function f divides a positive real number K then $f(x+mK) = f(x)$ for all real numbers x and integers m.

Mathematicians have proved that if $f\colon \mathbb{R} \to \mathbb{R}$ is any sufficiently well-behaved function from $\mathbb{R}$ to $\mathbb{R}$ with the property that $f(x+2\pi) = f(x)$ for all real numbers x then f may be represented as an infinite series of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \tag{25}$$

In particular it follows from theorems proved by Dirichlet in 1829 that a function $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies $f(x+2\pi) = f(x)$ for all real numbers x can be represented as a trigonometrical series of this form if the function is bounded, with at most finitely many points of discontinuity, local maxima and local minima in the interval $[-\pi, \pi]$, and if

$$f(x) = \tfrac{1}{2}\left(\lim_{h\to 0+} f(x+h) + \lim_{h\to 0+} f(x-h)\right)$$

at each value x at which the function is discontinuous (where $\lim_{h\to 0+} f(x+h)$ and $\lim_{h\to 0+} f(x-h)$ denote the limits of $f(x+h)$ and $f(x-h)$ respectively as h tends to 0 from above).

Fourier in 1807 had observed that if a sufficiently well-behaved function could be expressed as the sum of a trigonometrical series of the above form, then

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, && (26)\\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, && (27)\\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx && (28)\end{aligned}$$

for each positive integer n. These expressions for the coefficients a_n and b_n may readily be verified on substituting the trigonometic series for the function f (equation (25)) into the integrals on the right hand side of the equation, provided that one is permitted to interchange the operations of integration and summation in the resulting expressions.

Now it is not generally true that the integral of an infinite sum of functions is necessarily equal to the sum of the integrals of those functions. However if the function f is sufficiently well-behaved then the trigonometric series for the function f will converge sufficiently rapidly for this interchange of integration and summation to be valid, so that

$$\int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} a_m \cos mx\right) dx = \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx\,dx \quad \text{etc.}$$

If we interchange summations and integrations in this fashion and make use of the trigonometric integrals provided by Theorem 7.1, we find that

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,dx &= a_0\pi + \int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} a_m \cos mx\right) dx + \int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} b_m \sin mx\right) dx \\ &= a_0\pi + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx\,dx \\ &= a_0\pi, \end{aligned}$$

Also

$$
\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&\quad = \tfrac{1}{2}a_0 \int_{-\pi}^{\pi} \cos nx\,dx + \int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} a_m \cos mx\,\cos nx\right) dx \\
&\qquad\quad + \int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} b_m \sin mx\,\cos nx\right) dx \\
&\quad = \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx\,\cos nx\,dx + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx\,\cos nx\,dx \\
&\quad = a_n\pi, \\
&\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&\quad = \tfrac{1}{2}a_0 \int_{-\pi}^{\pi} \sin nx\,dx + \int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} a_m \cos mx\,\sin nx\right) dx \\
&\qquad\quad + \int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} b_m \sin mx\,\sin nx\right) dx \\
&\quad = \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx\,\sin nx\,dx + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx\,\sin nx\,dx \\
&\quad = b_n\pi.
\end{aligned}
$$

A trigonometric series of the form (25) with coefficients a_n and b_n given by the integrals (26), (27) and (28) is referred to the *Fourier series* for the function f. The coefficients defined by the integrals (26), (27) and (28) are referred to as the *Fourier coefficients* of the function f.

Example Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$
f(x) = \begin{cases} \frac{1}{2} & \text{if } x = m\pi \text{ for some integer } m; \\ 1 & \text{if } 2m\pi < x < (2m+1)\pi \text{ for some integer } m; \\ 0 & \text{if } (2m-1)\pi < x < 2m\pi \text{ for some integer } m. \end{cases}
$$

This function f has the property that $f(x) = f(x + 2m\pi)$ for all real numbers x and integers m, and can be represented by a Fourier series. The coefficients a_n and b_n of the Fourier series are given by the formulae

$$
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n>0),\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n>0),
\end{aligned}
$$

Now $f(x) = 0$ if $-\pi < x < 0$, and $f(x) = 1$ if $0 < x < \pi$. Therefore $a_0 = 1$, and

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx = \frac{1}{n\pi}\left[\sin nx\right]_0^\pi\\
&= 0 \qquad (n>0),\\
b_n &= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx = \frac{1}{n\pi}\left[-\cos nx\right]_0^\pi\\
&= \frac{1}{n\pi}(1-\cos n\pi) = \frac{1}{n\pi}(1-(-1)^n)\\
&= \begin{cases} \dfrac{2}{n\pi} & \text{if } n \text{ is odd and } n>0,\\ 0 & \text{if } n \text{ is even and } n>0, \end{cases}
\end{aligned}
$$

(We have here made use of the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for all integers n.) Thus

$$
\begin{aligned}
f(x) &= \frac{1}{2} + \sum_{\substack{n\,\text{odd}\\ n>0}} \frac{2}{n\pi}\sin nx\\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}\sin\left((2k-1)x\right)
\end{aligned}
$$

7.4 Fourier Series of Even and Odd Functions

Definition A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *even* if $f(x) = f(-x)$ for all real numbers x. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *odd* if $f(x) = -f(-x)$ for all real numbers x.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be an integrable function. Then

$$
\int_{-\pi}^{0} f(x)\,dx = \int_{0}^{\pi} f(-x)\,dx, \tag{29}
$$

$$
\int_{-\pi}^{0} f(x)\cos nx\,dx = \int_{0}^{\pi} f(-x)\cos nx\,dx, \tag{30}
$$

$$
\int_{-\pi}^{0} f(x)\sin nx\,dx = -\int_{0}^{\pi} f(-x)\sin nx\,dx \tag{31}
$$

(The first of these identities may be verified by making the substitution $x \mapsto -x$ and then interchanging the two limits of integration. The second and the third follow from the first on replacing $f(x)$ by $f(x)\cos nx$ and $f(x)\sin nx$ and noting that $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$.) It follows that the Fourier coefficients of f are given by the following formulae:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} (f(x)+f(-x))\,dx, && (32) \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} (f(x)+f(-x))\cos nx\,dx, && (33) \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} (f(x)-f(-x))\sin nx\,dx && (34) \end{aligned}$$

for all positive integers n.

Of course $f(x) + f(-x) = 2f(x)$ and $f(x) - f(-x) = 0$ for all real numbers x if the function $f\colon \mathbb{R} \to \mathbb{R}$ is even, and $f(x) + f(-x) = 0$ and $f(x) - f(-x) = 2f(x)$ if the function $f\colon \mathbb{R} \to \mathbb{R}$ is odd. The following results follow immediately,

Theorem 7.2 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an even periodic function whose period divides 2π. Suppose that the function f may be represented by a Fourier series. Then the Fourier series of f is of the form*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

for all positive integers n.

Theorem 7.3 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an odd periodic function whose period divides 2π. Suppose that the function f may be represented by a Fourier series. Then the Fourier series of f is of the form*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

for all positive integers n.

7.5 Fourier Series for General Periodic Functions

Let $f: \mathbb{R} \to \mathbb{R}$ be a periodic function whose period divides l, where l is some positive real number. Then $f(x + l) = f(x)$ for all real numbers x. Let

$$g(x) = f\left(\frac{lx}{2\pi}\right) \quad \text{so that} \quad f(x) = g\left(\frac{2\pi x}{l}\right).$$

Then $g: \mathbb{R} \to \mathbb{R}$ is a periodic function, and $g(x + 2\pi) = g(x)$ for all real numbers x. If the function f is sufficiently well-behaved (and, in particular, if the function f is bounded, with only finitely many local maxima and minima and points of discontinuity in any finite interval, and if $f(x)$ at each point of discontinuity is the average of the limits of $f(x + h)$ and $f(x - h)$ as h tends to zero from above) then the function g may be represented by a Fourier series of the form

$$g(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

The coefficients of this Fourier series are then given by the formulae

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\,du, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos nu\,du, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin nu\,du
\end{aligned}$$

for each positive integer n. If we make the substitution $u = \dfrac{2\pi x}{l}$ in these integrals, we find that

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{l}\right), \tag{35}$$

where

$$\begin{aligned}
a_0 &= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} g\left(\frac{2\pi x}{l}\right)\,dx \\
&= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} f(x)\,dx, \\
a_n &= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} g\left(\frac{2\pi x}{l}\right)\cos\left(\frac{2n\pi x}{l}\right)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} f(x)\cos\left(\frac{2n\pi x}{l}\right)\,dx,\\
b_n &= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} g\left(\frac{2\pi x}{l}\right)\sin\left(\frac{2n\pi x}{l}\right)\,dx\\
&= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} f(x)\sin\left(\frac{2n\pi x}{l}\right)\,dx
\end{aligned}
$$

for all positive integers n. Note that these integrals are taken over a single period of the function, from $-\frac{1}{2}l$ to $+\frac{1}{2}l$. It follows from the periodicity of the integrand that these integrals may be replaced by integrals from c to $c+l$ for any real number c, and thus

$$
\begin{aligned}
a_0 &= \frac{2}{l}\int_{c}^{c+l} f(x)\,dx, &(36)\\
a_n &= \frac{2}{l}\int_{c}^{c+l} f(x)\cos\left(\frac{2n\pi x}{l}\right)\,dx, &(37)\\
b_n &= \frac{2}{l}\int_{c}^{c+l} f(x)\sin\left(\frac{2n\pi x}{l}\right)\,dx &(38)
\end{aligned}
$$

for all positive integers n. (Indeed, if $h\colon \mathbb{R} \to \mathbb{R}$ is any integrable function with the property that $h(x+l) = h(x)$ for all real numbers x, and if p and q are real numbers with $p \leq q \leq p+l$ then

$$
\begin{aligned}
\int_p^{p+l} h(x)\,dx &= \int_p^q h(x)\,dx + \int_q^{p+l} h(x)\,dx\\
&= \int_{p+l}^{q+l} h(x)\,dx + \int_q^{p+l} h(x)\,dx = \int_q^{q+l} h(x)\,dx.
\end{aligned}
$$

Repeated applications of this identity show that

$$
\int_p^{p+l} h(x)\,dx = \int_q^{q+l} h(x)\,dx
$$

for all real numbers p and q.)

Example Let k be a positive real number. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$
f(x) = \begin{cases} e^{k(x-m)} & \text{if } m < x < m+1 \text{ for some integer } m;\\ \frac{1}{2}(e^k+1) & \text{if } x \text{ is an integer.} \end{cases}
$$

This function is periodic, with period 1, and may be expanded as a Fourier series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x + \sum_{n=1}^{\infty} b_n \sin 2n\pi x,$$

where

$$\begin{aligned} a_0 &= 2\int_0^1 f(x)\,dx, \\ a_n &= 2\int_0^1 f(x)\cos 2n\pi x\,dx \qquad (n > 0), \\ b_n &= 2\int_0^1 f(x)\sin 2n\pi x\,dx \qquad (n > 0). \end{aligned}$$

Note that $f(x) = e^{kx}$ if $0 < x < 1$. We see therefore that

$$a_0 = 2\int_0^1 e^{kx}\,dx = \frac{2}{k}\left[e^{kx}\right]_0^1 = \frac{2}{k}(e^k - 1).$$

Now if k and ω a positive real numbers then

$$\begin{aligned} \int e^{kx}\cos\omega x\,dx &= \frac{k}{k^2+\omega^2}e^{kx}\cos\omega x + \frac{\omega}{k^2+\omega^2}e^{kx}\sin\omega x + C, \\ \int e^{kx}\sin\omega x\,dx &= \frac{k}{k^2+\omega^2}e^{kx}\sin\omega x - \frac{\omega}{k^2+\omega^2}e^{kx}\cos\omega x + C, \end{aligned}$$

where C is a constant of integration. (These identities may be verified by differentiating the expressions on the right hand side.) We find therefore that

$$\begin{aligned} a_n &= 2\int_0^1 e^{kx}\cos 2n\pi x\,dx \\ &= \left[\frac{2k}{k^2+4n^2\pi^2}e^{kx}\cos 2n\pi x + \frac{4n\pi}{k^2+4n^2\pi^2}e^{kx}\sin 2n\pi x\right]_0^1 \\ &= \frac{2k}{k^2+4n^2\pi^2}(e^k - 1) \\ b_n &= 2\int_0^1 e^{kx}\sin 2n\pi x\,dx, \\ &= \left[\frac{2k}{k^2+4n^2\pi^2}e^{kx}\sin 2n\pi x - \frac{4n\pi}{k^2+4n^2\pi^2}e^{kx}\cos 2n\pi x\right]_0^1 \\ &= -\frac{4n\pi}{k^2+4n^2\pi^2}(e^k - 1), \end{aligned}$$

for each positive integer n, since $\cos 2n\pi = 1$ and $\sin 2n\pi = 0$ when n is an integer. Thus

$$\begin{aligned} e^{kx} &= \frac{1}{k}(e^k - 1) + \sum_{n=1}^{\infty} \frac{2k}{k^2 + 4n^2\pi^2}(e^k - 1)\cos 2n\pi x \\ &\quad - \sum_{n=1}^{\infty} \frac{4n\pi}{k^2 + 4n^2\pi^2}(e^k - 1)\sin 2n\pi x \end{aligned}$$

for all real numbers x satisfying $0 < x < 1$.

7.6 Sine Series

Let $f\colon [0,l] \to \mathbb{R}$ be a function defined on the interval $[0,l]$, where $[0,l] = \{x \in \mathbb{R} : 0 \leq x \leq l\}$. Suppose that $f(0) = f(l) = 0$. Let $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ be the function defined such that

$$\tilde{f}(x) = f(x - 2nl) \text{ if } 2nl \leq x \leq (2n+1)l \text{ for some integer } n$$

and

$$\tilde{f}(x) = -f((2n+2)l - x) \text{ if } (2n+1)l \leq x \leq (2n+2)l \text{ for some integer } n.$$

The function $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is an odd function with the property that $\tilde{f}(x+2l) = \tilde{f}(x)$ for all real numbers x. Indeed it is easily seen that $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is the unique odd function with this property which agrees with the function f on the interval $[0,l]$.

If the function f is sufficiently well-behaved (and, in particular, if the function f is bounded, with at most finitely many local maxima and minima and points of discontinuity, and has the property that $f(x)$ at each point of discontinuity is the average of the limits of $f(x+h)$ and $f(x-h)$ as h tends to zero from above) then the function $\tilde{f}$ may be represented as a Fourier series. This Fourier series is of the form

$$\tilde{f}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where

$$\begin{aligned} b_n &= \frac{1}{l}\int_{-l}^{l} \tilde{f}(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\int_{0}^{l} \tilde{f}(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \end{aligned}$$

for all positive integers n. (This follows from equations (35) and (38) on replacing l by $2l$, and then using the fact that $\tilde{f}(-x) = -\tilde{f}(x)$ for all real numbers x.)

Therefore every sufficiently well-behaved function $f\colon [0,l] \to \mathbb{R}$ which satisfies $f(0) = f(l) = 0$ may be represented in the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \tag{39}$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right)\, dx \tag{40}$$

for each positive integer n.

Example Let l be a positive real numbers, and let $f\colon [0,l] \to \mathbb{R}$ be the function defined by $f(x) = x(l-x)$ (where $0 \le x \le l$). This function can be expanded in a sine series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

Using the method of integration by parts, and the result that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for all integers n, we find then

$$\begin{aligned}
b_n &= \frac{2}{l} \int_0^l x(l-x) \sin \frac{n\pi x}{l}\, dx = -\frac{2}{n\pi} \int_0^l x(l-x) \frac{d}{dx}\left(\cos \frac{n\pi x}{l}\right)\, dx \\
&= -\frac{2}{n\pi} \left[x(l-x) \cos \frac{n\pi x}{l} \right]_0^l + \frac{2}{n\pi} \int_0^l (l-2x) \cos \frac{n\pi x}{l}\, dx \\
&= \frac{2}{n\pi} \int_0^l (l-2x) \cos \frac{n\pi x}{l}\, dx \\
&= \frac{2l}{n^2\pi^2} \int_0^l (l-2x) \frac{d}{dx}\left(\sin \frac{n\pi x}{l}\right)\, dx \\
&= \frac{2l}{n^2\pi^2} \left[(l-2x) \sin \frac{n\pi x}{l} \right]_0^l - \frac{2l}{n^2\pi^2} \int_0^l \left(-2 \sin \frac{n\pi x}{l}\right)\, dx \\
&= \frac{4l}{n^2\pi^2} \int_0^l \sin \frac{n\pi x}{l}\, dx = -\frac{4l^2}{n^3\pi^3} \left[\cos \frac{n\pi x}{l} \right]_0^l
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4l^2}{n^3\pi^3}(1-\cos n\pi) = \frac{4l^2}{n^3\pi^3}(1-(-1)^n) \\
&= \begin{cases} \dfrac{8l^2}{n^3\pi^3} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
$$

Thus

$$
f(x) = \sum_{\substack{n\,\mathrm{odd}\\ n>0}} \frac{8l^2}{n^3\pi^3} \sin \frac{n\pi x}{l}.
$$

or (setting $n = 2k-1$ for each positive integer k),

$$
x(l-x) = \sum_{k=1}^{\infty} \frac{8l^2}{(2k-1)^3\pi^3} \sin \frac{(2k-1)\pi x}{l} \qquad (0 \le x \le l).
$$

7.7 Cosine Series

Let $f\colon [0,l] \to \mathbb{R}$ be a function defined on the interval $[0,l]$, where $[0,l] = \{x \in \mathbb{R} : 0 \le x \le l\}$. Let $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$
\tilde{g}(x) = f(x - 2nl) \text{ if } 2nl \le x \le (2n+1)l \text{ for some integer } n
$$

and

$$
\tilde{g}(x) = f((2n+2)l - x) \text{ if } (2n+1)l \le x \le (2n+2)l \text{ for some integer } n.
$$

The function $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ is an even function with the property that $\tilde{g}(x+2l) = \tilde{g}(x)$ for all real numbers x. Indeed it is easily seen that $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ is the unique even function with this property which agrees with the function f on the interval $[0,l]$.

If the function f is sufficiently well-behaved then the function $\tilde{g}$ may be represented as a Fourier series. This Fourier series is of the form

$$
\tilde{g}(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right),
$$

where

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} \tilde{g}(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\
&= \frac{2}{l}\int_{0}^{l} \tilde{g}(x)\cos\left(\frac{n\pi x}{l}\right)\,dx
\end{aligned}
$$

for all non-negative integers n. (This follows from equations (35), (36) and (37) on replacing l by $2l$, and then using the fact that $\tilde{g}(-x) = \tilde{g}(x)$ for all real numbers x.)

Therefore every sufficiently well-behaved function $f\colon [0,l] \to \mathbb{R}$ may be represented in the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \tag{41}$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \tag{42}$$

for each positive integer n.

Example Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = x \quad \text{if } 0 \le x \le 1.$$

This function may be represented as a cosine series of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

where

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 x\,dx = 1,$$

and where

$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx$$

for all positive integers n. Using the method of integation by parts, and making use of the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for all integers n, we find that

$$\begin{aligned}
a_n &= 2\int_0^1 x\cos n\pi x\,dx = \frac{2}{n\pi}\int_0^1 x\frac{d}{dx}\left(\sin n\pi x\right)\,dx \\
&= \frac{2}{n\pi}\left[x\sin n\pi x\right]_0^1 - \frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx = -\frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx \\
&= \frac{2}{n^2\pi^2}\left[\cos n\pi x\right]_0^1 = -\frac{2}{n^2\pi^2}\left(1-(-1)^n\right) \\
&= \begin{cases} -\dfrac{4}{n^2\pi^2} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Thus

$$x = \frac{1}{2} - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2} \cos n\pi x \quad \text{when } 0 \le x \le 1.$$

Remark The function $\tilde{g}$ defined by

$$\tilde{g}(x) = \frac{1}{2} - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2} \cos n\pi x$$

for all real numbers x is an even periodic function, with period equal to 2, which coincides with the function $f\colon [0,1] \to \mathbb{R}$ on the interval $[0,1]$, where $f(x) = x$ for all real numbers x satisfying $0 \le x \le 1$. It follows that

$$\tilde{g}(x) = |x - 2m| \quad \text{whenever } m \text{ is an integer and } 2m-1 \le x \le 2m+1.$$

Remark Setting $x = 1$ in the identity

$$x = \frac{1}{2} - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2} \cos n\pi x \quad \text{when } 0 \le x \le 1,$$

we find that

$$1 = \frac{1}{2} - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2} \cos n\pi = \frac{1}{2} + \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2}$$

and thus

$$\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

But

$$\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \left(1 - \frac{1}{4}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$