Course 2BA1: Hilary Term 2003
Section 6: Ordinary Differential Equations

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6 Ordinary Differential Equations

6.1 Ordinary Differential Equations

An *ordinary differential equation* is an equation of the form

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0,$$

where F is some function of $n+2$ real variables. Such an equation determines a variable y as a function of an independent variable x. Such a differential equation is said to be of order n if it involves no derivatives of y of orders higher than n and if the nth derivative of y occurs non-trivially in the equation.

A differential equation of order n may often be expressed in the form

$$\frac{d^n y}{dx^n} = G\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

in which the nth derivative of y is expressed as a function of y itself and the first $n-1$ derivatives of y.

We shall investigate the solutions of certain specific types of differential eqations.

6.2 Homogeneous Linear Differential Equations

A homogeneous linear differential equation of order n is a differential equation of the form

$$a_0(x)y + \sum_{j=1}^{n} a_j(x)\frac{d^j y}{dx^j} = 0,$$

where $a_0, a_1, \ldots, a_n$ are functions of the independent variable x, and where a_n is not the zero function. We shall assume furthermore that these functions $a_0, a_1, \ldots, a_n$ are suitably well-behaved (i.e., that they satisfy appropriate continuity and differentiability conditions). We may express such a differential equation in the form

$$Ly = 0,$$

where L is the differential operator of order n given by

$$a_0(x) + \sum_{j=1}^{n} a_j(x)\frac{d^j}{dx^j}.$$

Such a differential operator takes any n-times differentiable function y of x to the function Ly, where

$$Ly = a_0(x)y + \sum_{j=1}^{n} a_j(x)\frac{d^j y}{dx^j}.$$

If y is any n-times differentiable function of x, and if A is any real number, then $L(Ay) = A(Ly)$. Also $L(y_1 + y_2) = Ly_1 + Ly_2$ for any n-times differentiable functions y_1 and y_2. It follows that

$$L(A_1y_1 + A_2y_2) = A_1(Ly_1) + A_2(Ly_2)$$

for all n-times differentiable functions y_1 and y_2, and for all real numbers A_1 and A_2. It follows that if y_1 and y_2 are solutions of the homogeneous linear differential equation $Ly = 0$ then so is $A_1y_1 + A_2y_2$ for any real numbers A_1 and A_2.

Definition Functions $y_1, y_2, \ldots y_n$ of the variable x are said to be *linearly dependent* if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0.$$

Functions $y_1, y_2, \ldots, y_n$ are said to be *linearly independent* if they are not linearly dependent.

It follows directly from this definition that two functions y_1 and y_2 of x are linearly independent if neither is a constant multiple of the other.

It can be proved that if $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of a homogeneous linear differential equation

$$a_0(x)y + \sum_{j=1}^{n} a_j(x)\frac{d^jy}{dx^j} = 0,$$

of order n, where $a_0, a_1, \ldots, a_n$ are well-behaved functions of x, and if a_n is everywhere non-zero, then any solution y of this differential equation is of the form

$$y = A_1y_1 + A_2y_2 + \cdots + A_ny_n$$

where $A_1, A_2, \ldots, A_n$ are arbitrary constants.

Example Let

$$y_1 = e^x, \quad y_2 = e^{2x}, \quad y_3 = e^{3x}.$$

The functions y_1, y_2 and y_3 are linearly independent, since there do not exist real numbers c_1, c_2 and c_3 which are not all zero and which have the property that $c_1e^x + c_2e^{2x} + c_3e^{3x} = 0$. The functions y_1, y_2 and y_3 are also solutions of the homogeneous linear differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0.$$

The general solution of this differential equation is therefore of the form

$$y = A_1e^x + A_2e^{2x} + A_3e^{3x},$$

where A_1, A_2, A_3 are arbitrary constants.

6.3 Inhomogeneous Linear Differential Equations

A inhomogeneous linear differential equation of order n is a differential equation of the form

$$a_0(x)y + \sum_{j=1}^{n} a_j(x)\frac{d^j y}{dx^j} = f(x),$$

where $a_0, a_1, \ldots, a_n$ and f are functions of the independent variable x, and where a_n and f are not the zero function. If y_P is a particular solution of such a differential equation then all solutions of this equation are of the form $y_P + y_C$, where y_C satisfies the corresponding homogeneous differential equation

$$a_0(x)y_C + \sum_{j=1}^{n} a_j(x)\frac{d^j y_C}{dx^j} = 0.$$

The solution y_P of the inhomogeneous differential equation is referred to as a *particular integral* of that equation, and the function y_C is referred to as a *complementary function*.

6.4 Linear Differential Equations of the First Order

We shall describe a method for solving differential equations of the form

$$\frac{dy}{dx} + p(x)y = r(x).$$

Such an equation is a homogeneous linear first order differential equation if $r(x) = 0$ for all x. It is inhomogeneous if the function r is not everywhere zero.

Consider the function $q(x)$ where

$$q(x) = \exp\left(\int p(x)\,dx\right).$$

(Here $\exp u = e^u$ for all real numbers u, and $\int p(x)\,dx$ denotes some indefinite integral of the function p.) On applying the Chain Rule and the Fundamental Theorem of Calculus, we find that

$$\frac{d}{dx}q(x) = \exp\left(\int p(x)\,dx\right)\frac{d}{dx}\int p(x)\,dx = q(x)p(x).$$

Thus

$$p(x) = \frac{q'(x)}{q(x)},$$

where

$$q'(x) = \frac{dq(x)}{dx}.$$

It follows that a function y of x is a solution of the differential equation

$$y'(x) + p(x)y(x) = r(x).$$

if and only if

$$q(x)y'(x) + q'(x)y(x) = q(x)r(x).$$

But

$$q(x)y'(x) + q'(x)y(x) = \frac{d}{dx}\left(q(x)y(x)\right).$$

It follows that the function y satisfies the differential equation

$$y'(x) + p(x)y(x) = r(x)$$

if and only if

$$q(x)y(x) = \int q(x)r(x)\,dx + C,$$

where C is a constant of integration. The general solution of the differential equation

$$\frac{dy}{dx} + p(x)y = r(x).$$

is thus given by

$$y(x) = \frac{1}{q(x)}\int q(x)r(x)\,dx + \frac{C}{q(x)}.$$

The function q is referred to as an *integrating factor* for the differential equation.

Example Consider the differential equation

$$\frac{dy}{dx} + cy = 0,$$

where c is a constant. If we set $p(x) = c$ then a suitable indefinite integral of the constant function p is the function sending x to cx, and thus we may take $q(x) = e^{cx}$ as an integrating factor for the differential equation. The general solution of the differential equation is then given by

$$y(x) = \frac{C}{q(x)} = Ce^{-cx}.$$

Example Consider the differential equation

$$\frac{dy}{dx} + cy = x.$$

The general solution then has the form

$$y(x) = \frac{1}{q(x)} \int q(x) r(x)\, dx + \frac{C}{q(x)},$$

where $q(x) = e^{cx}$ and $r(x) = x$. Using the method of Integration by Parts, we find that

$$\begin{aligned} \int_0^x q(s) r(s)\, ds &= \int_0^x s e^{cs}\, ds = \left[\frac{1}{c} s e^{cs}\right]_0^x - \frac{1}{c} \int_0^x e^{cs}\, ds \\ &= \frac{x}{c} e^{cx} - \frac{1}{c^2}(e^{cx} - 1). \end{aligned}$$

Using this function as an indefinite integral of $q(x)r(x)$, we find that the general solution of the differential equation is given by

$$\begin{aligned} y(x) &= \frac{1}{e^{cx}} \left(\frac{x}{c} e^{cx} - \frac{1}{c^2}(e^{cx} - 1) \right) + \frac{C}{e^{cx}} \\ &= \frac{x}{c} - \frac{1}{c^2}(1 - e^{-cx}) + Ce^{-cx}. \end{aligned}$$

where C is an arbitrary constant. We may write this general solution in the simpler form

$$y(x) = \frac{x}{c} - \frac{1}{c^2} + Ae^{-cx},$$

where A is an arbitrary constant. The constants A and C in these two forms of the general solution are related by the equation

$$A = C + \frac{1}{c^2}.$$

6.5 Homogeneous Linear Differential Equations of the Second Order with Constant Coefficients

We discuss the general solution of differential equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

where a, b and c are real numbers, and $a \neq 0$. We write

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}.$$

We shall show the solutions of the differential equation $ay'' + by' + cy = 0$ are determined by the roots of the *auxiliary polynomial* $as^2 + bs + c$ determined by the differential equation.

We begin our investigation of the solutions of these differential equations by showing that functions specified by the equations

$$y = e^{rx}, \quad y = xe^{rx}, \quad y = e^{px}\sin qx, \quad y = e^{px}\cos qx,$$

satisfy differential equations of the given form for appropriate values of the constants a, b and c.

First suppose that $y = e^{rx}$. Then

$$y' = re^{rx} = ry, \quad y'' = r^2e^{rx} = r^2y,$$

and hence

$$ay'' + by' + cy = (ar^2 + br + c)y.$$

It follows that $y = e^{rx}$ is a solution of the differential equation $ay''+by'+cy = 0$ if and only if $ar^2 + br + c = 0$.

Next suppose that $y = xe^{rx}$. Then

$$y' = (rx + 1)e^{rx}, \quad y'' = (r^2x + 2r)e^{rx},$$

and hence

$$ay'' + by' + cy = (ar^2 + br + c)xe^{rx} + (2ar + b)e^{rx}.$$

It follows that $y = xe^{rx}$ is a solution of the differential equation $ay''+by'+cy = 0$ if and only if

$$ar^2 + br + c = 0 \text{ and } 2ar + b = 0.$$

Next suppose that $y = e^{px}\sin qx$. Then

$$y' = pe^{px}\sin qx + qe^{px}\cos qx$$

and

$$y'' = (p^2 - q^2)e^{px}\sin qx + 2pqe^{px}\cos qx.$$

It follows that

$$ay'' + by' + cy = \left(a(p^2 - q^2) + bp + c\right) e^{px}\sin qx + (2apq + bq)e^{px}\cos qx.$$

The right hand side of this equation must be zero for all real values of x in order that $y = e^{px}\sin qx$ satisfy the differential equation $ay'' + by' + cy = 0$. It follows that this differential equation is satisfied if and only if

$$a(p^2 - q^2) + bp + c = 0 \text{ and } 2apq + bq = 0.$$

Finally suppose that $y = e^{px}\cos qx$. Then

$$ay'' + by' + cy = \bigl(a(p^2 - q^2) + bp + c\bigr)\, e^{px}\cos qx - (2apq + bq)e^{px}\sin qx.$$

It follows that the differential equation $ay'' + by + cy = 0$ is satisfied by the function $y = e^{px}\cos qx$ if and only if

$$a(p^2 - q^2) + bp + c = 0 \text{ and } 2apq + bq = 0.$$

These results may be interpreted with the aid of standard results concerning the roots of quadratic polynomials which are collected in the following theorem.

Lemma 6.1 *Let a, b and c be real numbers, with $a \neq 0$. The roots of the quadratic polynomial $as^2 + bs + c$ are given by the formula*

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

A real number r is a repeated root of this polynomial if and only if

$$ar^2 + br + c = 0 \textit{ and } 2ar + b = 0.$$

A complex number of the form $p + iq$ for some real numbers p and q is a root of the quadratic polynomial $as^2 + bs + c$ if and only if

$$a(p^2 - q^2) + bp + c = 0 \textit{ and } 2apq + bq = 0.$$

Proof The formula for the roots of a quadratic polynomial is well-known. A real number r is a repeated root of the quadratic polynomial if and only if

$$as^2 + bs + c = a(s - r)^2 = as^2 - 2ars + ar^2,$$

and therefore r is a repeated root of the polynomial if and only if

$$b = -2ar, \quad c = ar^2.$$

But

$$ar^2 + br + c = (2ar + b)r + (c - ar^2).$$

and therefore r is a repeated root of the polynomial if and only if $ar^2+br+c = 0$ and $2ar + b = 0$.

If p and q are real numbers, and if $s = p + iq$ then $s^2 = p^2 - q^2 + 2ipq$ and hence

$$as^2 + bs + c = a(p^2 - q^2) + bp + c + i(2apq + bq),$$

and therefore $as^2 + bs + c = 0$ if and only if

$$a(p^2 - q^2) + bp + c = 0 \text{ and } 2apq + bq = 0,$$

as required. ∎

Let a, b, c, p, q and r be real numbers with $a \neq 0$. We see that $y = e^{rx}$ is a solution of the differential equation $ay'' + by' + cy = 0$ if and only if r is a root of the quadratic polynomial $as^2 + bs + c$. Moreover $y = xe^{rx}$ is a solution of this differential equation if and only if r is a repeated root of this quadratic polynomial. Also $y = e^{px} \sin qx$ and $y = e^{px} \cos qx$ are solutions of this differential equation if and only if $p + iq$ is a root of the polynomial $as^2 + bs + c = 0$.

Two solutions y_1 and y_2 of a linear differential equation are linearly independent if and only if neither is a constant multiple of the other. It is a fact that if y_1 and y_2 are two linearly independent solutions of a homogeneous linear second order ordinary differential equation whose coefficients are well-behaved functions of x then the general solution of that equation is of the form $y = Ay_1 + By_2$, where A and B are arbitrary constants.

The results that we have already verified enable us to write down the general solutions of second order differential equations of the form $ay'' + by' + cy = 0$.

Theorem 6.2 *Let a, b and c be real numbers, with $a \neq 0$. The solutions of the differential equation*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

are determined by the roots of the auxiliary polynomial $as^2 + bs + c$ as follows:—

(i) *if $b^2 > 4ac$ then the auxiliary polynomial $as^2 + bs + c$ has two real roots r_1 and r_2, and the general solution of the differential equation is given by*

$$y = Ae^{r_1 x} + Be^{r_2 x},$$

where A and B are constants;

(ii) *if $b^2 = 4ac$ then the auxiliary polynomial $as^2 + bs + c$ has a repeated root r, and the general solution of the differential equation is given by*

$$y = (Ax + B)e^{rx},$$

where A and B are constants;

(iii) *if $b^2 < 4ac$ then the auxiliary polynomial $as^2 + bs + c$ has two non-real roots $p + iq$ and $p - iq$ (where p and q are real numbers), and the general solution of the differential equation is given by*

$$y = e^{px} \left(A \sin qx + B \cos qx\right),$$

where A and B are constants.

Proof In each case the specified general solution is a linear combination of two functions which are linearly independent and which we know to be solutions of the differential equation. $\blacksquare$

Example Consider the differential equation

$$\frac{d^2y}{dx^2} - 11\frac{dy}{dx} + 24y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 - 11s + 24$. This polynomial has two real roots with values 3 and 8. The general solution of this differential equation is therefore of the form

$$y = Ae^{3x} + Be^{8x},$$

where A and B are arbitrary real constants.

Example Consider the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 + 4s + 4$. This polynomial has a repeated real root with values -2. The general solution of this differential equation is therefore of the form

$$y = (Ax + B)e^{-2x},$$

where A and B are arbitrary real constants.

Example Consider the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 - 4s + 5$. This polynomial has a a pair of non-real roots with values $2 + i$ and $2 - i$. The general solution of this differential equation is therefore of the form

$$y = Ae^{2x}\sin x + Be^{2x}\cos x,$$

where A and B are arbitrary real constants.

6.6 Inhomogeneous Linear Differential Equations of the Second Order with Constant Coefficients

We now discuss the general solution of an *inhomogenous linear differential equation of the second order with constant coefficients.* Such a differential equation is of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where a, b and c are real numbers, and $a \neq 0$.

Suppose that y_P is some function of the variable x which satisfies this differential equation. Let y be any twice-differentiable function of the variable x, and let $y_C = y - y_P$. Then

$$\begin{aligned} a\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C &= a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy - a\frac{d^2y_P}{dx^2} - b\frac{dy_P}{dx} - cy_P \\ &= a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy - f(x). \end{aligned}$$

It follows that the function y satisfies the inhomogeneous differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

if and only if y_C satisfies the corresponding homogeneous differential equation

$$a\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0,$$

We see therefore that, once a particular solution y_P of the inhomogeneous differential equation has been found, any other solution of the inhomogeneous differential equation may be obtained by adding to y_P a solution y_C of the corresponding homogeneous differential equation. The function y_P is referred to as a *particular integral* of the inhomogeneous differential equation, and the function y_C is referred to as the *complementary function.* Any solution y of the given inhomogeneous differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

is the sum of the particular integral y_P, which satisfies the same differential equation, and a complementary function y_C, which satisfies the corresponding homogeneous linear differential equation

$$a\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = x^2.$$

We first find a particular integral of this equation. Examination of this equation shows that it might be sensible to look for a particular integral which is a quadratic polynomial in x of the form $px^2 + qx + r$, where the coefficients p, q and r are chosen appropriately. Now if $y = px^2 + qx + r$ then

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 10px^2 + (10q + 14p)x + 10r + 7q + 2p.$$

If the right hand side of this equation is to equal x^2, then p, q and r must be chosen so as to satisfy the equations

$$10p = 1, \quad 10q + 14p = 0, \quad 10r + 7q + 2p = 0.$$

The solution of these equations is given by

$$p = \frac{1}{10}, \quad q = -\frac{7}{50}, \quad r = -\frac{39}{500}.$$

We conclude that a particular integral y_P of the differential equation is given by

$$y_P = \frac{1}{10}x^2 - \frac{7}{50}x - \frac{39}{500}.$$

The complementary function y_C must satisfy the differential equation

$$\frac{d^2y_C}{dx^2} + 7\frac{dy_C}{dx} + 10y_C = 0.$$

The roots of auxiliary polynomial $s^2 + 7s + 10$ associated to this differential equation are -2 and -5. The complementary function y_C is then of the form

$$y_C = Ae^{-2x} + Be^{-5x}.$$

where A and B are arbitrary real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = x^2$$

is then

$$y = \frac{1}{10}x^2 - \frac{7}{50}x - \frac{39}{500} + Ae^{-2x} + Be^{-5x}.$$

Remark Suppose that one is seeking a particular integral of an inhomogeneous differential equation of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where $f(x)$ is a polynomial in x, and $c \neq 0$. There will exist a particular integral y_P of the form $y_P = g(x)$, where $g(x)$ is a polynomial in x of the same degree as $f(x)$. Let

$$f(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n, \quad g(x) = q_0 + q_1x + q_2x^2 + \cdots + q_nx^n,$$

If we equate coefficients of powers of x on both sides of the differential equation

$$a\frac{d^2}{dx^2}g(x) + b\frac{d}{dx}g(x) + cg(x) = f(x),$$

we obtain a system of simultaneous linear equations which determine the coefficients $q_0, q_1, \ldots, q_n$ of the polynomial $g(x)$ in terms of the coefficients $p_0, p_1, \ldots, p_n$ of the polynomial $f(x)$. This enables us to find a particular integral of the differential equation.

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x.$$

First we seek a particular integral of this equation. Now

$$\text{if } y = \sin x \text{ then } y'' - 6y' + 9y = 8\sin x - 6\cos x,$$

$$\text{if } y = \cos x \text{ then } y'' - 6y' + 9y = 8\cos x + 6\sin x.$$

Thus if

$$y_P = \frac{1}{50}\left(4\sin x + 3\cos x\right)$$

then $y_P'' - 6y_P' + 9y_P = \sin x$, and thus y_P is a particular integral of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x.$$

The complementary function y_C is then a solution of the corresponding homogeneous differential equation $y_C'' - 6y_C' + 9y = 0$. The associated auxiliary polynomial $s^2 - 6s + 9$ has a repeated root, whose value is 3. The complementary function y_C is then given by $y_C = (Ax + B)e^{3x}$, where A and B are real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x$$

is then given by

$$y = \frac{1}{50}\left(4\sin x + 3\cos x\right) + (Ax + B)e^{3x}.$$

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = xe^{3x}.$$

Examination of this differential equation suggests that it might be sensible to look for a particular integral of the form $y_P = (p + qx)e^{3x}$, where p and q are appropriately chosen real constants. Now if $y_P = (p + qx)e^{3x}$ then

$$y_P' = (3p + q + 3qx)e^{3x}, \quad y_P'' = (9p + 6q + 9qx)e^{3x},$$

and thus

$$y_P'' - 2y_P' + 5y_P = (8p + 4q + 8qx)e^{3x}.$$

Thus $y_P'' - 2y_P' + 5y_P = xe^{3x}$ if and only if $p = -\frac{1}{16}$ and $q = \frac{1}{8}$. A particular integral y_P of the differential equation is thus given by

$$y_P = \frac{1}{16}(2x - 1)e^{3x}.$$

The complementary function y_C satisfies the differential equation $y_C'' - 2y_C' + 5y_C = 0$. The roots of the associated auxiliary polynomial $s^2 - 2s + 5$ are $1 + 2i$ and $1 - 2i$. The complementary function y_C is therefore of the form

$$y_C = Ae^x \sin 2x + Be^x \cos 2x.$$

where A and B are arbitrary real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = xe^{3x}$$

is thus given by

$$y = \frac{1}{16}(2x - 1)e^{3x} + Ae^x \sin 2x + Be^x \cos 2x.$$

6.7 Initial Value Problems

In an *initial value problem* concerning a second order differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

the value of the solution $y(x_0)$ and its first derivative $y'(x_0)$ are prescribed for some value x_0 of the independent variable x. Such a problem may be solved by first finding the general solution of the differential equation, and then choosing the constants in this general solution so as to ensure that the solution y and its first derivative y' have the required values when $x = x_0$.

Example Let us find the solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = xe^{3x}.$$

for which $y(0) = 0$ and $y'(0) = 1$. We have seen that the general solution of this differential equation is given by

$$y = \frac{1}{16}(2x - 1)e^{3x} + Ae^x \sin 2x + Be^x \cos 2x.$$

The derivative of this general solution is then given by

$$y' = \frac{1}{16}(6x - 1)e^{3x} + (A - 2B)e^x \sin 2x + (B + 2A)e^x \cos 2x.$$

We therefore require that

$$0 = y(0) = -\frac{1}{16} + B, \quad 1 = y'(0) = -\frac{1}{16} + B + 2A,$$

and thus

$$A = \frac{1}{2}, \quad B = \frac{1}{16}.$$

The solution of the initial value problem is thus given by

$$y = \frac{1}{16}(2x - 1)e^{3x} + \frac{1}{2}e^x \sin 2x + \frac{1}{16}e^x \cos 2x.$$

6.8 Boundary Value Problems

In an *boundary value problem* concerning a second order differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

the value of the solution is prescribed for two values x_0 and x_1 of the independent variable x. Such a problem may be solved by first finding the general solution of the differential equation, and then choosing the constants in this general solution so as to ensure that the solution y has the required value when $x = x_0$ and when $x = x_1$.

Example Let us find the solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x.$$

for which $y(0) = 0$ and $y(\pi) = 0$. We have seen that the general solution of this differential equation is given by

$$y = \frac{1}{50}(4\sin x + 3\cos x) + (Ax + B)e^{3x}.$$

The values of the general solution at $x = 0$ and $x = \pi$ are given by

$$y(0) = \frac{3}{50} + B, \quad y(\pi) = -\frac{3}{50} + (A\pi + B)e^{3\pi}.$$

We are seeking the solution with $y(0) = y(\pi) = 0$. We therefore require that

$$A = \frac{3}{50\pi}e^{-3\pi} + \frac{3}{50\pi}, \quad B = -\frac{3}{50}.$$

The solution of the boundary value problem is therefore obtained by substituting these values of A and B into the general solution of the differential equation.

6.9 Higher Order Linear Differential Equations with Constant Coefficients

We describe the general solution of a homogeneous linear differential equation

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0y = 0$$

with constant coefficients $a_0, a_1, \ldots, a_n$. The general solution of such a differential equation is of the form

$$A_1 y_1 + A_2 y_2 + \cdots + A_n y_n,$$

where $A_1, A_2, \ldots, A_n$ are arbitrary real constants and $y_1, y_2, \ldots, y_n$ are linearly independent solutions determined by the roots of the associated *auxiliary polynomial*

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0.$$

We shall describe the form of the solutions determined by real and by non-real roots of the auxiliary polynomial.

A real root r of this auxiliary polynomial of multiplicity m determines m linearly independent solutions of the differential equation. These are given by $y = x^{j-1} e^{rx}$ for each integer j between 1 and m. In particular $y = e^{rx}$ is a solution of the differential equation for each real root r of the auxiliary polynomial.

A pair of non-real roots $p+iq$ and $p-iq$, each of multiplicity m, determines $2m$ linearly independent solutions of the differential equation, with a pair

$$y = x^{j-1} e^{px} \sin qx \text{ and } y = x^{j-1} e^{px} \cos qx$$

of solutions for each integer j between 1 and m.

Example Consider the differential equation

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$$

The auxiliary polynomial of this differential equation is the cubic polynomial $s^3 - 3s^2 + 3s - 1$. This polynomial may be written in the form $(s-1)^3$, and thus has a repeated root with value 1 and multiplicity 3. Therefore the general solution of the differential equation is given by

$$y = (A + Bx + Cx^2)e^x.$$

This may be verified by showing that the functions e^x, xe^x and x^2e^x are solutions of the differential equation. (These functions are obviously linearly independent.)

The general solution y of an inhomogenous linear differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

of order n may be expressed as the sum of a particular integral y_P, which satisfies the given equation, and a complementary function y_C, which satisfies the homogeneous differential equation

$$a_n \frac{d^n y_C}{dx^n} + a_{n-1} \frac{d^{n-1} y_C}{dx^{n-1}} + \cdots + a_1 \frac{dy_C}{dx} + a_0 y_C = 0.$$

If $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of this homogeneous equation, then the general solution of the inhomogeneous equation is of the form

$$y = y_P + A_1 y_1 + A_2 y_2 + \cdots + A_n y_n,$$

where $A_1, A_2, \ldots, A_n$ are arbitrary constants.