Course 2BA1: Michaelmas Term 2003
Assignment II

Conall O'Brien

01734351

conall@conall.net

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Question

Prove that

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$$

Proof

Let $D = A \setminus (B \setminus C)$ and $E = (A \setminus B) \cup (A \cap C)$

If $x \in D$, then $x \in A$ and $x \in B \setminus C$, since $x \in D$, $x \notin B$ and $x \notin C$. $x \in A \setminus B$, since $x \in A$ and $x \notin B$. $x \in A \cap C$ and since $x \in A$ and $x \notin C$. $x \in (A \setminus B) \cup (A \cap C)$, ie $x \in E$.

If $x \in E$. Therefore $x \in A \setminus B$ or $x \in A \cap C$. If $x \in A \setminus B$, $x \in A$ and $x \notin B$. If $x \in A$ or $x \in C$, $x \notin B$. Therefore $x \in A \setminus (B \setminus C)$, since $x \in A$. Therefore $x \in D$.

$D \subset E$ and $E \subset D$.

Therefore $D = E$ and so $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

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(i)

xPy for all x, $y \in \mathbb{R}$ when $y = xk^2$. $k \in \mathbb{Z}$.

Reflexive?

Therefore xPx is true if and only if $x = xk^2$.

$$1 = k^2$$

$$1 = k$$

$$1 \in \mathbb{Z}$$

Therefore P is reflexive.

Symmetric?

$xPy = yPx$ if and only if $y = xk^2$ and $x = yl^2$. $xRy = yPx$ when k, $l \in \mathbb{Z}$.

$$y = (yl^2)k^2$$

$$y = yl^2k^2$$

$$1 = l^2k^2$$

$$1 = kl$$

if $l = 3$ and $k = \frac{1}{3}$, then $1 = 1 \times \frac{1}{3}$. However when $k = \frac{1}{3}$, $k \notin \mathbb{Z}$.
Therefore P is not symmetric.

Transitive?

xPy and yPz are true, so $y = xk^2$ and $z = yl^2$. Therefore xPz is true, if $z = xj^2$ is also true.

$$(yl^2) = xj^2$$

$$k^2l^2 = j^2$$

$$kl = j$$

Therefore j, k, $l \in \mathbb{Z}$. Thus P is transitive.

Anti-Symmetric?

$x =$ if xPy and yPx are true, if and only if $y = xk^2$ and $x = yl^2$.

$$y = (yl^2)k^2$$

$$y = yl^2k^2$$

$$1 = l^2k^2$$

$$1 = lk$$

If $k = 2$ and $l = \frac{1}{2}$, then $1 = 1$. However, when $l = \frac{1}{2}$, $k \notin \mathbb{Z}$. Hence P is not anti-symmetric.

Conclusion

P is neither an equivalence relation, nor a partial order, since it is neither symmetric nor anti-symmetric.

(ii)

xQy if x, $y \in \mathbb{R}$ if and only if $y^3 = x^3 - x + y$.

Reflexive?

xQx is true if and only if $x^3 = x^3 - x + x$.

$$x^3 = x^3$$

Therefore, Q is reflexive.

Symmetric?

$xQy = yQx$ if and only if $y^3 = x^3 - x + y$ and $x^3 = y^3 - y + x$.

$$y^3 = (y^3 - y + x) - x + y$$

$$y^3 = y^3 - y + x - x + y$$

$$y^3 = y^3$$

Therefore Q is symmetric.

Transitive?

If xQy and yQz are true, then $y^3 = x^3 - x + y$ and $z^3 = y^3 - y + z$ are also true. Therefore is $z^3 = x^3 - x + z$ true, hence xQz true as well?

$$(y^3 - y + z) = x^3 - x + z$$

$$(x^3 - x + y) - y + z = x^3 - x + z$$

$$x^3 - x + z = x^3 - x + z$$

Therefore Q is transitive.

Anti-Symmetric?

If xQy and yQx are true, and hence $y^3 = x^3 - x + y$ and $x^3 = y^3 - y + x$, is $x = y$ true?

$$y^3 = (y^3 - y + x) - x + y$$

$$y^3 = y^3 - y + x - x + y$$

$$y^3 = y^3$$

Therefore, if $y = y$, $y \neq x$, thus Q is not anti-symmetric.

Conclusion

Q is a not a partial order since it is not anti-symmetric. It is not an equivalence relation however, since it is reflexive, symmetric and transitive.

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(i)

$f : [-1, 1] \to [-2, 2]$ for $f(x) = x^3 + x$, $x \in [-1, 1]$.

$$f'(x) = 3x^2 + 1 = 0$$

Therefore $f(x)$ is a strictly increasing function.

$$f(-1) = (-1)^3 + (-1)$$

$$f(-1) = -2$$

$$f(1) = (1)^3 + (1) = 2$$

$$f(1) = 2$$

$$f(-1) \neq f(1)$$

Therefore f is injective.

$$f(-1) = -2$$

$$f(1) = 2$$

Therefore f is surjective and hence f is bijective, since it is injective and surjective. Hence it is invertible.

(ii)

$g : (-1,1) \to \mathbb{R}$ with $g(x) = \frac{1}{1-x^2}$ for all values of $x \in (-1,1)$.

$$g(0) = \frac{1}{1-(0)^2} = 1$$

$$g(0.5) = \frac{1}{1-0.25} = \frac{1}{0.75} = \frac{4}{3}$$

$$g(-0.5) = \frac{1}{1-0.25} = \frac{1}{0.75} = \frac{4}{3}$$

Therefore g is not injective since $0.5 \neq -0.5$.

However, $g(0.5) = \frac{4}{3}$, $g(-0.5) = \frac{4}{3}$. $\frac{4}{3} \in \mathbb{R}$. Therefore g is surjective.
Therefore g is not bijective, because it is not injective. Hence it is not invertible.