3 Graph Theory

3.1 Undirected Graphs

An *undirected graph* can be thought of as consisting of a finite set V of points, referred to as the *vertices* of the graph, together with a finite set E of *edges*, where each edge joins two distinct vertices of the graph.

We now proceed to formulate the definition of an undirected graph in somewhat more formal language.

Let V be a set. We denote by V_2 the set consisting of all subsets of V with exactly two elements. Thus, for any set V,

$$V_2 = fA$$

Definition Let $(V; E)$ be a graph with m vertices and n edges. Let the vertices be ordered as $v_1; v_2; \ldots; v_m$, and let the edges be ordered as $e_1; e_2; \ldots; e_n$. The *incidence matrix* for such a graph then takes the form

a_{11} a_{12} a_{1n} a_{21} a_{22} $\ldots$ a_{2n}

$\vdots$ $\vdots$ $\vdots$

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Then the adjacency matrix for this graph is the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

3.4 Complete Graphs

V is said to be *Vomplete* V and only if, $VV; v$

Definition A graph $(V; E$

Let $(V;E)$ be a graph, and let V' be a subset of V. Let

$$E' = \{\{a;b\} \in E : a \in V' \text{ and } b \in V'\};$$

(so that $\in$

Corollary 3.4 *Let $(V; E)$ be a k-regular graph. Then $k|V| = 2|E|$, where $|V|$ denotes the number of vertices and $|E|$ denotes the number of edges of the graph.*

Proof If the graph is k-regular then the suE

if and only if there exists a walk in the graph from a to b.

Lemma 3.7 *Let $(V; E)$ be an undirected graph. Then the relation on the set V of vertices of the graph* [illegible]

These subgraphs are *disjoint* since $V_i \cap V_j = \emptyset$ and $E_i \cap E_j = \emptyset$ if $i \neq j$. Moreover the graph (V_i, E_i) is the restriction of the graph (V, E) to V_i (also describable as the graph induced on V_i by (V, E)) for $i = 1, 2, \ldots, k$.

The subgraphs (V_i, E_i) of (V, E) are referred to as the *components* (or *connected components*) of the graph (V, E).

individually. Moreover properties of any one component do not affect those of any other, since no edge of the graph passes from any one component of the graph to any other.

3.12 Circuits

Definition Let (V, E) be a graph. A *walk* $v_0 v_1 v_2 \ldots v_n$ in the graph is said to be *closed* if $v_0 = v_n$.

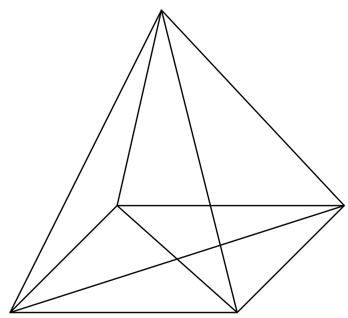
Thus a walk in a graph is closed if and only if it starts and ends at the same vertex.

Definition Let (V, E) be a graph. A *circuit* in the graph is a non-trivial closed trail in the graph.

least one of the vertices $v_0, v_1, \ldots, v_{m-2}$ is incident to v_m; let that vertex be v_k, where $0 \leq k \leq m-2$. Then $v_k v_{k+1} \ldots v_m v_k$ is a simple circuit in the graph. Thus a graph with no isolated or pendant vertices always contains a simple circuit. ∎

Theorem 3.11 *Let u and v be vertices of a graph, where $u \neq v$. Suppose that there exist at least two distinct paths in the graph from u to v. Then the graph contains at least one simple circuit.*

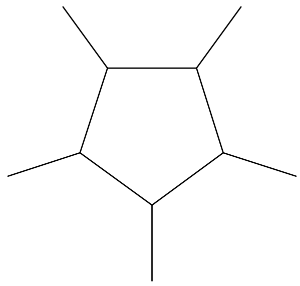
Proof Let $a_0 a_1 a_2 \ldots a_m$ and $b_0 b_1 b_2 \ldots b_n$ be two distinct paths in the graph with a_0



If $v = v_0$, and if the trail is not closed (i.e., if $v_m \neq v_0$), then the edges of the trail incident to v are the edge $v_0 v_1$ together with the edges $v_{i-1} v_i$ and $v_i v_{i+1}$

Lemma 3.16 *Let* $v\,w$

Now any vertex belonging to V_1 is incident to at least one edge traversed by the trail. But then all edges incident to a vertex belonging to V_1 must be traversed by the trail. But then any vertex of V adjacent to a vertex in V_1 must itself belong to V_1, and thus no edge can join a vertex in V_1 to a rtex in V_2. If the set33 11.955 Tf 210.8163.765 0 Td[(V)]TJ/F31 7.9



Theorem 3.22 *Every forest contains at least one isolated or pendant vertex.*

Proof

Theorem 3.25 *Given two distinct vertices of a tree, there exists a unique* consist of the remaining edges of the circuit traversing the edge vw.) Moreover every vertex in V could be joined to v by a walk whose edges belong to E^0, and could therefore be joined either to v or to w by a walk whose edges belong to E^{00}

the m m matrix (b