1BA5 Electrotechnology - Lecture Notes

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Contents

1 About this document **5**

2 Course Description **6**
2.1 Section I - Review of the mathematics necessary for this course . . . 6
2.2 Section II - Review of the physics necessary for this course . . . 6
2.3 Section III - Electric Field Theory . . . 6
2.4 Section IV - Materials (-science) . . . 6
2.5 Section V - Electric Circuit Components . . . 7
2.6 Section VI - Electric Circuits . . . 7

3 Mathematics Review **8**
3.1 Complex numbers . . . 8
3.1.1 Operation . . . 8
3.1.2 Graphical representation of a complex number 'The complex plane' . . . 8
3.1.3 The Triangular Inequality . . . 9
3.1.4 The complex exponential . . . 9
3.1.5 The coordinates . . . 10
3.1.6 Polar representation of a complex number . . . 10
3.1.7 Multiplication and division . . . 10
3.1.8 Addition and subtraction . . . 10
3.2 Vectors . . . 11
3.2.1 Addition and subtraction . . . 11
3.2.2 The Unit Vector . . . 11
3.2.3 The scalar (or 'dot') product of two vectors . . . 11
3.2.4 The vector (or 'cross') product . . . 12
3.2.5 Properties of the vector and scalar product . . . 12
3.2.6 Representation of a vector using a coordinate system . . . 12
3.2.7 Equality of vectors . . . 13
3.3 Matrix Theory . . . 13
3.3.1 Matrix transpose . . . 14
3.3.2 Diagonal Matrices . . . 14
3.3.3 Equality of matrices . . . 14
3.3.4 Matrix addition . . . 14
3.3.5 Properties of matrix addition . . . 14
3.3.6 Multiplication of a matrix by a scalar . . . 15
3.3.7 Properties of scalar multiplication . . . 15
3.3.8 Matrix multiplication . . . 15
3.3.9 Properties of matrix multiplication . . . 15
3.3.10 The inverse of a matrix . . . 16
3.4 Calculus and differential equations . . . 17
3.4.1 Differentiation . . . 17
3.4.2 Rules for differentiation . . . 17
3.4.3 Integration . . . 18

3.4.4 Techniques of integration . . . 18
3.4.5 General tips on performing integration . . . 19
3.4.6 Differential equations . . . 19
3.4.7 Terminology . . . 20
3.4.8 Solving differential equations . . . 21
3.4.9 Applications of differential equations . . . 23

4 Physics Review **25**
4.1 Part A - Mechanics . . . 25
4.1.1 Linear motion . . . 25
4.1.2 Momentum . . . 26
4.1.3 Conservation of momentum . . . 26
4.1.4 Force . . . 26
4.1.5 Newton's Laws of Motion . . . 26
4.1.6 Law I . . . 27
4.1.7 Law II . . . 27
4.1.8 Law III . . . 27
4.2 Part B - Electrical energy and work . . . 28
4.2.1 Principle of conservation of energy . . . 28
4.2.2 Forms of energy . . . 28
4.3 Part C - Basic atomic theory . . . 28

5 Electric and Magnetic Field Theory **30**
5.1 Electric Fields . . . 30
5.1.1 Electric Charge . . . 30
5.1.2 Coulomb's Law . . . 30
5.1.3 Definitions . . . 31
5.1.4 Ampère's Law . . . 31
5.2 Magnetic Fields . . . 32
5.2.1 The Lorentz Force Law . . . 32
5.2.2 The direction of the magnetic field . . . 32
5.2.3 Magnetic flux . . . 32
5.2.4 The electric displacement vector and magnetic field strength . . . 32
5.2.5 The significanse of ϵ and μ . . . 33
5.2.6 The orthogenality of electric and magnetic fields . . . 33
5.3 Maxwell's Equations . . . 34
5.3.1 History . . . 34
5.3.2 The Law of Faraday and Lang (applied to solenoids) . . . 34
5.3.3 Ampère's Law . . . 35
5.3.4 Gauss' Law I . . . 35
5.3.5 Gauss's Law II . . . 35
5.3.6 Examples . . . 36
5.4 Statics and dynamics . . . 40

6 Materials (-science) **41**
6.1 Conductors . . . 41
6.2 Dielectrics . . . 41
6.3 Semi-conductors . . . 41
6.3.1 The energy level diagram . . . 42
6.3.2 Semi-conductor doping . . . 43
6.3.3 n-type semi-conductors . . . 43
6.3.4 p-type semi-conductors . . . 43

7 Electric Circuit Components **44**
7.1 Preamble . . . 44
7.1.1 Current . . . 44
7.1.2 Notes on current . . . 44
7.1.3 Voltage . . . 44
7.1.4 Electric Power . . . 44
7.2 The Resistor . . . 45
7.2.1 Series and Parallel Resistors . . . 46
7.2.2 The Potential Divider . . . 47
7.2.3 The impedance of a Resistor . . . 48
7.3 The Capacitor . . . 49
7.3.1 Power delivered to a capacitor . . . 50
7.3.2 Capacitors in series and parallel . . . 50
7.3.3 Applications . . . 51
7.3.4 The impedance of a capacitor . . . 52
7.4 The Inductor . . . 53
7.4.1 Power delivered to an inductor . . . 54
7.4.2 Inductors in series and parallel . . . 54
7.4.3 Uses of inductors . . . 55
7.4.4 The impedance of an inductor . . . 56
7.5 Semi-conductor Diodes . . . 56
7.5.1 The Zener Diode . . . 57
7.5.2 Dynamic resistance of a diode . . . 57
7.5.3 Light Emitting Diodes (LED's) . . . 57
7.5.4 Photo diodes . . . 57
7.6 Transformers . . . 58
7.6.1 Principle of operation of transformers . . . 58
7.7 Ideal and practical sources . . . 58
7.7.1 Voltage sources . . . 59
7.7.2 Current sources . . . 59
7.8 Earthing and grounding . . . 60
7.8.1 Earthing . . . 60
7.8.2 Grounding . . . 60
7.9 Instrumentation . . . 61

8 Analysis of electric circuits in steady state **62**
8.1 Preamble . . . 62
8.1.1 A.C. vs. D.C. Analysis . . . 62
8.2 Terminology . . . 62
8.3 Kirchoff's Laws . . . 63
8.3.1 Kirchoff's Current Law (K.C.L.) . . . 63
8.3.2 Kirchoff's Voltage Law (K.V.L.) . . . 63
8.4 Source Transformation . . . 63
8.5 The Principle of Superposition . . . 64
8.6 Complex Notation . . . 64
8.6.1 Complex Sources . . . 64
8.6.2 Complex elements . . . 65
8.7 Impedance Series and Parallel . . . 66
8.8 Thévnin's Theorem (A.C. or D.C.) . . . 67
8.8.1 Procedure to get Thévnin's Equivalent Circuit . . . 68
8.9 Norton's Theorem (A.C. or D.C.) . . . 68
8.10 Generalized means of analysing electric circuits . . . 70
8.10.1 Mesh Analysis . . . 70
8.10.2 Nodal Analysis . . . 73
8.11 Power . . . 74
8.11.1 D.C. Power . . . 74

8.11.2 A.C. Power . . . 74
8.11.3 Root Mean Square Current and Voltage . . . 74
8.11.4 Complex Power . . . 75
8.11.5 Power Factor . . . 76
8.11.6 Power factor correction . . . 76
8.12 Phasor Diagrams . . . 77
8.12.1 The Power Triangle . . . 77
8.13 The maximum power tranfer theorem . . . 80
8.14 Resonant Circuits . . . 81
8.14.1 Background . . . 81
8.14.2 The series resonant circuit . . . 82
8.14.3 The Quality Factor Q . . . 82
8.14.4 Impedance vs. Frequency . . . 83
8.14.5 Variation of V_R, V_L and V_c with frequency in a series RLC circuit . . . 84

9 Transient Analysis of RC and RL circuits **85**
9.1 Transient analysis of an RC circuit . . . 85
9.1.1 Charging phase of an RC circuit . . . 86
9.1.2 Discharge phase of an RC circuit . . . 87
9.2 Transient analysis of an RL circuit . . . 87
9.2.1 Charging phase . . . 88
9.2.2 Discharge Phase . . . 88

Chapter 1

About this document

This document contains all the lecture notes of the 1BA5 of Computer Science, BA., Trinity College, Dublin, as lectured by Dr. O'Nuallain. The notes are a literal representation of the notes given in class, including diagrams, graphs and circuits.

Note that the course overview in the first chapter is not actually correct. This is because the course structure was changed halfway the course. But since this document is a direct copy of the notes, this inconsistency is still visible in this document. Never mind ;-)

This document has been created using Latex for the typesetting, GNUPlot for the graphs and M4 macro package, dpic and the CircuitMacros 5.0 for the circuits. The compiled version is available in PDF (Portable Document Format) and PS (PostScript).

Dr. O'Nuallain has **not** been involved in the creation of this document, nor has he given his approval of the information contained in this document. Therefore you are **not** guaranteed that the information in this document is 100% correct.

In fact, even though I tried my best to avoid errors, there probably are quite a number of typos, mistakes, and even some missing pieces of information. Might you find any, please send a correction to me:

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Of course, suggestions are always welcome as well.

To conclude, I hope the notes are of any use to you.

Edsko de Vries

Chapter 2

Course Description

Objective To give students of Computer Science an understanding of the operating principles of electrical hardware.

2.1 Section I - Review of the mathematics necessary for this course

1. Complex numbers (page 8)
2. Vectors (page 11)
3. Matrices (page 13)
4. Calculus and differential equations (page 17)

2.2 Section II - Review of the physics necessary for this course

1. Concepts of speed, velocity, acceleration, force, work, power, energy, meas, charge, etc.
2. Newton's Laws

2.3 Section III - Electric Field Theory

1. Maxwell's Equations in one dimension
2. Statics
3. Dynamics
4. The nature of electomagnetic radiation (in particular light)

2.4 Section IV - Materials (-science)

1. Molecules theory / atomic theory
2. Conditions
3. Dielectrics (insulators)
4. Semiconductors

2.5 Section V - Electric Circuit Components

1. Resistors
2. Capacitors
3. Inductors
4. Transformers
5. Semiconductor diodes and transistors

2.6 Section VI - Electric Circuits

1. D.C. Circuits
2. A.C. Circuits

Chapter 3

Mathematics Review

For a discussion of the mathematical section see:

Calculus with analytic geometry
by J.B. Freighly
Addison Wesley

3.1 Complex numbers

Recall $z = x + iy; x, y \in \mathbb{R}, z \in \mathbb{C}$
where $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of imaginairy numbers
Let $y = 0 \to z = x$; we declare $\mathbb{R} \in \mathbb{C}$

Re $\{z\} = x$ (the 'real part of z'
Im $\{z\} = iy$ (the 'imaginairy part of z')

Note $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2 \wedge y_1 = y_2$, where 'iff' means 'if and only iff'.

3.1.1 Operation

If $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, then

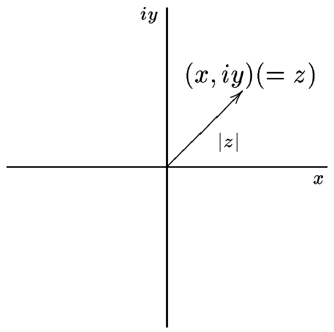
$$\begin{aligned}
z_1 \pm z_2 &= (x_1 \pm x_2) + i(y_1 \pm y_2) \\
z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\
&= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \\
\frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\
&= \frac{x_1 x_2 + y_1 y_2}{{x_2}^2 + {y_2}^2} + \frac{i(x_2 y_2 - x_1 y_2)}{{x_2}^2 + {y_2}^2}
\end{aligned}$$

Definition If $z = x + iy$ we define the complex conjugate $\overline{z}$ as $\overline{z} = x - iy$. $\overline{\overline{z}} = x$.

Definition The modulus / absolute value / magnitude of $z = |z| = \sqrt{x^2 + y^2}$

3.1.2 Graphical representation of a complex number 'The complex plane'

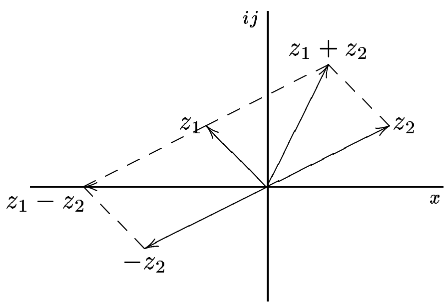
$z = x + iy$ is identified on the complex plane by the point (x, y)



Note Note the following rules:

1. Re $\{z\}$ = Re $\{\overline{z}\}$
2. Im $\{z\}$ = -Im $\{\overline{z}\}$
3. $|z|^2 = x^2 + y^2 = z\overline{z}$
4. Re $\{z\} = \frac{1}{2}(z + \overline{z})$
5. Im $\{z\} = \frac{1}{2i}(z - \overline{z})$

3.1.3 The Triangular Inequality



Definition From the diagram we can obtain: $|z_1 + z_2| \leq |z_1| + |z_2|$. This is the *Triangular inequality.*

Exercise Prove the triangular inequality

3.1.4 The complex exponential

Definition $e^2 = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Definition $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!}$

Definition $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

Let $z = iz$; then $(iz)^2 = i^2 z^2 = -1z^2 = -z^2$, then

$$
\begin{aligned}
e^{iz} &= i + iz + \frac{(iz)^2)}{2!} + \frac{(iz)^3}{3!} + \cdots \\
&= 1 + iz + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \\
&= (1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots) + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots)
\end{aligned}
$$

Recognising here the power series for $\cos^2$ and $\sin^2$, we can write the last equation as:

Definition $e^{iz} = \cos^2 + i \sin^2$. This is *Euler's Formula*

If $z = x + iy$, it is reasonable to expect that $e^z = e^{x+iy} = e^x e^{iy}$.

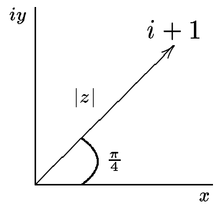
3.1.5 The coordinates

Let $x = r\cos\Theta, y = r\sin\Theta; r = |z|$, then

$$\begin{aligned} z &= x + iy = r\cos\Theta + ir\sin\Theta \\ &= r(\cos\Theta + i\sin\Theta) \\ &= re^{i\Theta} \end{aligned}$$

Θ is measured in radians, and is taken to the positive in the counter-clockwise direction from the +ve (positive) x-axis and negative in the clockwise direction, and is called the *argument of the complex number.*
Since $\frac{\sin}{\cos}(\Theta + 2n\pi) = \frac{\sin}{\cos}(\Theta); n \in \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers), there are infinitely many possibilities for $\Theta = \arg(z)$.

Example $z = 1 + i$
$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$
$\arg(z) = \frac{\pi}{4} = \frac{\pi}{4} + 2n\pi$



So, in polar form:

$$\begin{aligned} z &= |z|(\cos\Theta + i\sin\Theta) \\ &= \sqrt{2}(\cos\tfrac{4}{\pi} + i\sin\tfrac{\pi}{4}) \end{aligned}$$

Example $z = 2i$

$$\begin{aligned} 2i &= 2(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}) \\ &= 2\{\cos(\tfrac{\pi}{2} + 2n\pi) + i\sin(\tfrac{\pi}{2} + 2n\pi)\}, n \in \mathbb{Z} \end{aligned}$$

3.1.6 Polar representation of a complex number

From $e^{i\Theta} = \cos\Theta + i\sin\Theta$ $(\Theta \in \mathbb{R})$, the polar representation of a complex number can be written in the form:

$$\begin{aligned} z &= a + ib = r(\cos\Theta + i\sin\Theta) = re^{i\Theta} \\ \Theta &= \arg(z) = ba^{-1}(\tfrac{b}{a}) \\ r &= \mathrm{mod}(z) = |z| = \sqrt{a^2 + b^2} \end{aligned}$$

Example $-1 - i = \sqrt{2}e^{i\frac{5\pi}{4}}$

3.1.7 Multiplication and division

Multiplication and division of complex numbers expressed in polar form is easy:

$$\begin{array}{lllll} \text{Let:} & z_1 & = & a + ib & = \sqrt{1}e^{i\Theta_1} \\ & z_2 & = & c + id & = \sqrt{2}e^{i\Theta_2} \\ \text{Then} & \dfrac{z_1}{z_2} & = & \dfrac{\sqrt{1}\,e^{i\Theta_1}}{\sqrt{2}\,e^{i\Theta_2}} & = \dfrac{r_1}{r_2}e^{i(\Theta_1 - \Theta_2)} \\ & z_1 z_2 & = & r_1 r_2 e^{i(\Theta_1 + \Theta_2)} & \end{array}$$

3.1.8 Addition and subtraction

Convert to cartesian form $(x + iy)$ and proceed.

3.2 Vectors

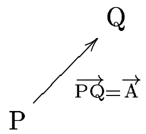
Definition A 'vector' is the name given to a quantity which in order to be completely specified has to be given a magnitude and a direction.

Example Wind (Force 3, N.W.)
Velocity (60 km/ph, east)
Force (10 N, upwards)
Acceleration due to gravity ($9.81ms^{-1}$ towards the center of the earth)

Definition A 'scalar' is the name given to a quantity which is completely specified given a magnitude only.

Example Speed (60 km/ph)
Length (6m)
Temperature (26°)

A vector is represented graphically by a direction line segment $\overrightarrow{PQ}$ or $\overrightarrow{R}$ where the length is proportional to the magnitude (modulus) of the vector and whose direction is the same as that of $\overrightarrow{PQ}$.

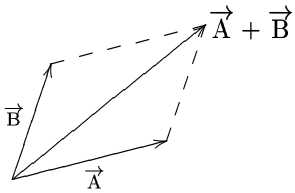


Note Note the absence of co-ordinate axes. The reason for this is that the origin is immaterial.

We denote a vector quantity with an 'arrow' superscript, say $\overrightarrow{A}$. The magnitude will be denoted by either A or |A|.

3.2.1 Addition and subtraction

The sum of vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is given by the diagonal of the parallelogram constructed with these vectors as adjacent sides.



Subtraction taken place in the same fashion ($\overrightarrow{A} - \overrightarrow{B} = \overrightarrow{A} + -\overrightarrow{B}$, compare to section 3.1.3, page 9).

3.2.2 The Unit Vector

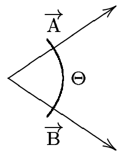
The unit vector is a vector of magnitude 1. The unit vector in the direction of $\overrightarrow{A}$ is defined as $\frac{\overrightarrow{A}}{|\overrightarrow{A}|}$ where $\overrightarrow{A}$ is specified by a cartesian (or polar) coordinate system.

3.2.3 The scalar (or 'dot') product of two vectors

The scalar product of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is denoted with a dot ($\cdot$) and is defined as:

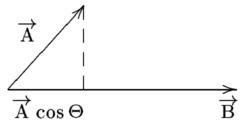
Definition $\overrightarrow{A} \cdot \overrightarrow{B} = |\overrightarrow{A}||\overrightarrow{B}| \cos \Theta$

where Θ is the $\angle$ subtracted by $\overrightarrow{A}$ and $\overrightarrow{B}$.



The scalar product can be viewed as a projection of one vector onto another.

Example



3.2.4 The vector (or 'cross') product

The vector product of two vectors is denoted with a cross ($\times$) and is defined as:

Definition $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin \Theta \hat{N}$

$\hat{N}$ is the unit vector perpendicular to $\vec{A}$ and $\vec{B}$.
The direction of $\hat{N}$ is given by the "right hand rule". That is, close your right hand and stick your thumb out. If $\vec{A}$ and $\vec{B}$ are given in a clockwise direction (in the sense of your fingers) then $\hat{N}$ will be in the direction of your thumb.

3.2.5 Properties of the vector and scalar product

The scalar product

The scalar product is commutitive, distributive, but not associative:

$$\begin{array}{lcll} \vec{A} \cdot \vec{B} & = & \vec{B} \cdot \vec{A} & \text{(commutative)} \\ \vec{A}(\vec{B} + \vec{C}) & = & \vec{A}\vec{B} + \vec{A}\vec{B} & \text{(distributive)} \\ (\vec{A} \cdot \vec{B}) \cdot \vec{C} & \neq & \vec{A} \cdot (\vec{B} \cdot \vec{C}) & \text{(not associative)} \end{array}$$

The scalar product does *not* form a closed set. I.e., if $\vec{A}$ and $\vec{B}$ are vectors, $\vec{A} \cdot \vec{B}$ is *not* a vector (but a scalar).

The vector product

The vector product is distributive, but *not* commutative or associative:

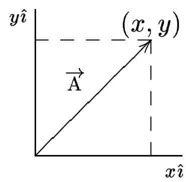
$$\left(\begin{array}{lcl} \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} & & \\ \vec{A} \times \vec{B} & \neq & \vec{B} \times \vec{A} \\ \vec{A} \times \vec{B} & = & -(\vec{B} \times \vec{A}) \\ \vec{A} \times (\vec{B} \times \vec{C}) & \neq & (\vec{A} \times \vec{B}) \times \vec{C} \end{array} \right)$$

The vector product forms a closed set. I.e., if $\vec{A}$ and $\vec{B}$ are vectors, $\vec{A} \times \vec{B}$ is a vector.

3.2.6 Representation of a vector using a coordinate system

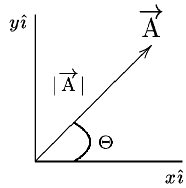
A vector may be represented in cartesian (x, y) form or polar $(r(\Theta))$ form in conjunction with the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ along the 'x' and 'y' axes respectively.

Cartesian representation of a vector



$\overrightarrow{\mathrm{A}} = x\hat{\imath} + y\hat{\jmath} \equiv (x\hat{\imath}, y\hat{\jmath}) \equiv (x, y)$.

Polar representation of a vector



$|\overrightarrow{\mathrm{A}}|,\ \angle\Theta\hat{\mathrm{a}}\ (= |\overrightarrow{\mathrm{A}}\angle\Theta\hat{\mathrm{a}})$

3.2.7 Equality of vectors

Two vectors $\overrightarrow{\mathrm{A}}$ and $\overrightarrow{\mathrm{B}}$ are equal iff $|\overrightarrow{\mathrm{A}}| = |\overrightarrow{\mathrm{B}}|$ and $\angle\mathrm{A} = \angle\mathrm{B}$ (i.e., magnitude and direction are equal).

Note Using a cartesian co-ordinate system, it can be easily shown that where $\overrightarrow{\mathrm{A}} = a\hat{\mathrm{i}} + b\hat{\mathrm{j}} \equiv (a, b)$ and $\overrightarrow{\mathrm{B}} = c\hat{\mathrm{i}} + d\hat{\mathrm{j}} \equiv (c, d)$:

$$\begin{aligned}
\overrightarrow{\mathrm{A}} \cdot \overrightarrow{\mathrm{B}} &= ac + bd \\
\overrightarrow{\mathrm{A}} \times \overrightarrow{\mathrm{B}} &= \hat{\mathrm{k}}\begin{vmatrix} a & b \\ c & d \end{vmatrix} \\
&= \hat{\mathrm{k}}(ad - bc)
\end{aligned}$$

where $\hat{\mathrm{k}}$ is $\perp \hat{\imath}, \hat{\jmath}$ (i.e., $\hat{\mathrm{k}}$ is the unit vector in the 'z' direction).

3.3 Matrix Theory

Definition A matrix is a rectangular array of numbers consisting of rows and columns. An m×n matrix has m rows and n columns for m, n > 0.

We write it like this:

$$\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
\vdots & & & & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{pmatrix}$$

E.G., a_{32} is the entry in the third row and second column.
More briefly: $\mathrm{A} = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$.
The size of A is m×n; a_{ij} are the entries.

Definition We denote matrices with capitals (A, B, X, etc.) A matrix of size m×m is referred to as a *square matrix.*

3.3.1 Matrix transpose

A is m×n, A = (a_{ij}), $\mathrm{A}^{\mathrm{T}} = (a_{ji})$.
I.e., to obtain the transpose of a matrix A (A^{T}), we interchange the rows and columns.

Example If $\mathrm{A} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 4 & -1 \end{pmatrix}$ then $\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & -1 \end{pmatrix}$

A^{T} is an n x m matrix. $(\mathrm{A}^{\mathrm{T}})^{\mathrm{T}} = \mathrm{A}$. $(\mathrm{A}+\mathrm{B})^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}} + \mathrm{B}^{\mathrm{T}}$.

3.3.2 Diagonal Matrices

A matrix A = (a_{ij}) is diagonal if:

1. It is square
2. All off-diagonal entries are zero

I.e., $a_{ij} = 0 \ \forall \ i, j \wedge i \neq j$. The diagonal entries can be zero, but do not need to be.

3.3.3 Equality of matrices

Matrices A = (a_{ij}) and B = (b_{ij}) are equal iff:

1. A and B are the same size
2. $a_{ij} = b_{ij} \ \forall \ i, j$

I.e., the entries is corresponding positions of like sized matrices are all equal.

3.3.4 Matrix addition

Only like-sizes matrices may be added. The sum is obtained by adding each entry in a matrix with the corresponding entry in the other.
In general: Given A = (a_{ij}), B = (b_{ij}): A + B = $(a_{ij} + b_{ij})$
Same goes for subtraction since A - B = A + (-B). For the meaning for -B see below.

3.3.5 Properties of matrix addition

Let A, B, C be m×n matrices.

1. Closure: A + B is an m×n matrix
2. Commutative: A + B = B + A
3. Associative: (A + B) + C = A + (B + C)
4. Distributive: A(B + C) = AB + AC
5. ∃ an m×n matrix '0_{mn}' or simply '0', with each entry equal to zero; the *zero-matrix.* 0 + A = A + 0 = A ∀ matrices A.
6. Given A, ∃ -A, where $-(a_{ij}) = (-a_{ij})$. A + (-A) = (-A) + A = 0 (the *additive inverse* of A).

Note Cartesia co-ordinates can be considered matrices (e.g., (-2,1) or (-3,2)). Sometimes the following representation is used: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, conveying the same meaning. This notation is especially used for vectors.

3.3.6 Multiplication of a matrix by a scalar

A + A + A + A + A = 5A.

Given a real number $k \in \mathbb{R}$ and a matrix (a_{ij}) we define kA to be the m×n matrix where the entries are (ka_{ij}).

3.3.7 Properties of scalar multiplication

$k \in \mathbb{R}$, A, B are m×n matrices

1. Closure: kA is an m×n matrix
2. Distributive: $k(\text{A} + \text{B}) = k\text{A} + k\text{B}$ (*Also:* $(k_1 + k_2)\text{A} = k_1\text{A} + k_2\text{A}$)
3. Associative: $k_1(k_2\text{A}) = (k_1k_2)\text{A}$
4. $1 \cdot \text{A} = \text{A}$

3.3.8 Matrix multiplication

If A in m x **n** and B is **n** x k, multiplication of A and B goes as follows:

Example A: 2 x 3, B: 3 x 3

A $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -5 \end{pmatrix}$, B $\begin{pmatrix} 6 & -1 & 3 \\ 0 & 2 & 4 \\ 1 & 1 & -3 \end{pmatrix}$, then:

$$\begin{aligned} \text{AB} &= \begin{pmatrix} 1\cdot 6 + 2\cdot 0 + 4\cdot 1 & -1\cdot -1 + 2\cdot 2 + 4\cdot 1 & 1\cdot 3 + 2\cdot 4 + -3 \\ 0\cdot 6 + 1\cdot 0 + -5\cdot 1 & 0\cdot -1 + 1\cdot 2 + -5\cdot 1 & 0\cdot 3 + 1\cdot 4 + -5\cdot -3 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 7 & -1 \\ -5 & -3 & 19 \end{pmatrix}, \text{ which is an 2 x 3 matrix.} \end{aligned}$$

Two matrices that can be multiplied are said to be *compatible.* If A = 2 x 4 and B = 4 x 6, then AB $\exists$, but BA $\nexists$.

3.3.9 Properties of matrix multiplication

1. If A = m×n and B = n×k then AB is m×k
2. Matrix multiplication is not always commutative; AB $\neq$ BA could happen:
   $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 7 \end{pmatrix}$, but $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 4 \end{pmatrix} \neq \begin{pmatrix} 0 & 3 \\ 0 & 7 \end{pmatrix}$.
3. The distributive law of matrix multiplication: If A = t×$\underline{\text{m}}$, B = $\underline{\text{m}}$×n and C = $\underline{\text{m}}$×n then A(B + C) is defined as and equal to AB + AC.
4. The associative law of matrix multiplication: Let A (a_{ij}) be m×n and B = (b_{ij}) be n×p then AB is the m×p matrix C = (c_{ij}) where
   $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{1m}b_{mj}$:
   $$\begin{pmatrix} & \cdots & \cdots & \\ a_{i1} & a_{i2} & \cdots & a_{im} \\ & \cdots & \cdots & \end{pmatrix} \cdot \begin{pmatrix} & b_{1j} & \\ \vdots & b_{2j} & \vdots \\ \vdots & \vdots & \vdots \\ & b_{mj} & \end{pmatrix} = c_{ij}$$
   In general: $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$
   Let A and B as above, and D = $(d_{ij}) \equiv$ p×q, then AB(D) = A(BD).

5. If A is an n×n (square) matrix, consider 1_n, the n×n matrix with '1' in every diagonal entry and '0' everywhere else. Then $1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $1_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, etc.

   1_n (the *identity matrix*) satisfies $A1_n = A \; \forall$ n×n matrices A.

6. Inverse matrices: Let A ≡ n×m. If ∃ an n×m matrix B such that $AB = BA = 1_n$ we say that A is *invertible* or *non-singular* and B is the *inverse* of A.

   $B = A^{-1}$ and $A = B^{-1}$ since $(A^{-1})^{-1} = A$.

   Matrix inverses - where they exist - are unique. EG.: If $AB = BA = 1_n$ and $AC = CA = 1_n$ then $B = B1_n = B(AC) = (BA)C = C1_n = C$ and hence $B = C$.

7. If $A = B \not\Rightarrow A = 0$ and/or $B = 0$.

   EG.: $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

   Reversing the order of the multiplication above, it can be shown that if $AB = 0 \not\Rightarrow BA = 0$.

3.3.10 The inverse of a matrix

Definition If $A = (a_{ij}) \equiv$ n×n then the determinant $\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1m}A_{1m}$ where A_{ij} is the *cofactor* of a_{ij} and is the determinant of the (n-1)×(n-1) matrix. Obtain A_{ij} by deleting row i and column j and then multiply by $(-1)^{i+j}$ (alternate signs).

Example The following is an example of matrix co-factors and determinants.

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix},\ B = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Then the co-factors for A are:

$$\begin{array}{ll} A_{11} = (-1)^{1+1} \cdot 4 = 4 & A_{12} = (-1)^{1+2} \cdot -3 = 3 \\ A_{21} = (-1)^{2+1} \cdot 1 = -1 & A_{22} = (-1)^{2+2} \cdot 2 = 2 \end{array}$$

Similarly (but in two steps), the co-factors for B are:

$$B_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 \quad B_{12} = (-1)^{1+2}\begin{vmatrix} -1 & -1 \\ 0 & 3 \end{vmatrix} = 3 \quad B_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} = 1$$

$$B_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix} = 3 \quad B_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9 \quad B_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -1 \\ 0 & -1 \end{vmatrix} = 3$$

$$B_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 3 & -1 \end{vmatrix} = 1 \quad B_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} = 3 \quad B_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8$$

Then $\det A = 2 \cdot 4 + 1 \cdot 3 = 11$ and $\det B = 3 \cdot 8 + -1 \cdot 3 + 0 \cdot 1 = 21$

Theorem (Theorem given without proof): If $\det A \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det A}\left[A_{ij}\right]^T$. If $\det A = 0$, A does *not* have an inverse.

Example A and B as above.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^T = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 8 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 8 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 8 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 8 \end{pmatrix}$$

$$\mathrm{B}^{-1} = \tfrac{1}{\mathrm{detB}} \begin{pmatrix} 8 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 8 \end{pmatrix} = \tfrac{1}{21} \begin{pmatrix} 8 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 8 \end{pmatrix}$$

Exercise Check that $\mathrm{AA}^{-1} = 1$.

3.4 Calculus and differential equations

3.4.1 Differentiation

Given a continuous real valued function $f(x)$ on the interval $[a, b]$, we say that $f(x)$ is differentiable on $[a, b]$ if:

$$\lim_{h\to\infty} = \frac{f(x+h) - f(x)}{h} \ \exists \ \forall \ x \in [a, b]$$

We call this limit function $f'(x) \equiv \frac{\mathrm{d}f(x)}{\mathrm{d}x}$ the *derivative* of $f(x).f''(x) \equiv \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$.
If $y = f(x)$ we write $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ or indeed $y'(x)$.

3.4.2 Rules for differentiation

1. $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) + g(x)\} = \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{\mathrm{d}g(x)}{\mathrm{d}x}$

   Hence differentiation is a **linear operator**.
2. Deriverative of a product: $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) \cdot g(x)\} = f(x)g'(x) + f'(x)g(x)$
3. Deriverative of a quotient: $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \wedge g(x) \neq 0$
4. Deriverative of a function of a function: $\frac{\mathrm{d}}{\mathrm{d}x} f\{g(x)\} = f'\{g(x)\} \cdot g'(x)$
5. Deriverative of a constant ('c'): $\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0 \ \forall \ c$
6. Deriverative of a mononomial: $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1} \ \forall \ n$
7. Deriverative of a polynomial: Follows from rule 1 and 6
8. Deriverative of trigonometric functions:
   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x) \ \forall \ x \neq \frac{(2n+1)\pi}{2}$ where $n \in \mathbb{Z}$ and $\sec(x) = \frac{1}{\cos(x)}$
9. Deriverative of a exponential function: $\frac{\mathrm{d}}{\mathrm{d}x} e^x \equiv \frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = e^x \equiv \exp(x)$
10. Deriverative of a logarithmic function: $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\log e^x = \frac{1}{x}$ for $x > 0$

Example

$$f(x) = \sin^2\left\{\frac{(2x^3+3x)\cos x}{e^{(x^3-\tan x)}}\right\}$$

$$f'(x) = 2\sin\left\{\frac{(2x^3+3x)\cos x}{e^{(x^3-\tan x)}}\right\} \cdot \cos\left\{\frac{(2x^3+3x)\cos x}{e^{(x^3-\tan x)}}\right\} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{(2x^3+3x)\cos x}{e^{(x^3-\tan x)}}\right\}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{(2x^3+3x)\cos x}{e^{(x^3-\tan x)}}\right\} = \frac{e^{(x^3-\tan x)}\frac{\mathrm{d}}{\mathrm{d}x}(2x^3+3x)\cos x - (2x^3+3x)\cos x \cdot \frac{\mathrm{d}}{\mathrm{d}x}e^{(x^3-\tan x)}}{\left(e^{(x^3-\tan x)}\right)^2}$$

$\frac{\mathrm{d}}{\mathrm{d}x}(2x^3+3x)\cos x = (6x^2+3)\cos x + (2x^3+3x)(-\sin x)$

$\frac{\mathrm{d}}{\mathrm{d}x}e^{x^3-\tan x} = e^{x^3-\tan x}\frac{\mathrm{d}}{\mathrm{d}x}x^3 - \tan x = e^{x^3-\tan x}(3x^2-\sec^2 x)$

$$f'(x) = 2\sin\left\{\frac{(2x^3+3x)\cos x}{e^{(x^3-\tan x)}}\right\}\cdot\cos\left\{\frac{(2x^3+3x)\cos x}{e^{(x^3-\tan x)}}\right\}\cdot$$
$$\frac{e^{(x^3-\tan x)}\left\{(6x^2+3)\cos x+(2x^3+3x)(-\sin x)\right\}-(2x^3+3x)\cos x\cdot e^{x^3-\tan x}(3x^2-\sec^2 x)}{\left(e^{(x^3-\tan x)}\right)^2}$$

3.4.3 Integration

Given a continuous real valued function $f(x)$ on the interval $[a,b]$ we say $f(x)$ is integrable on $[a,b]$ if:

$$\lim_{\Delta x_n\to 0}\sum_{n=a}^{n=b} f(x)\Delta x_n \ \exists\ \forall\ x\in[a,b]$$

We call this limit function $\int_a^b f(x)\mathrm{d}x$ the *definite integral* of $f(x)$ with respect to x over the interval $[a,b]$. If $a=b=\infty$ we get the *indefinite integral.*
So: $\int f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = y(x)+c$.

$\int_a^b f(x)\mathrm{d}x = g(x)\Big|_a^b = g(b)-g(a)$

Note $f(x)\Big|_a^b = f(b)-f(a)$
$\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=1}$: say $\frac{\mathrm{d}y}{\mathrm{d}x}=h(x)$ then $\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=1}=h(1)$

3.4.4 Techniques of integration

Substitution

Example $\int(30x^5+100x)^{10}(3x^4+2)\mathrm{d}x$
Let $U=30x^5+100x$
$\frac{\mathrm{d}U}{\mathrm{d}x}=150x^4+100=50(3x^4+2)$
Then $\int(30x^5+100x)^{10}(3x^4+2)\mathrm{d}x = \int U^{10}\frac{\mathrm{d}y}{\mathrm{d}x\cdot 50}\cdot\mathrm{d}x = \frac{1}{50}\int U^{10}\mathrm{d}u = \frac{1}{50}\cdot\frac{U^{11}}{11} = \dfrac{(30x^5+100x)^{11}}{50\cdot 11}+c$

Example $\int\frac{f'(x)}{f(x)}\mathrm{d}x$
Let $U=f(x)$, then $\frac{\mathrm{d}u}{\mathrm{d}x}=f'(x)$ $(\mathrm{d}u=f'(x)\mathrm{d}x)$
$\int\frac{f'(x)}{f(x)}\mathrm{d}x = \int\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x}\mathrm{d}x = \int\frac{1}{u}\mathrm{d}u = \log_e|U| = \ln|f(x)|+c$

Example $\int\tan x\mathrm{d}x = \int\dfrac{\sin x}{\cos x}\mathrm{d}x = -\int\dfrac{-\sin x}{cosx}\mathrm{d}x = -\log_e|\cos x|+c$

Trigonometric substitution

Example $\int\dfrac{\mathrm{d}x}{x^2+a^2}; a\neq 0$
Let $x=a\tan\Theta$, then $\frac{\mathrm{d}x}{\mathrm{d}\Theta}=a\sec^2\Theta$.
$x^2+a^2=a^2(\tan^2\Theta+1)=a^2\sec^2\Theta$
$\int\dfrac{\mathrm{d}x}{x^2+a^2} = \int\dfrac{a\sec^2\Theta}{a^2\sec^2\Theta}\mathrm{d}\Theta = \frac{1}{a}\int 1d\Theta = \frac{\Theta}{a} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\ \left\{\Theta=\tan^{-1}\frac{x}{a}\right.$

Note Note the trigonometric identity $\tan^2\Theta+1=\sec^2\Theta$ (for $\cos^2\Theta+\sin^2\Theta=1$).

Integration by parts

Recall If U = U(x) and V=V(x), then $\frac{\mathrm{d}}{\mathrm{d}x}UV = U\frac{\mathrm{d}}{\mathrm{d}x}V + V\frac{\mathrm{d}}{\mathrm{d}x}U$
So, $\int \frac{\mathrm{d}}{\mathrm{d}x}UV\mathrm{d}x = \int V\frac{\mathrm{d}}{\mathrm{d}x}U\mathrm{d}x + \int U\frac{\mathrm{d}}{\mathrm{d}x}V\mathrm{d}x = UV$.

We write this as $\int u\ \mathrm{d}v = uv - \int v\mathrm{d}u$.
This method is used where we have the product of functions of different types.
EG.: $\int x\sin x\mathrm{d}x$; $\int e^x\cos x\mathrm{d}x$, etc.

Example $\int x\cos x\mathrm{d}x$
Try $x = u$; $\cos x\mathrm{d}x = \mathrm{d}u$
$\frac{\mathrm{d}}{\mathrm{d}x}v = \cos x, v = \sin x$
$\frac{\mathrm{d}}{\mathrm{d}x}u = 1, \mathrm{d}u = \mathrm{d}x$
Hence: $\int x\cos x\mathrm{d}x = x\sin x - \int \sin x\mathrm{d}x = x\sin x + \cos x + c$

Example $\int x\cdot e^x$
Try $u = e^x, \mathrm{d}v = x\mathrm{d}x$
$\mathrm{d}u = e^x\mathrm{d}x, v = \dfrac{x^2}{2}$
Hence $\int u\mathrm{d}v = \dfrac{e^x\cdot x^2}{2} - \int \dfrac{x^2}{2}\cdot e^x\mathrm{d}x$

Now try: $u = x, \mathrm{d}v = \mathrm{d}x$; $\mathrm{d}v = e^x\mathrm{d}x, v = e^x$
Hence $\int u\mathrm{d}v = x\cdot e^x - \int e^x\mathrm{d}x = xe^x - e^x + c$

From (i) and (ii) we see that we can arrive at a formula for $\int \dfrac{x^2}{2}e^x\mathrm{d}x$, so our original substitution was not in vain.

Note In general, integration is more difficult that differentiation. The rules for integration are the same for those as differentiation, replacing the '$\frac{\mathrm{d}}{\mathrm{d}x}$' operator with '$\int \mathrm{d}x$'. There is however no formula for the integral of a product or a quotient.

3.4.5 General tips on performing integration

1. If the integral is not simple (ie., that of a familiar function ($\cos x, e^x$, etc.), try substitution
2. If that fails, try integration by parts. Unless you see the product of two types of functions, use substitution first.
3. Remember 'LIATE' for the order of integration by parts where L = log, I = inverse, A = algebra, T = trig and e = exponent / exponential. Pick u as the first occuring in the list, and dv and the other.

3.4.6 Differential equations

Introduction

Differential equations are equations containg the differential operator(s): $\frac{\mathrm{d}}{\mathrm{d}x}, \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \frac{\mathrm{d}}{\mathrm{d}t}$, etc. These equations arise regularly in mechanics, field theory, electronics, chemical engineering, etc.

Example Simple examples of differential equations are:

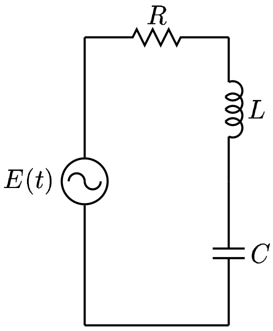
Speed	$\frac{\mathrm{d}u}{\mathrm{d}x} = C$ (EG, 60 km/h)
Acceleration	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = a$ (EG, $\frac{10\ \mathrm{km/h}}{\mathrm{h}}$)

More complicated examples

The MCk system $M\dfrac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + C\dfrac{\mathrm{d}y(t)}{\mathrm{d}t} + ky(t) = F(t)$

This equation describes the oscillation of a body on a spring where m is the mass of the body, C the damping constant, k the spring modulus at rest.
$F(t)$ is the applied force on the body, causing it to oscillate when released.

The RLC-circuit $L\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + R\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{i(t)}{C} = E(t)$
In electronic circuits a similar differential equation governs the behaviour of circuits of resistors, capacitors and inductors.
The equation describes the current flow $i(t)$ through a circuit consisting of a voltage source $E(t)$ and a series connected resistor, inductor and capacitor.



We will be examining these types of circuits and their governing differential equations in section VI (page 62).

3.4.7 Terminology

Differential equations terminology

Consider the following differential equations:

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + y^k \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^m + y^j = Q(x)$$

The highest derivative order n determines the *order* of the differential equation. The highest derivative power m determines the *degree* of the differential equation.
If j (the higest power of y) is 0 or 1, the differential equation is termed *linear* . If $j \neq 0, 1$ the differential equation is termed *non-linear* .

Functions terminology

A function $f(x, y)$ is said to be *seperable* if it can be written as $f(x, y) = g(x)h(y)$ where $g(x)$ is a function of x only and $h(g)$ is a function of y only.

Example $f(x) = x^5 y^4$ is seperable.

A function $f(x, y)$ is said to be *homogeneous* if $f(tx, ty) = f(x, y)\ \forall\ t \in \mathbb{R}^{\backslash\{0\}}$.

Example $f(x) = \sin\sqrt{\frac{x+y}{2x-3y}}$ is homogeneous.

Solutions terminology

A solution is said to be *explicit* if it can be written in the form: $y = f(x)$ where $f(x)$ is a function of x only.

Example $y = \sin(x^2 + 4)$

A solution is said to be *implicit* if it can be written in the form: $f(x, y) + g(x, y) + \cdots = k; k \in \mathbb{R}$.

Example $y^3 x^3 + y^5 x^3 + \sin(x, y) = 10$

3.4.8 Solving differential equations

Type I - Seperate first order

$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ where $f(x, y)$ is seperable.

This is an easy type to solve. The equation becomes:

$$\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$

$$\begin{aligned}\text{Hence}\quad & \int \frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x && = \int g(x)\mathrm{d}x \\ \text{i.e.}\quad & \int \frac{\mathrm{d}y}{h(y)} && = \int g(x)\mathrm{d}x + c\end{aligned}$$

So finding the solution is reduced to finding two anti-deriverates (integrals). The arbitrary constant is essential here and turns out to have an interpretation. In practise we can write:

$$\begin{aligned} & \frac{1}{h(y)}\mathrm{d}y && = g(x)\mathrm{d}x \\ \text{So}\quad & \int \frac{\mathrm{d}y}{h(y)} && \int g(x)\mathrm{d}x + c\end{aligned}$$

The interpretation of c: the solution represents a family of curves (given by varying c's).

Given a particular value of (x, y), this will fix c and give a definite solution. This particular value of (x, y) is referred to as the *initial conditions* or *boundary condition* .

Example Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+y}$

Seperable because $e^{x+y} = e^x e^y$. Write the equation as:

$$e^{-y}\mathrm{d}y = e^x\mathrm{d}x$$

$$\begin{aligned}\text{Hence}\quad & -e^{-y} && = e^x + c \\ \text{So}\quad & x && = \ln|-e^{-y} + c| \quad \text{(explicit solution)}\end{aligned}$$

Example Find the general solution of the differential equation: $\frac{\mathrm{d}y}{\mathrm{d}x} = (1 + x)y^2$ and find the particular solution for which $y = 1$ when $x = 1$.

This equation is a seperable type:

$$\begin{aligned} & \frac{1}{y^2}\mathrm{d}y && = \left(1 + \tfrac{1}{x}\right)\mathrm{d}x \\ & \int \frac{1}{y^2}\mathrm{d}y && = \int \left(1 + \tfrac{1}{x}\right)\mathrm{d}x \\ \text{Hence}\quad & -\tfrac{1}{y} && = x + \ln|x| + c \quad \text{(general solution)}\end{aligned}$$

$y = 1$ if $x = 1$; $-1 = 1 + 0 + c; c = -2$.

The particular solution is $-\frac{1}{y} = x + \ln x - 2$.

Type II - Homogeneous first order

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ where $f(x, y)$ is a homogeneous function. Equations of this type can be transformed into equations of a seperable type by a change of variable.
Let $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$

Note If $f(x,y)$ is homogeneous, then $f=(\frac{x}{x},\frac{y}{x})=f(1,\frac{y}{x})=f(1,v)$

The equation then becomes $x\frac{\mathrm{d}v}{\mathrm{d}x}+v=f(1,v)$, which can be written in seperable form:

$\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{1}{x}\Big\{f(1,v)-v\Big\}$

which can be solved using previous methods.

Example Find the general solution of the equation

$x(2x-y)\frac{\mathrm{d}y}{\mathrm{d}x}=(x+y)^2$

And find the particular solution given $y=0$ when $x=1$.

First, check for homogenomity. $\frac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+y)^2}{x(2x-y)}=f(x,y)$. $f(tx,ty)=f(x,y)\ \forall\ t\in\mathbb{R}^{\setminus\{0\}}$, so the equation is homogeneous.

Use the substitution $y=vx$ to get $x\frac{\mathrm{d}v}{\mathrm{d}x}+v=\dfrac{\left(1+\frac{y}{x}\right)^2}{\left(2-\frac{y}{x}\right)}=\dfrac{(1+v)^2}{2-v}$ or alternatively $x\frac{\mathrm{d}v}{\mathrm{d}x}=\dfrac{1+2v+v^2}{2-v}-v=\dfrac{1+2v^2}{2-v}$.

Hence: $\int\dfrac{2-v}{1+2v^2}\mathrm{d}v=\int\dfrac{1}{x}\mathrm{d}x$ (i.e., of seperable type).

Solving this given the general implicit solution:

$\sqrt{2}\tan^{-1}\Big(\dfrac{y\sqrt{e}}{x}\Big)-\frac{1}{4}\ln\Big(1+\dfrac{2y^2}{x^2}\Big)=\ln|x|+c$

To find the general particular solution for which $y=0$ when $x=1$, substitute the values: $0-0=0+c$, so $c=0$. So the particular solution is:

$\sqrt{2}\tan^{-1}\Big(\dfrac{y\sqrt{e}}{x}\Big)-\frac{1}{4}\ln\Big(1+\dfrac{2y^2}{x^2}\Big)=\ln|x|$

Type III - Linear first order

Use the integration factor method.

Consider $\frac{\mathrm{d}y}{\mathrm{d}x}+p(x)y=Q(x)$ where $P(x),Q(x)$ are functions of x only.
One method of solution is to find a function $\mu(x)$ so that if we multiply both sides of the differential equation by $\mu(x)$, it becomes:

$\frac{\mathrm{d}}{\mathrm{d}x}\big(\mu(x)y\big)=\mu(x)Q(x)$

Then the solution will be:

$y=\dfrac{\int\mu(x)Q(x)\mathrm{d}x+c}{\mu(x)}$

So how do we find $\mu(x)$?

1. $\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\mu(x)p(x)y=\mu(x)Q(x)$
2. But $\frac{\mathrm{d}}{\mathrm{d}x}\big(\mu(x)y\big)=\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}\mu(x)}{\mathrm{d}x}y=\mu(x)Q(x)$
3. 1. and 2. must be identical, so we need $\frac{\mathrm{d}\mu(x)}{\mathrm{d}x}=\mu(x)p(x)$
4. So $\dfrac{1}{\mu(x)}\frac{\mathrm{d}}{\mathrm{d}x}\mu(x)=p(x)$
   $\ln|\mu(x)|=\int p(x)\mathrm{d}x$

5. So $\mu(x) = e^{\int p(x)\mathrm{d}x}$

Then $y = \dfrac{\int \mu(x)Q(x)\mathrm{d}x + c}{\mu(x)}$

Example $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = x^2$

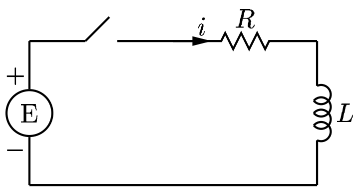
So, $p(x) = \frac{1}{x}, \mu(x) = e^{\int p(x)\mathrm{d}x} = e^{\int \frac{\mathrm{d}x}{x}} = e^{\ln|x|} = x$. Then $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3$. The general solution is:

$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3; xy = \frac{x^4}{4} + c$

3.4.9 Applications of differential equations

Application I - The RL circuit

Setup:



The electric current $i(t)$ (amps) flowing through a coil of inductance L (Henrys) and a resistor of R (ohms) as a result of an applied voltage E (volts), satisfies the differential equation:
$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) = E$ where L, R, E are constants.
Time t is measured in seconds; $i(t) = 0$ for $t = 0$.
So, $\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{R}{L}i(t) = \frac{E}{L}$
The integrating factor $\mu(t) = e^{\int \left(\frac{R}{L}\right)\mathrm{d}t} = e^{\left(\frac{R}{L}\right)t}$. Multiply accross by $\mu(t)$ to get $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\left(\frac{R}{L}\right)t} \cdot i(t)\right) = \frac{E}{L}\left(e^{\left(\frac{R}{L}\right)t} \cdot i(t)\right)$
This has the general solutions:

$e^{\left(\frac{R}{L}\right)t} \cdot i(t) = \frac{E}{L} \cdot \frac{L}{i^2} \cdot e^{\left(\frac{R}{L}\right)t} + c.$

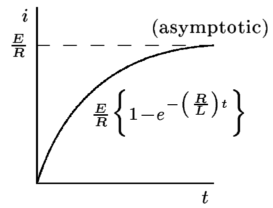
I.e. $i(t) = \frac{E}{R} + ce^{-\left(\frac{R}{L}\right)t}$

$i(t) = 0$ for $t = 0$, so $p = \frac{E}{R} + c, c = -\frac{E}{R}$.

Hence the particular solution is:

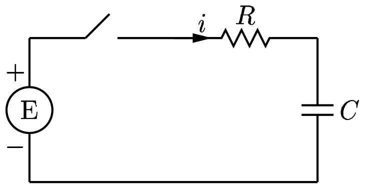
$i(t) = \frac{E}{R}\left\{1 - e^{-\left(\frac{R}{L}\right)t}\right\}$
Letting $t \to \infty$ then $i(t) \to \frac{E}{R}$ (steady state value of the current).

Plot of solution:



Application II - the RC circuit

Setup:



The potential difference $V_c(t)$ across a capacitor of capacitence c Ferods in series with a resistance of R ohms and a voltage (DC) supply of E volts satisfies the following differential equation:

$\frac{\mathrm{d}V_c(t)}{\mathrm{d}t} + \frac{V_c(t)}{RC} = \frac{E}{RC}$ where C, R, E are onstants.

Initial condition: $V_c(t) = 0$ when $E = 0$. Find the solution for $V_c(t)$.

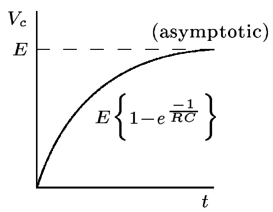
Integration factor method: $\begin{array}{lll} \frac{\mathrm{d}V_c(t)}{\mathrm{d}t} & +\frac{V_c(t)}{RC} & = \frac{E}{RC} \\ \frac{\mathrm{d}y}{\mathrm{d}x} & +p(x)y & = Q(x) \end{array}$

Hence the general solution is:

$$\begin{aligned} V_c(t) &= \frac{\int e^{\int\left(\frac{1}{RC}\right)\mathrm{d}t}\frac{E}{RC}\mathrm{d}t + c}{e^{\left(\frac{1}{RC}\right)\mathrm{d}t}} \\ &= \frac{\int e^{\frac{t}{RC}}\frac{E}{RC}\mathrm{d}t + c}{e^t RC} \\ &= E\{1 - e^{-\frac{t}{RC}}\} \end{aligned}$$

We established the value of c using the initial value.

Plot of the solution:



Chapter 4

Physics Review

For a discussion on the physics section, see:

Senior Physics
by George Porter

4.1 Part A - Mechanics

4.1.1 Linear motion

Definition The displacement of a point from another point is its distance from that point in a particular direction. Displacement is a vector quantity. It's units are the meter 'm'.

Example The displacement of B from D is 129 km due W ('as the crow flies'). The distance of 'B' from Dublin by road is 180 km (due to the curvature of the road).
If I drive to B from D and back, the distance I have travlled is 380 km but my displacement from D is 0 km.

Example I walk 10 km due West of my home, I then walk 5 km due North. The total distance I have travalled is 15 km. My displacement from home is $\sqrt{125}$ km WNW.

Definition **Velocity** is the rate of change of displacement with respect to time. Velocity is a vector quantity. It's unit's are the meter per second ms^{-1}. Let displacement be denoted by the symbol s, then velocity $v = \frac{\text{d}s}{\text{d}t}$.

Example The displacement of S from C is 265 km N. The distance by road is 330 km. A motorist travels from C to S in 5 hours.

Then is average speed is $\dfrac{\text{distance}}{\text{time}} = \dfrac{330}{5} = 66$ km/h.
It's average velocity is $\dfrac{\text{displacement}}{\text{time}} = \dfrac{265}{5} = 53$ km/h.

Definition **Acceleration** is the rate of change of velocity with respect to time. Acceleration is a vector quantity. It's unit is the $\frac{\text{m/s}}{\text{s}} = \text{ms}^{-2}$. Mathematically acceleration a is:

$a = \frac{\text{d}v}{\text{d}t} = \frac{\text{d}^2 s}{\text{d}t^2}$

Example The velocity of a car changes from 10 m/s E into 25 m/s E in 10 s. Find it's average acceleration a.

$a = \dfrac{\text{change in velocity}}{\text{time}} = \dfrac{15}{10} = 1.5\text{ms}^{-2}$ E.

Definition The **mass** of a body is a measure of its ability to resists changes in its velocity. Mass m is a scalar quantity and its units are the kilogram kg. **Alternative definition** The mass of a body is a measure of the quantity of matter in it.

Example It takes 10 times the force to move object A from rest as it does to move object B (in the same environment). Hence the mass of object A is 10 times that of object B.

Example Sample A of a particular substance is 10 times the volume of substance B (in the same environment). Hence the mass of object A is 10 times that of object B (and it will take 10 times the force to move object A from rest as it does object B).

Note Important notes on mass:

1. The mass of a body does *not* depend on where it is. Hence a body of 10 kg on earth will have a mass of 10 kg on the moon, Jupiter or outer space.

2. The colloquial use of the term "weight" should not be confused with mass. "Weight" is a measure of force and its units are the Newton (N). The weight of a body weighing 10 N on earch would weigh 1N on the moon; 90 N on Jupiter and 0N in outer space.

4.1.2 Momentum

Definition The momentum of a body is the product of its mass and its velocity. Momentum is a vector quantity. Mathematically momentum $P = mv$, where m is the mass and v the velocity. The unit of momentum is the kg $\text{ms}^{-1} \equiv \text{kg}\ \frac{\text{m}}{\text{s}}$.

Example A truck travelling at 30 km/h hits a man and kills him. A peddle thrown at 30 km/h at the man has no impact. Hence impact is momentum.

$P \propto m$
$P \propto v$
$P \propto mv$

So, $P = cmv$, where c is a constant. Define $c = 1$, we get P = mv.

4.1.3 Conservation of momentum

The principle of conservation of momentum states that in any interaction involving a system of bodies, the total momentum before the interaction equals the total momentum afte rthe interaction.

Example A bullet of mass 0.02kg strikes a block of wood of mass 2.0kg, which is at rest.

After the collision the bullet and the block move together. Assuming there is a neglegible interaction (friction) between the block and its support, calculate the velocity of the block and the bullet immediately after impact.
Solution: total momentum before = total momentum after. Therefore:

$$\begin{aligned}
(v_{bullet} \cdot m_{bullet}) + (v_{block} \cdot m_{block}) &= (m_{bullet+block} \cdot v) \\
(400 \cdot 0.02) + (0 \cdot 2.0) &= (2.02 \cdot v) \\
8 &= 2.02v \\
v &= 3.96\ \text{ms}^{-1} \text{ in the direction of the bullet.}
\end{aligned}$$

4.1.4 Force

Definition A force is that which causes acceleration. Force (F) is a vector quantity and its unit is the Newton (N).

4.1.5 Newton's Laws of Motion

Law I

The velocity of a body does not change unless a resultant external force acts on it.

Law II

When a resultant external force acts on a body, the rate of change of the body's momentum is proportional to the force and takes place in the direction of the force.

Law III

In any interaction between two bodies A and B, the force exerted by A on B is equal in magnitude but opposite in direction to the force exerted by B on A.

4.1.6 Law I

Example Consider a car travelling at a constant velocity of 50 km/h E. Fact: there is no net force acting on the car. In other words, the forward force given by the engine is equal and opposite to the resistance given by the air, friction of mechanical parts (wheel etc.) and friction of the road surface. The driver presses the accelerator and the car accelerates to a new speed of 70 km/h. During this time a resultant (net) force is exerted on the car in the forward direction.
As the velocity increases, so do the resistance factors given above, until the new forward force equals the forces of resistance and the velocity of the car steadies to 70 km/h.

Example A man steps from a spacecraft in deep space (= far away from stars and planets). If he in unattached to the craft he will continue to move away from the spacecraft at a constant velocity indefinitely (or until the gravity of some nearby star or planet pulls him in), because there is no resistance to his motion.

Exercise Why does he not continue with a constant acceleration?

Note In this scenario the spacecraft will move in a direction opposite to the man and will continue in that direction with a constant velocity, because of the principle of conservation of momentum.

4.1.7 Law II

Mathematically:

$\mathrm{F} \propto \frac{\mathrm{dP}}{\mathrm{dt}}$ where P is the momentum.

If the mass of the body is constant, then $\mathrm{F} \propto \mathrm{m}\dfrac{\mathrm{dv}}{\mathrm{dt}}$ i.e. $\mathrm{F} \propto \mathrm{ma}$. So, $\mathrm{F} = \mathrm{cma}$ where c is a constant. Define $\mathrm{c} = 1$, then:

Definition $\mathrm{F} = \mathrm{ma}$

Hence the unit of force is thte Newton (N). One N is that force which gives a mass of 1 kg an acceleration of ms^{-2}. So $1\ \mathrm{N} = 1\ \mathrm{kg\ ms}^{-2}$.

Example Find the acceleration produced by a force of 10 N SW, acting on a mass of 20 kg.

$\mathrm{F} = \mathrm{ma}$, $10 = 20\mathrm{a}$, $\mathrm{a} = 6.5\ \mathrm{ms}{-}2$.

4.1.8 Law III

This laws says in effect that forces occur in pairs.

Example A block of wood rests on a table. The block exerts a force downwards (due to it's own weight). The table exerts an equal and opposite force upwards. Hence, the resultant force is zero (vector addition) – there is no motion ($\overrightarrow{\mathrm{F}_1} + \overrightarrow{\mathrm{F}_2} = 0$). The the block is very heavy, the table is unable to sustain its weight i.e., provide sufficient upward force. Hence the table breaks and the block falls through with a constant acceleration.

Exercise

Why with a constant acceleration and not with constant velocity?

Note Read "Senior Physics" about Newton's Law's of Gravitation ($F = G\,\frac{M_1M_2}{d^2}$).

4.2 Part B - Electrical energy and work

Definition **Work** is done when a force moves a body. Work is a scalar quantity and its unit is the Joule. Mathematically:

$$W = \vec{F} \cdot \vec{S}$$

1 J is the work done when a force of 1 N causes a displacement of 1 m in the same direction of the force. So 1 J = 1 NM.

Definition **Energy** is the ability to do work. Energy is essentially "stored work"'. Its unit is therefore the Joule.

4.2.1 Principle of conservation of energy

Energy cannot be created or destroyed.

4.2.2 Forms of energy

Energy has many forms (sound, chemical, electrical, etc.) There are two important categories of enery:

1. Kinetec energy
2. Potential energy

Kinetic energy

Kinetic energy is the energy of a moving body. We calculate the energy by calculating the work done in coming to rest.

Potential energy

This is the energy a body has due to its state or position (eg. a wound up spring or a rock at the edge of a cliff). Again, the potential energy is calculated by calculating the work done by the body in reaching a stable position (eg. the unwinding of the spring or the impact of the rock on the ground).

4.3 Part C - Basic atomic theory

All matter is composed of tiny particles called atoms. They in turn are made up of even smaller particles (sub-atomic particles), namely:

1. Neutrons
2. Protons
3. Electrons

Protons and electrons are attracted to one another (this physical force of attraction is much greater than and supplemental to the gravitation force that exists between them).

We say that protons are positively charged and electrons are negatively charged, so there is an electrical force of attraction between them.

The magnitude of charge of a proton is equal to that of an electron. Neutrons are uncharged. Normally atoms are uncharged, i.e. de number of protons equals the number of electrons.

Neutrons and protons exist tightly bound together and form the nucleon of the atom. Electrons 'revolve' around this nuclean at various distances (strictly speaking this is not true, but close enough...).

The mass of an electron is $9.11 \cdot 10^{-28}$ g and that of a proton is $1.672 \cdot 10^{-24}$ g (which roughly equals that of a neutron). Hence the mass of a proton (or neutron) is roughly 2000 times that of an electron (e^-).

Protons, neutrons and electrons are considered to be spherical in shape. Their radius is approximately in the order of magnitude $2 \cdot 10^{-15}$ m.

For the hydrogen atom (which has one proton, one electron and no neutrons), the orbit followed by the electron is $\approx 5\dot{1}1^{-11}$ m. From these dimensions it follows that the setup is equivalent to a marble revolving around another marble 0.25 mile away!

Different atoms make up different elements. So, Hydrogen has is made up of hydrogen atoms and copper matal is made up of copper atoms.

There are 110 elements known to man. Different atoms will have various numbers of of electrons which reside in concentric shells around the nucleons. The first shell, which is closest to the nucleons, can contain a maximuum of 2 electrons. If an atom should have 3 electrons (i.e. Lithium, Li) the third atom will reside in the next (2(ᴷ) shell, which can contain a maximum of 8 electrons.

The equation $2n^2$ determines the maximum number of electrons a shell can contain where n is the number of the shell. Hence the 4th shell can contain a maximum of $2 \cdot 4^2$ electrons.

Since the nucleons contain positivily charged protons, an attractive force exists between them and the electrons that revolve around them. As the distance between the nucleons and the shell increases, the binding force decreases until it reaches its lowest level at the outermost shell.

Due to these weaker binding forces, less energy must be expended to remove an electron from its othermost shell than from its inner shell.

Combinations of atoms are referred to as molecules. EG. H_2O, CO_2 (respectively water and carbon dioxide). Carbon dioxide is composed of two atoms of oxygen chemically bound with one atom of carbon. Water exists of one atom of oxygen chemically bound with two hydrogen atoms.

This chemical binding is a result of atoms 'sharing' electrons in their outermost shells. Materials which are composed of combinations of atoms are referred to as compounds, of which there are very many.

Chapter 5

Electric and Magnetic Field Theory

This section, and all subsequent sections, are potential exam material.

5.1 Electric Fields

5.1.1 Electric Charge

Recall from atomic theory that protons and electrons are said to be charged. Furthermore, protons are said to be positively charged and electrons are said to be negatively charged. A force of attraction exists between unlike charges and a force of repulsion exists between like charges. This force is given by Coulomb's Law.

5.1.2 Coulomb's Law

The force between any two point charges is proportional to the product of the charges and inversly proportional to the square of the distance between them.

Mathematically $\vec{\mathrm{F}} = \frac{1}{4\pi\epsilon} \cdot \frac{Q_1 Q_2}{r^2}\hat{\mathrm{r}}$

Where Q represents charge, $\hat{\mathrm{r}}$ is the unit vector in the direction of $\vec{\mathrm{F}}$ and ϵ is a function of the material in which the charge reside. It is known as the *electrical permittivity* of the medium.

Note ϵ_0 is the permittivity of free space (i.e. vacuum) and is equal to $8.854 \cdot 10^{-12}\mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2} \equiv \mathrm{FM}^{-1}$ (Farads per metre). Air is normally taken to have this permittivity.

ϵ_r is the relative permittivity of a medium and is defined as $\epsilon_r = \frac{\epsilon}{\epsilon_0}$.

Example Calculate the magnitude of the force in the air between two points charges of 4 nC each if the distance between them is 5 mm.

$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q_1 Q_2}{r^2} = \frac{1}{4\pi \cdot 8.9 \cdot 10^{-12}} \cdot \frac{(4 \cdot 10^{-9})^2}{(5 \cdot 10^{-3})^2} = 5.7mN$$

Lets say $\vec{\mathrm{F}}$ was required — then $\vec{\mathrm{F}}$ = 5.7 mN repulsive.

If multiple charges exist, the force on any one of the charges equals the vector sum of the forces exerted on it by the other charges.

I.e. for the total force on a charge Q, $\vec{\mathrm{F}} = \frac{1}{4\pi\epsilon} Q \sum_i \frac{q_i}{{r_i}^2}\hat{\mathrm{r}}_\mathrm{i}$

5.1.3 Definitions

When the wind blows from one point to another, we say there is a pressure difference between two points.
When heat (energy) flows from one point to another we say there is a temperature difference between two points.
Likewise, when charge flows from one point to another, we say there is a *potential* difference between two points, i.e. de difference in potential energy.

Definition The **potential difference** or p.d. for short between two points is the work done in bringing unit charge from one point to another.
Potential difference is a scalar and its unit is the Volt (V).

Definition The potential difference between two points is 1 **Volt** (V) if the work done in bringing a charge of 1 C from one point to the other is equal to 1 J.

A term closely related to potential difference is electromotive force (e.m.f.)

Definition The **e.m.f.** is any closed loop is the work done in bringing unit charge around the loop.

Definition When there is a potential difference between two points an **electric field** is said to exist between them.

Definition The **electric field strength** or intensity at a point is the force per unit positive charge at that point.

Mathematically $\overrightarrow{\mathrm{E}} = \dfrac{\overrightarrow{\mathrm{F}}}{Q}$

The electric field strength is a vector quantity, and its unit is the NC^{-1} or Vm^{-1}.

Definition Where an electric field exists, the **electric flux** through an area is defined as:

$$Q_e = \frac{\hat{\mathrm{n}} \cdot \overrightarrow{\mathrm{E}}}{\overrightarrow{\mathrm{E}}} A$$

Where A is the area and $\hat{\mathrm{n}}$ is the unit vector normal to the area.

When charge (or charged particles) move from one point to another we have electric current (I)

Definition The **electric current** is the rate of change of charge with respect to time.

Mathematically $I = \dfrac{\mathrm{d}Q}{\mathrm{d}t}$

The unit of electric current is the Ampère. Hence one Ampère of current flows where the rate of change of charge is Cs^{-1}. Current is a scalar quantity.

5.1.4 Ampère's Law

Ampère discovered that when currents flow they exert a force of attraction or repulsion an each other. This force cannot be explained by Coulomb's Law. Currents attract each other when flowing in the same direction and repel each other when flowing in opposite direction.
This pheomenon is used to define the unit of electrical current, the A (Ampère or Amp for short).

Definition The **Ampère** is that constant current which, maintained in two straight parallel conductors of infinite length, of negligible cross section, and placed 1 m apart in a vacuum, causes each to exert a force of $1 \cdot 10^{-7}$ N per metre length on the other.

From this follows the definition of the unit of charge, the Coulomb.

Definition The **Coulomb** is the quantity of charge transferred when a current of 1 A flows for 1 s.

Ampère's summarized his discovery as followed:

When charges move they exert a force on one another. Another way of looking at this phenomenon is to say that a moving charge radiates a field which exerts a force an another charge moving in that field.

5.2 Magnetic Fields

Definition From Ampère's discovery we may state: Moving electric charge is said to radiate a **magnetic field**.

Definition The magnetic flux density ($\overrightarrow{\mathrm{B}}$) at a point is the force per unit positive charge (electric) where the charge is moving at a velocity of 1 ms^{-1} perpendicular to the direction of the magnetic field.
The magnetic flux density is a vector quantity. The unit of magnetic flux density is the Wb (Weber) m^{-2}.
Magnetic flux density is used to characterize the magnetic field.

5.2.1 The Lorentz Force Law

When a charge q moves with a velocity v through a magnetic field of magnetic flux density $\overrightarrow{\mathrm{B}}$, it experiences a force $\overrightarrow{\mathrm{F}}$ given by $\mathrm{F} = q\overrightarrow{\mathrm{u}} \times \overrightarrow{\mathrm{B}}$.

Extension

If there exists simultaneously electric and magnetic fields then the total force on a moving charge is $\overrightarrow{\mathrm{F_T}} = \overrightarrow{\mathrm{F_E}} + \overrightarrow{\mathrm{F_B}} = q(\overrightarrow{\mathrm{E}} + \overrightarrow{\mathrm{V}} \times \overrightarrow{\mathrm{B}})$.

5.2.2 The direction of the magnetic field

The direction of the magnetic field (radiated by a moving charge) is given by the right hand rule (or more precisely, by Maxwell's Equations).

Rule: Choose right fist and extend thumb. Then were $\hat{\mathrm{i}}$ is in the direction of the moving charge (current), the magnetic field ($\overrightarrow{\mathrm{B}}$) will be in the direction given by the sence of rotation of the fingers.

Note Magnetic fields form closed loops

Note In the case of the electric field, the force exerted on a charge is in the direction of the electric field. An electric field therefore 'begins' at a positive charge and 'ends' at a negative charge.

In the case of the magnetic field, the force exerted on the moving charge is perpendicular to the direction of the magnetic field (and to its velocity).

5.2.3 Magnetic flux

Where a magnetic field exists, the magnetic flux through an area is defined as:

$$\varphi_m = \hat{\mathrm{n}} \cdot \overrightarrow{\mathrm{B}} A$$

(*see* Definition of electric flux, page 31)

5.2.4 The electric displacement vector and magnetic field strength

Other vectors are used to characterize electric and magnetic fields.

1. $\overrightarrow{\mathrm{D}}$ The electric displacement vector, unit cm^{-2}
2. $\overrightarrow{\mathrm{H}}$ The magnetic field strength, unit Am^{-1}

These are related to the vectors $\overrightarrow{\mathrm{E}}$ and $\overrightarrow{\mathrm{B}}$ (in linear media) via the following equations:

1. $\overrightarrow{\mathrm{D}} = \epsilon\overrightarrow{\mathrm{E}}$
2. $\overrightarrow{\mathrm{B}} = \mu\overrightarrow{\mathrm{H}}$

Where μ is a constant for the medium and is called the magnetic permeability. μ_0 is the magnetic permeability of free space and is equal to $4\pi \cdot 10^{-7}$ HM^{-1} (Henrys per metre).

The relative permeability is defined as $\mu_r = \dfrac{\mu}{\mu_0}$.
$\overrightarrow{\mathrm{D}}$ is also known as the *electric flux density*. It can be considered as the amount of electric flux per unit area.
$\overrightarrow{\mathrm{H}}$, the magnetic field strength, can be understood as the force exerted on a unit magnetic charge. Note however, magnetic charges do *not* (!) but are sometimes used to explain concepts. Where magnetic charges are used, the unit given them is the Weber (Wb).
Note also that a magnetic charge will experience a force parallel to the magnetic field.
A moving electric charge (which *does* exist) will experience a force perpendicular to the magnetic field.

5.2.5 The significanse of ϵ and μ

The electric field strength $\overrightarrow{\mathrm{E}}$ and the magnetic flux density $\overrightarrow{\mathrm{B}}$ due to a charge or a moving charge will vary depending on the medium. ϵ and μ characterize the ability of a medium to sustain an electric and magnetic field respectively. The smaller ϵ and larger μ mean the medium can better sustain electric and magnetic fields (*see also* Coulombs' Law, page 30, and the Lorentz Force Law, page 32).

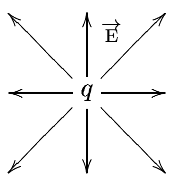
Materials which sustain electric fields well are known as *conductors*. Those which don't are known as *insulators* or *dielectrics*. Examples of good conductors are copper, iron and carbon. Examples of good dielectrics are a vacuum, plastic, and wood.

Materials which sustain magnetic fields well are known as *magnetic materials*. Those which don't are known as *non-magnetic materials* . Examples of magnetic materials are iron, nickel and cobalt. Examples of non-magnetic materials are copper, lead and water.

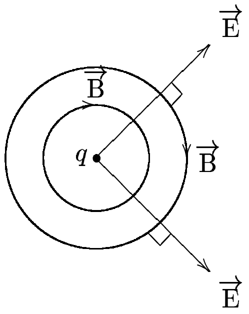
5.2.6 The orthogenality of electric and magnetic fields

Consider a charge moving through vacuum. We know from experiment:

That the electric field extends radially.



The magnetic field extends in a circular fashion given by the right hand rule.



$\vec{V}$ into the page

If we superimpose the two diagrams, it is clear that $\vec{B}$ is perpendicular to $\vec{E}$ at all points.

5.3 Maxwell's Equations

Maxwell's Equations are the laws of

1. Faraday and Lang
2. Ampère
3. Gauss
4. Gauss, Biot and Savart

5.3.1 History

In 1873 Maxwell (a Scottish mathematician) published his now famous treatise "Electricity an Magnetism". In it he wrote the above four laws in 3D vector differential form.

Having done this, he deduced that Ampère's Law was not consistent with the principle of conservation of charge as given by the "Continuity Equation", and so he amended Ampère's Law. The result was revolutionary; his work predicted:

1. Electromagnetic waves
2. Light velocity
3. Displacement current (e.g. A.C. current through a capacitor)

His work describes *all* electriomagnetic phenomena as we know them. The above four equations relate electric and magnetic fields to each other and to crrent / charge distributions.

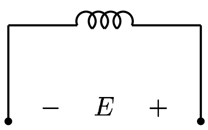
We will now examine each of Maxwell's Equations as they apply to electric circuits with the mathematical tools we have (i.e. 1D calculus).

5.3.2 The Law of Faraday and Lang (applied to solenoids)

When a changing magnetic field threads a conducting loop, an e.m.f. E (don't confuse with $\vec{E}$ for magnetic fields!) is induced in the loop. If there are n such loops, the equation becomes:

$$E = \frac{n\mathrm{d}\varphi_m}{\mathrm{d}t}$$

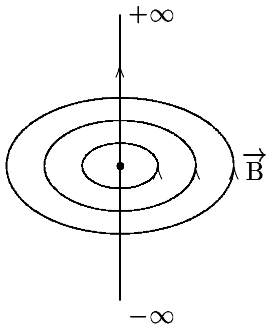
Diagram



Where the magnetic field inside the solenoid is varying. The inducted voltage $E(t)$ will have a polarity such that it will oppose the change in flux. I.e., the induced e.m.f. will cause an induced current $i(t)$ to flow in the solenoid. This current in turn will have its own magnetic field. From the right hand rule we see this magnetic field $\overrightarrow{\mathrm{B}_2}(t)$ will oppose $\overrightarrow{\mathrm{B}_1}(t)$.

5.3.3 Ampère's Law

Applied to an infinite straight wire $\overrightarrow{\mathrm{B}} = \frac{\mu I}{2\pi R}(\hat{\imath} \times \hat{\mathrm{r}})$, relates the magnetic field to current for an infinite straight wire. Hence the magnetic field is said to decrease 'like $\frac{1}{R}$', where R is the distance from the wire.



5.3.4 Gauss' Law I

This law states that the total electric flux threatening an arbitrary closed surface equals the charge enclosed. I.e. $\varphi_{e,\text{tot}} = Q$.

Take for example a point charge surrounded by a sphere of radius r.
The electric flux density D is $D = \frac{1}{4\pi} \cdot \frac{Q}{r^2}$.

$$
\begin{aligned}
\text{Then } \varphi_{e,\text{tot}} &= DA \text{ (since } \hat{\mathrm{d}} \text{ is parallel to } \hat{\mathrm{n}} \text{)} \\
&= \frac{Q}{4\pi r^2} \cdot 4\pi r^2 \\
&= Q
\end{aligned}
$$

5.3.5 Gauss's Law II

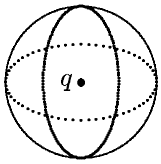
The fourth of Maxwell's equations is normally attributed to either Gauss or Biot and Savart. It is as follows:

The total magnetic flux threading an arbitrary closed surface is *zero* , i.e. $\varphi_{m,total} = 0$. Hence magnetic charge does not exist.

The Biot-Savart Law is as follows:

$$\overrightarrow{\mathrm{B}} = \frac{\mu_0 \cdot q(\overrightarrow{\mathrm{u}} \times \overrightarrow{\mathrm{r}})}{4\pi |r|^3}$$

Which relates the magnetic field to the velocity of a *discrete* charge.



A spherical field decreases 'like $\frac{1}{r^3}$'.

Using 3D calculus it is possible to show that the laws of Ampère and Biot-Savart are equivalent to Gauss' Second Law, but this is not within the scope of this course.

5.3.6 Examples

Faradaw's Law

Example A magnetic flux of 400 μWb passing through a coil of 1200 turns is reversed in 0.10 sec. Calculate the average value of the e.m.f. induced in the coil.

Change in flux $= D\mu_m = -800$ μWb. Hence e.m.f. $E = \frac{1200 \cdot (-800\mu\text{Wb})}{0.1} = 9.6\text{V}$.
This answer must be positive, because the wires are carrying opposite currents: $\overrightarrow{\mathrm{B_a}}$ is parallel to $\overrightarrow{\mathrm{B_b}}$ between the wires, and in the same direction.

From Ampère's Law:

$$\begin{aligned} \mathrm{B}_a &= \mathrm{B}(x) &&= \frac{\mu_0 \cdot I}{2\pi x} \\ \mathrm{B}_b &= \mathrm{B}(D-x) &&= \frac{\mu_0 \cdot I}{2\pi(D-x)} \end{aligned}$$

Hence $\mathrm{B}_{tot} = \mathrm{B}_a + \mathrm{B}_b = \frac{\mu_0 \cdot I}{2\pi}\left\{\frac{D}{x(D-x)}\right\}$

Ampère's Law

Example A coil of 200 turns is wound uniformly over a wooden ring, having a mean circumference of 600 mm and a uniform cross-section area of 500 mm^2. If the current through the coil is 4A, calculate:

1. The magnetic field strength
2. The flux density
3. The total flux

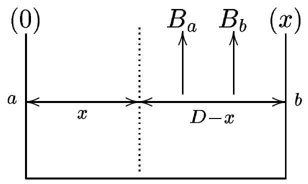
The mean circumference is 0.6m:

$H = 4 \cdot \frac{200}{0.6} = 1333\ \text{AM}^{-1}$

Flux density $= \mu_e H = 1675$ μT (Testa)

Cross sectional area = 500 mm^2. Total flux $= 1675 \cdot 500 \cdot 10^{-6} = 0.8375$ μWb.

Example Two very long, straight, thin copper wires, placed distance D apart, are carrying equal currents I in opposite directions. If the wires are placed perpendicular to the page, find the expression for the magnetic field B in terms of the distance x from the wire 'a' as shown in the figure.



(Sorry, I don't have the rest of this example. Anybody cares to tell me?)

Example A mild steel ring having a cross-selected area of 500 mm^2 and a mean circumference of 400 mm has a coil of 200 turns wound uniformly around it.

Calculate:

1. The reluctance of the ring
2. THe current required to produce a flux of 800 units in the ring

Flux density in the ring $= \dfrac{800 \cdot 10^{-6}}{500 \cdot 10^{-6}} = 1,6 \quad \text{T}.$
μ_r for mild steel $= 380$
Reluctance $= \dfrac{0.4}{380} \cdot 4\pi \cdot 10^{-7} \cdot 5 \cdot 10^{-4} = 1.699 \cdot 10^6 \text{A Wb}^{-1}$
$800 \cdot 10^{-6} = \dfrac{\text{m.m.f.}}{1.699 \cdot 106}$
m.m.f. $= 1342$ A
$I = \dfrac{1342}{200} = 16.7 \quad \text{A}.$

Notes on Ampère's Law Examples

In example I: $H = \dfrac{NI}{2\pi R}$

Consider Ampère's Law $B = \dfrac{\mu I}{2\pi R}$ so $I = 2\pi RH$ — which gives the field strength at distance R from I.

But the general form of Ampère's Law states that the integral of $\overrightarrow{\text{H}}$ around any closed loop equals the current enclosed. I.e., $\int_l H \cdot al = I$. Hence H $\times$ circumference = I enclosed. But I encloed = NI where N is the number of turns and I is the current in each turn. So: $H = \dfrac{NI}{2\pi R}$

We will shortly prove this formula using the Biot-Savart Law.

In Example III: The quantity NI is known as the magnetomative force (m.m.f.).
Hence, reluctance $= \dfrac{\text{m.m.f.}}{\varphi_m}$. So, reluctance $= \dfrac{l}{\mu A}$.

The Lorentz-Force Law

Example Calculate the force on a straight 0.5m conductor carrying 1A, placed perpendicular to a magnetic field of 2T.

$\overrightarrow{\text{F}} = Q\overrightarrow{\text{V}} \times \overrightarrow{\text{B}}, I = \dfrac{\text{d}Q}{\text{d}t}$ so $Q = \int_0^t I\text{d}t$
$\overrightarrow{\text{F}} = \int_0^t I\text{d}t\dfrac{\text{d}s}{\text{d}t} \times \overrightarrow{\text{B}} = \int_0^t I\text{d}s \cdot B$ (for $\hat{\text{s}}$ is perpendicular to $\hat{\text{b}}$) $= BIL = 2 \cdot 1 \cdot 0.5 = 1N$, perpendicular to $\hat{\text{s}}$ and $\hat{\text{b}}$

Example If a force of 1 N perpendicular to $\hat{\text{s}}$ and $\hat{\text{b}}$ is applied to a straight 0.5 m conductor carrying 1A positioned perpendicular to a magnetic field of 2T, calculate the current induced in the conductor.

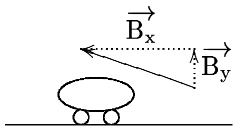
$\overrightarrow{\text{F}} = BIL$, so $I = 1\text{A}$.

Notes of the Lorentz Force Law Examples

Example I. The fact that a current is induced in a conductor moving through a magnetic field follows from the principle of conservation of energy.

Example II. Since a current is induced in a conductor moving through a magnetic field it follows that an e.m.f. has been induced in the conductor. This e.m.f. is given by Faraday's Law: $E = \frac{\mathrm{d}\varphi_m}{\mathrm{d}t}$. Here $\mathrm{d}\frac{\varphi_m}{\mathrm{d}t}$ represents the rate of which the conductor cuts or traverses a flux.

Example Calculate the e.m.f. generated in the exile of a car travelling at 80 km/h where the length of the exile is 2m and the vertical component of the earth's magnetic field is 40 μT.



$80 \text{ km/h} = 22.2 \text{ ms}^{-1}$
$\overrightarrow{\mathrm{B}_\perp} = 40 \cdot 10^{-1} \text{ T}$

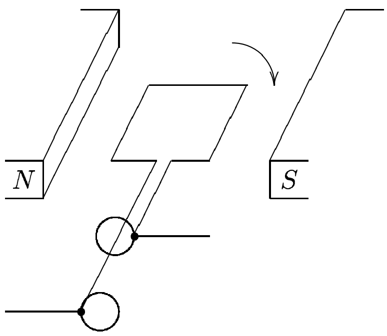
(Sorry, don't have the rest of this example either. Please mail me so I can update this document)

Notes on Faraday's Law and the Lorentz Force

Electrical Power Generation

The fact that an e.m.f. is induced in a conductor threading a magnetic field, is the principle of *all* electric power generators.

Simple example:



Electric Motors

Electic motors operate on exactly the same principle as generators. If I apply an e.m.f. to the open terminals, a current will flow in the conductor causing the loop to rotate in the magnetic field.

The Biot-Savart Law

Example

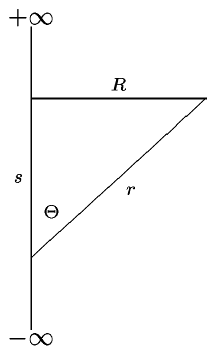
1. Show the equivalence between the Biot-Savart Law and Ampère's Law for an infinite straight wire
2. Find the magnetic field $\overrightarrow{\mathrm{B}}$ as a function of the distance along the axis of a current I carrying loop of radius R
3. Hence determine the magnetic field at the centre of an infinite straight solenoid carrying n turns per unit length

1.

$$\vec{B} = \frac{\mu}{4\pi} \cdot \frac{q(\vec{v} \times \vec{r})}{|r|^3}$$

In differential form: $dB = \frac{\mu}{4\pi} \frac{dq\frac{d\vec{s}}{dt} \times \vec{r}}{|r|^3} = \frac{\mu I}{4\pi} \cdot \frac{d\vec{s} \times \vec{r}}{|r|^3}$

Now consider an infinite straight wire:

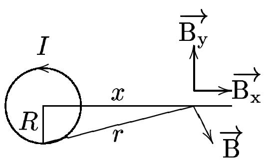


By Pythagoras: $r^2 = R^2 + s^2$ where R is the straight line distance from the wire to the point where we are finding the field.

Also: $\sin(\Theta) = \frac{R}{\sqrt{R^2 + s^2}}$

Hence $B = \frac{\mu}{4\pi} IR \int_{-\infty}^{+\infty} \frac{ds}{(R^2 + s^2)^{\frac{3}{2}}} = \frac{\mu I}{2\pi R}$

2.



$d\vec{B} = \frac{\mu I}{4\pi} \cdot \frac{d\vec{s} \times d\vec{r}}{|r|^3}$ (where $\vec{r} = |r|\hat{r}$) $= \frac{\mu I}{4\pi} \cdot \frac{ds}{x^2 + R^2} \cdot \frac{R}{\sqrt{x^2 + R^2}}$

Hence $\vec{B} = \frac{\mu I}{4\pi} \cdot \frac{2\pi R^2}{(x^2 + R^2)^{\frac{3}{2}}} = \frac{\mu I}{2} \cdot \frac{R}{(x^2 + R^2)^{\frac{3}{2}}}$ along x (because the y-components of $\vec{B}$ are always cancelled in the loop). I.e., $ds \mapsto 2\pi r$.

3.

We have $B\hat{x} = \frac{\mu I}{2} \cdot \frac{R^2}{(x^2 + R^2)^{\frac{3}{2}}}$ for a single loop.

Then $B_{tot}\hat{x} = \frac{\mu I_n}{2} \int_{-\infty}^{+\infty} \frac{R^2}{(x^2 + R^2)^{\frac{3}{2}}} dx = \mu n I$ for n loops per unit length.

Hence $\hat{x} = nI$, which to a good approximation for a finite length solenoid is:

$H_{tot}\hat{x} = \frac{NI}{L}$ where N is the total number of loops and L the length of the solenoid.

It can be shown that this is not only the magnetic field strength along the axis of a solenoid, but also at every point within the solenoid — but this is beyond the scope of this program.

5.4 Statics and dynamics

We have covered static fields; i.e. electrostatics and magnetostatics. Dynamic fields are electromagnetic fields where the electric field $\vec{E}$ is a function of time.

Dynamic fields are very important in nature. The reason for this is that energy is transformed from one point to another via these fields. E.g. the heat of the sun, wireless communication, etc.

However, the mathematics necessary to describe dynamic fields are beyond the scope of this program.

Chapter 6

Materials (-science)

6.1 Conductors

A conductor is a material which on application of an E.M.F. facilitates current (charge transfer) well. The reason for this is simple: conductors are composed of atoms (or molecules) which for whatever reason have electrons in the outer shell that are loosley bound to the nucleons — which means that on application of an E.M.F. these electrons are easily 'freed' and will move in the direction of the relatively positive charge, while the relatively negative charge replenishes the supply of electrons. Hence current!

The best conductors are metals and among the best of them is copper, which is why copper is normally used in electric wiring.

The reason metals tend to make good conductors is because:

1. They are generally large atoms (lot of electrons available for conduction and lots of shells)
2. They have a relatively large number of electrons in their outer orbit which as a result of their position are loosely bound to the nucleons

Note These are *general* observations and there are exceptions.

Hence good conductors are a 'sea' of electrons available for conduction.

Example Consider silver (Ag) — a good conductor. 108 g of Ag contains $6.02 \cdot 10^{23}$ atoms (1 mole). The density of Ag is 10.5 g / cm^3. Hence 1 cm^3 of Ag contains $\dfrac{6.02 \cdot 10^{23}}{10.5} \approx 6.0{\cdot}10^{22}$ atoms. Statistically speaking, each atomm of Ag has one electron available for conduction (at room temperature). Therefore 1 cm^3 of Ag has $6.0 \cdot 10^{22}$ 'free' electrons.

6.2 Dielectrics

A dielectric is a material which on application of an E.M.F. facilitates current poorly. The reason for this is simple: they are composed of atoms (or molecules) which (for whatever reason) have their electrons bound tightly to the nucleons (or 'bonds' in the case of molecules), which means that on application of an E.M.F. there are very few electrons available for conduction.

6.3 Semi-conductors

Semi-conductors are materials which on application of an E.M.F. facilitate a 'moderate' current. Semi-conductors are a class of materials which fall between conductors and insulators. Examples of semi-conductors are Silicon (Si) and Germanium (Ge).

As opposed to conductors, where approximately one electron per atom is available, in semiconductors this figure is roughly one electron for every 10^{10} atoms.

Example 28.1 g of Silicon contains $6.02 \cdot 10^{25}$ atoms. The density of Si is 2.33 g / cm^3. Hence 1 cm^3 of Si contains $5 \cdot 10^{22}$ atoms and there are $5 \cdot 10^{12}$ electrons per cm^3 available for conduction.

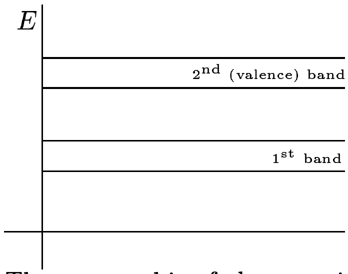
When an E.M.F. is applied across a semiconductor 'free' electrons move towards the anode (relatively positive charge). The 'space' left behind (referred to as a 'hole') has a small positive charge and is replenished by electrons from the cathode side (relatively negative charge). These electrons in turn leave a hole, which in turn is replenished, etc. etc.

This process of charge transfer can be *viewed* as a process of either electrons moving from the cathode to the anode, or holes moving from the anode to the cathode — or both. For the purpose of explaining succintly the operation of semi-conductor devices, we view the current through a semi-conductor as a process consisting of 'free' electrons moving to the anode *and* 'holes' moving to the cathode. In a pure semi-conductor (i.e. a semi-conductor conposed of one element) the number of 'free' electrons equals the number of 'holes' when an E.M.F. is applied.

6.3.1 The energy level diagram

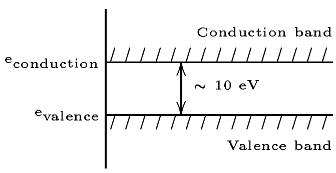
We use an energy level diagram to explain the relative availability of electrons for conduction. We have seen that the availability of electrons for conduction depends on the material in question. The kinetic energy of an electron $k_e \propto T$.

As we have seen, electrons occupy discrete 'orbits' at varying distances from the nucleon. It follows that electrons in the outer orbits have greater kinetic energy than these in the inner orbits. Hence electrons are said to occupy 'energy' levels. There will be some variation in the kinetic energies of electrons in the same shell. Hence the energy level diagram is composed of bands.



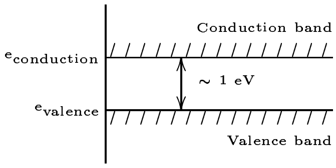
The outer orbit of electrons is associated with the 'valence' band on the energy level diagram. It is from this band that electrons will make themselves available for conduction. In order to be fully available for conduction, an electron must occupy an energy band at a higher level than the valence band. This band is known as the conduction band.
If an electron occupies this band then a tiny E.M.F. will cause it te move; i.e., it is no longer bond to the nucleons. The relative conductivity of conductors, insulators and semi-conductors can be expressed graphically using an energy level diagram.

Dielectrics

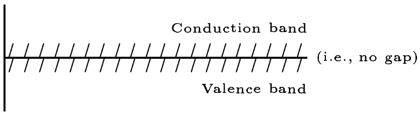


1 eV $\approx 1.6 \cdot 10^{-19}$ J. This is the energy gained by an electron when accelerated through a p.d. of 1 V. $e_{\text{conduction}}$ in the diagram points to the minimum energy an electron needs to be able to get to the conduction band (likewise is e_{valence} the *maximum* energy to stay in the valence band).

Semi-conductors



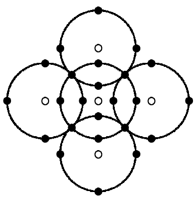
Conductors



6.3.2 Semi-conductor doping

A process known as doping, which is the addition of an 'impurity' element to a semiconductor, is used to enhance the conductivity of the semiconductor itself.

Consider silicon. It's atomic number is 14, so it has 4 electrons in it's valence orbit. All atoms have a propensity to seek eight electrons in their outer shell for $n > 1$. This is the reason for the chemical binding: atoms share valence electrons so that both make up this compliment in their outer shells. Pure silicon does this by establishing the following relationship among it's atoms (only nucleon ($\circ$) and valence orbit with the valence electrons ($\bullet$) drawn):



Here the centre Si atom has eight electrons in the valence orbit. The outer atoms here will in turn share their remaining 3 electrons with other atoms so that all will have their full complement of electrons in the outer shell. This arrangment gives the crystalline structure of Si.
A pure semi-conductor (eg. composed of Si only) is known as an intrinsic semi-conductor. For most applications, there are not enough free electrons and holes in an intrinsic semiconductor to produce a usable current. Doping means adding impurity atoms to increate either the number of free electrons or the number of holes. When a semi-conductor has been doped, it is called an extrinsic semi-conductor.

6.3.3 n-type semi-conductors

To get extra conduction band electrons we can add pentavalent atoms (5 electrons in the valence orbit). The structure of the resulting compound has 'extra' electrons which tend to be pushed away from the outer shell, and are therefore more readily available for conduction. Pentavalent atoms include Arsenic, Atimony and Phosphorus.

In this way, the number of free electrons exceed the number of holes when an E.M.F. is applied. Hence here the electrons are referred to as the majority carriers and holes as the minority carriers. A semi-conductor doped with a pentavalent substance is known as a n-type semi-conductor (n for negative).

6.3.4 p-type semi-conductors

Another way of enhancing the conductivity of an intrinsic semi-conductor is to add a trivalent substance (3 atoms in the outer shell). The resulting substance effectively has an exceed of 'holes'. Examples of trivalent atoms are Aluminium and Gullium. A semi-conductor doped by a trivalent impurity is known as a p-type semi-conductor (p for positive). Hence here the majority carriers are holes and the minority carriers are electrons.

Chapter 7

Electric Circuit Components

7.1 Preamble

7.1.1 Current

Definition The **current** $i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t}$

7.1.2 Notes on current

1. Current 'flows' between two points as a result of a potential difference between these points
2. Current 'flows' from a (relatively) positive potential to a relatively negative one
3. Current is (normally) the flow of negative charged electrons (except in the case of displacement current e.g. through a capacitor). Hence a positive potential is associated with an accumulation of negative charge in electric circuits.

I.e.:

Where E is a battery and R a resistor.

7.1.3 Voltage

Definition The **voltage** $V(t) = \frac{\mathrm{d}w(t)}{\mathrm{d}q}$ where w is the work done.

7.1.4 Electric Power

Electric power is defined as the rate of energy expenditure: $p(t) = \frac{\mathrm{d}w(t)}{\mathrm{d}t}$. Hence the unit of power $[p] = \frac{J}{s} = w$ (watt). $\mathrm{d}w(t) = p(t)\mathrm{d}t = v(t)\mathrm{d}q \mapsto p(t) = v(t)\frac{\mathrm{d}q}{\mathrm{d}t}$

So $p(t) = v(t)i(t)$

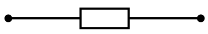
7.2 The Resistor

A resistor is a two-terminal element which impedes the flow of electric current through it by converting some of the electrical energy into heat. The degree to which it impedes the current flow is characterized by it's resistance. Every material posesses this property to some extent. The relationship between the terminal voltage of a resistor and the current flowing through it is given by Ohm's Law:

$v(t) = Ri(t)$

This empirical law states that the voltage drop across the resistor is directly proportional to the current flowing through it. $[R] = \frac{V}{A} = \Omega$ (Ohm). The symbol for a resistor is:

Or, alternatively:



So, in the circuit of page 44:

$V(t) = Ri(t),\ p(t) = v(t)i(t)$ so $p(t) = \frac{V^2(t)}{R} = i^2(t)$

Example Consider a heater which is connected to some fixed voltage source (e.g. mains). The power of this heater (i.e., the rate at which heat is produced) is given by:

$$P_{\text{heater}} = \frac{V^2(t)}{R_{\text{heater}}}$$

It is obvious that a smallar resistance is needed for a greater heating effect. Such heaters are common in household appliances such as heaters, toasters, electric hair-dryers etc.

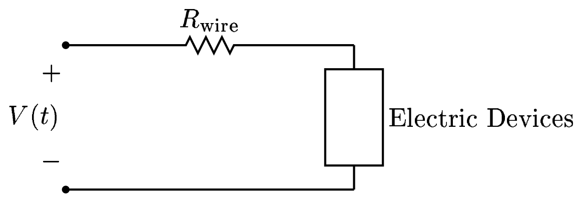
Note If $R \to 0 \Rightarrow p \to \infty$ (burnout of circuit). I.e., either the resistor will burn or the wire will melt. Therefore protect with fuses or circuit breakers.

Example Consider the transmission of electric power through power lines. In this case there is a fixed amount of power intended for transmission:

$p(t) = v(t)i(t)$
There is also power lost due to the resistance of the wire (R_{wire}) of the transmission line. Hence the power lost is:

$$p_{\text{lost}} = R_{\text{wire}} i^2(t)$$

There is two solutions. You can reduce R_{wire} but that is very expensive. A better solution is to make $i(t)$ as small as possible. Since p_{trans} is fixed, the voltage $v(t)$ must be high in order to transmit the same power with the desired small current. The latter approach is the basic idea behind high-voltage transmission lines.



Note You may ask:

$$p_{\text{wire}} = \frac{V(t)}{R_{\text{wire}}}$$

which seems to contradict the second suggestion above. However, it doesn't. $V(t)$ is the potential difference between the positive and negative terminals, i.e. across the wire resistance *and* the load. Hence:

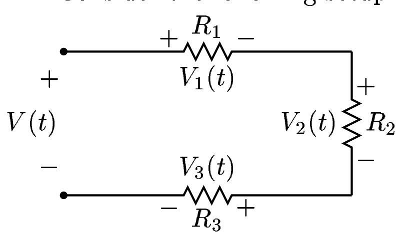
$$p_{\text{wire}} \neq \frac{V_{\text{supply}}(t)}{R_{\text{wire}}}, \text{ but in fact } p_{\text{wire}} = \frac{V_{\text{wire}}}{R_{\text{wire}}}$$

I.e., $V_{\text{supply}}(t) \neq V_{\text{wire}}(t)$

7.2.1 Series and Parallel Resistors

Series Resistors

Consider the following setup:

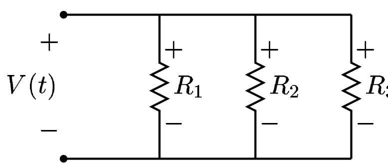


$$\begin{aligned}
&\text{Then} && V_1(t) = i(t)R_1 \\
& && V_2(t) = i(t)R_2 \\
& && V_3(t) = i(t)R_3 \\
&\text{But} && V_1(t) + V_2(t) + V_3(t) = V_T(t) \\
&\text{So} && V_T(t) = i(t)(R_1 + R_2 + R_3)
\end{aligned}$$

So, for n resistor in series: $R_T = R_1 + R_2 + \cdots + R_n$

Parallel Resistors

Consider the following setup:

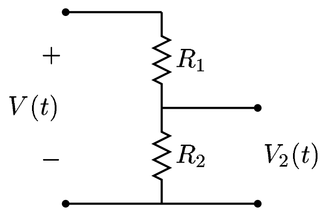


$$\begin{aligned}
&\text{Then} && i(t) = i_1() + i_2(t) + i_3(t) \\
& && V(t) = i_1(t)R_1 = i_2(t)R_2 = i_3(t)R_3 = i(t)R_T \\
&\text{So} && i_1(t) = \frac{V(t)}{R_1} \\
& && i_2(t) = \frac{V(t)}{R_2} \\
& && i_3(t) = \frac{V(t)}{R_3} \\
& && i(t) = \frac{V(t)}{R_T} \\
&\text{And} && \frac{V(t)}{R_t} = \frac{V(t)}{R_1} + \frac{V(t)}{R_2} + \cdots + \frac{V(t)}{R_n}
\end{aligned}$$

So, for n resistors in parallel: $\dfrac{1}{R_t} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \cdots + \dfrac{1}{R_n}$

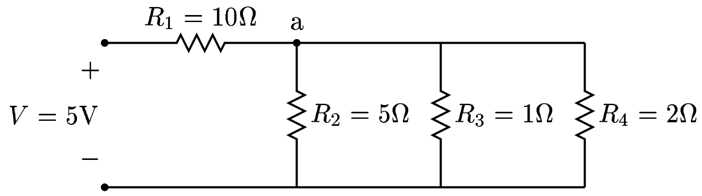
7.2.2 The Potential Divider

An important application of resistors is the potential divider. It is used to obtain a lower potential (voltage) from a higher one. It consists of an E.M.F. applied across two series resistors. I.e.:



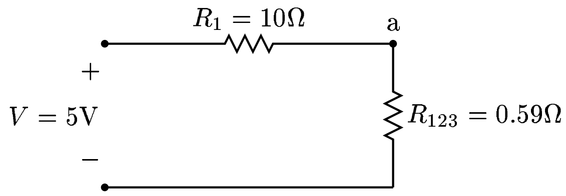
$V(t) = i(t)(R_1 + R_2)$, but $V_2(t) = i(t)R_2$, so $V_2 = V(t)\left\{\dfrac{R_2}{R_1 + R_2}\right\}$

Example Find the current I and the voltage at point 'a' in the following circuit:

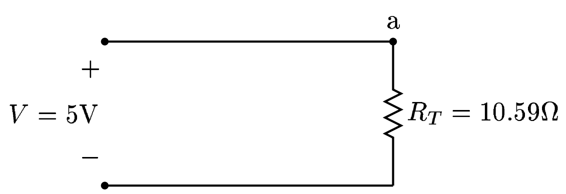


To analyse resistance circuits, work from right to left, i.e. back to the supply.

Consider $R_2 \parallel R_3 \parallel R_4$; $R_{123} = (\frac{1}{5} + \frac{1}{1} + \frac{1}{2})^{-1} = 0.59\Omega$. I.e.:



$R_T = 10\Omega + 0.59\Omega = 10.59\Omega$. I.e.:



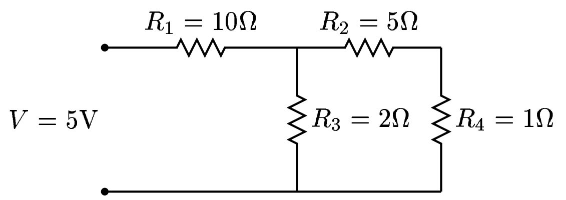
$I = \dfrac{V}{R} = \dfrac{5}{10.59}$ A $= 0.47$ A. Since we know I, then $V_a = 0.59 \cdot 0.47 = 0.27\text{V}$.

Alternatively, using the potential divider formula:

$$V_a = V\left\{\frac{0.59}{10 + 0.59}\right\} = 0.27V$$

Example Consider the following setup:

Find the current I.



Any purely resistor circuit can be reduced to a supply in series with a single resistor. The problem is to conculate the resistance R of this resistor.

R_2 in series with R_4 so $R_{24} = R_2 + R_4 = 6\Omega$
$R_3 \parallel R_{24}$ so $\dfrac{1}{R_{324}} = \dfrac{1}{R_3} + \dfrac{1}{R_{24}}$, $R_{324} = 1.5\Omega$
R_1 in series with R_{324} so $R_T = R_1 + R_{324} = 11.5\Omega$

$I = \dfrac{V}{R} = \dfrac{5}{11.5}$

7.2.3 The impedance of a Resistor

As we have seen, the relationship between the terminal voltage and current for a resistor is given by:

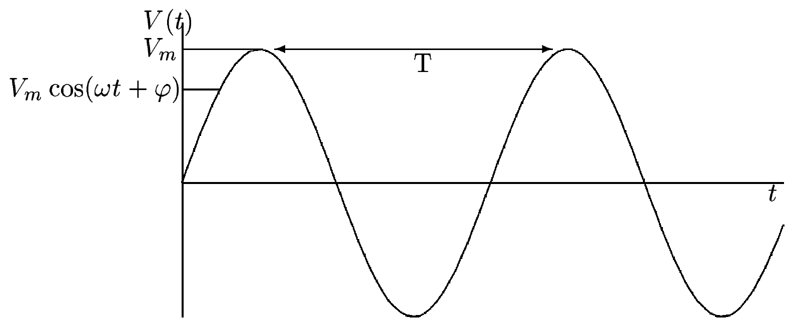
$V(t) = Ri(t)$ (Ohm's Law)

The quantity R is the resistance. This quantity indicates the degree to which the device impedes the flow of current. Greater $R \to$ less current and vice versa.

$V(t) = V_m \cos(\omega t + \varphi)$

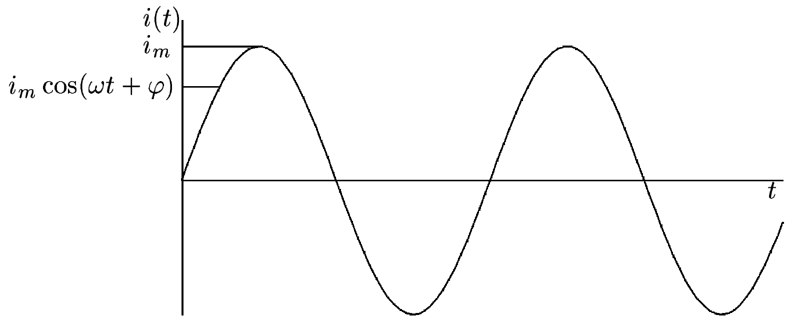
Where	V_m	is a constant
	ω	is the frequency in rads
	t	is the time
	φ	is the phase displacement

Graphically:



The period of the waveform is T (sec). The frequency $\dfrac{1}{T}$ (s^{-1} or Hz) or $\dfrac{2\pi}{T} = 2\pi f$ (rad s^{-1}). Then $i(t) = RV_m \cos(\omega t + \varphi) = I_m \cos(\omega t + \varphi)$.

Graphically:



In this setup the terminal voltage oscillates positive and negative. Hence alternating current (A.C.), as opposed to direct curent (D.C.) in previous examples, where we had a constant voltage supply.

Definition The **impedance** of a current device is defined as $Z = \dfrac{V_m}{I_m}$.

In the case of a resistor, this is equal to R:

$$Z_R = \frac{V_m \cos(\omega t + \varphi)}{I_m \cos(\omega t + \varphi)} = R$$

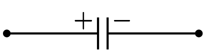
Example Consider the setup of page 47, but replace the 5 D.C. supply with a A.C. one, and find $i(t)$. The analysis is the same as for the D.C. example, so $i(t) = \dfrac{5}{11.5}\cos(\omega t + \varphi)$ A.

7.3 The Capacitor

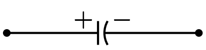
A capacitor is a two-terminal element which stores energy in it's electric field. It's ability to do this is characterized by it's capacitance C. A capacitor is formed by two conductors seperated by a distance. Sometimes a dielectric material may be placed between the conductors to provide a means of seperation and to enhance the capacitance. When a voltage source is applied, both conductors of the capacitor possess charges of equal magnitude but opposite sign. These charges create an electric field bewteen the conductors.

Definition The **capacitance** is defined as the ratio between the charge and the voltage, i.e. $C = \dfrac{q(t)}{V(t)}$.

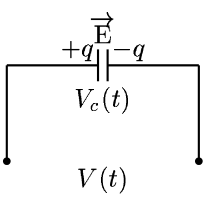
The symbol for the capacitor is



Or alternatively



The energy is stored in an electric field between the two plates of the capacitor:



Although the capacitance is the ratio of charge to voltage, it does not depend directly on either of these two variables. The capacitance is determined only by the dimensions of the conductors, the seperating distance between them and the permittivity of the dielectric material between the conductors. So:

$C = \dfrac{\epsilon A}{d}$

Where	ϵ	is the absolute electric permittivity
	A	is the area of one plate
	d	is the distance between the plates

Fabrication methods for capacitors include parallel plates with ceromic, mica, air, etc. dielectrics and rolled electrolytic paper of foil.

In order to obtain the terminal relationship between voltage and current in a capacitor we differentiate:

$$\begin{aligned} q(t) &= CV(t) \\ \frac{\mathrm{d}q(t)}{\mathrm{d}t} &= C\frac{\mathrm{d}V(t)}{\mathrm{d}t} \\ i(t) &= C\frac{\mathrm{d}V(t)}{\mathrm{d}t} \end{aligned}$$

From this equation it is clear that current will only 'travel' through a capacitor if there is a changing terminal voltage. It is important to note that when current 'flows' through a capacitor there is no movement of charge between the plates. Rather, the changing potential across the dielectric *induces* current flow on either side of the plates by virtue of the electric field set up between the plates. Reduction of the positive charge on the left hand side induces a reduction of the negative charge on the right hand side; hence current flows (remember current $= i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t}$). Therefore capacitors will allow A.C. current to flow, but will block D.C. current. Because the current through a capacitor does not involve movement of charge, this current is referred to as *displacement* current.

The unit of capacitance is the Farad (F), defined as the capacitance of a capacitor which contains one Coulomb of charge on each plate when a potential difference or voltage of 1 V is applied.

7.3.1 Power delivered to a capacitor

Power $\equiv p(t) = v(t)i(t)$. Note that this power is dissipated on heat in a resistor, however all of it is stored in the $\overrightarrow{\mathrm{E}}$ field of the capacitor. However:

$$\begin{aligned} i(t) &= C\frac{\mathrm{d}V(t)}{\mathrm{d}t} \\ p(t) &= CV(t)\frac{\mathrm{d}V(t)}{\mathrm{d}t} \\ &= \frac{C}{2}\cdot\frac{\mathrm{d}V^2(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}w(t)}{\mathrm{d}t} &= \frac{C}{2}\cdot\frac{\mathrm{d}V^2(t)}{\mathrm{d}t} \\ w(t) &= \frac{C}{2}V^2(t) \end{aligned}$$

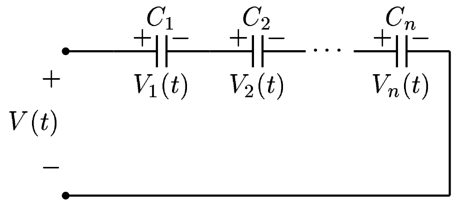
But in a capacitor 'work done' means 'energy stored', so the energy that is stored $w(t) = \frac{C}{2}V^2(t)$.

7.3.2 Capacitors in series and parallel

In series

Consider the following setup:
The charge is the same on each capacitor, i.e. $Q(t) = Q_1(t) = Q_2(t) = Q_n(t)$. Also $V(t) = V_1(t) + V_2(t) + \cdots V_n(t)$.

Hence the total capacitance $C = \dfrac{Q(t)}{V(t)}$.

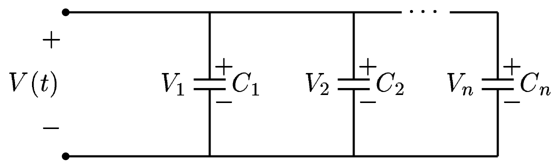


$$\begin{aligned}\frac{1}{C_T} &= \frac{V(t)}{Q(t)}\\ &= \frac{V_1(t)}{Q(t)} + \frac{V_2(t)}{Q(t)} + \cdots + \frac{V_n(t)}{Q(t)}\\ &= \frac{V_1(t)}{Q_1(t)} + \frac{V_2(t)}{Q_2(t)} + \cdots + \frac{V_n(t)}{Q_n(t)}\end{aligned}$$

So $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$

In parallel

Consider the following setup:

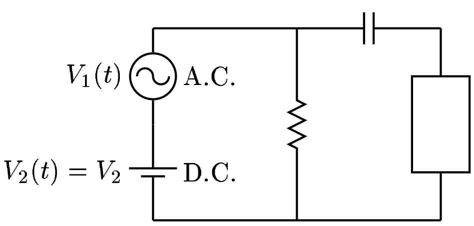


$$\begin{aligned}Q_T(t) &= Q_1(t) + Q_2(t) + \cdots + Q_n(t)\\ V(t) &= V_1(t) = V_2(t) = V_n(t)\\ C_T &= \frac{Q_1(t)}{V(t)} + \frac{Q_2(t)}{V(t)} + \cdots + \frac{Q_n(t)}{V(t)}\\ &= \frac{Q_1(t)}{V_1(t)} + \frac{Q_2(t)}{V_2(t)} + \cdots + \frac{Q_3(t)}{V_3(t)}\end{aligned}$$

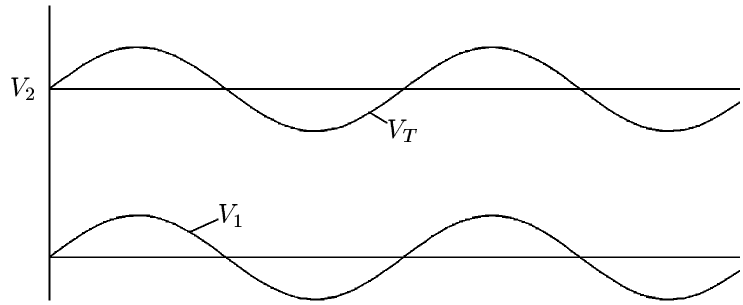
So $C_T = C_1 + C_2 + \cdots + C_n$

7.3.3 Applications

Capacitors are widely used in filters. Consider the following setup:



Here we have an A.C. supply superimposed by a D.C. supply. Hence $V_T(t) = V_1(t) + V_2(t) = V_1(t) + V_2$. I.e.:



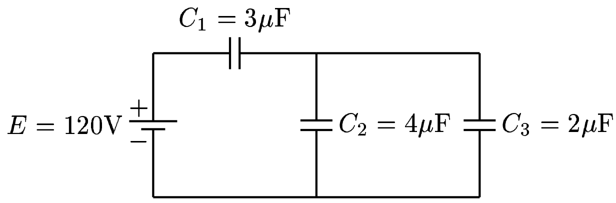
However, V_{load} will be at the same level at V_1 due to the capacitor.

Exercise What whould happen if the resistor and the capacitor were interchanged?

Example What is the capacitance of two circular discs of radius 1 cm seperated by a distance of 1 mm? The dielectric between the plates is aluminia ($\epsilon_r = 6$).

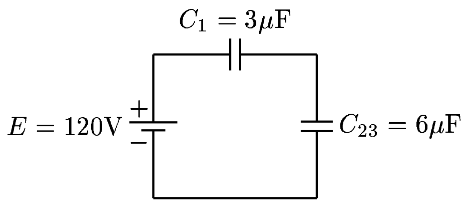
$$C = \frac{\epsilon A}{d} = \frac{\epsilon_r}{\epsilon_0}\frac{\pi r^2}{d} = \frac{6}{8.854 \cdot 10^{-12}} \cdot \frac{\pi(10^{-2})^2}{10^{-3}} = 16.68 \text{ pF}.$$

Example Find the voltage across and charge on each capacitor of the following network:



$C_2 \parallel C_3$ so $C_{23} = C_2 + C_3 = 6\mu$F
C_1 in series with C_{23} so $\dfrac{1}{C_T} = \dfrac{C_1 C_T}{C_1 + C_T} = 2\mu$F
$Q_T = C_T V = 2 \cdot 10^{-6} \cdot 120 = 240\mu$F.

Above network is equivalent to:



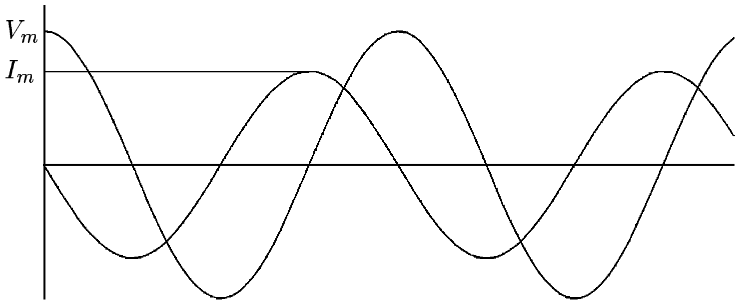
$Q_T = Q_1 = Q_{23}$ so $Q_1 = 240\mu$C
$V_1 = \dfrac{Q_1}{C_1} = \dfrac{240 \cdot^{-4}}{3 \cdot 10^{-6}} = 80$V
$V_{23} = \dfrac{Q_{23}}{C_{23}} = \dfrac{240 \cdot frm[o]--0^{-6}}{6 \cdot 10^{-4}} = 40$V
$Q_2 = C_2 V_T = 4 \cdot 10^{-6} \cdot 40 = 160\mu$C
$Q_3 = C_2 V_T = 2 \cdot 10^{-6} \cdot 40 = 80\mu$C

7.3.4 The impedance of a capacitor

Recall that the impedance of a current element is the ratio of the terminal voltage to the current.

$$\begin{array}{lrcl} \text{Let} & V(t) & = & V_m \cos(\omega t + \varphi) \\ & i(t) & = & c\dfrac{\mathrm{d}v}{\mathrm{d}t} \\ \text{so} & i(t) & = & -C_\omega V_m \sin(\omega t + \varphi) \\ & & = & C_\omega V_m \cos(\omega t + \varphi + \frac{\pi}{2}) \\ & & = & I_m \cos(\omega t + \varphi + \frac{\pi}{2}) \end{array}$$

Diagramatically:



Where $V_m = \dfrac{I_m}{\omega c}$ (magnitude relation). Note also how there is a phase shift between the current and voltage of a capacitor. I.e., the current leads the voltage by $\frac{\pi}{2}$ rad. Because of this, we must complement our impedance value to take this into account. We do this by writing:

$V(t) = Z_c i(t)$ where $Z_c = \dfrac{1}{\omega c}(\frac{\pi}{2}$ phase shift). $\dfrac{1}{\omega c}$ is called the *capacitive reactance* .

7.4 The Inductor

An inductor is a two-terminal element which stores energy in the magnetic field, and it's ability to do this is characterized by it's inductance value L. An inductor is a coil of wire consisting of many closely spaced turns — a solenoid. In order to enhance the inductance, an iron or ferite core is usually inserted in the coil because of the high μ_r of their substances. The direction of the magnetic field is given by the right hand rule.

Definition The **inductance**, denoted L, is by definition the ratio of the total magnetic flux 'linking' the inductor turns to the current flowing through the inductor.

Mathematically: $L = \dfrac{\varphi_r(t)}{i(t)}$

However, the inductance does not depend on either the flux linkage or the current. It depends solely on the design of the inductor itself. The relationship is to a good approximation:

$$L = \frac{N^2 \mu A}{l}$$

Where	N	is the number of turns
	μ	is the absolute magnetic permeability
	A	is the cross sectional area
	l	is the length of the solenoid

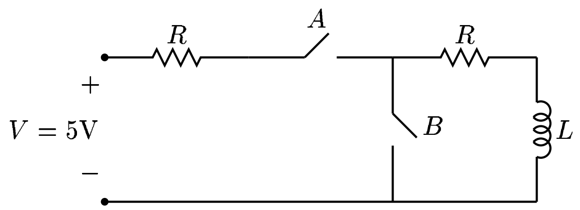
The unit of inductance is the Henry (H). It is defined as the inductance of an inductor which has 1 A of current in its coil and 1 Wb magnetic flux in its core.
The electrical symbol for the inductor is:

To determine the terminal relationship between voltage and current in the inductor we use Faraday's Law:

$$V_L(t) = \frac{\mathrm{d}\varphi_T(t)}{\mathrm{d}t} = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$$

This equation means that the terminal voltage is proportional to the rate of change of current in the coil. If the current is not changing, there will be *no* potential difference across the inductor. In other words, a *changing* current in the coil *induces* a potential difference across the coil. The polarity of the potential difference is such as to *oppose* the change in current.

Consider the following setup:



If I switch the 5V supply by closing switch A, a potential difference is setup across the inductor, opposing the supply voltage. If I then connect the terminals of the inductor by simutaneously opening switch A and closing switch B (I have to not allowed enough time for the current to stabilize in stap 1), a new potential difference will be set up across the inductor with polarity opposite to that of the previous induced potential difference.

7.4.1 Power delivered to an inductor

The power delivered to an inductor is:

$$P(t) = V(t)i(t) = L(t)\frac{\mathrm{d}i(t)}{\mathrm{d}t} = \frac{L}{2}\frac{\mathrm{d}c^2(t)}{\mathrm{d}t}$$

This energy is stored in the $\overrightarrow{\mathrm{H}}$ -field of the inductor. The energy stored at any time t is:

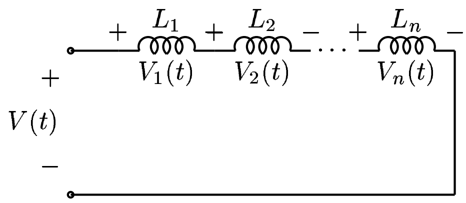
$$P(t)\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \frac{L}{2}\frac{\mathrm{d}c^2(t)}{\mathrm{d}t}, \text{ so } w(t) = L\frac{i^2(t)}{2} = E(t)$$

where $E(t)$ is the stored energy.

7.4.2 Inductors in series and parallel

Inductors in series

Consider the following setup:



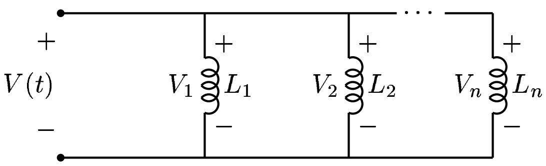
$$V(t) = V_1(t) + V_2(t) + \cdots + V_n(t)$$

Note $V(t)$ is the terminal voltage which is not necessarily equal to the supply or applied voltage, as the terminal voltage will be the sum of the applied and induced voltages. However, $\frac{\mathrm{d}i(t)}{\mathrm{d}t}$ is the same throughout.

Then: $L_T = \frac{\mathrm{d}i(t)}{\mathrm{d}t} = L_1\frac{\mathrm{d}i(t)}{\mathrm{d}t} + L_2\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \cdots L_n\frac{\mathrm{d}i(t)}{\mathrm{d}t} \therefore L_T = L_1 + L_2 + \cdots + L_n$

Inductors in parallel

Consider the following setup:



$i(t) = i_1(t) + i_2(t) + \cdots + i_n(t)$
$\frac{\mathrm{d}i(t)}{\mathrm{d}t} = \frac{\mathrm{d}i_1(t)}{\mathrm{d}t} + \frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + \cdots + \frac{\mathrm{d}i_n(t)}{\mathrm{d}t}$
$\frac{V(t)}{L_T} = \frac{V_1(t)}{L_1} + \frac{V_2(t)}{L_1} + \cdots + \frac{V_n(t)}{L_1}$
Since $V(t) = V_1(t) = V_2(t) = \cdots = V_n(t)$, $\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$

7.4.3 Uses of inductors

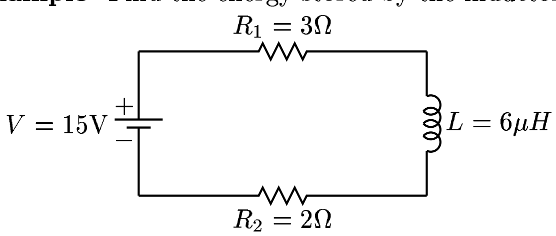
Inductors are used in resonant (tuning) circuits and to model the behaviour of electrical machines which contain many solenoids.

Example The current through an 10 mH inductor is given by:
$i(t) = t^2e^{-\frac{t}{10}}$ for $t > 0$

Find:
a The voltage across the inductor
b The power delivered to the inductor
c The energy stored in the inductor

a $V(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = 0.04te^{-\frac{t}{10}} - 0.002te^{-\frac{t}{10}} = 0.04te^{-\frac{t}{10}}\left(1 - \frac{t}{20}\right)$ V
b $P(t) = V(t)i(t) = 0.04te^{-\frac{t}{10}}\left(1 - \frac{t}{20}\right) = 2t^2e^{-\frac{t}{10}}$
c $w(t) = L\frac{i^2(t)}{2} = 0.02t^4e^{-\frac{t}{5}}$ W

Example Find the energy stored by the inductor in the following circuit, when the current has stabilized:



When the current has stabilized, ie. 'V' has been switched in long enough such that $\frac{\mathrm{d}i}{\mathrm{d}t} = 0$, then $i(t) = \frac{V}{R_1 + R_2} = 3$A (constant).

The $w_{\text{stored}} = \frac{1}{2}Li^2 = \frac{1}{2}(6 \cdot 10^{-3})(3)^2 = 27$ mJ.

7.4.4 The impedance of an inductor

Let $V(t) = V_m \cos(\omega t + \varphi)$. $i(t) = \frac{1}{L}\int V(t)\mathrm{d}t = \frac{1}{L}\int V_m \cos(\omega t + \varphi)\mathrm{d}t = \frac{V_m}{\omega L}\sin(\omega t + \varphi)$.
Then $i(t) = I_m \cos(\omega t + \varphi - \frac{\pi}{2})$

We see therefore that the magnitude and phase of the current and termina voltage are related by:

$$\begin{array}{rll} & \angle V_m & = \angle \omega L I_m \text{ (where } \angle \text{ indicates phase)} \\ \text{and} & V(t) & = i(t) - \frac{\pi}{2} \end{array}$$

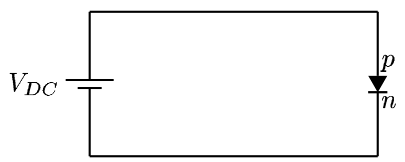
These expression indicate that the magnitudes are related by a factor ωL and the current 'lags' the voltage by $\frac{\pi}{2}$

This information can be expressed as:

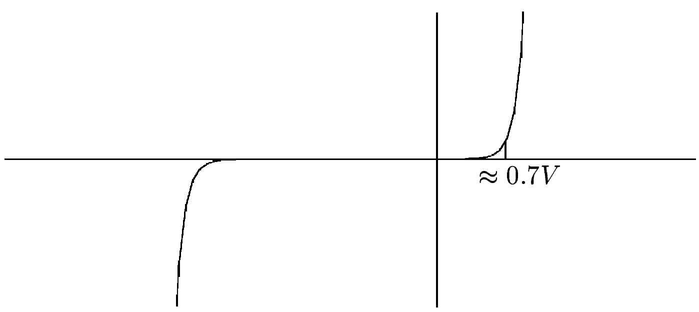
$V(t) = Z_L i(t)$ (Z_L is the impedance of the inductor) where $Z_L = \omega L \angle - \frac{\pi}{2}$. ωL is referred to as the inductance reactance and $\angle - \frac{\pi}{2}$ is referred to as the phase shift.

7.5 Semi-conductor Diodes

It is possible to produce the following crystal of half p-type and half n-type material. The 'depletion layer' at the junction of the p- and n-type materials is an area devoid of 'free electrons' and 'holes'. The reason for the existance of this region is that some 'free electrons' from the n-side will diffuse to the left and combine with the holes there, and vice versa.



Because in the depletion layer there are no free electrons or holes, the depletion layer itself will limit the diffusion process, because it is a poor conductor of current. If however we apply a variable D.C. voltage source as in the above circuit we will notice the following current profile as we increase the voltage from zero.



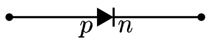
I.e. at $V \approx 0.7$V (for Silicon, for Germanium it is about 0.3V) we see an 'exponential' rise in the current. So, at $V = 0.7$V the potential difference between the terminals is enough to *force* carriers across the depletion layer or in other words, the depletion layer breaks down and current flows freely.

When I reverse the polarity of the source, practically no current will flow due to an expanded depletion layer region until the potential difference is great enough to overcome this relatively large depletion layer (which at this stage is the whole diode).

The forward characteristic is well approximated by the following equation:

$$I = I_0\Big(e^{\left(\frac{v}{\frac{kt}{e}}\right)} - 1\Big)$$

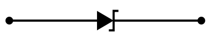
In all, a semi-conductor diode is a current valve — allowing current to flow in one diretion and not in the other in the range of -24V - 0.7V (typically).
The electrical symbol for a diode is:



7.5.1 The Zener Diode

Small signal diodes are never intentionally operated in the breakdown region beause this may damage them. The Zener-diode is a silicon diode that the manufacturer has optimized for operation in the breakdown voltage. Sometimes called the breakdown diode, the Zener diode is the backbone of voltage regulators — circuits that hold load voltage constant despite large changes in the line voltage and load impedance.

By varying the doping level of Silicon, a manufacturer can produce Zener-diodes with breakdown voltages in the rage 1 to 200V. Like the ordinary Silicon-diode, it's forward 'knee' voltage is usually about 0.9V. The electrical symbol for the Zener-diode is:

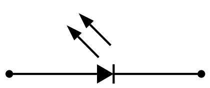


7.5.2 Dynamic resistance of a diode

Take the ration of voltage to current, i.e. $\frac{V}{I}$ and you get the dynamic resistance of the diode. Unlike resistors, diodes do not exhibit a linear relationship between voltage and current. The dynamic resistance increases as the voltage increases. In any case the forward dynamic resistance is low. The reserve dynamic resistance is usually very high.

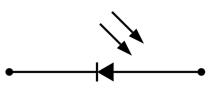
7.5.3 Light Emitting Diodes (LED's)

LED's are diodes which emit light in forward bias. The symbol of a LED is:



7.5.4 Photo diodes

Photo diodes are diodes which conduct current on application of light, which has the effect of stimulation electrons to cross the potential barrier at the depletion region. The symbol of a photo diode is:

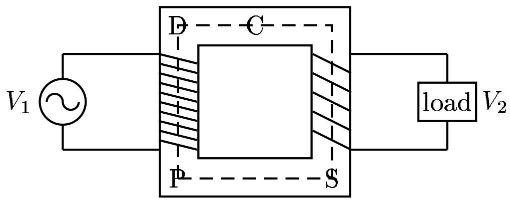


7.6 Transformers

One of the main advantages of A.C. over D.C. is the ease with which alternating voltage can be increased or reduced. This is especially important for transmission. For instance, the general practise is to generate at voltages of about 10 - 25 kV and step-up by means of transformers to higher voltages for transmission (usually ≈ 110 kV in Ireland). At suitable points other transformers are isntalled to step the voltage down to voltages suitable for motors, lamps, heaters, etc — the mains voltage ≈ 380-450 V peak.

7.6.1 Principle of operation of transformers

The following diagram shows the general arrangment of a transformer. The steal core C is laminated to reduce loss due to eddy currents induced by the alternating magnetic flux.



Here $V_1 I_1 \approx V_2 I_2$ and therefore $P_1 \approx P_2$ (there is about 2% power loss). Coil P is connected to the supply and is therefore termed the primary. Coil S is connected to the load ad is referred to as the secundary.

The alternating voltage applied to P produces an alternating $\vec{\mathrm{B}}$ flux in the steal core. The mean path of this flux is the dotted line D.

If the whole of the flux produced by P passed through S (or 'threads' S), the E.M.F. induced in each turn is the same for P and S. Hence if N_1 and N_2 are the number of turns in P and S respectively, then:

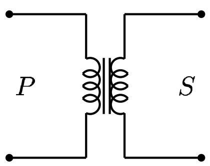
$$\frac{\text{Total E.M.F. in } S}{\text{Total E.M.F. in } P} = \frac{N_2}{N_1} \cdot \frac{\text{E.M.F. per turn in } S}{\text{E.M.F. per turn in } P}$$

I.e. $\dfrac{V_2}{V_1} = \dfrac{N_2}{N_1}$ and hence $\dfrac{I_1}{I_2} = \dfrac{V_2}{V_1}$

Transformers can also be constructed using other type cores, or even air. Obviously ferrite based materials are best because μ is very large for these materials.

Electrical Symbol

The electrical symbol for a transformer is:



Where the double line in the middle denotes an iron or ferrite core.

7.7 Ideal and practical sources

There are two types of sources available to us in electrics.

1. Voltage sources
2. Current sources

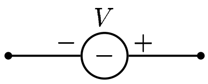
7.7.1 Voltage sources

An *ideal* voltage source will provide a stable voltage (D.C. or A.C.) regardless of the nature of the current it supplies.

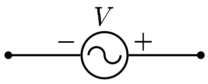
Example If I connect a battery across a low resistance, say $\frac{1}{2}\Omega$, there is more current flowing through the the circuit than the battery is able to replenish at its terminals — the voltage drops. I.e., a battery is a non-ideal voltage source.

By definition: The ideal voltage source is a two-terminal element with the property that the voltage across the terminals is specified at every instant in time. The voltage does not depend on the current (positive or negative) flowing through the source.

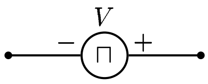
The electrical symbol for an ideal D.C. voltage source is:



A.C:



A.C. (square wave):



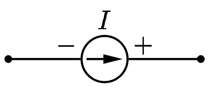
7.7.2 Current sources

The ideal current soruce will provide a stable current (D.C. or A.C.) regardless of the nature of the circuit.

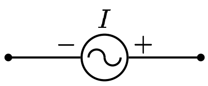
Example In the example of the inductor connected to a series and shut switch, we attempted to ramp up the current linearly from 0 to 10A by applying a voltage source. In practise there would be some deviation from this linear profile because of the nature of the circuit. In fact, the desired linear profile would need a current source.

By definition: An ideal current source is the two-terminal element with the property that the current flowing through the device is specified at every instant in time. The current does not depend on the voltage across.

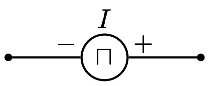
The electrical symbol for an ideal D.C. current source is:



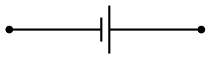
A.C:



A.C. (square wave):



In practise sources are not ideal. This deviation is modelled using an ideal source in series with a small resistance (the internal resistance of the source).
Then a battery (a non-ideal voltage source)



is equivalent to

Usually R is small enough to be ignored — that is, it the source is well designed.

7.8 Earthing and grounding

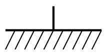
7.8.1 Earthing

Earthing is a safety feature of electrical appliances. Metal parts, not used in the internal circuiting, are connected to the earth wire, which is green / yellow striped (the 'live' and 'neutral' are red and black or brown and blue respectively). If due to a circuit fault any of the metal parts become 'live', the potentially lethal effects to a person coming in contact with it are avoided by having the parts connected to earth via the earth wire.

Since the earth is of electrical potential zero, almost all current will flow to the earth as the resistance of the earth wire is very small compared to that of the human body.

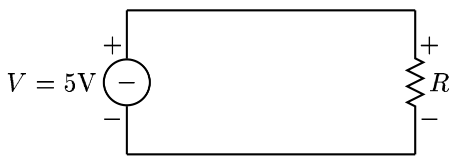
Other examples of earthing are a lightning rod or static dischargers on cars.

The symbol for earth is:



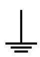
7.8.2 Grounding

Grounding has an entirely different meaning to earthing and it is essential to understand its meaning, as it is of great practical importance. Consider the following setup:



In this isolated setup there is a potential difference of 5V across the supply and R. The absolute potentials (with respect to earth) may however be different (EG. 10V and 15V, 500V and 505V, etc.). The same current i will flow regardless. For simplicity of analysis we assign the return or 'bottom' wire a potential of zero — ground. We call this the ground wire or reference. A good analogy would be with catography where terrain height is measured with respect to sea level. It should be noted that ground does not necessarily have earth's zero absolute potential.

The electrical symbol for ground is:



Convention

You will have noticed the assignment of the positive and negative terminals to the resistor in the above circuit. We note that electrical current is deemed to flow from the positive terminal of the source to the negative. In reality it is negatively charged electrons which flow from the negative terminal to the positive.

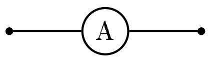
Practically speaking, there is no difference in how one views this phenomenon. But the convention stands — ie., current flows from positive to negative. Hence the assignment of the positive and negative terminals to the resistor in the circuit. This could well be *any* component.

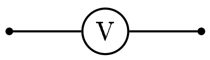
7.9 Instrumentation

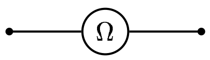
Read Boylestad / Senior Physics on the following subjects:

1. The moving coil galvanometer
2. The ampmeter
3. The voltmeter
4. The ohmmeter

The symbols for the above are (excluding the galvanometer):

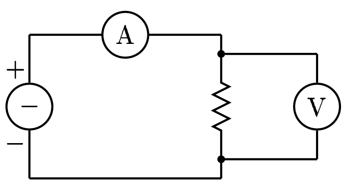




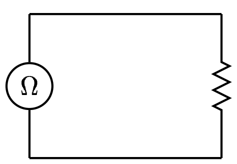


respectively.
To apply them in a circuit, wire them up as follows:

In circuit:



Out of circuit:



Chapter 8

Analysis of electric circuits in steady state

8.1 Preample

A circuit (A.C. or D.C.) is said to be in steady state if and only if there exists a T such that $V(t) = V(t+T)$ and $i(t) = i(t+T)$ for all $t > t_0$ throughout the circuit.

It is important to note that in the special (but very important) case of circuits with sinusoidal inputs, the steady-state behaviour will also be sinusoidal in nature. Sinusoidal sources are important in electrotechnology because electic generators give us this type of wave form 'naturally'. Hence much of electrotechnology is concerned with the operation of circuits with A.C. sinusoidal inputs and outputs.

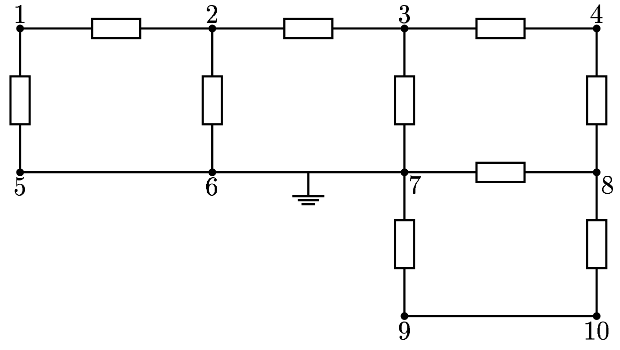
8.1.1 A.C. vs. D.C. Analysis

Analysis of A.C.is somewhat more complicated than for D.C. However, these is little need to distinguish between the two if we view D.C. as a special case of A.C. (ie. $\omega = 0$). Impedance values Z, though defined for A.C., reduce to the D.C. equivalent using the above transformation ($\omega = 0$):

AC	**DC**	
$Z_r = R$	$Z_r = R$	
$Z_c = \frac{1}{\omega C}\angle -\frac{\pi}{2}$	$Z_c = \infty$	(ie. open circuit in steady state)
$Z_L = \omega L \angle \frac{\pi}{2}$	$Z_L = 0$	(ie. short circuit in steady state)

8.2 Terminology

Consider the following circuit or electrical network:



The boxes denote circuit elements which may be sources or components. Points 1 to 10 are referred to as 'nodes'. Section [1,2], [2,3], [3,7], etc., are referred to as branches, and [1,2,6,5] or [2,3,6,7] are referred to as loops.

8.3 Kirchoff's Laws

8.3.1 Kirchoff's Current Law (K.C.L.)

Kirchoff's Current Law states that the algebraic sum or electric components at any node of an electric circuit is equal to zero at every instant in time.

On other words, at any node the total current entering a node equals the total current leaving the node. Current entering the node is given a psotive sign and current leaving the node is given a negative sign. Hence the algebraic sum is zero.

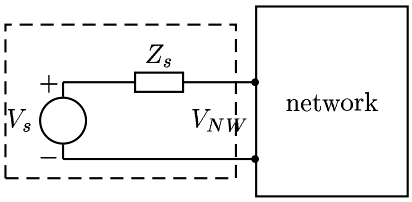
8.3.2 Kirchoff's Voltage Law (K.V.L.)

The algebraic sum of branch voltages around any loop of an electric circuit is equal to zero at very instant in time.

Ie. $\sum_{loop} V_{branch} = 0$

8.4 Source Transformation

Current sources and voltage sources can be interchanged using the following transformation. Consider the following setup: A non-ideal (or practical) V-source feeding an electrical network:

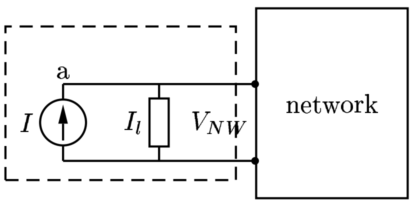


non-ideal V-src

Then by K.V.L.:

$V_s = IZ_S + V_{NW}$

Now consider this setup, a non-ideal I-source feeding a n.w.:



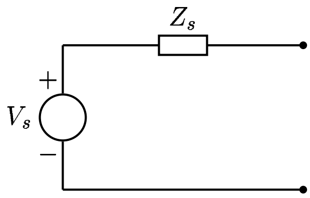
non-ideal V-src

I_l is referred to as the leakage current. It is a feature of all current sources resulting in a lessening of it's ability to provide a 'constant' current.

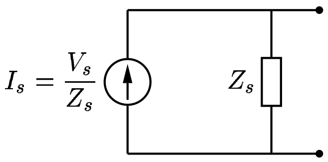
By K.C.L. at Node A $I_s = I + I_l$. Hence $I_s = I + \dfrac{V_{NW}}{Z_s}$.
For the voltage source and the current source to be equivalent, V_{NW} and I must be equivalent. Therefore:

$V_{NW} = V_s - IZ_{s_V}$
$V_{NW} = (I_s - I)Z_{s_I}$

For these two equations to be equivalent Z_{s_I} and Z_{s_V} must be equal, and $V_s = Z_s I_s$. Hence:



is equivalent to



This fact can be especially useful to us when we are analysing circuits containing current and voltage sources.

8.5 The Principle of Superposition

The principle of superposition states that the current through or voltage across a linear bilateral network is equal to the algebraic sum of the currents and voltages produced independently by each source. Ideal sources and ideal circuit elements are said to be linear (for example, to voltage sources immediately after each other in a network is the same as one voltage source with the sum of the voltages of the other two). Bilateral means that there is no change in the behaviour or characteristics of elements if the polarity of the supply is reserved (eg. a diode is *not* a bilateral element).

To set a voltage source to 'zero', we replace it with a short circuit. To set a current source to zero, we replace it with an open circuit.

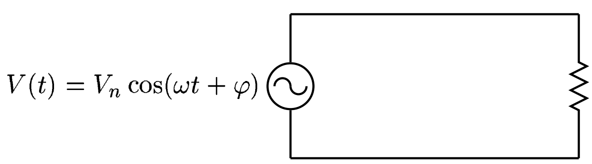
In short, the principle of superposition allows us to analyze a circuit with multiple sources by summing the effect of each source acting on the circuit on its own.

8.6 Complex Notation

In analysing steady state A.C. circuits the principle of superposition allows us to use complex numbers to represent sources, voltages, currents and impedances. There are two major advantages to this. Complex numbers give information on amplitude and phase — essential for A.C. analysis. Besides, the complex exponential is easily differentiated, integrated, multiplied and divided.

8.6.1 Complex Sources

Consider the following circuit:



$$
\begin{aligned}
V(t) &= V_m\cos(\omega t+\varphi); V_m \in \mathbb{R} \\
&= \text{Re}\left\{V_m e^{j(\omega t+\varphi)}\right\} \text{ (from Euler's formula } (j=\sqrt{-1})) \\
&= \text{Re}\left\{V_m e^{j\varphi} e^{j\omega t}\right\} \\
&= \text{Re}\left\{\hat{\text{V}} e^{j\omega t}\right\}
\end{aligned}
$$

$\hat{\text{V}} = V_m e^{j\varphi}$ is referred to as a *phasor*.
For simplicity, we can ignore the Re{ } notation and simply describe the supply voltage in the above circuit as $V(t) = \hat{\text{V}}e^{j\omega t}$.

$$I = \frac{\hat{\text{V}}e^{j\omega t}}{R} = \frac{\hat{\text{V}}}{R}\cos\omega t + \frac{\hat{\text{V}}}{R}j\ \sin(\omega t)$$

In the case of $\varphi = 0$: $I = \dfrac{V_m}{R}\cos\omega t + j\dfrac{V_m}{R}\sin(\omega t)$

From the principle of superposition the analysis of this or any linear circuit will remain unaffected given the 'complex supply'. The observable result of the analysis will remain unchanged and is the real part of the result. Ie. for the above:

$$i(t) = \frac{V(t)}{R} = \frac{\hat{\text{V}}e^{jwt}}{R}$$

In reality we observe $i(t) = \text{Re}\{\frac{\hat{\text{V}}}{R}e^{jwt}\}$. Using complex sources $\hat{\text{V}}e^{j\omega t}$ and / or $\hat{\text{I}}e^{j\omega t}$ leads us to complex values for the impedance of circuit elements.

8.6.2 Complex elements

Resistor

$$
\begin{array}{lrcl}
 & V\cos(\omega t+\varphi) & = & RI\cos(\omega t+\varphi) \\
 & Ve^{j\omega} & = & RIe^{j\omega} \\
\text{or} & \hat{\text{V}} & = & \hat{\text{Z}}\hat{\text{I}} \\
\text{where} & \hat{\text{Z}} & = & R
\end{array}
$$

Capacitor

$$
\begin{array}{lrcl}
 & V\cos\left(\omega t+\varphi+\frac{\pi}{2}\right) & = & \dfrac{I}{\omega C}\cos(\omega t+\varphi) \\
 & Ve^{j\left(\varphi+\frac{\pi}{2}\right)} & = & \dfrac{I}{\omega C}e^{j\omega} \\
 & Ve^{j\omega}e^{j\frac{\pi}{2}} & = & \dfrac{I}{\omega C}e^{j\omega} \\
 & \hat{\text{V}} & = & \hat{\text{Z}}\hat{\text{I}} \\
\text{where} & \hat{\text{Z}} & = & \dfrac{1}{j\omega C} \text{ (note that } \dfrac{1}{j} = -j)
\end{array}
$$

Inductor

$$
\begin{array}{lrcl}
 & V\cos(\omega t+\varphi) & = & \omega LI\cos\left(\omega t+\varphi+\frac{\pi}{2}\right) \\
 & \hat{\text{V}} & = & \hat{\text{Z}}\hat{\text{I}} \\
\text{where} & \hat{\text{Z}} & = & j\omega L
\end{array}
$$

The values of complex impedance are the same as those given by analysis using real numbers. The only difference is the appearance of $\pm j = \pm\sqrt{-1}$ in the impedance quantities. This indicates the phase shift of $\pm\frac{\pi}{2}$ between the current and the voltage through the impedance.

Note $j = e^{j\frac{-\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2}$. The j value in the impedance indicates the current *lags* the voltage by $\frac{\pi}{2}$ rad and $-j$ or $\frac{1}{j}$ in the impedance values indicates the current *leads* the voltage by $\frac{\pi}{2}$ rad.

Note The unit assigned to all impedances is the Ohm ('Ω').

8.7 Impedance Series and Parallel

It is easily shown from the formulae for Z_r, Z_c and Z_L that for impedances in series:

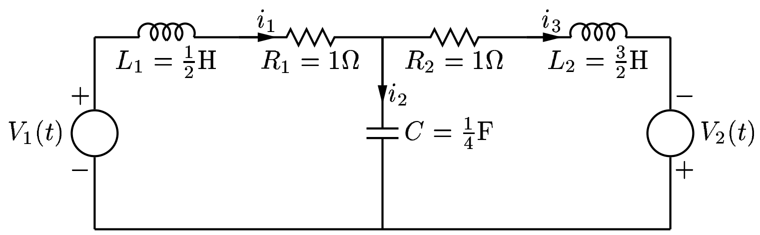
$$Z_{tot} = Z_1 + Z_2 + \cdots + Z_n$$

And for impedances in parallel:

$$Z_{tot} = \left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}\right)^{-1}$$

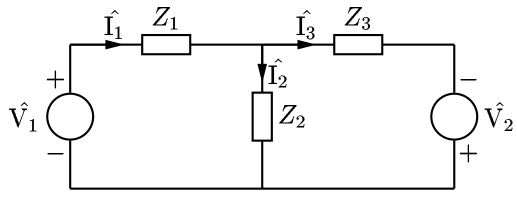
Exercise Prove this

Example Consider the following setup:

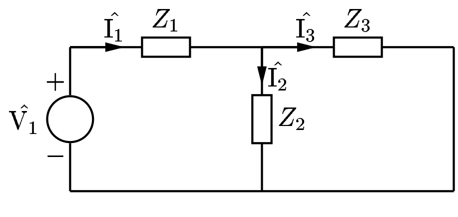


$V_1(t) = \cos(2t) \therefore V \equiv \cos(\omega t)$
$V_2(t) = \sqrt{2}\cos\left(2t - \frac{\pi}{2}\right) \therefore V \equiv A\cos(\omega t + \varphi)$ where A is the amplitude and $\omega t + \varphi$ is the phase shift.

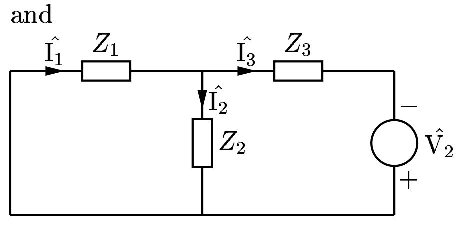
To find all the branch currents (in steady state), simplify the circuit using phasor / impedance notation. Hence the equivalent circuit is:



Following the principle of superposition, we analyse the following two circuits independently:



and



$$\text{Then} \quad \begin{aligned} \hat{I}_1 &= \hat{I}_{1a} + \hat{I}_{1b} \\ \hat{I}_2 &= \hat{I}_{2a} + \hat{I}_{2b} \\ \hat{I}_3 &= \hat{I}_{3a} + \hat{I}_{3b} \end{aligned}$$

Note
$$
\begin{aligned}
Z_3 &= R_2 + j\omega L_2 = 1 + 3j\,\Omega \\
V_1(t) &= \cos(2t) \therefore \hat{\mathrm{V}}_1 = 1 \text{ V} \\
V_2(t) &= \sqrt{2}\cos\left(t - \tfrac{\pi}{4}\right) = \sqrt{2}e^{-j\frac{\pi}{4}} \\
&= 1 - j \text{ V}
\end{aligned}
$$

Then
$$
\begin{aligned}
\hat{\mathrm{I}}_{1a} &= \frac{\hat{\mathrm{V}}_1}{Z_1 + \dfrac{Z_3 \cdot Z_2}{Z_3 + Z_2}} \\
\hat{\mathrm{I}}_{2a} &= \frac{\hat{\mathrm{I}}_{1a} \cdot Z_3}{Z_3 + Z_2} \\
\hat{\mathrm{I}}_{3a} &= \frac{\mathrm{I}_{1a} \cdot Z_2}{Z_3 + Z_2} \\
\hat{\mathrm{I}}_{1b} &= \frac{\mathrm{I}_{3b} \cdot Z_2}{Z_3 + Z_2} \\
\hat{\mathrm{I}}_{2b} &= \frac{\mathrm{I}_{3b} \cdot Z_1}{Z_3 + Z_2} \\
\hat{\mathrm{I}}_{3b} &= \frac{\hat{\mathrm{V}}_2}{Z_3 + \dfrac{Z_2 \cdot Z_1}{Z_1 + Z_2}}
\end{aligned}
$$

Which yields

$\hat{\mathrm{I}}_{1a} = \dfrac{1+j}{6}$, $\hat{\mathrm{I}}_{1b} = \dfrac{-(1+j)}{3} \therefore \hat{\mathrm{I}}_1 = \dfrac{-(1+j)}{6}$

$\hat{\mathrm{I}}_{2a} = \dfrac{1+3j}{6}$, $\hat{\mathrm{I}}_{2b} = \frac{1}{3} \therefore \hat{\mathrm{I}}_2 = \dfrac{-1+3j)}{6}$

$\hat{\mathrm{I}}_3 = \dfrac{-4j}{6}$

To get $i_1(t), i_2(t)$ and $i_3(t)$, we use $i(t) = \mathrm{Re}\,\{\hat{\mathrm{I}}e^{j\omega t}\}$. This gives:

$i_1(t) = \frac{\sqrt{2}}{6}\cos(2t - 135°)$ A

$i_2(t) = \frac{\sqrt{10}}{6}\cos(2t + 108.4°)$ A

$i_3(t) = \frac{2}{3}\cos(2t - 90°)$ A

Exercise Repeat the above example for $V_2(t) = \sqrt{2}\cos(4t - 45°)$ V.

Answer:

$i_1(t) = \frac{\sqrt{2}}{6}\cos(2t + 45°) + \frac{\sqrt{10}}{15}\cos(4t - 18.4°)$ A

$i_2(t) = \frac{\sqrt{10}}{6}\cos(2t + 71.6°) - \frac{\sqrt{2}}{3}\cos(4t + 135°)$ A

$i_3(t) = \frac{1}{3}\cos(2t - 90°) + \frac{2\sqrt{5}}{15}\cos(4t - 116.6°)$ A

8.8 Thévnin's Theorem (A.C. or D.C.)

Thévnin's Theorem states that any active, linear network can be replaced by an equivalent circuit consisting of an ideal voltage source and a series impedance (A.C.) or resistance (D.C.). I.e. a non-ideal voltage source!

Note

AC		DC
R	$\rightarrow$	R
$\dfrac{1}{j\omega C}$	$\rightarrow$	∞
$j\omega L$	$\rightarrow$	0

Note An *active* network is a network containing voltage and / or current sources. A *passive* network is a network containing circuit elements only.

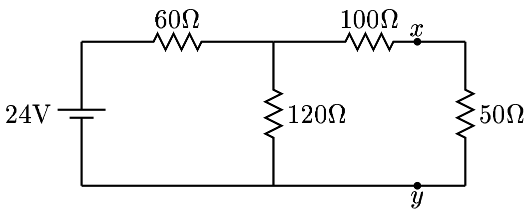
8.8.1 Procedure to get Thévnin's Equivalent Circuit

1. Remove the load impedance and mark the terminals a and b
2. Calculate R_{TH} or Z_{TH} by setting all sources to zero and finding the resulting impedance
3. Calculate E_{TH} by returning all sources to their original position and finding the open circuit voltage between a and b (if necessary, use the principle of superposition)
4. Draw the Thévnin equivalent circuit which is E_{TH} in series with R_{TH} or Z_{TH}
5. Finally replace the load impedance

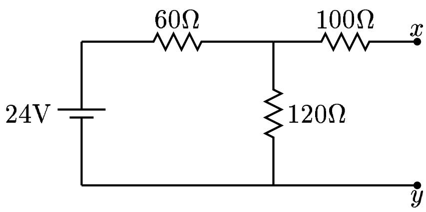
8.9 Norton's Theorem (A.C. or D.C.)

Norton's Theorem states that any active network can be replaced by an ideal current source in parallel with an impedance (A.C.) or a resistance (D.C.). This theorem follows from Thévnin's Theorem and the source convertion).

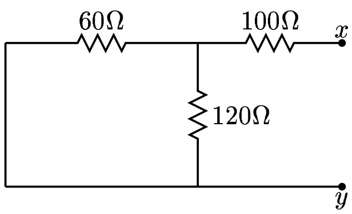
Example (D.C.) Find the Thévnin equivalent circuit between x and y:



Step 1:

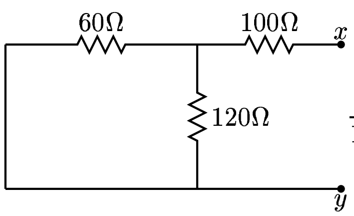


Step 2:



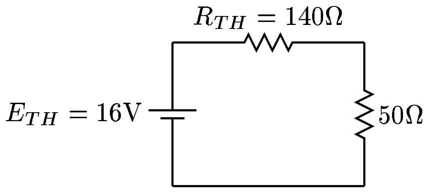
$R_{TH} = 140\Omega$

Step 3:

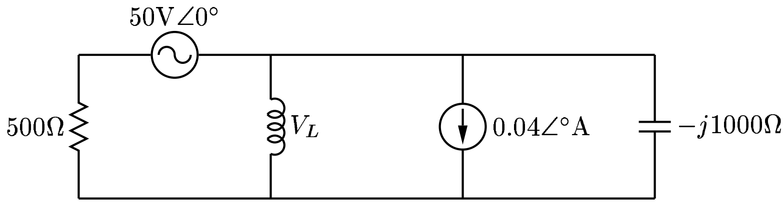


$$\frac{120 \cdot 24}{120 + 60} = 60\text{V}$$

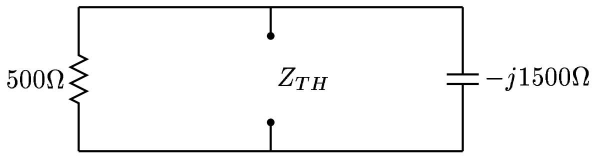
Answer (the Thévnin equivalent circuit):



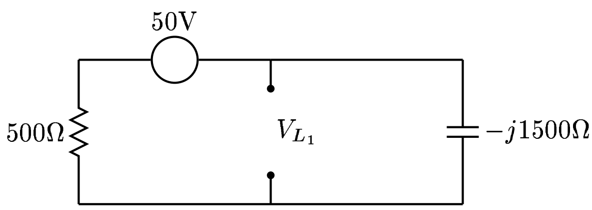
Example (A.C.) The find the voltage across the inductor (in polar form) by replacing the circuit external to the inductor with its Thévnin's equivalent.



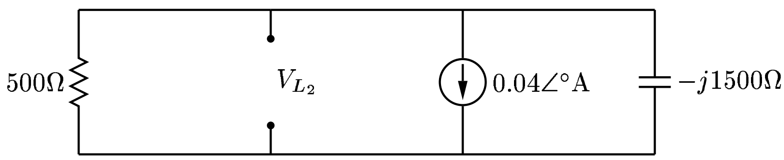
Removing the sources and the load yields:



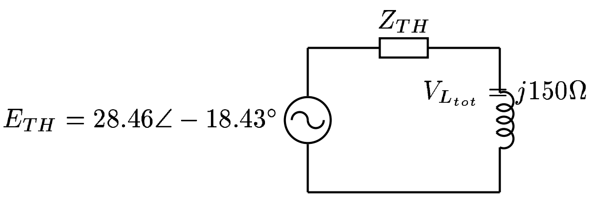
Then:



and



Hence $Z_{TH} = 450 - j1500\Omega$ and $V_{L_{tot}} = V_{L_1} + V_{L_2}$. Then the result is:



$$
\begin{aligned}
Z_{TH} &= 500\Omega || -j1500\Omega \\
&= \frac{(500\angle 0^\circ)(1500\angle -90^\circ)}{500-j1500} \\
&= \frac{7.5\cdot 10^5\angle -90^\circ}{1581\angle -71.57^\circ} \\
&= 474.38\angle -18.43^\circ\ \Omega \\
&= 450-j150\Omega \text{ (see polar representation of complex numbers)}
\end{aligned}
$$

$$
\begin{aligned}
V_{L_1} &= \frac{1500\angle -90^\circ}{500-j1500}\cdot 0\angle 0^\circ \text{ V} \\
&= 45-j15 \text{ V (potential divider)}
\end{aligned}
$$

$$
\begin{aligned}
V_{L_2} &= (0.04\angle 0^\circ)(474.38\angle -18.43^\circ) \\
&= 18-j6 \text{ V}
\end{aligned}
$$

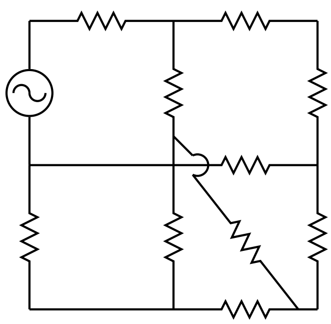
$$
V_L = V_{L_1}+V_{L_2}
$$

8.10 Generalized means of analysing electric circuits

Two methods may be used to determine branch currents and node voltages in planer circuits: Mesh Analysis (or 'Formal Approach' as Boylestad refers to it), and Nodal Analysis. See Boylestad for examples.

8.10.1 Mesh Analysis

By definition a mesh is a loop that does not contain other loops. Mesh analysis is useful only for planer circuits — that is, circuits that can be drawn in the plane in such a way that elements do not cross.
An example of a *non*-planer circuit is:

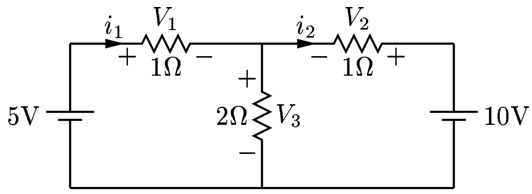


Hence a planer circuit necessarily partitions the plane into meshes.

Procedure

1. Assign mesh currents variables to each mesh
2. Apply K.V.L. to each mesh
3. Express voltages across the impedances in terms of the mesh currents
4. Solve the resulting simultaneous equations for the mesh currents
5. From the mesh currents, determine the branch currents
6. Determine the node voltages from the branch currents

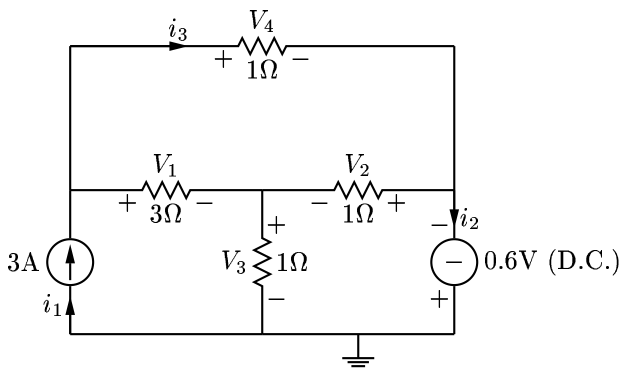
Example (D.C.) Find the current through the 2Ω resistor in the following circuit using mesh analysis:



By K.V.L.:
$V_1 + V_3 = 5$
$V_2 + V_3 = 10$
By Ohm's Law:
$1i_1 + 2(i_1 - i_2) = 5 \therefore 3i_1 - 2i_2 = 5$
$-i_2 + 2(i_1 - i_2) = 10 \therefore 2i_1 - 3i_2 = 10$
Solving for i_1 and i_2 gives:
$i_1 = -1\text{A}$, $i_2 = -4\text{A}$
Hence the current through the 2Ω resistor is:
$i_1 - i_2 = 3\text{A}$

When there are current sources in the network the analysis is sometimes simplified (though not always) somewhat in step 4. A current source sometimes *gives* the solution for a mesh current — as will be seen in the following exercise.

Exercise (D.C.) Using mesh analysis, determine the potendial difference across the 3Ω resistor.



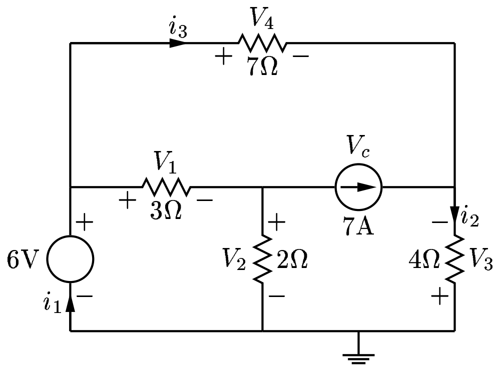
Since the only mesh current that flows through the current source is Z, then by inspection:

$i_1 = 3\text{A}$

So the answer is $V_{3\Omega} = 0.5\text{V}$

Sometimes the existance of a current source does not help greatly:

Example (D.C.) Using mesh analysis, find the voltage drop across the 3Ω resistor.



By K.V.L.:

$V_1 + V_2 - 6 = 0$
$V_2 - V_3 - V_c = 0$
$V_4 - V_1 - V_c = 0$

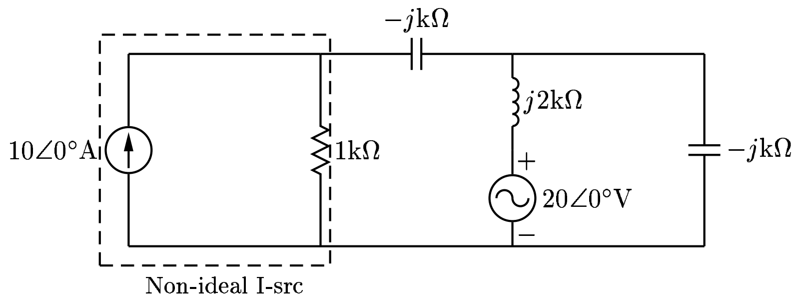
Eliminating V_c gives:

$V_2 - V_3 - V_4 + V_1 = 0$
$5i_1 - 6i_2 - 10i_3 = 0$
$5i_1 - 2i_2 - 3i_3 = 6$

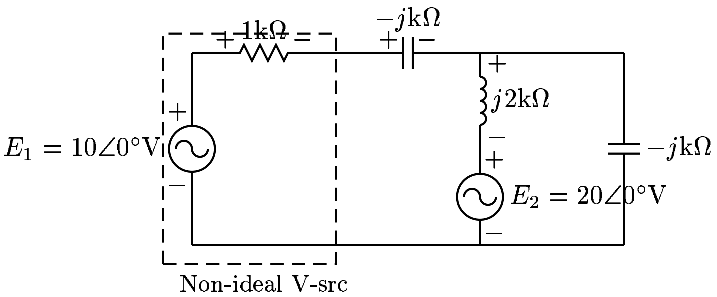
But $I_2 - I_3 = 7\mathrm{A}$

$\therefore V_{3\Omega} = 20\mathrm{V}$

Example (A.C.) Use mesh analysis to find the voltage across the inductor in the following circuit:



By source convertion we get:



Note The source convertion was not necessary to analyse the circuit, but it was elegant!

Let $Z_1 = 1\mathrm{k}\Omega, Z_2 = -j\mathrm{k}\Omega, Z_3 = j2\mathrm{k}\Omega, Z_4 = -j\mathrm{k}\Omega$. Then from K.V.L.:

$i_1(Z_1 + Z_2) + (i_1 - i_2)Z_2 + E_2 = E$
$(i_2 - i_1)Z_3 + i_2 \cdot Z_4 = E_2$

Solving gives $V_L = 22.8\angle -164.74°$ V

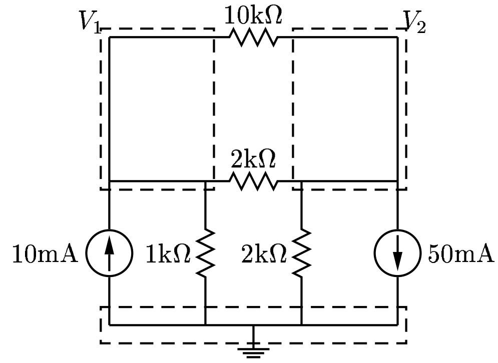
8.10.2 Nodal Analysis

Procedure

1. Define all nodes using independent variables, taking ground as reference ($\equiv 0$)
2. Assign all branches with a direction for the current
3. Apply K.C.L. at each node
4. Express each current in terms of the neighbouring node voltages
5. Solve the resulting system of simultaneous equations

Note The presence of voltage-sources sometimes simplifies analysis in step 3 as voltages sources sometimes *give* the node voltage solution.

Example (D.C.) Solve for all unknown voltages and currents using nodal-analysis:



From K.C.L.:

$$\frac{V_1 - 0}{1\text{k}} + \frac{V_1 - V_2}{10\text{k}} + \frac{V_1 - V_2}{2\text{k}} = 0.01$$

$$\frac{V_1 - V_2}{10\text{k}} + \frac{V_1 - V_2}{2\text{k}} = \frac{V_2}{2\text{k}} + 0.05$$

Solving for V_1 and V_2 gives:

$V_1 = -13.57$ V

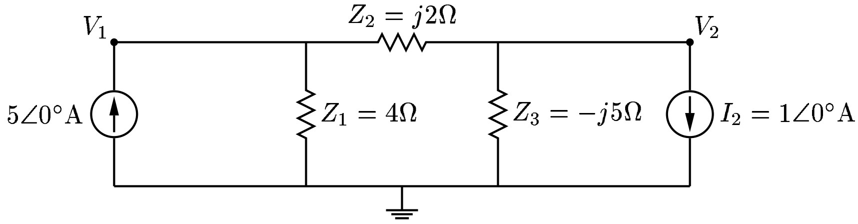
$V_2 = -52.86$ V

Hence:

$$i_{10\text{k}\Omega} = \frac{V_1 - V_2}{10\text{k}} = 3.93 \text{ mA}$$

$$i_{1\text{k}\Omega} = \frac{V_1}{1\text{k}} = -13.57 \text{ mA}$$

Example (A.C.) Using nodal analysis, find the current through the inductor in the following circuit:



By K.C.L.:

Node 1: $i_1 = \dfrac{V_1}{Z_1} + \dfrac{V_1 - V_2}{Z_2}$

Node 2: $\dfrac{V_1 - V_2}{Z_2} = \dfrac{V_2}{Z_3} + i_2$

Solving for V_1 and V_2 gives:

$V_1 = 4 - 8 - j6.4$ V

$V_2 = 8 - j14$ V

Hence

$$I_L = \frac{V_1 - V_2}{Z_2} = 4.12\angle 22.88^\circ \text{ A}$$

8.11 Power

By definition the instanteneous electric power supplied to a network is:

$P(t) = V(t)i(t)$

8.11.1 D.C. Power

The instanteneous power equals the *average* D.C. power:

$P(t) = V(t)i(t)$
$P = VI$

8.11.2 A.C. Power

$P(t) = V(t)i(t)$

Average power is $P_{av} = \dfrac{1}{T}\int_0^T P(t)\mathrm{d}t \; \dfrac{1}{T}\int_0^T V(t)i(t)\mathrm{d}t$

In the case of alternating currents and voltages $P_av = 0$. However, the work done is positive. Hence another measure is needed to quanity A.C. power.

8.11.3 Root Mean Square Current and Voltage

$P_{av} = \dfrac{1}{T}\int_0^T V(t)i(t)\mathrm{d}t = \dfrac{Z}{T}\int_0^T i^2(t)\mathrm{d}t$

By definition, the root-mean-square (R.M.S.) value of a time varying function $y(t)$ is:

$y_{rms} = \sqrt{\dfrac{1}{T}\int_0^T y^2(t)\mathrm{d}t}$

Hence: $I_{rms} = \sqrt{\dfrac{1}{T}\int_0^T i^2(t)\mathrm{d}t}$, $P_{av} = ZI_{rms}^2$

The R.M.S. value of an A.C. quantity gives the 'effective' D.C. value which would effect the same average power.

Also: $P_{av} = \dfrac{1}{Z}\left(\dfrac{1}{T}\int_0^T v^2(t)\mathrm{d}t\right)$, $V_{rms} = \sqrt{\dfrac{1}{T}\int_0^T v^2(t)\mathrm{d}t} \therefore P_{av} = \dfrac{V_{rms}^2}{Z}$

R.M.S. vs. peak value (for sinusoids)

There is an important relationship between the peak value of sinusoidal currents and voltages and their R.M.S. values.

To derive this relation consider:

$i(t) = I_m \cos(\omega t + \varphi)$
$i^2(t) = I_m^2 \cos^2(\omega t + \varphi) = I_m^2\{1 + \cos(2\omega t + 2\varphi)\}$

Using the definition of R.M.S.:

$$i_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\mathrm{d}t} = \frac{1}{T}\int_0^T I_m^2\{1 + \cos(2\omega t + 2\varphi)\}\mathrm{d}t$$

But $\int_0^T \cos(2\omega t + 2\varphi)\mathrm{d}t = 0$

Hence $I_{rms} = \dfrac{I_m}{\sqrt{2}}$

Using the same line of reasoning we get the voltage relation

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

8.11.4 Complex Power

Complex power is used to quantify A.C. power. It is defined as:

$\hat{\mathrm{S}} = \frac{1}{2}\hat{\mathrm{V}}\hat{\mathrm{I}}^*$

Where	$\hat{\mathrm{V}}$	is the voltage phasor $V_m e^{j\varphi_v}$
	$\hat{\mathrm{I}}^*$	is the conjugate of the current phasor $I_m e^{-j\varphi_i}$

Hence $\hat{\mathrm{S}} = \frac{1}{2}V_m I_m e^{j(\varphi_v - \varphi_i)} = V_{rms}I_{rms}e^{j\varphi}$

I.e., $\hat{\mathrm{S}}$ is itself a phasor, where $\varphi = \varphi_v - \varphi_i$ — the phase difference between the voltage and the current.

$\hat{\mathrm{S}} = V_{rms}I_{rms}\cos\varphi + jV_{rms}I_{rms}\sin\varphi$

The *apparent* power S is $|\hat{\mathrm{S}}| = V_{rms}I_{rms}$ and is measured in Volt-Ampères (VA). The apparent power is the average power that *would* be delivered to the load if it were purely resistive.

$P = V_{rms}I_{rms}\cos\varphi (= S\cos\varphi)$ is known as the *real* or *active* power, and is measured in Watts (W). The real power is the average power delivered and dissipated by the load through its resistive components (assuming the resistance of the connecting wires to be negligible).

$Q = V_{rms}I_{rms}\sin\varphi (= S\sin\varphi)$ is known as the *imaginary* or *reactive* power, and is measured in Volt-Ampères-Reactive (VAR). The reactive power is the average power exchanged between the source and the reactive components of the load. Recall capacitors and inductors (ie., reactive components) do not dissipate energy — that is done only by resistors! Reactive components instead *store* energy. Since the instanteneous power $P(t) = V(t)i(t)$ is alternating, these components absorb energy and return energy to the source in an alternation fashion.

$\cos\varphi$ is known as the *power factor* and is dimensionless.

Note We defined $\hat{\mathrm{S}} = \frac{1}{2}\hat{\mathrm{V}}\hat{\mathrm{I}}^*$ instead of $\hat{\mathrm{S}} = \frac{1}{2}\hat{\mathrm{V}}\hat{\mathrm{I}}$ since the latter representation would give incorrect values for real and reactive power. It would however give the correct value for the apparent power.

8.11.5 Power Factor

The power factor p.f. is a feature of every reactive circuit. $0 < p.f. < 1$ since p.f. $= cos\varphi$. The power factor is of significant economic importance where the distribution and consumption of A.C. power is concerned.

To see this, consider the following: $P_{av} = V_{rms}I_{rms} = \frac{\text{work done}}{\text{time}}$

If the load is reactive (ie., its impedance has an imaginary component), then:

$P_{av} = V_{rms}I_{rms}\cos\varphi + jV_{rms}I_{rms}\sin\varphi$
The useful work done $V_{rms}I_{rms}\cos\varphi$ will then be less than the work done if the circuit were purely resistive.

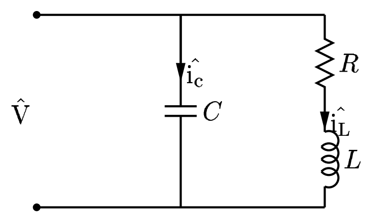
Recall the angle φ in the p.f. represents the phase shift between the voltage and current which happens only in a resistive circuit. Hence reactive components tend to reduce the amount of power available to do work, the remaining power continuously exchanged between the source and the reactive components!
Consequently power utility companies have to generate more power to supply the same useful average power to a customer with a p.f. < 1 than would be required if the p.f. were 1. As a result, power companies penalise industrial users of power whose p.f. is low.

However, customers with reactive loads can compensate in order to raise their power factor — ideally to unity. This procedure is called power factor correction.

8.11.6 Power factor correction

The above discussion suggests it is important to raise the power factor of the load. Normally, the overall reactive component of an industrial load is inductive, since those loads consist of many motors, which are by their nature inductive (ie., motors consist internally of multi-turn coils or wire).

Here we will examine how to correct the power factor of an inductive load. This is best done by connecting a capacitance parallel with the load.



From K.C.L.:

$\hat{\mathrm{i}} = \hat{\mathrm{i}}_{\mathrm{c}} + \hat{\mathrm{i}}_{\mathrm{L}}$

Also:

$\hat{\mathrm{i}}_{\mathrm{c}} = j\omega C\hat{\mathrm{V}}$

$\hat{\mathrm{i}}_{\mathrm{L}} = \dfrac{\hat{\mathrm{V}}}{R + j\omega L}$

(Multiply by complex conjugate of the denominator)

$= \hat{\mathrm{V}}\left\{\dfrac{R}{R^2+\omega^2L^2} - j\dfrac{\omega^2}{R^2+\omega^2L^2}\right\}$

Hence

$\hat{\mathrm{I}} = \dfrac{R}{R^2+\omega^2L^2}\hat{\mathrm{V}} + j\left(\omega C - \dfrac{\omega L}{R^2+\omega^2L^2}\right)\hat{\mathrm{V}}$

If capacitance C is chosen to be $C = \dfrac{L}{R^2+\omega^2L^2}$, then

$\hat{\mathrm{I}} = \dfrac{R}{R^2+\omega^2L^2}\hat{\mathrm{V}}$

Ie., there is no phase shift and the power factor is now 1.

Note We connected the capacitor in parallel with the load since if we connected it in series, the voltage across the load would have been affected.

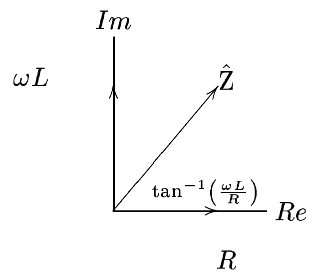
8.12 Phasor Diagrams

Phasors, being complex numbers, can be represented on the complex plane. It is customary to represent them as vectors in the complex plane.

Example Consider $\hat{\mathrm{Z}} = R + j\omega L$

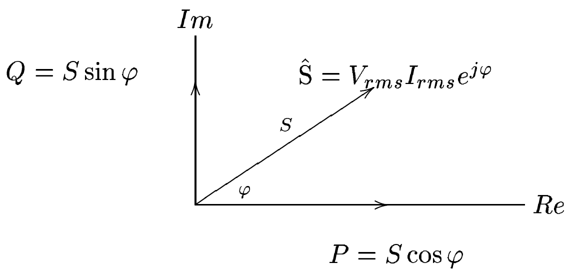
Then $\hat{\mathrm{Z}} = |Z|e^{j\tan^{-1}\left(\frac{\omega L}{R}\right)}$, $\hat{\mathrm{R}} = R\angle 0°, \hat{\mathrm{x}}_{\mathrm{L}} = \omega L$.

Phasor diagram:



8.12.1 The Power Triangle

The power triangle is given via the representation of the phasor $\hat{\mathrm{S}}$ and its real reactive components on the complex plane.



Example The potential difference across and the current through a circuit are represented by $100 + j200$V and $10 + j5$A respectively. Calculate the active and reactive power.

Active power	$= (100 \cdot 10) + (200 \cdot 5) = 2000$ W
Reactive power	$= (200 \cdot 10) + (100 \cdot 5) = 1500$ VAR

Alternatively:

$100 + j200 = 223.6\angle 63°26'$ V

$10 + j5 = 11.18\angle 26°34'$ A

Hence the phase difference is

$63°26' - 26°34' = 36°52'$

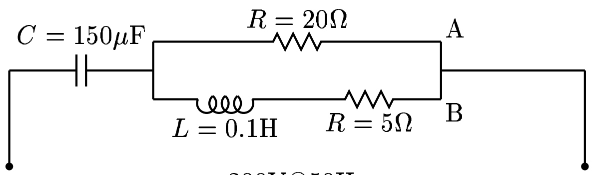
p.f. $\cos(36°52')$

Hence

Active power $= 223.6 \cdot 11.18 \cos(36°52') = 2$ kW

Reactive power $= 223.6 \cdot 11.18 \sin(36°52') = 1.5$ kVAR

Example A network is arranged as indicated. Calculate the value of the current in each branch and its phase with respect to the supply voltage, and draw the complete phasor diagram.



200V@50Hz

Solution:

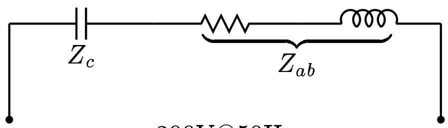
$Z_a = 20 + j0\Omega$

$Z_b = 5 + j314 \cdot 0.1 = 5 + j31.4\Omega$ (Note that 50 Hz = 314 rads^{-1}

Then $Z_{ab} = 13.78 + j7.8$

$Z_c = -j\dfrac{10^6}{314 \cdot 150} = -j21.2\Omega$

Hence the circuit can be replaced by the following equivalent:



200V@50Hz

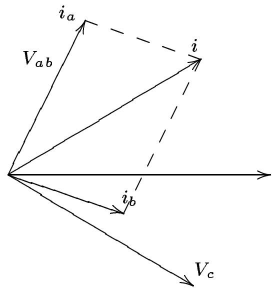
The total impedance $Z = 13.78 + j7.8 - j212.2 = 13.78 - j13.4 = 19.22\angle - 44°12'\Omega$. If the supply voltage $V = 200\angle 0°$V then $I = \dfrac{200\angle 0°}{19.22\angle - 44°12'} = 10.4\angle 44°12'$ A. Ie., the supply current is leading the supply voltage by $44°12'$. However, $Z_{ab} = 13.78 + j7.8 = 15.85\angle 29°30'\Omega$, so $V_{ab} = 10.4\angle 44°12' \cdot 15.85\angle 29°30' = 164.8\angle 73°42'$ V.

$Z_{ab} = 20 + j0 = 20\angle 0°\Omega$ so $I_a = \dfrac{164.8\angle 73°42'}{20\angle 0°} = 8.24\angle 73°.42'$ A. Ie., the current through branch A is 8.24 A *leading* the supply voltage by $73°42'$!

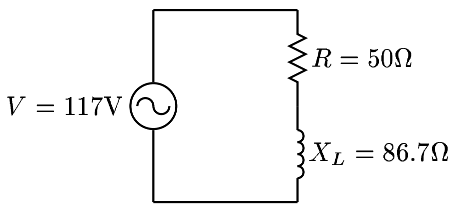
Similarly for $Z_b = 5 + j31.4 = 31.8\angle 80°58'\Omega$, $I_b = \dfrac{164.8\angle 73°42'}{31.8\angle 80°58'} = 5.18\angle - 7°16'$ A. Ie., the current through branch B is 5.18 A *lagging* the supply voltage by $7°16'$.

The impedance of C $Z_c = -j21.2 = 21.2\angle -90°\Omega$ so the potential difference across C $V_c = I_c Z_c = 10.4\angle 44°12' \cdot 21.2\angle 90° = 220\angle - 45°48'$ V.

Phasor diagram:



Example Consider the following circuit:



Calculate:

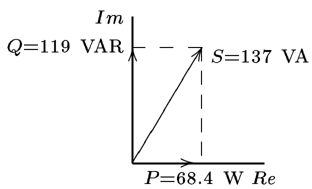
1. The power factor of the load
2. The real and reactive power — draw the power triangle
3. The value of a suitable reactance placed in parallel with the load to bring the power factor to unity

Solution:

1.
$Z_{tot} = R + jX_L = 50 + j86.7\Omega = 100\angle 60°$
Power factor $= \cos\varphi = \cos 60 = 0.5$ (lagging)

2.
Current drawn by the load: $I = \dfrac{V}{Z} = \dfrac{117\angle 0°}{100\angle 60°} = 1.17\angle - 60°$ A.
Hence the load power is $117 \cdot 1.17 \cos 60° = 68.4$ W
Reactive power is $117 \cdot 1.17 \sin 60° = 119$ VAC

Power triangle:



3.
Connecting a capacitor in parallel with the load which uses 119 VAR will correct the power factor to unity. The value of the capacitor is:

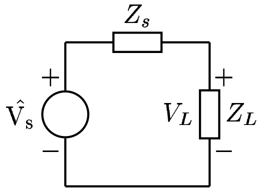
$$C = \frac{L}{R^2 + \omega^2 L^2} = \frac{0.23}{50^2 + 86.7^2} = 23\ \mu\text{F}.$$

Alternatively:

If C uses 119 VAR, then: $X_L = \dfrac{V^2}{Q} = \dfrac{117^2}{119} = 115\Omega \therefore C = \dfrac{1}{\omega X_L} = \dfrac{1}{377 \cdot 115} = 23\ \mu\text{F}.$

8.13 The maximum power tranfer theorem

Consider the following circuit:



$$\hat{I}_L = \frac{\hat{V}}{Z_s + Z_L} \therefore \hat{I}_L = \frac{\hat{V}_s^*}{(Z_s + Z_L)^*}$$

$$\hat{V}_L = \frac{\hat{V}_s Z_L}{Z_s + Z_L}$$

$$P_{av} = \mathrm{Re}\left\{\tfrac{1}{2}\hat{V}_L\hat{I}_L^*\right\} = \tfrac{1}{2}|\hat{V}_s^2| \cdot \frac{\mathrm{Re}\{Z_L\}}{|Z_s + Z_L|^2}$$

$$P_{av} = \tfrac{1}{2}V_m^2 \frac{R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

Here P_{av} clearly achieves its maximum if $X_L = -X_s$ (for fixed R_L).

Now we find the value of R_L for which P_{av} achieves its maximum:

$$P_{av} = \tfrac{1}{2}V_m^2 \frac{R_L}{(R_s + R_L)^2}$$

$$\frac{\mathrm{d}P_{av}}{\mathrm{d}R_L} = \tfrac{1}{2} \frac{V_m^2(R_s + R_L)^2 - 2R_L(R_s + R_L)}{(R_s + R_L)^4} = \tfrac{1}{2}V_m \frac{R_s - R_L}{(R_s + R_L)^3}$$

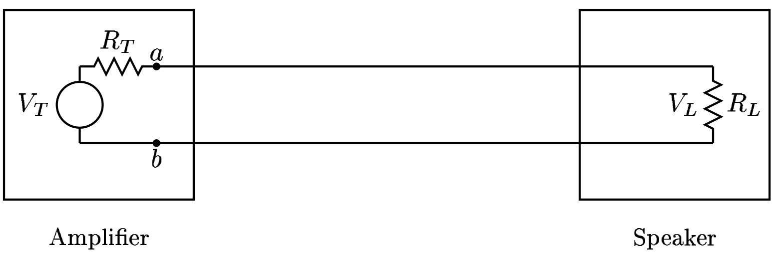
Here P_{av} achieves its maximum at $R_L = R_s$.

Combining these two results: Maximum power transfer occurs when:

$Z_L = Z_s^*$

The process whereby R_L is chosen to be equal to R_s and so effect maximum power transfer is known as *impedance matching.*

Example A modal of a stereo amplifier and a speaker is shown in the following diagram:



Suppose, not knowing that the internal resistance of the amplifier $R_T = 8\Omega$, we have connected a mismatched speaker with $R_L = 16\Omega$. How much more power could be delivered to the speaker if it's resistance were matched with the amplifier?

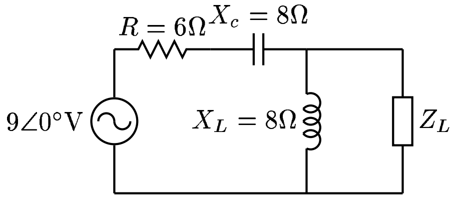
Solution:
For the 16Ω speaker: $V_L = \dfrac{R_L}{R_L + R_T} = \tfrac{2}{3}V_T$, $P_T = \dfrac{V_L^2}{R_L} = \dfrac{4}{9R_L}V_T^2 = (0.0278)V_T^2$

If the amplifier and speaker were matched, then:

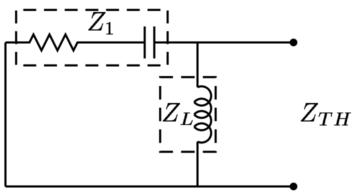
$R_L = R_T \therefore V_L = \dfrac{V_T}{2}$
$P_{L,matched} = \dfrac{V_T^2}{4R_L} = 0.03125V_T^2$

Hence the power increase with impedance matching would be $\dfrac{0.03125 - 0.0278}{0.08125} = 12.5\%$.

Example Find the load impedance for maximum power transfer to the load:



Use Thévnin's Theorem:



$Z_1 = R - jX_c = 6 - j8 = 10\angle - 53°13'$
$Z_L = jX_L = j8$
$\therefore Z_{TH} = \dfrac{Z_1 Z_2}{Z_1 + Z_2} = \dfrac{(10\angle - 53°13')(8\angle 90°)}{6 - j8 + j8} = 10.66 + j8$
$Z_L = 13.3\angle - 36°87' = 10.66 - j8$

$V_{TH} = V_L,\ V_{TH} = \dfrac{Z_L V_s}{Z_L + Z_1} = 12\angle 90°$
$P_{max} = \dfrac{V_{TH}^2}{4\Omega} = 3.38$ W.

8.14 Resonant Circuits

In our analysis of the maximum power transfer theorem we saw for final R_s and R_L that maximum power was transferred from an ideal source when the power factor of the load was unity. In our analysis of the concept of the power factor, we noted that at p.f. = 1, in a reactive circuit, power oscillates between the inductive and capacitive components only — ie., there is no exchange of power with the source.

When the power factor is unity, a reactive circuit is said to *resonate*. Since reactance (and hence p.f.) is a function of frequency, it is clear that there is a particular frequency corresponding to this resonance condition: the resonant frequency. This feature of resonant circuits is very useful in electronic engineering, signal processing and telecommunications.

It can be used to remove particular unwanted frequencies from a signal or it can be used to select a signal transmitting information at a specific frequency from other signals being transmitted at other frequencies.

8.14.1 Background

When an electromagnetic wave impinges on a metal, it induces an alternating electric current on the surface of the metal. Hence given a piece of metal, different frequency alternating electric currents are flowing on its surface due to the E.M. radiation present in the environment. Ie., the metal now acts as a source. Were we to connect the source to a resonant circuit, the power oscillating between L and C will be a maximum at the resonant frequency.

8.14.2 The series resonant circuit

Consider the following (RLC) circuit:

where the resistor represents losses (mostly in the inductor).

The impedance of the circuit is a minimum at resonance (p.f. = 1):
$Z_T = R + j(X_L - X_c)$

Hence series resonancewill occur when $X_L = X_c$. The total impedance at resonance is then:

$Z_T = R$

Hence the current through and voltage across the circuit are in phase — ie., the power factor is unity at resonance.

Resonant frequency (series circuit)

The resonant frequency is derived from the condition for resonance:
$Z_L = Z_c$, $X_L = X_c$, $\omega L = \dfrac{1}{\omega C}$
Hence the resonant frequency $\omega_s = \dfrac{1}{\sqrt{LC}}$ rads^{-1} or $f_s = \dfrac{1}{2\pi\sqrt{LC}}$ Hz.

8.14.3 The Quality Factor Q

Recall from the maximum power transfer theorem that maximum power is transferred when $Z_s = Z_L$. Since E.M. radiation acts as an ideal source (ie., $Z_s = 0$), maximum power will be transferred when $Z_r = 0$.

Due to resistive loses in the antenna, the capacitor and most notably in the inductor, there exists some resistance in series with L and C, which serves to reduce to power transferred to the resonant components.

The measure of the quality of a resonant circuit is a measure of its resistance — the lower the better. The quality factor is defined as:

$Q_s = \dfrac{\text{reactivepower}}{\text{realpower}}$ — at resonance.

Substituting for an inductive reactance:

$Q_s = \dfrac{i^2 X_L}{i^2 R} \therefore Q_s = \dfrac{\omega_s L}{R}$

or for a capacitive reactance:

$Q_s = \frac{X_C}{R} = \frac{1}{\omega_s LCR}$

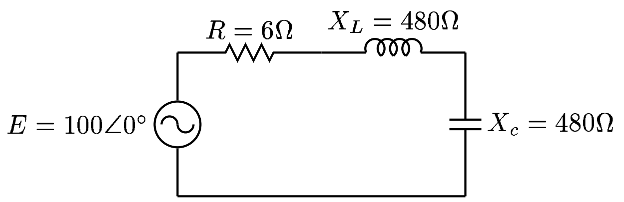
Note The inductive and capacitive reactances are equal at resonance. But $\omega_s = \dfrac{1}{\sqrt{LC}} \therefore Q_s = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$

Voltages across L and C at resonance

Inductor $\quad V_L = \frac{X_L E}{Z_T} = Q_s E$

Capacitor $\quad V_c = \frac{X_c E}{Z_T} = Q_s E$

Example Determine the voltage across the inductor and capacitors in the following circuit:



$Q_s = \frac{X_c}{R} = 80$ — This is a high–Q circuit!
$V_L = V_c = Q_s E = 80 \cdot 1000 = 8000$ V

This is a potentially lethal voltage.

8.14.4 Impedance vs. Frequency

The total impedance of a series RLC circuit is:

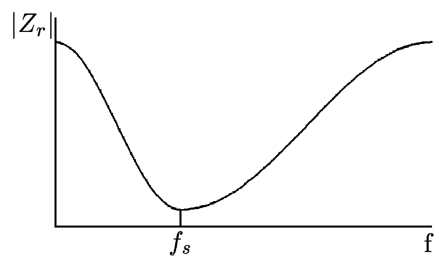
$Z_T = R + j(X_L - X_c)$

The magnitude of the impedance vs. frequency is determined from:

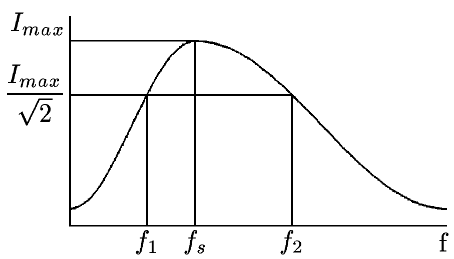
$|Z_r(f)| = \sqrt{R^2(f) + \{X_L(f) - X_c(f)\}^2}$

Where $\quad X_L = 2\pi f L$
$\qquad\quad\; X_c = 2\pi f C$

Plot of $|Z_r(f)|$ vs. frequency f:



Ie., in a series resonant the total impedance of the circuit is a minimum (but not zero!) at resonance. Hence for a fixed E, the current i through the circuit varies like:



From the plot of i vs. f we note there is a range of frequencies about f_s where the current is near its maximum. These frequencies (f_1, f_2), corresponding to $\frac{1}{\sqrt{2}}$ of the maximum current, are called the band frequencies, cut-off frequencies or half-power frequencies.
The range of frequencies between the two is referred to as the bandwidth of the circuit. The greater Q_s, the smaller the circuit bandwidth — ie., the circuit becomes more selective.

Note that the curve need not be symmetrical. For $Q_s \geq 10$, the curve is approximately symmetrical about the axes defined by the resonant frequency.

At the half-power frequency, the current is $\frac{1}{\sqrt{2}}$ times its maximum value, which implies the magnitude of the circuit impedance is $\sqrt{2}$ times its resonant value:

$Z_T = \sqrt{2}R$
$\therefore \sqrt{2}R = \sqrt{R^2 + (X_L - X_c)^2}$
$2R^2 = R^2 + (X_L - X_c)^2$
$R = X_L - X_c$ — at resonance.

Off resonance, there are two possibilities. $X_L > X_c$ (which corresponds to f_2) and $X_L < X_c$. In the former case:

$$\omega_L^2 - \frac{R}{L}\omega_L - \frac{1}{LC} = 0$$

Solving this quadratic gives:

$$f_2 = \tfrac{1}{2\pi}\left(\frac{R}{2L} + \tfrac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right) \text{ Hz}$$

By similar analysis it can be shown with $X_c > X_L$:

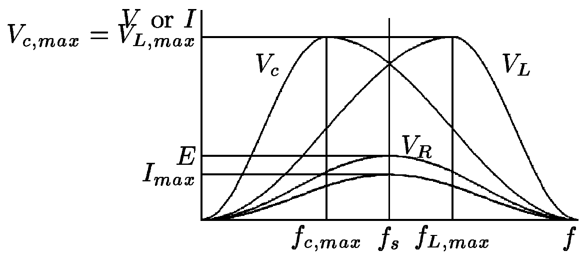
$$f_1 = \tfrac{1}{2\pi}\left(\frac{-R}{2L} + \tfrac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right) \text{ Hz}$$

Hence the bandwidth $f_2 - f_1 = \dfrac{R}{2\pi L}$ Hz. But $Q_s = \dfrac{\omega_s L}{R}$, so:

Bandwidth $= \dfrac{f_s}{Q_s}$. (*see* Boysestad for more details)

8.14.5 Variation of V_R, V_L and V_c with frequency in a series RLC circuit

From previous theory it follows that the variation of component voltages with frequency varies like:



(Note that $f_{c,max} \neq f_1$ and $f_{L,max} \neq f_2$). V_L and V_c are not centered at f_s due to losses in L and C.

Chapter 9

Transient Analysis of RC and RL circuits

Transient analysis is the analysis of the behaviour of a system in the time immediately following a change in its circumstance.
Transient analysis is what takes place before steady state analysis — which we have covered.

Example We push a rock over a cliff. The transient behaviour of the rock is its fall, break-up below and rolling to rest. The steady state response is its behaviour after it has settled.

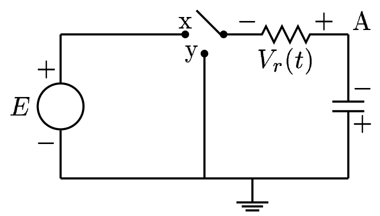
In electric circuits, transient analysis refers to the behaviour of electric circuits when the voltage or current through the circuit is suddenly changed due to an external influence. We shall examine the response of RC and RL circuits to a pulse from a voltage source. This is equivalent to analyzing a circuit when the DC power has been switched in.

To perform transient analysis on RC and RL circuits, it is necessary to use differential equations, since the behaviour of reactive components are so defined.
The behavriour of currents and voltages in RC and RL circuits take the form of 1^{st} order integral and differential equations and are as a result referred to as 1^{st} order circuits.

The behaviour of i and V in an RLC circuit is described using 2^{nd} order differential equations — hence these circuits are referred to as 2^{nd} order circuits.

9.1 Transient analysis of an RC circuit

Consider the following setup:



At time $t = 0$, the switch is set to position x and current flows in the RC loop, charging the capacitor C in the process until C becomes fully charged, such that the potential difference across it is E and current stops flowing — ie., steady state has been reached.

Note Steady state does not necessarily imply a system stops functioning. If we used a sinusoidal forcing function above, the transient behaviour would be sinusoidal plus decaying exponential, and the steady state condition would be *sinusoidal*.

9.1.1 Charging phase of an RC circuit

Apply K.C.L. at node A at $t > 0$ giving:

$$C\frac{\mathrm{d}V_c(t)}{\mathrm{d}t} + \frac{V_c(t) - E}{R} = 0$$
$$\frac{\mathrm{d}V_c(t)}{\mathrm{d}t} + \frac{V_c(t)}{RC} = \frac{E}{RC} \quad (1^{st} \text{ order differential equation})$$

The initial condition: $V_c(t) = 0$ at $t = 0$

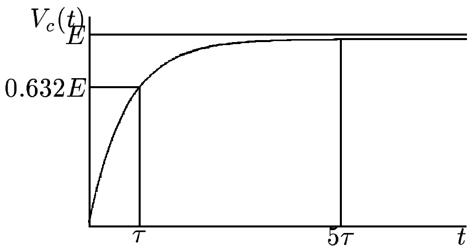
This will give us the particular solution we require. We can use the integrating factor method to find $V_c(t)$:

$$\frac{\mathrm{d}V_c(t)}{\mathrm{d}t} + \frac{V_c(t)}{RC} = \frac{E}{RC}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = Q(x)$$
$$y(x) = \frac{\int \mu(x)Q(x)\mathrm{d}x + k}{\mu(x)}; k \in \mathbb{R}$$

where $\mu(x) = e^{\int (p(x)\mathrm{d}x}$

Hence the general solution to the differential equation is:

$$V_c(t) = \frac{\int e^{\int \left(\frac{t}{RC}\right)\mathrm{d}t}\frac{E}{RC}\mathrm{d}t + k}{e^{\left(\frac{t}{RC}\right)\mathrm{d}t}} = \frac{\int e^{\left(\frac{t}{RC}\right)}\frac{E}{RC}\mathrm{d}t + k}{e^{\left(\frac{t}{RC}\right)}}; k \in \mathbb{R}$$
$$\therefore V_c(t) = E\left\{1 - e^{-\left(\frac{t}{RC}\right)}\right\} \text{ where } V_c(t) = 0, t = 0$$



$\tau = RC$ is referred to as the time constant or 'rise time' of the circuit. At $t = \tau$, the voltage across the capacitor has reached 63.2% of its final value. To see this, substitute $t = RC$ into the equation for $V_c(t)$:

$$V_c(RC) = E\left\{1 - e^{-\left(\frac{RC}{RC}\right)}\right\} = E\{0.632\}$$

For $t \geq 5\tau$, the capacitor is effectively fully charged (having reached 99.3% of its final value). Ie., for $t \geq 5\tau$, the circuit is in steady state. τ gives an indication of the speed at which the capacitor charges in an RC circuit.

By varying the value of τ, we note that a smaller τ corresponds to a more reapid charging of the capacitor — and vice versa. This effect is easily explained since clearly the larger C, the greater the time needed to charge it fully. The larger R implies the same since R limits the current available to C for charging.

The current through the capacitor is given by:

$$i_c(t) = C\frac{\mathrm{d}V_c(t)}{\mathrm{d}t} = C\frac{\mathrm{d}E\left\{1 - e^{-\left(\frac{t}{RC}\right)}\right\}}{\mathrm{d}t}$$

$\therefore i_c(t) = \frac{E}{R} e^{-\left(\frac{t}{RC}\right)}$

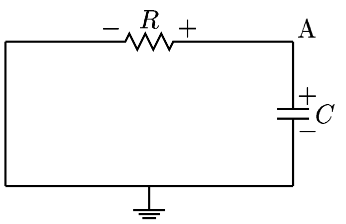
Then the plot of $i_c(t)$ would be the mirror image of the plot of $V(t)$. At $t = \tau = RC$, the current is at 36.8% of its initial value (ie., it has dropped 63.2% from its initial value). At $t \geq 5\tau$, the current has effectively stopped flowing (at 0.67% of its initial value).

Whoops — bit missing here. If somebody has the notes between this and the next subsection (Discharge phase of an RC circuit), please mail me.

9.1.2 Discharge phase of an RC circuit

We have examined the charging phase of the RC circuit. This corresponded to switching the the switch to position x above. Now we'll examine what occurs when the switch is set to position y.

Clearly, with the switch in position y the charged capacitor will discharge through R. The supply is now isolated and we are looking at the following circuit:



The initial condition is $V_c(t) = E, t = 0$. Apply K.C.L. at node A:

$$\frac{\mathrm{d}V_c(t)}{\mathrm{d}t} = -V_c(t)R$$
$$\frac{\mathrm{d}V_c(t)}{\mathrm{d}t} + \frac{V_c(t)}{RC} = 0$$

Integrating factor method $\left(\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = Q(x)\right)$:

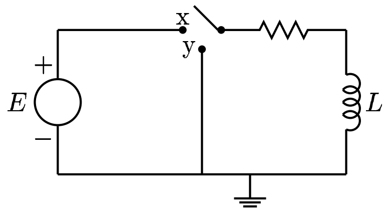
$Q(x) = 0 \therefore p(x) = \frac{-1}{RC}$

Then $V_c(t) = Re^{-\left(\frac{t}{RC}\right)} = Ee^{-\left(\frac{t}{RC}\right)}$ (as $V_c(t) = E, t = 0$)

Since there is now no applied voltage clearly $V_r(t) = -Ee^{-\left(\frac{t}{RC}\right)}$ and $i_c(t) = i_R(t) = \frac{-E}{R} e^{-\left(\frac{t}{RC}\right)}$

9.2 Transient analysis of an RL circuit

Consider the following setup:
At $t = 0$, close the switch.



9.2.1 Charging phase

Apply K.V.L. around the loop:

$$V_L(t) + V_r(t) = E$$
$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) = E$$

Initial condition $i(t) = 0, t = 0$

Solving usingthe integrating factor method gives:

$i(t) = \frac{E}{R}\left\{1 - e^{-\left(\frac{R}{L}\right)t}\right\}$ (*see* notes on differential equations).

Hence $V_L(t) = Ee^{-\left(\frac{R}{L}\right)t}$ $(V_L(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t})$

Clearly $V_r(t) = E - V_c(t) = E\left\{1 - e^{-\left(\frac{R}{L}\right)t}\right\}$

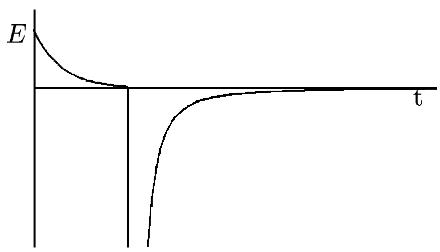
Define $\tau = \frac{L}{R}$ from the RL circuit and we note the same as before with RC circuits. Ie., $i(t)$ has reached about 63% of its total final value at $t = \tau$ and the circuit is in steady state at $t = 5\tau$.

9.2.2 Discharge Phase

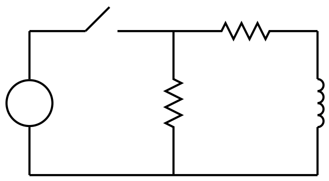
The discharge phase of an RL circuit is anomalous since moving the switch from position x to y will cause a discontinuity in the current. This has important implications for the inductor.

This means that a large voltage will be induced in the inductor, which will cause sparking at the switch at the moment the switch is moved from x to y.

This discharge of energy will result in the discharge phase of an RL circuit not appearing as the 'mirror-image' of the charging phase (which was the case with the RC circuit). Moving the switch to position y will give a discharge phase looking like:

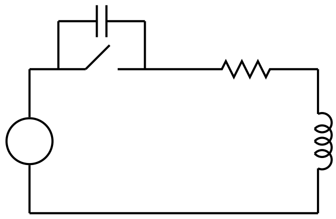


Sparking at the switch can be prevented by having the circuit set up as:



Since when the swtich is opened, the potential difference across R_2 will be E for a brief moment and so the transition of the current flow $\frac{E}{R} \to 0$ will not be instanteneous. This setup is described in Boylestad.

Also, a capacitor could be connected in parallel with the switch:



This serves to absorb the energy discharge on switching — thus preventing sparking.

Index

A.C., *see* Current, Alternating
Absolute value
  Of complex numbers, *see* Complex numbers, Modulus
Acceleration, 22
Ampère, 28
Analysis
  Mesh, 67
  Nodal, 70
  Transient, 82
Anode, 39
Atomic theory, 25
Atoms, 25
  Pentavalent, 40
  Trivalent, 40

Band
  Conduction, 39
  Valence, 39
Bandwidth, *see* Circuit, Bandwidth
Bilateral, 61
Binding
  Chemical, 26, 40

$\mathbb{C}$, 5
Capacitance, 46
  Unit of, *see* Farad (F)
Capacitive reactance, 50
Capacitors, 46
  In parallel, 48
  In series, 47
Carrier
  Majority, 40
  Minority, 40
Cathode, 39
Charge
  Electric, 27
Circuit
  Bandwidth, 80
  Planer, 67
  RC, 20
  Resonant, 78
    Series, 79
  Rise Time, 83
  RL, 19
  RLC, 16
  Thévnin's Equivalent, 65
  Time constant, 83
Complex numbers, 5
  Argument, 7
  Conjugate, 5
  Graphical representation, 5
  Modulus, 5
  Operation, 5
    In polar form, 7
  Polar representation, 7
  Rules, 6
Compounds, 26
Condition
  Boundary, *see* Initial condition
  Initial, 18
Conductor, 38
  Semi-$\sim$, *see* Semi-conductor
Conductors, 30
Coordinates
  As matrices, 11
  Polar, 7
Coulomb, 28
Current, 41
  Alternating, 46
  Direct, 46
  Displacement, 47
  Electric, 28

D.C., *see* Current, Direct
Derivative, 14
Diagram
  Energy Level $\sim$, 39
Dielectric, 38
Dielectrics, 30
Differential equations, 16
  Degree, 17
  Linear, 17
  Non-linear, 17
  Order, 17
  Solving
    Homogeneous first order (type II), 18
    Linear first order (type III), 19
    Seperate first order (type I), 17
  Terminology, 17
Differentiation, 13
  Rules, 14
Diodes, 53
  Light Emitting, 54

  Zener, 54
Displacement, 22
Doping, 40
Dynamic resistance
  Of diodes, 54

Electric Displacement Vector, 29
Electrons, 25
  Max. number of in a shell, 26
Energy, 25
  Forms of, 25
  Kinetic, 25
  Potential, 25
eV, 39

Farad (F), 47
Field
  Electric, 28
    Strength, 28
  Magnetic, 29
    Direction, 29
    Strength, 29
Filters
  With capacitors, 48
Flux
  Electric, 28
  Electric ~ Density ($\overrightarrow{\mathrm{D}}$), 30
  Magnetic, 29
  Magnetic ~ Density ($\overrightarrow{\mathrm{B}}$), 29
Force, 23
  Electromotive (e.m.f.), 28
Formula
  Euler's, 6
Frequency
  Band, 80
  Cut-off, *see* Frequency, Band
  Half-power, *see* Frequency, Band
  Resonant, 78
    Series circuit, 79
Functions
  Homogeneous, 17
  Seperable, 17
  Terminology, 17

Henry (H), 50
Holes, 39

Impedance
  Matching, 77
  Of a capacitor, 49
  Of a resistor, 45
Inductance, 50
  Unit of, *see* Henry (H)
Inductor, 50
Inductors
  In parallel, 52
  In series, 51
Insulators, 30
Integral
  Definite, 15
  Indefinite, 15
Integration, 14
  Techniques, 15
    Integration by parts, 15
    Substitution, 15
    Trigonometrix substitution, 15

K.C.L., *see* Law, Kirchoff's Current
K.V.L., *see* Law, Kirchoff's Voltage

Law
  Ampère's, 32
  Biot-Savart, *see* Gauss' Law II
  Coulomb's, 27
  Gauss' ~ I, 32
  Gauss' ~ II, 32
  Kirchoff's Current, 60
  Kirchoff's Voltage, 60
  Lorentz Force, 29
LED, *see* Diodes, Light Emitting
Linear (sources, elements), 61

Magnetic
  Charge, 30
  Material, 30
  Non-~ Material, 30
Magnitute
  Of complex numbers, *see* Complex numbers, Modulus
Mass, 22
Matrix, 10
  Addition, 11
    Properties, 11
  Co-factor, 13
  Compatible, 12
  Determinant, 13
  Diagonal, 10
  Equality, 11
  Identity matrix, 12
  Inverse, 12, 13
  Multiplication
    By a scalar, 11
    By another matrix, 12
    Properties, 11, 12
  Square, 10
  Subtraction, 11
  Transpose, 10
MCk system, 16
Modulus
  Of complex numbers, *see* Complex numbers, Modulus
Molecules, 26

Momentum, 23

n-type, *see* Semi-conductor, n-type
Network
    Active, 64
    Passive, 64
Neutrons, 25
Newton's Laws, 23
    I, 24
    II, 24
    III, 24
Numbers
    Imaginairy, 5

Part
    Imaginairy, 5
    Real, 5
Permeability
    Magnetic, 30
    Of free space, 30
    Relative, 30
Permittivity
    Electrical, 27
    Of free space, 27
    Relative, 27
Phase shift, 53
Phasor, 62
    Diagram, 74
Potential Difference, 28
Potential Divider, 44
Power, 71
    A.C., 71
    Active, 72
    Apparent, 72
    Complex, 72
    D.C., 71
    Electric, 41
    Factor, 72
        Correction, 73
    Imaginary, *see* Power, Reactive
    Reactive, 72
    Real, *see* Power, Active
    Triangle, 74
Power series, 6
Principle
    of conservation of energy, 25
    of conservation of momentum, 23
    Of superposition, 61
Protons, 25

Quality factor, 79

$\mathbb{R}$, 5
R.M.S., *see* Root Mean Square
Reactiance
    Inductance, 53
Resistance, 42
Resistors, 42
    Parallel, 43
    Series, 43
Resonance, 78
Right hand rule, 9
Rise time, *see* Circuit,Rise time
Root Mean Square, 71
    vs. Peak Value (for sinusoids), 72

Scalar, 8
Semi-conductor, 38
    Extrinsic, 40
    Intrinsic, 40
    n-type, 40
    Pure, 39
Solutions
    Explicit, 17
    Implicit, 17
Source
    Current
        Ideal, 56
    Transformation, 60
    Voltage
        Ideal, 56
Sub-atomic particles, 25

Theorem
    Maximum power transfer, 77
    Norton's, 65
    Thévnin's, 64
Transformers, 55
Triangular Inequality, 6
Trigonometric identity, 15

Vectors, 8
    Addition, 8
    As matrices, 11
    Equality, 10
    Graphical representation, 8
        In cartesian form, 9
        In polar form, 10
    Product
        Cross, *see* Dot product
        Dot, *see* Scalar product
        Properties, 9
        Scalar, 8
        Vector, 9
    Subtraction, 8
    Unit vector, 8
Velocity, 22
Volt, 28
Voltage, 41

Weber, 29
Weight, 23
Work, 25